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Existence of r-self-orthogonal Latin squares ☆

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Abstract

Two Latin squares of order v are r-orthogonal if their superposition produces exactly r distinct ordered pairs. If the second square is the transpose of the first one, we say that the first square is r-self-orthogonal, denoted by r-SOLS(v). It has been proved that for any integer $v \geqslant 28$, there exists an r-SOLS(v) if and only if $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$. In this paper, we give an almost complete solution for the existence of r-self-orthogonal Latin squares.

Keywords: Latin square; r-Orthogonal; r-Self-orthogonal; Transversal

1. Introduction

Two Latin squares of order v, $L=(l_{ij})$ and $M=(m_{ij})$, are said to be *r-orthogonal* if their superposition produces exactly r distinct pairs, that is

$$|\{(l_{ij}, m_{ij}) : 0 \leqslant i, j \leqslant v-1\}| = r.$$

Belyavskaya (see [2–4]) first systematically treated the following question: for which integers v and r does a pair of r-orthogonal Latin squares of order v exist? Evidently, $v \leqslant r \leqslant v^2$, and an easy argument establishes that $r \notin \{v+1, v^2-1\}$. In papers by Colbourn and Zhu [8], Zhu and Zhang [19,20], this question has been completely answered, and the final result is in the following theorem.

Theorem 1.1 (*Zhu and Zhang [20, Theorem 2.1]*). *For any integer $v \geqslant 2$, there exists a pair of r-orthogonal Latin squares of order v if and only if $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$ with the exceptions of v and r shown in Table 1.*

In a pair of r-orthogonal Latin squares of order v, if the second square is the transpose of the first one, we say that the first square is *r-self-orthogonal*, denoted by r-SOLS(v). When an r-SOLS(v) exists, we can simply list only one square for a pair of r-orthogonal Latin squares of order v.

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Table 1

order v	Genuine exceptions of r
2	4
3	5, 6, 7
4	7, 10, 11, 13, 14
5	8, 9, 20, 22, 23
6	33, 36

For the existence of an r-SOLS(v), we have the necessary condition in Theorem 1.1, i.e., $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$. It is well-known that an SOLS(v) exists if and only if $v \neq 2, 3, 6$ (see, for example, [18]). This solves the case of $r = v^2$. For the case of $r = v$, we take the symmetric Latin square

$$L = (a_{ij}), \quad a_{ij} = i + j \pmod v, \quad i, j \in Z_v.$$

It is easily seen that L is a v-SOLS(v), and we have the following theorem.

Theorem 1.2. *There exist v-SOLS(v) for all integer $v > 0$, and v^2-SOLS(v) for all integer $v > 0$, $v \neq 2, 3, 6$.*

So, we can focus on the cases $v + 1 < r < v^2 - 1$ for the existence of an r-SOLS(v).

For small orders, we have the following results.

Theorem 1.3 (*Zhu and Zhang [20]*). *For order $v = 4$, there is only one $r \in [v+1, v^2-1]$, namely $r = 9$, such that an r-SOLS(v) exists.*

For $v = 5$ and $v + 1 < r < v^2 - 1$, there is an r-SOLS(5) for $r \in \{7, 10, 11, 13, 14, 15, 17, 19, 21\}$ only.

For $v = 6$ and $v + 1 < r < v^2 - 1$, there is an r-SOLS(6) for $r \in [8, 31]$ only.

For $v = 7$ and $v + 1 < r < v^2 - 1$, there is an r-SOLS(7) for all $r \in [9, 47] \setminus \{46\}$ only.

r-SOLS(8) for all $r \in [10, 62]$ are listed at the web site http://www.cs.uiowa.edu/~hzhang/sr/. So we have

Theorem 1.4. *There exists an r-SOLS(8) for every $r \in [8, 64] \setminus \{9, 63\}$.*

Zhu and Zhang [20, Conjecture 3.1] conjectured that there is an integer v_0 such that for any $v \geqslant v_0$, there exists an r-SOLS(v) for any $r \in [v, v^2] \setminus \{v+1, v^2-1\}$. The authors [14] have shown that $v_0 \leqslant 28$.

Theorem 1.5 (*Xu and Chang [14, Theorem 6.4]*). *For any integer $v \geqslant 28$, there exists an r-SOLS(v) if and only if $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$.*

In this paper, we investigate the existence of r-SOLS(v) for the remaining v, $9 \leqslant v \leqslant 27$.

2. Direct constructions

Let S be a set and L and M be two Latin squares based on S. If the superposition of L and M yields every ordered pair in $S \times S$, then L and M is said to be a pair of mutually orthogonal Latin squares, and denoted by MOLS$(|S|)$, where $|S|$ is the cardinality of S.

Let $\mathscr{H} = \{H_1, H_2, \ldots, H_k\}$ be a set of nonempty subsets of S. A *holey* (or, *incomplete*) *Latin square* having *hole set* $\mathscr{H}$ is an $|S| \times |S|$ array, L, indexed by S, which satisfies the following properties:

(1) every cell of L is either empty or contains a symbol of S,
(2) every symbol of S occurs at most once in any row or column of L,

(3) the subarrays $H_i \times H_i$ are empty for $1 \leqslant i \leqslant k$ (these subarrays are referred to as *holes*),
(4) symbol $x \in S$ occurs in row x or column y if and only if $(x, y) \in (S \times S) \backslash \bigcup_{i=1}^{k} (H_i \times H_i)$.

