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On Einstein–Kropina metrics ☆

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ABSTRACT

In this paper, a characteristic condition of Einstein–Kropina metrics is given. By the characteristic condition, we prove that a non-Riemannian Kropina metric $F = \frac{\alpha^2}{\beta}$ with constant Killing form β on an n-dimensional manifold M, $n \geqslant 2$, is an Einstein metric if and only if α is also an Einstein metric. By using the navigation data (h, W), it is proved that an n-dimensional ($n \geqslant 2$) Kropina metric $F = \frac{\alpha^2}{\beta}$ is Einstein if and only if the Riemannian metric h is Einstein and W is a unit Killing vector field with respect to h. Moreover, we show that every Einstein–Kropina metric must have vanishing S-curvature, and any conformal map between Einstein–Kropina metrics must be homothetic.

1. Introduction

Let F be a Finsler metric on an n-dimensional manifold M. F is called an Einstein metric with Einstein scalar σ if

$$Ric = \sigma F^2, \tag{1.1}$$

where $\sigma = \sigma(x)$ is a scalar function on M. In particular, F is said to be Ricci constant (resp. Ricci flat) if F satisfies (1.1) where $\sigma = \text{const}$ (resp. $\sigma = 0$).

Recently, some progress has been made on Finsler–Einstein metrics of (α, β) type. The (α, β)-metrics form an important class of Finsler metrics appearing iteratively in formulating Physics, Mechanics, Seismology, Biology, Control Theory, etc. (see [1,9,12]). Bao and Robles have shown that every Einstein–Randers metric of dimension $n(\geqslant 3)$ is necessarily Ricci constant. A 3-dimensional Randers metric is Einstein if and only if it is of constant flag curvature, see [3]. For every non-Randers (α, β)-metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ with a polynomial function $\phi(s)$ of degree greater than 2, Cheng has proved that it is an Einstein metric if and only if it is Ricci-flat [5].

The Kropina metric is an (α, β)-metric where $\phi(s) = 1/s$, i.e., $F = \alpha^2/\beta$, which was considered by Kropina firstly [7]. Such a metric is of physical interest in the sense that it describes the general dynamical system represented by a Lagrangian function (cf. [2]), although it has the singularity. Some recent progress on Kropina metrics has been made, e.g., see [9,12,13].

The purpose of this paper is to investigate Einstein–Kropina metrics $F = \frac{\alpha^2}{\beta}$, for which we shall restrict our consideration to the domain where $\beta = b_i(x)y^i > 0$. By using a complicated computation, we obtain the characteristic conditions of Einstein–Kropina metrics in Theorems 3.1 and 1.1, which generalize and improve the results of [10].

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For an (α, β)-metrics, the form β is said to be Killing (resp. closed) form if $r_{ij} = 0$ (resp. $s_{ij} = 0$). β is said to be a constant Killing form if it is a Killing form and has constant length with respect to α, equivalently $r_{ij} = 0$, $s_i = 0$. And accordingly, a vector field W in a Riemannian manifold (M, h) is said to be a constant Killing vector field if it is a Killing vector field and has constant length with respect to the Riemannian metric h.

For (α, β)-metrics with constant Killing form, by using the characteristic condition of Einstein–Kropina metrics, we have the following theorem.

Theorem 1.1. *Let $F = \frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric with constant Killing form β on an n-dimensional manifold M, $n \geqslant 2$. Then F is an Einstein metric if and only if α is also an Einstein metric. In this case, $\sigma = \frac{1}{4}\lambda b^2 \geqslant 0$, where $\lambda = \lambda(x)$ is the Einstein scalar of α. Moreover, F is Ricci constant when $n \geqslant 3$.*

Remark. Rezaei et al., also discussed Einstein–Kropina metrics with constant Killing form. Unfortunately, the computation and results in [10] are wrong. Theorem 1.1 is the corrected version of Theorem 4.6 and Corollary 4.9 of [10].

As is well known, a Finsler metric is of Randers type if and only if it is a solution of the navigation problem on a Riemannian manifold, see [4]. Inspired by this idea, we can prove that there is a one-to-one correspondence between a Kropina metric and a pair (h, W), where h is a Riemannian metric and W is a vector field on M with the length $\|W\|_h = 1$. And we call this pair (h, W) the navigation data of the Kropina metric (see Section 4 for details). The new perspective allows us to characterize Einstein–Kropina spaces as follows.

Theorem 1.2. *Let $F = \frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric on an n-dimensional manifold M, $n \geqslant 2$. Assume the pair (h, W) is it's navigation data. Then F is an Einstein metric if and only if h is an Einstein metric and W is a unit Killing vector field. In this case, $\sigma = \delta \geqslant 0$, where $\delta = \delta(x)$ is the Einstein scalar of h. Moreover, F is Ricci constant for $n \geqslant 3$.*

For the S-curvature with respect to the Busemann–Hausdorff volume form, we have the followings.

Theorem 1.3. *Every Einstein–Kropina metric $F = \frac{\alpha^2}{\beta}$ has vanishing S-curvature.*

Finally, we discuss conformal rigidity for Einstein–Kropina metrics.

Theorem 1.4. *Any conformal map between Einstein–Kropina spaces must be homothetic.*

The content of this paper is arranged as follows. In Section 2 we introduce essential curvatures of Finsler metrics, as well as notations and conventions. And we compute the Ricci curvature of Kropina metrics. The characterization of Einstein–Kropina metrics, i.e., Theorem 3.1, is obtained in Section 3. By using it, we obtain Theorem 1.1. And in Section 4 the navigation version of Theorem 3.1 (Theorem 1.2) is proved. In Section 5 we investigate the S-curvature of Kropina metrics and Theorem 1.3 is proved. In Section 6 the conformal rigidity for Einstein–Kropina metrics is given.

2. Ricci curvature of Kropina metrics

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficients of F, which are defined by

$$G^i := \frac{1}{4} g^{il}\left\{\left[F^2\right]_{x^k y^l} y^k - \left[F^2\right]_{x^l}\right\}.$$

For any $x \in M$ and $y \in T_xM \backslash \{0\}$, the Riemann curvature $\mathbf{R}_y := R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

Ricci curvature is the trace of the Riemann curvature, which is defined by

$$Ric := R^m{}_m.$$

By definition, an (α, β)-metric on M is expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a positive definite Riemannian metric, $\beta = b_i(x) y^i$ a 1-form. It is known that (α, β)-metric with $\|\beta_x\|_\alpha < b_0$ is a Finsler metric if and only if $\phi = \phi(s)$ is a positive smooth function on an open interval $(-b_0, b_0)$ satisfying the following condition:

$$\phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0, \quad \forall |s| \leqslant b < b_0,$$

see [6].

Let

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}=\frac{1}{2}(b_{i|j}-b_{j|i}),$$

where "|" denotes the covariant derivative with respect to the Levi-Civita connection of α. Denote

$$r^i{}_j:=a^{ik}r_{kj},\qquad r_j:=b^ir_{ij},\qquad r:=r_{ij}b^ib^j=b^jr_j,\qquad s^i{}_j:=a^{ik}s_{kj},\qquad s_j:=b^is_{ij},$$

where $(a^{ij}):=(a_{ij})^{-1}$ and $b^i:=a^{ij}b_j$. Denote $r^i:=a^{ij}r_j$, $s^i:=a^{ij}s_j$, $r_{i0}:=r_{ij}y^j$, $s_{i0}:=s_{ij}y^j$, $r_{00}:=r_{ij}y^iy^j$, $r_0:=r_iy^i$ and $s_0:=s_iy^i$.

Let G^i and $\bar{G}^i$ be the geodesic coefficients of F and α, respectively. Then we have the following lemma.