The *order* of L is $|S|$. Two holey Latin squares on symbol set S and hole set $\mathscr{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every ordered pair in $(S \times S) \backslash \bigcup_{i=1}^{k} (H_i \times H_i)$. We shall use the notation IMOLS$(v; h_1, h_2, \ldots, h_k)$ to denote a pair of orthogonal holey Latin squares on symbol set S and hole set $\mathscr{H} = \{H_1, H_2, \ldots, H_k\}$, where $v = |S|$ and $h_i = |H_i|$ for $1 \leqslant i \leqslant k$. If $\mathscr{H} = \emptyset$, we obtain an MOLS(v). If $\mathscr{H} = \{H_1\}$, we simply write IMOLS(v, h_1) for the orthogonal pair of holey Latin squares. Here an IMOLS stands for an *incomplete mutually orthogonal Latin squares*.

If L_1 and L_2 form an IMOLS$(v; h_1, h_2, \ldots, h_k)$ such that L_2 is the transpose of L_1, then L_1 is said to be a *holey* SOLS, and denoted by ISOLS$(v; h_1, h_2, \ldots, h_k)$. If $\mathscr{H} = \emptyset$, or $\{H_1\}$, then a holey SOLS is an SOLS(v), or ISOLS(v, h_1), respectively.

If $\mathscr{H} = \{H_1, H_2, \ldots, H_k\}$ is a partition of S, then an IMOLS is called a *frame* MOLS. The *type* of the frame MOLS is defined to be the multiset $\{|H_i| : 1 \leqslant i \leqslant k\}$. We shall use an "exponential" notation to describe types: Type $t_1^{n_1} t_2^{n_2} \cdots t_l^{n_l}$ denotes n_i occurrences of t_i, $1 \leqslant i \leqslant l$, in the multiset. We briefly denote a frame MOLS of type $t_1^{n_1} t_2^{n_2} \cdots t_l^{n_l}$ by FMOLS$(t_1^{n_1} t_2^{n_2} \cdots t_l^{n_l})$.

If L_1 and L_2 form an FMOLS (frame MOLS) such that L_2 is the transpose of L_1, then we call L_1 an FSOLS.

We observe that the existence of an SOLS(v) is equivalent to the existence of an FSOLS(1^v), and the existence of an ISOLS(v, h) is equivalent to the existence of an FSOLS $(1^{v-h} h^1)$.

Two holey Latin squares on symbol set S and hole set $\mathscr{H}$, say L_1 and L_2, are said to be *r-orthogonal* if their superposition yields r distinct ordered pairs. We shall use the notation r-IMOLS$(v; h_1, h_2, \ldots, h_k)$ to denote a pair of r-orthogonal holey Latin squares on symbol set S and hole set $\mathscr{H} = \{H_1, H_2, \ldots, H_k\}$, where $v = |S|$ and $h_i = |H_i|$ for $1 \leqslant i \leqslant k$. If $\mathscr{H} = \emptyset$, we obtain an r-MOLS(v). If $\mathscr{H} = \{H_1\}$, we simply write r-IMOLS(v, h_1) for the r-orthogonal pair of holey Latin squares.

If L_1 and L_2 form an r-IMOLS such that L_2 is the transpose of L_1, then we call L_1 an r-ISOLS.

The following construction is a modification of the starter-adder type constructions. The idea has been described by several authors including Horton [12], Hedayat and Seiden [10], Zhu [17], and Heinrich and Zhu [11].

Construction 2.1. *Let* $\mathbf{e} = (a_{00}, a_{01}, a_{02}, \ldots, a_{0(n-1)})$ *be a vector of length n with entries in* $\mathbb{Z}_n \cup X$, *where* $X = \{x_1, x_2, \ldots, x_u\}$ *is a set of u index symbols. Let* $\mathbf{f} = (a_{0x_1}, a_{0x_2}, \ldots, a_{0x_u})$ *and* $\mathbf{g} = (a_{x_10}, a_{x_20}, \ldots, a_{x_u0})$ *be vectors of length u with entries in* $\mathbb{Z}_n \backslash \{0\}$. *These vectors are used to construct an array* $A = (a_{ij})$ *of order* $n + u$ *with an empty subarray of order u having row and column indices and entries in* $\mathbb{Z}_n \cup X$. *The array is constructed as follows, where all the elements including indices are calculated modulo n, and* x_i*'s act as "infinite" elements.*

(1) *If* $a_{ij} \in \mathbb{Z}_n$, $0 \leqslant i, j \leqslant n-1$, *then* $a_{(i+1)(j+1)} = a_{ij} + 1$.
(2) *If* $a_{ij} \in X$, $0 \leqslant i, j \leqslant n-1$, *then* $a_{(i+1)(j+1)} = a_{i,j}$.
(3) *If* $0 \leqslant i \leqslant n-1$, *and* $j \in X$, *then* $a_{(i+1)j} = a_{ij} + 1$.
(4) *If* $0 \leqslant j \leqslant n-1$, *and* $i \in X$, *then* $a_{i(j+1)} = a_{ij} + 1$.

Let $D_1 = \{\pm(a_{0i} - a_{0(n-i)} - i) : a_{0i}, a_{0(n-i)} \in \mathbb{Z}_n, 1 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor\}$, $D_2 = \{\pm(a_{0x_j} - a_{x_j0}) : 1 \leqslant j \leqslant u\}$, *where the elements of* D_1 *and* D_2 *are calculated modulo n.*

Let

$$D = \begin{cases} D_1 \cup D_2 \cup \{0\}, & n \text{ is odd}, \\ D_1 \cup D_2 \cup \{0, n/2\}, & n \text{ is even} \end{cases}$$

and $r = n|D| + 2nu$. *If* $\{a_{0i} : 0 \leqslant i \leqslant n-1\} \cup \{a_{0x_j} : 1 \leqslant j \leqslant u\} = \{a_{0i} - i : 0 \leqslant i \leqslant n-1\} \cup \{a_{x_j0} : 1 \leqslant j \leqslant u\} = \mathbb{Z}_n \cup X$, *and* $a_{0,n/2} \notin X$ *when n is even, then* $A = (a_{ij})$ *is an r-ISOLS*$(n+u, u)$.