Lemma 2.1. *(See [8].) For an (α,β)-metric $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$, the geodesic coefficients G^i are given by*

$$G^i=\bar{G}^i+\alpha Qs^i{}_0+\Psi(r_{00}-2\alpha Qs_0)b^i+\frac{1}{\alpha}\Theta(r_{00}-2\alpha Qs_0)y^i,\tag{2.1}$$

where

$$Q:=\frac{\phi'}{\phi-s\phi'},\qquad \Psi:=\frac{\phi''}{2[\phi-s\phi'+(b^2-s^2)\phi'']},\qquad \Theta:=\frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi[\phi-s\phi'+(b^2-s^2)\phi'']}.$$

From now on we consider a special kind of (α,β)-metrics which is called Kropina-metric with the form

$$F=\alpha\phi(s),\qquad \phi(s):=s^{-1},\qquad s=\frac{\alpha}{\beta}.$$

Throughout the paper we shall restrict our consideration to the domain where $\beta=b_i(x)y^i>0$, so that $s>0$.

Now we get the Ricci curvature of Kropina metric by using Lemma 2.1.

Proposition 2.1. *For the Kropina metric $F=\frac{\alpha^2}{\beta}$, its geodesic coefficients are:*

$$G^i=\bar{G}^i-\frac{\alpha^2}{2\beta}s^i{}_0+\frac{1}{2b^2}\left(\frac{\alpha^2}{\beta}s_0+r_{00}\right)b^i-\frac{1}{b^2}\left(s_0+\frac{\beta}{\alpha^2}r_{00}\right)y^i.\tag{2.2}$$

Proof. By a direct computation, we can get (2.2) from (2.1). □

Proposition 2.2. *For the Kropina metric $F=\frac{\alpha^2}{\beta}$, the Ricci curvature of F is given by*

$$Ric=\overline{Ric}+T,\tag{2.3}$$

where $\overline{Ric}$ denotes the Ricci curvature of α, and

$$\begin{aligned}T={}&-\frac{\alpha^2}{b^4\beta}s_0r-\frac{r}{b^4}r_{00}+\frac{\alpha^2}{b^2\beta}b^ks_{0|k}+\frac{1}{b^2}b^kr_{00|k}+\frac{n-2}{b^2}s_{0|0}+\frac{n-1}{b^2\alpha^2}\beta r_{00|0}+\frac{1}{b^2}\left(\frac{\alpha^2}{\beta}s_0+r_{00}\right)r^k{}_k\\&-\frac{\alpha^2}{\beta}s^k{}_{0|k}-\frac{1}{b^2}r_{0|0}-\frac{2(2n-3)}{b^4}r_0s_0-\frac{n-2}{b^4}s_0^2-\frac{4(n-1)}{b^4\alpha^2}\beta r_{00}r_0+\frac{2(n-1)}{b^4\alpha^2}\beta r_{00}s_0+\frac{3(n-1)}{b^4\alpha^4}\beta^2r_{00}^2\\&+\frac{2n}{b^2}s^k{}_0r_{0k}+\frac{1}{b^4}r_0^2-\frac{\alpha^2}{b^2\beta}s^k{}_0r_k+\frac{n-1}{b^2\beta}\alpha^2s^k{}_0s_k-\frac{\alpha^4}{2b^2\beta^2}s^ks_k-\frac{\alpha^2}{b^2\beta}s^kr_{0k}-\frac{\alpha^4}{4\beta^2}s^j{}_ks^k{}_j.\end{aligned}\tag{2.4}$$

Proof. Let

$$T^i:=-\frac{\alpha^2}{2\beta}s^i{}_0+\frac{1}{2b^2}\left(\frac{\alpha^2}{\beta}s_0+r_{00}\right)b^i-\frac{1}{b^2}\left(s_0+\frac{\beta}{\alpha^2}r_{00}\right)y^i,$$

then

$$G^i=\bar{G}^i+T^i.$$

Thus the Ricci curvature of F is related to the Ricci curvature of α by

$$Ric=\overline{Ric}+2T^k{}_{|k}-y^jT^k{}_{.k|j}+2T^jT^k{}_{.j.k}-T^k{}_{.j}T^j{}_{.k},\tag{2.5}$$

where "|" and "." denote the horizontal covariant derivative and vertical covariant derivative with respect to the Berwald connection determined by $\bar{G}^i$ respectively.

Note that

$$\beta_{|k} = r_{0k} + s_{0k}, \qquad b^2{}_{|k} = 2(r_k + s_k), \qquad b^i{}_{|k} = r^i{}_k + s^i{}_k.$$

By a direct computation, we get

$$\begin{aligned}
2T^k{}_{|k} = {} & -\frac{2\alpha^2}{b^4\beta} s_0 r - \frac{2}{b^4} r_{00} r + \left(\frac{4}{b^4} - \frac{\alpha^2}{b^2\beta^2}\right) r_0 s_0 + \left(\frac{4}{b^4} + \frac{\alpha^2}{b^2\beta^2}\right) s_0^2 + \frac{4\beta}{b^4\alpha^2} r_{00} r_0 + \frac{4\beta}{b^4\alpha^2} r_{00} s_0 + \frac{\alpha^2}{b^2\beta} b^k s_{0|k} \\
& + \frac{1}{b^2} b^k r_{00|k} - \frac{2}{b^2} s_{0|0} - \frac{2\beta}{b^2\alpha^2} r_{00|0} + \frac{1}{b^2}\left(\frac{\alpha^2}{\beta} s_0 + r_{00}\right) r^k{}_k - \frac{2}{b^2\alpha^2} r_{00}^2 + \frac{\alpha^2}{\beta^2} s^k{}_0 (r_{0k} + s_{0k}) - \frac{\alpha^2}{\beta} s^k{}_{0|k}, \\
-y^j T^k{}_{.k|j} = {} & \frac{2}{b^4} r_0^2 - \frac{2(n-1)}{b^4} r_0 s_0 - \frac{1}{b^2} r_{0|0} - \frac{2(n+1)\beta}{b^4\alpha^2} r_{00} r_0 - \frac{2(n+1)\beta}{b^4\alpha^2} r_{00} s_0 \\
& + \frac{n+1}{b^2\alpha^2} r_{00}^2 + \frac{n+1}{b^2\alpha^2} \beta r_{00|0} - \frac{2n}{b^4} s_0^2 + \frac{n}{b^2} s_{0|0}, \\
2T^j T^k{}_{.j.k} = {} & \frac{1}{b^4}\left(\frac{\alpha^2}{\beta} s_0 + r_{00}\right) r - \frac{2(n+2)\beta}{b^4\alpha^2}\left(\frac{\alpha^2}{\beta} s_0 + r_{00}\right) r_0 - \frac{\alpha^2}{b^2\beta} s^k{}_0 r_k + \frac{2n}{b^4} s_0^2 \\
& + \frac{2(3n+2)\beta}{b^4\alpha^2} r_{00} s_0 + \left\{\frac{4(n+1)\beta^2}{b^4\alpha^4} - \frac{n+1}{b^2\alpha^2}\right\} r_{00}^2 + \frac{n\alpha^2}{b^2\beta} s^k{}_0 s_k + \frac{2(n+1)}{b^2} s^k{}_0 r_{0k}, \\
T^k{}_{.j} T^j{}_{.k} = {} & \left(\frac{\alpha^2}{b^2\beta^2} + \frac{n+2}{b^4}\right) s_0^2 + \frac{\alpha^2}{b^2\beta} s^k{}_0 s_k + \left(\frac{2}{b^2} + \frac{\alpha^2}{\beta^2}\right) r_{0k} s^k{}_0 + \frac{\alpha^2}{\beta^2} s_{0k} s^k{}_0 + \frac{2(n+4)}{b^4\alpha^2} \beta r_{00} s_0 \\
& - \left(\frac{\alpha^2}{b^2\beta^2} + \frac{4}{b^4}\right) r_0 s_0 + \frac{\alpha^4}{2b^2\beta^2} s^k s_k + \frac{1}{b^4} r_0^2 - \frac{6\beta}{b^4\alpha^2} r_{00} r_0 + \frac{\alpha^2}{b^2\beta} r_{0k} s^k \\
& + \left(-\frac{2}{b^2\alpha^2} + \frac{(n+7)\beta^2}{b^4\alpha^4}\right) r_{00}^2 + \frac{\alpha^4}{4\beta^2} s^i{}_j s^j{}_i.
\end{aligned}$$

Plugging all of these four terms into (2.5), we obtain (2.3). This completes the proof. □

Remark. For Riemann curvature and the Ricci curvature of (α,β)-metrics, Zhou gave some formulas in [15]. However, Cheng has corrected some errors of his formulas in [5]. To avoid making such mistakes, we use the definitions of Riemann curvature and Ricci curvatures to compute it.

From now on, "|" and "." denote the horizontal covariant derivative and vertical covariant derivative with respect to the Berwald connection determined by $\bar{G}^i$, respectively.

3. Equivalent equations of Einstein–Kropina metrics

The following lemma is necessary for the proof of theorems.

Lemma 3.1. *For (α,β)-metrics with $r_{00} = c(x)\alpha^2$, if α is an Einstein, i.e., $\overline{Ric} = \lambda(x)\alpha^2$ for some function $\lambda = \lambda(x)$, then the followings hold*

$$\begin{cases} s^i{}_{0|i} = (n-1)c_0 + \lambda\beta, \\ b^k s^i{}_{k|i} = (n-1)b^k c_k + \lambda b^2, \\ 0 = (n-1)b^k c_k + \lambda b^2 + s^k{}_{|k} + s^k{}_j s^j{}_k, \end{cases}$$

where $c_k := \frac{\partial c}{\partial x^k}$ and $c_0 := c_k y^k$.

Proof. Let β satisfy $r_{00} = c(x)\alpha^2$. Then

$$\begin{cases} b^j s^k{}_{j|i} = \left(b^j s^k{}_j\right)_{|i} - b^j{}_{|i} s^k{}_j = -s^k{}_{|i} - \left(r^j{}_i + s^j{}_i\right) s^k{}_j = -s^k{}_{|i} - c s^k{}_i - s^k{}_j s^j{}_i, \\ b^j s^k{}_{j|k} = -s^k{}_{|k} - s^k{}_j s^j{}_k. \end{cases} \tag{3.1}$$

Assume that α is an Einstein metric with Einstein scalar $\lambda(x)$. Since α is a Riemann metric, we have the Ricci identity, i.e., $b_{j|k|l} - b_{j|l|k} = b^s \bar{R}_{jskl}$, where $\bar{R}_{jskl}$ denotes the Riemann curvature of α. Contracting both sides of it with a^{jl}, we get

$$a^{jl}(b_{j|k|l}-b_{j|l|k})=b^l{}_{|k|l}-b^l{}_{|l|k}=\left(r^l{}_k+s^l{}_k\right)_{|l}-\left(r^l{}_l+s^l{}_l\right)_{|k}=-(n-1)c_k+s^l{}_{k|l}$$
$$=b^s a^{jl}\bar{R}_{jskl}=b^s\bar{R}_{sk}=\lambda b^s a_{sk}=\lambda b_k,$$

that is

$$s^l{}_{k|l}=(n-1)c_k+\lambda b_k. \tag{3.2}$$

This is equivalent to the following identity

$$s^k{}_{0|k}=(n-1)c_0+\lambda\beta.$$

Contracting (3.2) with b^k, we get $b^k s^l{}_{k|l}=(n-1)b^k c_k+\lambda b^2$. Comparing it with the second equation of (3.1), we obtain that

$$0=(n-1)b^k c_k+\lambda b^2+s^k{}_{|k}+s^k{}_j s^j{}_k.$$

This completes the proof. □

Using Proposition 2.2 and Lemma 3.1, we can obtain the necessary and sufficient conditions for Kropina metrics to be Einstein metrics.

Theorem 3.1. *Let $F=\frac{\alpha^2}{\beta}$ be the non-Riemann Kropina metric on an n-dimensional manifold M.*

1) *For $n=2$, F is an Einstein metric if and only if there exist scalar functions $c=c(x)$, $\lambda=\lambda(x)$ on M such that α and β satisfy the following equations*

$$\begin{cases} r_{00}=c\alpha^2,\\ \overline{Ric}=\lambda\alpha^2,\\ 0=\lambda b^2\beta-cs_0+b^k c_k\beta+b^k s_{0|k}-b^2 s^k{}_{0|k}+s^k{}_0 s_k. \end{cases} \tag{3.3}$$

2) *For $n\geqslant 3$, F is an Einstein metric if and only if there exist scalar functions $c=c(x)$, $f=f(x)$ on M such that α and β satisfy the following equations*

$$\begin{cases} r_{00}=c\alpha^2,\\ f\alpha^2=\overline{Ric}\,b^4+(n-2)\left\{b^2 s_{0|0}+b^2 c_0\beta-2c\beta s_0-s_0^2-c^2\beta^2\right\},\\ 0=\left\{(n-2)s_k s^k-b^2 s^k{}_{|k}-b^2 s^i{}_j s^j{}_i\right\}\beta+(n-3)b^2 cs_0+b^2 b^k s_{0|k}-b^4 s^k{}_{0|k}+(n-1)b^2 s^k{}_0 s_k, \end{cases} \tag{3.4}$$

where

$$f=-(n-2)b^2c^2-b^2b^k c_k+(n-2)s^k s_k-b^2 s^k{}_{|k}-b^2 s^i{}_j s^j{}_i. \tag{3.5}$$

In this case, $\sigma=-\frac{1}{2b^2}s^k s_k-\frac{1}{4}s^i{}_j s^j{}_i$ for $n\geqslant 2$.

Proof. Let $F=\frac{\alpha^2}{\beta}$ be an Einstein metric with Einstein scalar $\sigma(x)$. Multiplying both sides of (2.3) by $b^4\alpha^4\beta^2$ to remove the denominators, we provide the criterion for the Kropina metric to be an Einstein metric as follows

$$\begin{aligned}
0={}&3(n-1)\beta^4 r_{00}^2+(n-1)\left\{b^2 r_{00|0}-4r_{00}r_0+2r_{00}s_0\right\}\beta^3\alpha^2+\left\{\overline{Ric}\,b^4-r_{00}r+b^2 b^k r_{00|k}\right.\\
&\left.+(n-2)b^2 s_{0|0}+b^2 r_{00}r^k{}_k-b^2 r_{0|0}-(4n-6)r_0 s_0-(n-2)s_0^2+2nb^2 r_{0k}s^k{}_0+r_0^2\right\}\beta^2\alpha^4\\
&+\left\{-s_0 r+b^2 b^k s_{0|k}+b^2 s_0 r^k{}_k-b^4 s^k{}_{0|k}-b^2 s^k{}_0 r_k+(n-1)b^2 s^k{}_0 s_k-b^2 r_{0k}s^k\right\}\beta\alpha^6\\
&-b^2\left\{\frac{1}{2}s^k s_k+\frac{b^2}{4}s^i{}_j s^j{}_i+\sigma(x)b^2\right\}\alpha^8.
\end{aligned} \tag{3.6}$$

The above equation shows that α^2 divides $3(n-1)\beta^4 r_{00}^2$. Since α^2 is irreducible and β^5 can factor into linear terms, we have that α^2 divides r_{00}^2. Thus there exists a function $c(x)$ such that

$$r_{00}=c(x)\alpha^2, \tag{3.7}$$

which means that β is a conformal form with respect to α.