Example 2.2. Let $n = 11$, $u = 2$, $\mathbf{e} = (0, 10, 9, 5, x_1, 4, x_2, 1, 3, 2, 7)$, $\mathbf{f} = (6, 8)$, $\mathbf{g} = (3, 1)$. Then $D_1 = \{\pm 2, \pm 5, \mp 1\} = \{2, 9, 5, 6, 10, 1\}$, $D_2 = \{\pm 3, \pm 7\} = \{3, 8, 7, 4\}$, $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $r = 11 \times 11 + 2 \times 11 \times 2 = 165$. These vectors generate a 165-ISOLS(13, 2) (*ISOLS*(13, 2)) shown in Fig. 1, where a denotes 10.

$\underline{0}$	a	9	$\underline{5}$	x_1	4	x_2	1	3	2	7	6	8
8	1	0	a	6	x_1	5	x_2	2	4	3	7	9
4	9	2	1	0	7	x_1	6	x_2	3	5	8	a
6	5	a	3	2	1	8	x_1	7	x_2	4	9	0
$\underline{5}$	7	6	$\underline{0}$	4	3	2	9	x_1	8	x_2	a	1
x_2	6	8	7	1	5	4	3	a	x_1	9	0	2
a	x_2	7	9	8	2	6	5	4	0	x_1	1	3
x_1	0	x_2	8	a	9	3	7	6	5	1	2	4
2	x_1	1	x_2	9	0	a	4	8	7	6	3	5
7	3	x_1	2	x_2	a	1	0	5	9	8	4	6
9	8	4	x_1	3	x_2	0	2	1	6	a	5	7
3	4	5	6	7	8	9	a	0	1	2		
1	2	3	4	5	6	7	8	9	a	0		

Fig. 1. 165-ISOLS(13, 2).

Suppose that $A=(a_{ij})$ is an array. If $a_{sp}=a_{tq}$ and $a_{sq}=a_{tp}$, then

$$A(s,t;\,p,q)=\begin{pmatrix} a_{sp} & a_{sq} \\ a_{tp} & a_{tq} \end{pmatrix}$$

is a Latin sub-square of order 2 of A on the set $\{a_{sp},a_{sq}\}$. For example,

$$A(0,4;\,0,3)=\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$$

is a Latin sub-square of order 2 on set $\{0,5\}$ of the 165-ISOLS(13, 2) shown in Fig. 1.

If $A=(a_{ij})$ is a Latin square, and we alter A by interchanging the two columns of the $A(s,t;\,p,q)$, then we get a new Latin square A'. We say that we have given an *order-2-interchange* to A, and denote $A'=I_{(s,t;\,p,q)}(A)$, where $I_{(s,t;p,q)}$ denotes the order-2-interchange.

Fill the hole of the 165-ISOLS(13, 2) with

$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}$$

we then get a 167-SOLS(13) denoted by L, the missing pairs are (x_1,x_2) and (x_2,x_1), the repeated pairs are (x_1,x_1) and (x_2,x_2). It can be checked that $I_{(0,4;0,3)}(L)$ is a 160-SOLS(13) with missing pairs

$$(x_1,x_2),\ (x_2,x_1),\ (0,0),\ (5,6),\ (6,5),\ (5,x_1),\ (x_1,5),\ (0,2),\ (2,0)$$

and repeat pairs

$$(x_1,x_1),\ (x_2,x_2),\ (5,5),\ (0,6),\ (6,0),\ (0,x_1),\ (x_1,0),\ (5,2),\ (2,5).$$

Use the same method as above, we can further get (v^2-9)-SOLS(v) for $v\in\{15,16,17,18,20\}$ and (v^2-13)-SOLS(v) for $v\in\{17,20\}$. We list the vectors and the order-2-interchange in the following:

216-SOLS(15): $\mathbf{e}=(0,12,11,9,5,8,x_1,6,x_2,3,7,2,1)$,
$\mathbf{f}=(4,10)$, $\mathbf{g}=(8,5)$, $I_{(0,3;0,x_2)}$.

247-SOLS(16): $\mathbf{e}=(0,13,12,9,3,7,11,8,x_1,x_2,5,1,6,10)$,
$\mathbf{f}=(2,4)$, $\mathbf{g}=(3,7)$, $I_{(0,4;0,3)}$.

276-SOLS(17): $\mathbf{e} = (0, 13, 12, 11, 6, x_1, 5, 8, x_2, 1, x_3, 2, 7, 10)$,
$\mathbf{f} = (3, 4, 9)$, $\mathbf{g} = (7, 3, 4)$, $I_{(0,3;0,6)}$.

280-SOLS(17): $\mathbf{e} = (0, 14, 13, 12, 7, 11, 3, 8, x_1, x_2, 5, 10, 4, 2, 1)$,
$\mathbf{f} = (6, 9)$, $\mathbf{g} = (8, 5)$, $I_{(0,x_1;0,7)}$.

315-SOLS(18): $\mathbf{e} = (0, 15, 14, 11, 6, 10, x_1, 13, 9, x_2, 3, 2, 7, 1, 8, 12)$,
$\mathbf{f} = (4, 5)$, $\mathbf{g} = (3, 15)$, $I_{(0,4;0,3)}$.

387-SOLS(20): $\mathbf{e} = (0, 16, 15, 14, 13, 12, x_1, 8, 5, x_2, x_3, 6, 11, 4, 3, 2, 1)$,
$\mathbf{f} = (7, 9, 10)$, $\mathbf{g} = (5, 10, 3)$, $I_{(0,x_1;0,14)}$.

391-SOLS(20): $\mathbf{e} = (0, 17, 16, 15, 12, 8, 11, 6, 3, 10, x_1, x_2, 9, 5, 7, 4, 2, 1)$,
$\mathbf{f} = (13, 14)$, $\mathbf{g} = (6, 9)$, $I_{(0,5;5,13)}$.

So, we have the following lemma.

Lemma 2.3. *There exists a* $(v^2 - 9)$*-SOLS*(v) *for every* $v \in \{13, 15, 16, 17, 18, 20\}$, *and a* $(v^2 - 13)$*-SOLS*(v) *for every* $v \in \{17, 20\}$.

3. Recursive constructions

Construction 3.1 (*Filling in holes*). *Suppose there exists an ISOLS* $(v; h_1, h_2, \ldots, h_k)$ *with hole set* $\mathscr{H} = \{H_1, H_2, \ldots, H_k\}$ *such that* $H_i \cap H_j = \emptyset$ $(1 \leqslant i < j \leqslant k)$. *If there exist* r_i*-SOLS*(h_i) *for* $1 \leqslant i \leqslant k$, *then there exists an r-SOLS*(v) *for* $r = v^2 - \sum_{i=1}^{k} h_i^2 + \sum_{i=1}^{k} r_i$.

Proof. Fill in the ith hole of the ISOLS$(v; h_1, h_2, \ldots, h_k)$ with an r_i-SOLS(h_i) on set H_i for $1 \leqslant i \leqslant k$. □

To apply Construction 3.1, we need some "ingredients" provided in the following theorems.

Theorem 3.2 (Abel et al. [1, Theorem 2.10]). *There exists an ISOLS*(v, h) *for all values of v and h satisfying* $v \geqslant 3h+1$, *except for* $(v, h) = (6, 1), (8, 2)$ *and possibly for* $v = 3h + 2$, $h \in \{6, 8, 10\}$.

Theorem 3.3 (Zhang and Zhu [15, Lemma 2.2]). *There exists an ISOLS*$(9; 2, 2)$, *and an ISOLS* $(v; 2, 2)$, *an ISOLS*$(v; 2, 2, 2)$ *and an ISOLS*$(v; 2, 2, 2, 2)$ *for* $v \in \{10, 11\}$.

Theorem 3.4 (Zhang and Zhu [15, Theorem 7.1]). *Suppose a, n and b are positive integers and* $a \neq b$. *Then there exists an FSOLS*$(a^n b^1)$ *if and only if* $n \geqslant 4$ *and* $n \geqslant 1 + (2b/a)$, *except for* $(a, n, b) = (1, 6, 2)$ *and except possibly for* $(a, n, b) \in \{(t + 2, 6, 5a - 1/2), (t, 14, 13a - 1/2), (t, 18, 17a - 1/2), (t, 22, 21a - 1/2) : t \text{ is odd}\}$.

As an application of Construction 3.1, we give the following lemma.

Lemma 3.5. (1) *There exists a* $(v^2 - 2)$*-SOLS*(v) *for* $v \geqslant 7$ *and* $v \neq 8$; (2) *There exists a* $(v^2 - 3)$*-SOLS*(v) *for* $v \geqslant 25$ *and* $v \neq 26$.

Proof. (1) From Theorem 3.2 we know that there exists an ISOLS$(v, 2)$ for $v \geqslant 7$ and $v \neq 8$. Filling the hole with a symmetric Latin square of order 2 we obtain a $(v^2 - 2)$-SOLS(v).

(2) From Theorem 3.2 and Theorem 1.4 we know that there exists an ISOLS$(v, 8)$ for $v \geqslant 25$ and $v \neq 26$ and a 61-SOLS(8). Applying Construction 3.1 with $k = 1$, $h_1 = 8$ and $r_1 = 61$ we then obtain a $(v^2 - 3)$-SOLS(v). □

Let $L = (\ell_{ij})$ be an r-SOLS(v) and $P = \{(\ell_{ij}, \ell_{ji}) : 0 \leqslant i, j \leqslant v - 1\}$. It is obvious that $|P| = r$. We call P the *DOP set* (distinct ordered pairs set) of L.

The following recursive construction is referred to as *Inflation Construction*. It essentially "blows up" every cell of an initial p-SOLS(m) into a q-SOLS(n) or a q-MOLS(n) labelled by the element in that cell such that if one cell is filled with a certain q-SOLS(n), then its symmetric cell is filled with the same square, if one cell is filled with a certain square of a q-MOLS(n), then its symmetric cell is filled with the transpose of the second square. We mention the work of Brouwer and van Rees [7] and Stinson [13], which can be thought of as sources of Inflation Construction.

Construction 3.6 (*Inflation Construction*). *Suppose that there exists a p-SOLS(m) with DOP set* $\{(a_i, a_i) : 1 \leqslant i \leqslant k\} \cup \{(a_j, b_j), (b_j, a_j) : a_j \neq b_j, 1 \leqslant j \leqslant l\}$, *where* $k + 2l = p$. *If there exists a* q_1*-SOLS(n) for every* $q_1 \in R_1$, *and there exists a* q_2*-MOLS(n) for every* $q_2 \in R_2$ *when* $l > 0$, *then there exists an r-SOLS(mn) for* $r = \sum_{i=1}^{k} q_{1i} + 2\sum_{j=1}^{l} q_{2j}$, *where* $q_{1i} \in R_1, q_{2j} \in R_2$.