By (3.7), it is easy to get

$$\begin{cases} r_{00}=c\alpha^2,\quad r_{ij}=ca_{ij},\quad r_{0i}=cy_i,\quad r_i=cb_i,\quad r=cb^2,\quad r^i{}_j=c\delta^i{}_j,\\ r_{0k}s^k{}_0=0,\quad r_{0k}s^k=cs_0,\quad r_0=c\beta,\quad s^k{}_0 r_k=cs_0,\\ r_{00|k}=c_k\alpha^2,\quad r_{00|0}=c_0\alpha^2,\quad r^k{}_k=nc,\quad r_{0|0}=c_0\beta+c^2\alpha^2, \end{cases} \tag{3.8}$$

where $y_i:=a_{ij}y^j$.

Substituting all of these into (3.6) and dividing both sides by common factor α^4, we obtain

$$0=\overline{Ric}\,b^4\beta^2+(n-2)\{b^2s_{0|0}+b^2c_0\beta-2c\beta s_0-s_0^2-c^2\beta^2\}\beta^2+b^2\{(n-3)cs_0+(n-2)c^2\beta+b^kc_k\beta+b^ks_{0|k}-b^2s^k{}_{0|k}+(n-1)s^k{}_0s_k\}\beta\alpha^2-b^2\left\{\frac{1}{2}s^ks_k+\frac{b^2}{4}s^i{}_js^j{}_i+\sigma b^2\right\}\alpha^4. \tag{3.9}$$

Case I: $n=2$. (3.9) can be simplified as

$$0=\overline{Ric}\,b^2\beta^2+\{-cs_0+b^kc_k\beta+b^ks_{0|k}-b^2s^k{}_{0|k}+s^k{}_0s_k\}\beta\alpha^2-\left\{\frac{1}{2}s^ks_k+\frac{b^2}{4}s^i{}_js^j{}_i+\sigma b^2\right\}\alpha^4. \tag{3.10}$$

Thus there exists some function $\lambda=\lambda(x)$ such that

$$\overline{Ric}=\lambda\alpha^2, \tag{3.11}$$

i.e., α is an Einstein metric.

We plug (3.11) into (3.10). Then (3.10) is equivalent to

$$\begin{cases}\eta=\lambda b^2\beta-cs_0+b^kc_k\beta+b^ks_{0|k}-b^2s^k{}_{0|k}+s^k{}_0s_k,\\ 0=\beta\eta-\left\{\frac{1}{2}s^ks_k+\frac{b^2}{4}s^i{}_js^j{}_i+\sigma b^2\right\}\alpha^2.\end{cases} \tag{3.12}$$

From the second equation of (3.12), we know there exists some function $f=f(x)$ such that

$$\beta\eta=f\alpha^2, \tag{3.13}$$

where $f=\frac{1}{2}s^ks_k+\frac{b^2}{4}s^i{}_js^j{}_i+\sigma b^2$.

Now we consider (3.13) into two cases: 1) If $\eta=t\beta$ for some function $t=t(x)$ on M, then $tb_ib_j=fa_{ij}$. By the theory of matrix rank, we know that $t=f=0$. So $\eta=0$; 2) If $\eta\neq t\beta$ for any function $t=t(x)$ on M, then we just choose the suitable direction y, such that $\eta(y)=0$. For the positive definiteness of α, $\alpha(y)\neq0$, so we get $f=0$. All in all, $f=0$ and $\eta=0$.

Thus (3.12) is equivalent to

$$\begin{cases}0=\lambda b^2\beta-cs_0+b^kc_k\beta+b^ks_{0|k}-b^2s^k{}_{0|k}+s^k{}_0s_k,\\ \sigma=-\frac{1}{2b^2}s^ks_k-\frac{1}{4}s^i{}_js^j{}_i.\end{cases} \tag{3.14}$$

Conversely, if (3.3) holds, putting them into (2.3) yields $Ric=\sigma F^2$, where σ is given by the second equation of (3.14). Thus F is an Einstein metric.

Case II: $n\geqslant3$. From (3.9), we know there exists some function $f=f(x)$ such that

$$\overline{Ric}\,b^4+(n-2)\{b^2s_{0|0}+b^2c_0\beta-2c\beta s_0-s_0^2-c^2\beta^2\}=f\alpha^2. \tag{3.15}$$

Then (3.9) can be simplified as

$$0=\beta\{(n-3)b^2cs_0+(n-2)b^2c^2\beta+b^2b^kc_k\beta+b^2b^ks_{0|k}-b^4s^k{}_{0|k}+(n-1)b^2s^k{}_0s_k+f\beta\}-b^2\left\{\frac{1}{2}s^ks_k+\frac{b^2}{4}s^i{}_js^j{}_i+\sigma b^2\right\}\alpha^2. \tag{3.16}$$

Since α^2 can't be divided by β, we see that (3.16) is equivalent to the following equations

$$\begin{cases}0=(n-3)b^2cs_0+(n-2)b^2c^2\beta+b^2b^kc_k\beta+b^2b^ks_{0|k}-b^4s^k{}_{0|k}+(n-1)b^2s^k{}_0s_k+f\beta,\\ 0=\frac{1}{2}s^ks_k+\frac{b^2}{4}s^i{}_js^j{}_i+b^2\sigma.\end{cases} \tag{3.17}$$

Firstly, differentiating both sides of the first equation of (3.17) with respect to y^i yields

$$0=(n-3)b^2cs_i+(n-2)b^2c^2b_i+b^2b^kc_kb_i+b^2b^ks_{i|k}-b^4s^k{}_{i|k}+(n-1)b^2s^k{}_is_k+fb_i. \tag{3.18}$$

Contracting (3.18) with b^i gives

$$0=(n-2)b^4c^2+b^4b^kc_k-(n-2)b^2s^ks_k+b^4s^k{}_{|k}+b^4s^i{}_js^j{}_i+b^2f. \tag{3.19}$$

Removing the factor b^2 from (3.19), we obtain

$$f=-(n-2)b^2c^2-b^2b^kc_k+(n-2)s^ks_k-b^2s^k{}_{|k}-b^2s^i{}_js^j{}_i. \tag{3.20}$$

Plugging (3.20) into the first equation of (3.17) yields

$$0=\left\{(n-2)s_k s^k - b^2 s^k{}_{|k} - b^2 s^i{}_j s^j{}_i\right\}\beta + (n-3)b^2 c s_0 + b^2 b^k s_{0|k} - b^4 s^k{}_{0|k} + (n-1)b^2 s^k{}_0 s_k.$$

Secondly, by the second equation of (3.17), we obtain the Einstein scalar

$$\sigma = -\frac{1}{2b^2}s^k s_k - \frac{1}{4}s^i{}_j s^j{}_i. \tag{3.21}$$

Conversely, suppose (3.4) and (3.5) hold. Plugging them into (2.3), we conclude that F is an Einstein metric with Einstein scalar σ, which is given by (3.21). It completes the proof of Theorem 3.1. □

By Theorem 3.1, we can obtain Theorem 1.1, that is

Theorem 3.2. *Let $F=\frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric with constant Killing form β on an n-dimensional manifold M, $n\geqslant 2$. Then F is an Einstein metric if and only if α is also an Einstein metric. In this case, $\sigma=\frac{1}{4}\lambda b^2\geqslant 0$, where $\lambda=\lambda(x)$ is the Einstein scalar of α. Moreover, F is Ricci constant for $n\geqslant 3$.*