Proof. Start with the p-SOLS(m) as an initial square, replace each of the cells that form the ordered pairs (a_i, a_i) with a q_{1i}-SOLS(n) labelled by a_i. Replace each of the cells that form the ordered pair (a_j, b_j) and contains a_j with the first square of a q_{2j}-MOLS(n) labelled by a_j, and the symmetric cell contains b_j with the transpose of the second square labelled by b_j. We suppose that the input designs, q_1-SOLS(n) and q_2-MOLS(n), are all based on the same set. □

Corollary 3.7. *Suppose that there exists an SOLS(m). If there exist a* q_1*-SOLS(n) for every* $q_1 \in R_1$ *and a* q_2*-MOLS(n) for every* $q_2 \in R_2$, *then there exists an r-SOLS(mn) for* $r = \sum_{i=1}^{m} q_{1i} + 2\sum_{j=1}^{(m^2-m)/2} q_{2j}$, *where* $q_{1i} \in R_1, q_{2j} \in R_2$.

Proof. Applying Construction 3.6 with $p = m^2, k = m, l = (m^2 - m)/2$. □

The following recursive construction is a generalization of Corollary 3.7. It relies on information regarding the location of transversals in certain Latin squares. Suppose L is a Latin square on a symbol set S. A *transversal* is a set T of $|S|$ cells in L such that every symbol of S occurs in exactly one cell of T and the $|S|$ cells in T intersect each row and each column exactly once. A transversal T is *symmetric* if $(i, j) \in T$ implies $(j, i) \in T$. Two transversals T_1 and T_2 are called a *symmetric pair of transversals* if $(i, j) \in T_1$ if and only if $(j, i) \in T_2$. A set of transversals are said to be *disjoint* if they have no a cell in common.

Construction 3.8. *Let m be an even integer. Suppose that there exists an SOLS(m) with t disjoint symmetric transversals off the main diagonal. If there exist a* q_1*-SOLS(n+1) for every* $q_1 \in R_1$ *and a* q_2*-MOLS(n) for every* $q_2 \in R_2$; *there exists a* q_3*-MOLS(n+1) for every* $q_3 \in R_3$ *when* $t > 0$, *then there exists an r-SOLS(mn+t+1) for* $r = \sum_{i=1}^{m} (q_{1i} - 1) + 2\sum_{j=1}^{(m^2-tm-m)/2} q_{2j} + 2\sum_{k=1}^{tm/2} (q_{3k} - 1) + t + 1$, *where* $q_{1i} \in R_1, q_{2j} \in R_2, q_{3k} \in R_3$.

Proof. We suppose that the q_2-MOLS(n) is based on set $\mathbb{Z}_n$, the q_1-SOLS$(n+1)$ and the q_3-MOLS$(n+1)$ are based on the set $\mathbb{Z}_n \cup \{x\}$. Without loss of generality, we suppose that x is in the right bottom corner of the q_1-SOLS$(n+1)$ and the q_3-MOLS$(n+1)$. Delete x from the right bottom corner of the q_1-SOLS$(n+1)$. We then get a q_1-ISOLS$(n+1, 1)$ with DOP set containing (x, x) or a $(q_1 - 1)$-ISOLS$(n+1, 1)$ with DOP set not containing (x, x). Delete x from the right bottom corner of the q_3-MOLS$(n+1)$. We get a q_3-IMOLS$(n+1, 1)$ or $(q_3 - 1)$-IMOLS$(n+1, 1)$.

It is obvious that the cells on the main diagonal of the SOLS(m) form a symmetric transversal. Start with the SOLS(m) as an initial square, replace each of its cells with an $n \times n$ array labelled by the element in that cell. The array will be a q_{2j}-MOLS(n) if the cell is the jth cell not on the main diagonal and the t symmetric transversals.

If the cell is the ith one on the main diagonal, the array will be the upper left part of a q_{1i}-ISOLS$(n+1, 1)$ or $(q_{1i} - 1)$-ISOLS$(n+1, 1)$ on $\mathbb{Z}_n \cup \{x_0\}$ labelled by the element in that cell. The right column will be moved to the right part of the resultant square and the lower row will be moved to the lower part of the resultant square.

If the cell is the kth cell on the t symmetric transversals and it is on the ℓth symmetric transversal $(1 \leqslant \ell \leqslant t)$, the array will be the upper left part of a (labelled) q_{3k}-IMOLS$(n+1, 1)$ or $(q_{3k} - 1)$-IMOLS$(n+1, 1)$ on $\mathbb{Z}_n \cup \{x_\ell\}$. The right column of it will be moved to the right part of the resultant square and the lower row will be moved to the lower part of the resultant square.

Every element of the input design except x_0 and x_ℓ $(1 \leqslant \ell \leqslant t)$ is labelled by the element in the cell it replaced, but elements x_0 and x_ℓ $(1 \leqslant \ell \leqslant t)$ remains unchanged when labelling.

Then we get the upper left part of side mn of a holey Latin square of order $mn+t+1$ with hole of side $t+1$ and the hole set $\{x_0, x_1, \ldots, x_t\}$.

The right part of the holey Latin square of order $mn+t+1$ consists of columns $C_0, C_1, \ldots, C_t$, where C_0 comes from the labelled elements of the right column of the q_1-ISOLS$(n+1,1)$ or (q_1-1)-ISOLS$(n+1,1)$ on the main diagonal and C_ℓ $(1\leqslant \ell \leqslant t)$ comes from the right column with the labelled elements of the q_3-IMOLS$(n+1,1)$ or (q_3-1)-IMOLS$(n+1,1)$ on the ℓth symmetric transversal.