Proof. Assume that F is an Einstein metric. Substituting $r_{ij}=0$ and $s_i=0$ into (3.9) and removing the factor b^4, we get

$$0=\overline{Ric}\,\beta^2 - s^k{}_{0|k}\beta\alpha^2 - \left\{\frac{1}{4}s^i{}_j s^j{}_i + \sigma\right\}\alpha^4. \tag{3.22}$$

Thus $\overline{Ric}$ is divisible by α^2, i.e., there exists a function $\lambda(x)$ such that

$$\overline{Ric}=\lambda\alpha^2. \tag{3.23}$$

Putting (3.23) into (3.22) and dividing the common factor α^2, we conclude that

$$0=\left\{\lambda\beta - s^k{}_{0|k}\right\}\beta - \left\{\frac{1}{4}s^i{}_j s^j{}_i + \sigma\right\}\alpha^2. \tag{3.24}$$

By Lemma 3.1, we have $s^k{}_{0|k}=\lambda\beta$, $b^k s^i{}_{k|i}=\lambda b^2=-s^i{}_j s^j{}_i$. Thus (3.24) is equivalent to

$$\sigma=-\frac{1}{4}s^i{}_j s^j{}_i=\frac{1}{4}\lambda b^2. \tag{3.25}$$

For $\lambda b^2=b^k s^l{}_{k|l}=-s^i{}_j s^j{}_i=\|s_{ij}\|^2_\alpha\geqslant 0$, λ is nonnegative. Thus $\sigma=\frac{1}{4}\lambda b^2\geqslant 0$.

Conversely, assume $r_{ij}=s_i=0$ and α is an Einstein metric, i.e., $\overline{Ric}=\lambda(x)\alpha^2$. Then we have $s^k{}_{0|k}=\lambda\beta$, $b^k s^i{}_{k|i}=\lambda b^2=-s^i{}_j s^j{}_i$ by Lemma 3.1. Putting all of these and $r_{ij}=0$, $s_i=0$ into (2.3), we obtain $0=Ric-\sigma F^2$, where $\sigma=\frac{1}{4}\lambda b^2$. Hence F is an Einstein metric. It completes the proof of Theorem 3.2. □

Corollary 3.1. *Let $F=\frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric with $s_i=0$ on an n-dimensional manifold M, $n\geqslant 3$. If F and α are both Einstein metrics, then one of the followings holds*

1) *β is a constant Killing form. In this case, $\sigma=\frac{1}{4}\lambda b^2\geqslant 0$, where $\lambda=\lambda(x)$ denotes the Einstein scalar of α.*
2) *β is closed. In this case, $\sigma=0$, i.e., F is Ricci flat.*

Proof. Let $s_i=0$. Assume that Einstein scalars of α and F are λ and σ respectively, i.e., $\overline{Ric}=\lambda(x)\alpha^2$ and $Ric=\sigma(x)F^2$.

By Theorem 3.1, that F is an Einstein metric with $s_i=0$ is equivalent to

$$\begin{cases} r_{00}=c\alpha^2, \\ f\alpha^2=\lambda b^4\alpha^2+(n-2)\left\{b^2c_0-c^2\beta\right\}\beta, \\ 0=s^i{}_j s^j{}_i\beta+b^2 s^k{}_{0|k}, \\ f=-(n-2)b^2c^2-b^2b^kc_k-b^2s^i{}_js^j{}_i. \end{cases} \tag{3.26}$$

In this case, $\sigma=-\frac{1}{4}s^i{}_j s^j{}_i$.

For the same reason in discussing (3.13), the second equation of (3.26) is equivalent to

$$\begin{cases} b^2c_0 = c^2\beta, \\ f = \lambda b^4. \end{cases} \tag{3.27}$$

Differentiating both sides of the first equation of (3.27) by y^i yields

$$c^2 b_i = b^2 c_i. \tag{3.28}$$

Case I: $c(x) = \text{const}$. We have $c = 0$ by (3.28). So β is a constant Killing form. Thus by Theorem 1.1, we have $\sigma = \frac{1}{4}\lambda b^2 \geqslant 0$.
Case II: $c(x) \neq \text{const}$. We can rewrite (3.28) as

$$(c^{-1})_{|i} = -\frac{b_i}{b^2}.$$

So we have

$$(b^2c^{-1})_{|i} = (b^2)_{|i}c^{-1} + b^2(c^{-1})_{|i} = 2cb_ic^{-1} + b^2\left(-\frac{b_i}{b^2}\right) = b_i,$$

which means that $s_{ij} = 0$. Thus β is closed. From (3.26), we get $\sigma = 0$.
Note that by Lemma 3.1, the last two equations of (3.26) always hold. □

4. Kropina metrics through navigation description

In this section, we will algebraically derive an expression for F, and obtain another characterization of Einstein–Kropina metric.

Notice that we restrict our consideration to the domain where $\beta = b_i(x)y^i > 0$, which is equivalent to $W_0 = W_i(x)y^i > 0$. Let $h = \sqrt{h_{ij}(x)y^iy^j}$ be a Riemannian metric and $W = W^i\frac{\partial}{\partial x^i}$ a vector field on M. We can determine the Finsler metric $F = F(x, y)$ as follows

$$\left\|\frac{y}{F} - W\right\|_h = \sqrt{h_{ij}(x)\left(\frac{y^i}{F} - W^i\right)\left(\frac{y^j}{F} - W^j\right)} = 1.$$

It is equivalent to

$$\frac{h^2}{F^2} - 2\frac{W_0}{F} + \|W\|_h^2 = 1, \tag{4.1}$$

where $W_i := h_{ij}W^j$ and $W_0 := W_iy^i$.
Let $F = \frac{\alpha^2}{\beta}$. Solving (4.1) for h and W, we have that

$$0 = h^2\beta^2 - 2W_0\beta\alpha^2 + (\|W\|_h^2 - 1)\alpha^4. \tag{4.2}$$

Since $h^2\beta^2$ is divisible by α^2, we conclude that $h^2 = e^{2\rho}\alpha^2$ for some function $\rho = \rho(x)$ on M. Plugging it into (4.2) yields

$$0 = (e^{2\rho}\beta - 2W_0)\beta + (\|W\|_h^2 - 1)\alpha^2. \tag{4.3}$$

(4.3) is equivalent to

$$\begin{cases} \eta := e^{2\rho}\beta - 2W_0, \\ 0 = \eta\beta + (\|W\|_h^2 - 1)\alpha^2. \end{cases} \tag{4.4}$$

Now we consider second equation of (4.4) into two cases: 1) If $\eta = t\beta$ for some function $t = t(x)$ on M, then $tb_ib_j = (\|W\|_h^2 - 1)a_{ij}$. By the theory of matrix rank, we know that $t = \|W\|_h^2 - 1 = 0$. So $\eta = 0$; 2) If $\eta \neq t\beta$ for any function $t = t(x)$ on M, then we just choose the suitable direction y, such that $\eta(y) = 0$. For the positive definiteness of α, $\alpha(y) \neq 0$, so we get $\|W\|_h^2 - 1 = 0$. Above all, $\|W\|_h - 1 = 0$ and $\eta = 0$. So till now, we have

$$h_{ij} = e^{2\rho}a_{ij}, \qquad 2W_i = e^{2\rho}b_i \quad \text{and} \quad e^{2\rho}b^2 = 4. \tag{4.5}$$

Conversely, assume that $\|W\|_h = \sqrt{h_{ij}(x)W^iW^j} = 1$. Solving (4.1) for F, we obtain $F = \frac{h^2}{2W_0}$. Let $\alpha^2 = h^2$ and $\beta = 2W_0$. Thus $F = \frac{\alpha^2}{\beta}$ is a Kropina metric.