The lower part of the holey Latin square of order $mn+t+1$ consists of rows $R_0, R_1, \ldots, R_t$, where R_0 comes from the lower (labelled element) row of the q_1-ISOLS$(n+1,1)$ or (q_1-1)-ISOLS$(n+1,1)$ on the main diagonal, R_ℓ $(1\leqslant \ell \leqslant t)$ comes from the lower (labelled element) row of the q_3-IMOLS$(n+1,1)$ or (q_3-1)-IMOLS$(n+1,1)$ on the ℓth symmetric transversal.

Filling the hole of side $t+1$ of holey Latin square with a symmetric Latin square of order $t+1$ on the set $\{x_0, x_1, \ldots, x_t\}$, we then obtain an r-SOLS$(mn+t+1)$ for $r=\sum_{i=1}^{m}(q_{1i}-1)+2\sum_{j=1}^{(m^2-tm-m)/2} q_{2j}+2\sum_{k=1}^{tm/2}(q_{3k}-1)+t+1$. □

To apply the inflation constructions, we need some "ingredients" provided in the following theorem and lemmas.

Theorem 3.9 (*Bennet and Zhu [5,6], Du [9]*). *For all even m, $m \notin \{2, 6, 10, 14\}$, there exists an SOLS$(m)$ with $m-1$ disjoint symmetric transversals off the main diagonal.*

Lemma 3.10 (*Xu and Chang [14, Lemma 4.4]*). *If $v \geqslant 7$ is odd, then there exists a $(v+4)$-ISOLS$(v+1,1)$ on $\mathbb{Z}_v \cup \{x\}$ with DOP set $\{(0,0),(1,1),(0,2),(2,0)\}\cup\{(2i-1,2i),(2i,2i-1): 1\leqslant i \leqslant (v-1)/2\}\cup\{(x,x)\}$ and the hole set $\{x\}$.*

Lemma 3.11 (*Xu and Chang [14, Lemma 4.5]*). *Suppose that v is an integer and $v\geqslant 7$. For $r \in \{v+2, v+3\}$, there exists an r-SOLS(v) on $\mathbb{Z}_v$ with DOP set denoted by P; and there exists an $(r+1)$-ISOLS$(v+1,1)$ on $\mathbb{Z}_v\cup\{x\}$ with DOP set $P\cup\{(x,x)\}$ and the hole set $\{x\}$, where $|P|=r$.*

Lemma 3.12 (*Colbourn and Zhu [8, Lemma 2.4]*). *If there exists an r-MOLS(v), then there exists an r-MOLS(v) on $\mathbb{Z}_v$ with DOP set containing $\{(i,i): 0\leqslant i\leqslant v-1\}$.*

Lemma 3.13. *If there exists an r-MOLS(v), then there exists an r-IMOLS$(v,1)$ or $(r-1)$-IMOLS$(v,1)$ on $\mathbb{Z}_{v-1}\cup\{x\}$ with hole set $\{x\}$ and DOP set containing $\{(i,i): 0\leqslant i\leqslant v-2\}$.*

Proof. From Lemma 3.12 we have an r-MOLS(v) on $\mathbb{Z}_{v-1}\cup\{x\}$ with DOP set containing $\{(i,i): 0\leqslant i\leqslant v-2\}\cup\{(x,x)\}$. Denote the two squares of the r-MOLS(v) by $A=(a_{ij})$ and $B=(b_{ij})$, and suppose that $a_{st}=b_{st}=x$. Give the permutation $\pi=(s,t)$ to the row sets of A and B we get two Latin squares A' and B', respectively. Deleting x from the cells (t,t) of A' and B' we then obtain the desired r-IMOLS or $(r-1)$-IMOLS. □

Construction 3.14. *Let n be an integer and $k=1$ or 2. Suppose that: (1) there exists an r_1-MOLS$(n+1)$; (2) there exists an n-SOLS(n) on set $\mathbb{Z}_n$ with DOP set P_1 and an r_2-SOLS$(n+1)$ on set $Z_n\cup\{x\}$ with x in the right bottom corner and DOP set $P_2 \supset P_1\cup\{(x,x)\}$; (3) there exists an $(n+1)$-SOLS$(n+1)$ on set $\mathbb{Z}_n\cup\{x\}$ with x in the right bottom corner and DOP set $P_1\cup\{(x,x)\}$ when $k=2$. Then there exists an r-SOLS$(3n+k)$ for $r=2r_1+r_2-3+k$.*

Proof. Fig. 2 is a 3-SOLS(3) with three disjoint symmetric transversals $T_1=\{(0,1),(1,0),(2,2)\}$, $T_2=\{(0,0),(1,2),(2,1)\}$ and $T_3=\{(0,2),(1,1),(2,0)\}$. The DOP set is {(0,0), (1,2), (2,1)}.

0	1	2
2	0	1
1	2	0

Fig. 2. A 3-SOLS(3).

Since we have an r_1-MOLS$(n+1)$, from Lemma 3.13 we have an r_1-IMOLS$(n+1, 1)$ or (r_1-1)-IMOLS$(n+1, 1)$ on $\mathbb{Z}_n \cup \{x\}$ with DOP set containing $\{(i,i) : 0 \leqslant i \leqslant n-1\} \cup \{(x,x)\}$ and the hole set $\{x\}$. Without loss of generality, we suppose that the hole is in the right bottom corner. From Lemmas 3.12 and 3.13 we also have an n-MOLS(n) on the set $\mathbb{Z}_n$ with DOP set $P_1 = \{(i,i) : 0 \leqslant i \leqslant n-1\}$, and an $(n+1)$-IMOLS$(n+1, 1)$ on $\mathbb{Z}_n \cup \{x\}$ with DOP set $P_1 \cup \{(x,x)\}$ and the hole set $\{x\}$, respectively. Deleting x from the right bottom corner of the $(n+1)$-SOLS$(n+1)$ and r_2-SOLS$(n+1)$ we get an $(n+1)$-ISOLS$(n+1, 1)$ with DOP set $P_1 \cup \{(x,x)\}$, and an r_2-ISOLS$(n+1, 1)$ or (r_2-1)-ISOLS$(n+1, 1)$ with DOP set containing P_1, respectively.