Hence, we obtain the following theorem.

Theorem 4.1. *A Finsler metric F is of Kropina type if and only if it solves the navigation problem on some Riemannian manifold (M,h), under the influence of a wind W with $\|W\|_h = 1$. Namely, $F = \frac{\alpha^2}{\beta}$ if and only if $F = \frac{h^2}{2W_0}$, where $h^2 = e^{2\rho}\alpha^2$, $2W_0 = e^{2\rho}\beta$ and $e^{2\rho}b^2 = 4$.*

And we call such a pair (h,W) the navigation data of the Kropina metric F.

Remark. Similar navigation idea for Kropina metrics appeared in [13], where they unnaturally assumed that $\|W\|_h = 1$. As stated in [4], the navigation description for Randers metrics is guaranteed by the condition $\|W\|_h < 1$. In a sense, the navigation idea for Kropina metrics may be considered to be the limiting case of Randers metrics, as $\|W\|_h$ approaches to 1.

In order to prove Theorem 1.2, we first need to reexpress the Einstein–Kropina characterization of Theorem 3.1 in terms of the navigation data (h,W). To that end, it is helpful to first relate the covariant derivative $b_{i|j}$ of b (with respect to α) to the covariant derivative $W_{i;j}$ of W (with respect to h).

Let

$$\mathcal{R}_{ij} := \frac{1}{2}(W_{i;j} + W_{j;i}), \qquad \mathcal{S}_{ij} := \frac{1}{2}(W_{i;j} - W_{j;i}),$$
$$\mathcal{S}^i{}_j := h^{ik}\mathcal{S}_{kj}, \qquad \mathcal{S}_j := W^i\mathcal{S}_{ij}, \qquad \mathcal{R}_j := W^i\mathcal{R}_{ij}, \qquad \mathcal{R} := \mathcal{R}_j W^j,$$

where “;” denotes the covariant differentiation with respect to h.

By conformal properties, we have followings

$$r_{ij} = 2e^{-2\rho}\big(\mathcal{R}_{ij} - W^k\rho_k h_{ij}\big), \tag{4.6}$$
$$s_{ij} = 2e^{-2\rho}(\mathcal{S}_{ij} + \rho_i W_j - \rho_j W_i), \tag{4.7}$$

where $\rho_i = \frac{\partial\rho}{\partial x^i}$.

Lemma 4.1. *$r_{00} = c(x)\alpha^2$ is equivalent to $\mathcal{R}_{ij} = 0$. In this case, $W^k\rho_k = -\frac{1}{2}c$.*

Proof. Firstly, assume that $r_{00} = c(x)\alpha^2$. It is equivalent to $r_{ij} = ca_{ij}$. Contracting both sides of it with b^ib^j, we have $r = cb^2$.

By the third equation of (4.5), we have

$$0 = b^2\rho_k + r_k + s_k. \tag{4.8}$$

Contracting (4.8) with b^k yields

$$0 = b^2\rho_k b^k + r = 2b^2\rho_k W^k + cb^2.$$

So $W^k\rho_k = -\frac{1}{2}c$.

Then plugging (4.6) into $r_{ij} = ca_{ij}$, we get

$$ce^{-2\rho}h_{ij} = 2e^{-2\rho}\big(\mathcal{R}_{ij} - W^k\rho_k h_{ij}\big) = 2e^{-2\rho}\left(\mathcal{R}_{ij} + \frac{1}{2}ch_{ij}\right). \tag{4.9}$$

Obviously $\mathcal{R}_{ij} = 0$.

Conversely, by $\mathcal{R}_{ij} = 0$ and (4.6), we have

$$r_{ij} = -2e^{-2\rho}W^k\rho_k h_{ij}.$$

That is $r_{ij} = ca_{ij}$, where $c = c(x) = -2W^k\rho_k$. This completes the proof. □

Theorem 4.2. *Let $F = \frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric on an n-dimensional manifold M, $n \geqslant 2$. Assume the pair (h,W) is it's navigation data. Then F is an Einstein metric if and only if h is an Einstein metric and W is a unit Killing vector field. In this case, $\sigma = \delta \geqslant 0$, where $\delta = \delta(x)$ is the Einstein scalar of h. Moreover, F is Ricci constant for $n \geqslant 3$.*

Proof. Now assume that $F = \frac{h^2}{2W_0}$ is an Einstein metric. Then $\frac{h^2}{W_0}$ is also an Einstein–Kropina metric. By Theorem 3.1, we have $r_{00} = c(x)\alpha^2$ for $n \geqslant 2$. Then by Lemma 4.1, $\mathcal{R}_{ij} = 0$ holds. So W_0 is a unit Killing form with respect to h. Thus for the Kropina metric $\frac{h^2}{W_0}$, we know that it is Einstein and W_0 is a unit Killing form. Then according to Theorem 3.2, we know h

is also an Einstein metric. Conversely, assume h is an Einstein metric and W is a unit Killing vector field. Then W_0 is a unit Killing form with respect to h. By Theorem 3.2, we get $\frac{h^2}{W_0}$ is an Einstein metric and so is $F = \frac{h^2}{2W_0}$.

By Theorem 3.2, we obtain that the Einstein scalar of $\frac{h^2}{W_0}$ is

$$\frac{1}{4}\delta\|W\|_h^2 = \frac{1}{4}\delta = -\frac{1}{4}\mathcal{S}^i{}_j\mathcal{S}^j{}_i = \frac{1}{4}\|\mathcal{S}_{ij}\|_h^2 \geqslant 0,$$

where $\delta = \delta(x)$ is the Einstein scalar of h. Thus the Einstein scalar of $F = \frac{h^2}{2W_0}$ is $\sigma = \delta \geqslant 0$. It completes the proof of theorem. □

By Theorem 4.2, we can construct a vast Einstein–Kropina metrics by their navigation expressions, i.e., Riemannian Einstein metrics and unit Killing vector fields. Let h be n-dimensional Riemannian space of constant curvature μ. Denote $h = \|dx\|^2/H^2$, where $H := 1 + \frac{\mu}{4}\|x\|^2$ and $\|\cdot\|^2$ is the standard metric in Euclidean space. Then the general solutions of Killing vector field W with respect to h are

$$W_i(x) = \frac{1}{H^2}\left\{\sum_j Q_{ij}x^j + c_i - \frac{1}{4}\mu\|x\|^2 c_i + \frac{1}{2}\left[\sum_k \mu c_k x^k\right]x^i\right\}, \tag{4.10}$$

where $Q_{ij} = -Q_{ji}$ and c_i are $\frac{1}{2}n(n+1)$ constants, see [11]. So there exist lots of unit Killing vector fields. We list a special case here.

Example 4.1. Let M be an 3-dimensional unit sphere with standard metric h. Let

$$Q = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \qquad (c_1, c_2, c_3) = \pm(c, -b, a),$$

where $a^2 + b^2 + c^2 = 1$ and a, b, c are all non-zero constants. Define $W = W^i\frac{\partial}{\partial x^i}$ with the same form as in (4.10), where $W_i = h_{ij}W^j$. Then $\|W\|_h = 1$. Define $F = \frac{h^2}{2W_0}$, where $W_0 = W_i y^i$ and $W_0 = W_i(x)y^i > 0$. Thus F is an Einstein–Kropina metric.

For Ricci flat Kropina metric, we have the following.