This proof is similar to that of Construction 3.8. Here we say we fill a cell with an s-ISOLS$(n+1, n)$ or s-IMOLS$(n+1, n)$ means to fill the cell with the $n \times n$ upper left part of the input design, and move the right column to the right edge of the resultant square, the lower row to the lower edge of the resultant square. Every input design is labelled by the element in the cell it replaced, but x remains unchanged when labelling.

Start with the 3-SOLS(3). In T_1, we fill cell (0,1) with the first square of the r_1-IMOLS$(n+1, 1)$ or (r_1-1)-IMOLS$(n+1, 1)$, and cell (1,0) with the transpose of the second square. We fill cell (2,2) with the r_2-ISOLS$(n+1, 1)$ with DOP set P_2. We denote the hole set of the incomplete Latin squares by $\{x_1\}$.

In T_2, if $k=1$, we replace cell (1,2) with the first square of the n-MOLS(n) with DOP set P_1, cell (2,1) with the transpose of the second square, and cell (0,0) with the n-SOLS(n) with DOP set P_1; if $k=2$, we replace cell (1,2) with the first square of the $(n+1)$-IMOLS$(n+1, 1)$ with DOP set $P_1 \cup \{(x_2, x_2)\}$, cell (2,1) with the transpose of the second square, and cell (0,0) with the $(n+1)$-ISOLS$(n+1, 1)$ with DOP set $P_1 \cup \{(x_2, x_2)\}$. We denote the hole set of the incomplete Latin squares by $\{x_2\}$ here.

In T_3, we replace cell (2,0) with the first square of the n-MOLS(n) with DOP set P_1, cell (0,2) with the transpose of the second square, and cell (1,1) with the n-SOLS(n) with DOP set P_1.

Now we get an incomplete Latin square of order $3n+k$ with a hole of side k. Filling the hole with x_1 if $k=1$, with a symmetric Latin square on $\{x_1, x_2\}$ if $k=2$, we then obtain an r-SOLS$(3n+k)$ with $r = 2(r_1-1) + (|P_2|-1) + k = 2r_1 + r_2 - 3 + k$.

This completes the proof. □

4. Existence of r-SOLS(v) for $9 \leqslant v \leqslant 13$

Lemma 4.1. *There exists an r-SOLS*(9) *for every* $r \in [13, 78]$.

Proof. L_1 shown in Fig. 3 is an SOLS(9). Choose a proper permutation π to permute the rows of L_1 to obtain a new square L_1'. Then determine the cardinality of the DOP set of L_1'. The cardinality, r, and the corresponding row permutation π are listed in Table 2.

Give one or more proper order-2-interchanges to the Latin squares L_1 or L_2 in Fig. 3 in turn, we can get a new r-SOLS. We list r and the corresponding order-2-interchanges in Table 3, and simply write $s, t;\ p, q$ for $I_{(s,t;p,q)}$.

r-SOLS(9) for $r \in \{75, 76, 78\}$ are listed in Figs. 4 and 5. Filling each of the two holes of an ISOLS(9;2,2) from Theorem 3.3 with a symmetric Latin square of order 2 we obtain a 77-SOLS(9). □

$$L_1 = \begin{matrix} 0&2&1&4&3&6&7&8&5\\ 3&1&8&6&5&4&2&0&7\\ 4&5&2&7&1&8&0&3&6\\ 2&8&5&3&6&7&1&4&0\\ 5&0&7&1&4&2&8&6&3\\ 7&6&3&0&8&5&4&1&2\\ 8&7&4&5&0&3&6&2&1\\ 1&3&6&8&2&0&5&7&4\\ 6&4&0&2&7&1&3&5&8 \end{matrix} \qquad L_2 = \begin{matrix} 8&6&7&5&4&3&2&1&0\\ 7&8&5&6&3&4&1&0&2\\ 6&4&8&7&5&2&0&3&1\\ 4&7&6&8&1&0&5&2&3\\ 5&2&4&0&8&1&3&7&6\\ 2&5&3&1&0&8&6&4&7\\ 3&0&1&4&2&6&7&8&5\\ 0&1&2&3&7&5&8&6&4\\ 1&3&0&2&6&7&4&5&8 \end{matrix}$$

Fig. 3. An SOLS(9) and a 11-SOLS(9).