Corollary 4.1. *Let $F = \frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric on an n-dimensional manifold M, $n \geqslant 2$. If F is Ricci flat, then F is Berwald.*

Proof. Assume that F is Ricci-flat. By Theorem 4.2, we have $0 = \sigma = \delta = \|\mathcal{S}_{ij}\|_h^2$, which means that W_0 is closed. Thus W_0 is parallel with respect to h. So $G^i = \tilde{G}^i$, where $\tilde{G}^i$ denote the geodesic coefficients of h. Hence F is a Berwald metric. It completes the proof of Corollary 4.1. □

Finsler metrics, which are of constant flag curvature, are special cases of Einstein metrics. We have following results.

Corollary 4.2. *(See [14].) Let $F = \frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric on an n-dimensional manifold M, $n \geqslant 2$. F is of constant flag curvature K if and only if the following conditions hold*:

(1) *W is a unit Killing vector field.*
(2) *The Riemannian space (M, h) is of nonnegative constant curvature K.*

Proof. Suppose that F is of constant flag curvature K, i.e.,

$$R^i{}_k = K\left(F^2\delta^i{}_k - g_{ij}y^j y^k\right). \tag{4.11}$$

Then we have

$$Ric = \sigma F^2, \quad \sigma := (n-1)K = \text{const},$$

i.e., F is an Einstein metric. By Theorem 3.2, h is an Einstein metric, W_0 is a unit Killing form with respect to h and $\sigma = \delta \geqslant 0$, where $\delta = \delta(x)$ is the Einstein scalar of h. So $K \geqslant 0$.

By a direct computation, we can rewrite (4.11) as

$$K\frac{h^4}{4W_0^2}\left[\delta^i{}_k-\frac{2}{h^2}y^i\tilde{y}_k+\frac{y^iW_k}{W_0}\right]$$
$$=\tilde{R}^i{}_k-\frac{h^2}{W_0}\mathcal{S}^i{}_{0;k}+\frac{\tilde{y}_k}{W_0}\mathcal{S}^i{}_{0;0}-\frac{h^2}{2W_0^2}\mathcal{S}^i{}_{0;0}W_k+\frac{h^2}{2W_0}\mathcal{S}^i{}_{k;0}+\frac{h^2}{2W_0^2}\mathcal{S}^i{}_j\mathcal{S}^j{}_0\tilde{y}_k$$
$$-\frac{h^4}{4W_0^3}\mathcal{S}^i{}_j\mathcal{S}^j{}_0W_k-\frac{h^4}{4W_0^2}\mathcal{S}^i{}_j\mathcal{S}^j{}_k, \tag{4.12}$$

where $\tilde{y}_k:=h_{ik}y^i$. Multiplying both sides of (4.12) by $4W_0^3$ yields

$$0=4W_0^3\tilde{R}^i{}_k-4h^2W_0^2\mathcal{S}^i{}_{0;k}+4W_0^2\mathcal{S}^i{}_{0;0}\tilde{y}_k-2h^2W_0\mathcal{S}^i{}_{0;0}W_k+2h^2W_0^2\mathcal{S}^i{}_{k;0}+2h^2W_0\mathcal{S}^i{}_j\mathcal{S}^j{}_0\tilde{y}_k$$
$$-h^4\mathcal{S}^i{}_j\mathcal{S}^j{}_0W_k-h^4W_0\mathcal{S}^i{}_j\mathcal{S}^j{}_k-K\left[h^4W_0\delta^i{}_k-2h^2W_0y^i\tilde{y}_k+h^4y^iW_k\right]. \tag{4.13}$$

For division reason again, we can simplify (4.13) as

$$0=4W_0^2\tilde{R}^i{}_k-4h^2W_0\mathcal{S}^i{}_{0;k}+4W_0\mathcal{S}^i{}_{0;0}\tilde{y}_k-2h^2\mathcal{S}^i{}_{0;0}W_k+2h^2W_0\mathcal{S}^i{}_{k;0}+2Kh^2\left(W_0W^i-y^i\right)\tilde{y}_k$$
$$-Kh^4\left(W^iW_k-\delta^i{}_k\right)-Kh^2\left(h^2\delta^i{}_k-2y^i\tilde{y}_k\right)-Kh^4W^iW_k. \tag{4.14}$$

Contracting (4.14) with $\tilde{y}_i$ yields

$$\mathcal{S}_{k0;0}=K\left(W_0\tilde{y}_k-h^2W_k\right). \tag{4.15}$$

From it, we have

$$\begin{cases}\mathcal{S}^i{}_{0;k}=-\mathcal{S}^i{}_{k;0}-K\left(2W^i\tilde{y}_k-y^iW_k-W_0\delta^i{}_k\right),\\ \mathcal{S}^i{}_{0;0}=K\left(-h^2W^i+W_0y^i\right).\end{cases} \tag{4.16}$$

Plugging (4.15) and (4.16) into (4.14) yields

$$0=2W_0^2\tilde{R}^i{}_k+3h^2W_0\mathcal{S}^i{}_{k;0}+3Kh^2W_0W^i\tilde{y}_k-3Kh^2W_0y^iW_k-2Kh^2W_0^2\delta^i{}_k+2KW_0^2y^i\tilde{y}_k. \tag{4.17}$$

The above equation shows that h^2 divides $2W_0^2\tilde{R}^i{}_k-2Kh^2W_0^2\delta^i{}_k+2KW_0^2y^i\tilde{y}_k=2W_0^2(\tilde{R}^i{}_k-Kh^2\delta^i{}_k+Ky^i\tilde{y}_k)$. Thus there exists some function $d^i{}_k=d^i{}_k(x)$ on M such that

$$\tilde{R}^i{}_k-Kh^2\delta^i{}_k+Ky^i\tilde{y}_k=d^i{}_kh^2. \tag{4.18}$$

Contracting (4.18) with y^k yields $d^i{}_k=0$. Hence (4.18) can be simplified as $\tilde{R}^i{}_k=K(h^2\delta^i{}_k-y^i\tilde{y}_k)$, which means that h is of constant curvature K.

Converse is obvious. □

Remark. Yoshikawa et al. also studied Kropina metrics of constant flag curvature in terms of (h,W). Their computation is tedious. Corollary 4.2 is the revised version of Theorem 4 of [14], which does not restrict nonnegative constant curvature K.

5. *S*-curvature

Let (M,F) be an n-dimensional positive definite Finsler space, $n\geqslant 3$. Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for T_xM and $\{\theta^i\}_{i=1}^n$ the dual basis for T_x^*M. The Busemann–Hausdorff volume form is defined by

$$dV_F:=\sigma_F\theta^1\wedge\cdots\wedge\theta^n,$$

where

$$\sigma_F:=\frac{Vol(B^n(1))}{Vol\{(y^i)\in R^n\,|\,F(y^ie_i)<1\}},$$

Vol denotes the Euclidean volume and $Vol(B^n(1))$ denotes the Euclidean volume of the unit ball in $\mathbb{R}^n$. The Busemann–Hausdorff volume form dV_F determines a measure μ_{B-H} which is called the Busemann–Hausdorff measure.

Consider a Kropina norm $F=\frac{\alpha^2}{\beta}$ on M. We denote by $dV_F=\sigma_F\theta^1\wedge\cdots\wedge\theta^n$ and $dV_\alpha=\sigma_\alpha\theta^1\wedge\cdots\wedge\theta^n$ the volume forms of F and α, respectively. Let $\{e_i\}_{i=1}^n$ be an orthogonal basis for (T_xM,α). Thus $\sigma_\alpha=\sqrt{\det(a_{ij})}=1$. We may assume $\beta=by^1$. Then

$$\Omega := \{(y^i) \in R^n \mid F(y^i e_i) < 1\}$$

is a convex body in R^n and $\sigma_F := \frac{Vol(B^n(1))}{Vol(\Omega)}$. Ω is given by

$$\left\{\frac{2}{b}\left(y^1 - \frac{b}{2}\right)\right\}^2 + \sum_{\alpha=2}^{n}\left(\frac{2}{b}y^\alpha\right)^2 < 1.$$

Consider the following coordinate transformation $\psi : (y^i) \to (u^i)$

$$u^1 := \frac{2}{b}\left(y^1 - \frac{b}{2}\right), \qquad u^\alpha := \frac{2}{b}y^\alpha.$$

ψ sends Ω onto the unit ball $B^n(1)$ and the Jacobian of $\psi : (y^i) \to (u^i)$ is $(\frac{2}{b})^n$. Then

$$Vol(B^n(1)) = \int_{B^n(1)} du^1 \cdots du^n = \int_{\Omega} \left(\frac{2}{b}\right)^n dy^1 \cdots dy^n = \left(\frac{2}{b}\right)^n Vol(\Omega).$$

Thus

$$\sigma_F = \frac{Vol(B^n(1))}{Vol(\Omega)} = \left(\frac{2}{b}\right)^n.$$

Hence for a general basis $\{e_i\}_{i=1}^n$, we have

$$\sigma_F := \left(\frac{2}{b}\right)^n \sigma_\alpha, \quad \sigma_\alpha = \sqrt{\det(a_{ij})}.$$

Therefore

$$dV_F := \left(\frac{2}{b}\right)^n dV_\alpha.$$

Take an arbitrary standard local coordinate system (x^i, y^i). For a non-zero vector $y \in T_xM$, the distortion $\tau = \tau(x, y)$ is defined by

$$\tau := \ln \frac{\sqrt{g_{ij}(x, y)}}{\sigma_F(x)}.$$

F is Riemannian if and only if $\tau =$ constant. In general, τ is not a constant. However, it can be constant along any geodesic, but the Finsler metric is not Riemannian. Therefore, it is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM\backslash\{0\}$, let $c(t)$ be the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. The S-curvature S is defined by

$$S(x, y) := \frac{d}{dt}\left[\tau\left(c(t), \dot{c}(t)\right)\right]\Big|_{t=0}.$$

We can rewrite it as

$$S(x, y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial \ln \sigma_F}{\partial x^m}. \tag{5.1}$$

In this section we discuss the S-curvature with respect to the Busemann–Hausdorff volume measure μ_{B-H}.

Proposition 5.1. *For the Kropina metric $F = \frac{\alpha^2}{\beta}$, we have*

$$S(x, y) = \frac{n+1}{b^2}\left(r_0 - \frac{\beta}{\alpha^2}r_{00}\right). \tag{5.2}$$

Proof. By Proposition 2.1, we have

$$\frac{\partial G^m}{\partial y^m} = \frac{\partial \bar{G}^m}{\partial y^m} - \frac{n}{b^2}s_0 + \frac{1}{b^2}r_0 - \frac{n+1}{b^2\alpha^2}\beta r_{00}. \tag{5.3}$$

It is known that S-curvature of every Riemannian metric vanishes, i.e.,

$$0 = \frac{\partial \bar{G}^m}{\partial y^m} - y^m \frac{\partial \ln \sigma_\alpha}{\partial x^m}. \tag{5.4}$$

So plugging (5.3) and (5.4) into (5.1), we get (5.2). This proves the proposition. □

Theorem 5.1. *Every Einstein–Kropina metric $F = \frac{\alpha^2}{\beta}$ has vanishing S-curvature.*

Proof. Assume that F is an Einstein metric. By Theorem 3.1, we have $r_{00} = c\alpha^2$ for some scalar function $c = c(x)$ on M. Thus $r_0 = c\beta$. Plugging those into (5.2), we obtain $S = 0$. □

6. Conformal rigidity

In this section, we obtain a conformal rigidity result for Einstein–Kropina metrics.

Theorem 6.1. *Any conformal map between Einstein–Kropina spaces must be homothetic.*

Proof. Let $F = \alpha^2/\beta$, $\tilde{F} = \phi^{-1}F$ and $\tilde{F} = \tilde{\alpha}^2/\tilde{\beta}$. Then $\tilde{a}_{ij} = \phi^{-2}a_{ij}$ and $\tilde{b_i} = \phi^{-1}b_i$ hold. Let (h, W) and $(\tilde{h}, \tilde{W})$ be the navigation data of F and $\tilde{F}$, respectively. Suppose that $\tilde{h}_{ij} = e^{2\tilde{\rho}}\tilde{a}_{ij}$ and $h_{ij} = e^{2\rho}a_{ij}$ hold. So we have

$$\begin{cases} \tilde{b}^2 = \tilde{a}^{ij}\tilde{b}_i\tilde{b}_j = a^{ij}b_ib_j = b^2, \\ \tilde{h}_{ij} = e^{2\tilde{\rho}}\tilde{a}_{ij} = e^{2\tilde{\rho}}\phi^{-2}a_{ij} = e^{2(\tilde{\rho}-\rho)}\phi^{-2}h_{ij}, \\ 2\tilde{W}_i = e^{2\tilde{\rho}}\tilde{b}_i = e^{2\tilde{\rho}}\phi^{-1}b_i = 2e^{2(\tilde{\rho}-\rho)}\phi^{-1}W_i. \end{cases} \tag{6.1}$$

From (4.5) and the first equation of (6.1), we get that $\tilde{\rho} = \rho$. So the last two equations of (6.1) can be simplified as

$$\begin{cases} \tilde{h}_{ij} = \phi^{-2}h_{ij}, \\ \tilde{W}_i = \phi^{-1}W_i, \end{cases} \tag{6.2}$$

which means that two Riemannian metrics h and $\tilde{h}$ are conformal equivalent.

Firstly by conformal properties, we know that

$$\tilde{\gamma}^i_{jk} = \gamma^i_{jk} - \phi^{-1}\delta^i_j\phi_k - \phi^{-1}\delta^i_k\phi_j + \phi^{-1}\phi^i h_{jk},$$

where $\gamma^i{}_{jk}$ and $\tilde{\gamma}^i{}_{jk}$ are the coefficients of Levi-Civita connections of h and $\tilde{h}$, respectively, $\phi_k := \frac{\partial\phi}{\partial x^k}$ and $\phi^k := h^{ik}\phi_i$.

Let ";" and "," denote the covariant differentiation with respect to h and $\tilde{h}$, respectively. Thus we have

$$\tilde{W}_{j,k} = \frac{\partial\tilde{W}_j}{\partial x^k} - \tilde{W}_i\tilde{\gamma}^i_{jk} = \phi^{-1}W_{j;k} + \phi^{-2}\phi_jW_k - \phi^{-2}W_i\phi^ih_{jk}.$$

Hence

$$\tilde{W}_{j,k} + \tilde{W}_{k,j} = \phi^{-1}(W_{j;k} + W_{k;j}) + \phi^{-2}(W_j\phi_k + W_k\phi_j) - 2\phi^{-2}W^i\phi_ih_{jk}. \tag{6.3}$$

Assume that F and $\tilde{F}$ are both Einstein metrics. Thus by Theorem 1.2, we know that W and $\tilde{W}$ are both constant Killing vector fields. That is $0 = W_{j;k} + W_{k;j}$ and $0 = \tilde{W}_{j,k} + \tilde{W}_{k,j}$. Hence (6.3) can be rewritten as

$$0 = W_j\phi_k + W_k\phi_j - 2W^i\phi_ih_{jk}. \tag{6.4}$$

Contracting (6.4) with h^{jk} yields $W^i\phi_i = 0$. Putting it into (6.4) gets $0 = W_j\phi_k + W_k\phi_j$. Then contacting it with W^j yields $\phi_k = 0$, which means that $\phi =$ constant. It completes proof of Theorem 1.4. □

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