Table 2

r	π									r	π									r	π								
13	0	3	2	7	6	5	8	1	4	36	0	1	2	8	7	4	3	5	6	54	0	1	2	3	4	6	7	8	5
15	0	1	7	5	3	8	2	4	6	37	0	1	2	3	7	6	8	4	5	55	0	1	2	3	4	5	6	8	7
17	0	3	1	5	7	4	2	6	8	38	0	1	2	4	6	8	3	5	7	56	0	1	2	3	4	6	8	5	7
19	0	4	6	2	8	1	5	7	3	39	0	1	2	3	6	5	8	4	7	57	0	1	2	3	5	4	6	7	8
21	0	3	1	5	7	6	2	8	4	40	0	1	2	4	3	6	7	5	8	58	0	1	2	3	8	4	5	6	7
23	0	1	7	5	3	6	2	8	4	41	0	1	2	3	4	8	6	5	7	59	0	1	2	3	8	5	4	6	7
24	0	2	8	6	1	7	5	3	4	42	0	1	2	4	3	8	5	7	6	60	0	1	2	3	4	5	8	6	7
25	0	1	7	2	3	8	5	4	6	43	0	1	2	3	7	8	4	6	5	61	0	1	2	3	4	5	7	6	8
26	6	1	2	5	0	3	8	4	7	44	0	1	2	4	3	7	8	5	6	62	0	1	2	5	8	7	6	3	4
27	0	1	7	4	6	8	3	5	2	45	0	1	2	3	5	6	8	4	7	63	0	1	2	3	5	8	6	7	4
28	0	1	7	5	2	8	6	4	3	46	0	1	2	3	4	7	8	5	6	64	0	4	1	3	2	7	8	5	6
29	0	1	3	2	7	6	5	4	8	47	0	1	2	3	4	5	7	8	6	65	0	1	5	6	4	7	2	3	8
30	0	1	8	5	2	7	6	4	3	48	0	1	2	3	4	7	5	8	6	66	0	1	7	3	4	5	2	6	8
31	0	1	3	2	7	8	5	6	4	49	0	1	2	3	5	7	4	6	8	67	0	8	2	1	4	5	3	7	6
32	0	1	7	6	3	5	2	8	4	50	0	1	2	3	4	8	5	7	6	68	3	4	0	5	1	2	6	8	7
33	0	1	2	7	3	8	4	6	5	51	0	1	2	3	4	6	7	5	8	69	0	6	5	4	3	2	1	8	7
34	0	1	3	5	7	4	2	8	6	52	0	1	2	3	4	7	8	6	5	72	1	0	7	5	8	3	4	2	6
35	0	1	2	8	3	7	5	4	6	53	0	1	2	3	4	5	8	7	6	73	6	8	0	3	7	5	2	4	1

Table 3

r	Square	Order-2-interchanges		r	Square	Order-2-interchanges	
14	L_2	0, 7; 1, 7	1, 2; 0, 3	22	L_2	0, 7; 1, 7	1, 6; 3, 5
16	L_2	0, 7; 1, 7		70	L_1	0, 8; 7, 8	1, 2; 3, 8
18	L_2	0, 7; 1, 7	1, 5; 3, 6	71	L_1	0, 8; 7, 8	0, 7; 3, 8
20	L_2	0, 7; 1, 7	0, 1; 4, 5	74	L_1	0, 8; 7, 8	

0	2	1	4	5	6	3	8	7	0	8	7	6	5	4	3	2	1
3	1	7	0	8	2	5	6	4	6	1	4	2	0	7	8	3	5
4	3	2	6	7	1	8	5	0	5	6	2	4	7	8	0	1	3
5	6	4	7	3	0	1	2	8	7	5	8	3	2	0	1	6	4
8	4	0	5	2	7	6	1	3	8	7	3	0	4	1	2	5	6
2	7	5	3	6	8	0	4	1	3	4	1	8	6	5	7	0	2
6	0	3	8	1	4	2	7	5	1	3	6	7	8	2	5	4	0
7	8	6	1	0	5	4	3	2	4	2	0	5	1	3	6	7	8
1	5	8	2	4	3	7	0	6	2	0	5	1	3	6	4	8	7

Fig. 4. r-SOLS(9) for $r \in \{75, 76\}$.

Let k be an integer and T, T_1, T_2 be sets of integers. We define some set operations in the following:

$$kT = \{kt : t \in T\},$$

$$T_1 + T_2 = \{t_1 + t_2 : t_1 \in T_1, t_2 \in T_2\}$$

$$k \otimes T = \left\{ \sum_{i=1}^{k} t_i : t_i \in T \text{ for } 1 \leqslant i \leqslant k \right\}.$$

We simply write $k + T$ for $\{k\} + T$.

$$
\begin{array}{ccccccccc}
0 & 8 & 7 & 6 & 5 & 4 & 1 & 3 & 2 \\
7 & 1 & 8 & 4 & 6 & 3 & 0 & 2 & 5 \\
6 & 4 & 2 & 5 & 3 & 7 & 8 & 1 & 0 \\
8 & 0 & 4 & 3 & 7 & 2 & 5 & 6 & 1 \\
3 & 2 & 6 & 1 & 4 & 0 & 7 & 5 & 8 \\
2 & 7 & 0 & 8 & 1 & 5 & 3 & 4 & 6 \\
4 & 5 & 1 & 7 & 2 & 8 & 6 & 0 & 3 \\
1 & 3 & 5 & 0 & 8 & 6 & 2 & 7 & 4 \\
5 & 6 & 3 & 2 & 0 & 1 & 4 & 8 & 7
\end{array}
$$

Fig. 5. A 78-SOLS(9).

$$
L_1 = \begin{array}{cccccccccc}
0 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 9 & 7 \\
3 & 1 & 0 & 6 & 8 & 9 & 2 & 4 & 7 & 5 \\
4 & 8 & 2 & 0 & 6 & 7 & 9 & 5 & 1 & 3 \\
2 & 5 & 7 & 3 & 9 & 8 & 1 & 6 & 0 & 4 \\
1 & 7 & 8 & 5 & 4 & 0 & 3 & 9 & 6 & 2 \\
7 & 4 & 9 & 1 & 2 & 5 & 8 & 0 & 3 & 6 \\
8 & 9 & 3 & 7 & 5 & 2 & 6 & 1 & 4 & 0 \\
9 & 6 & 4 & 2 & 1 & 3 & 0 & 7 & 5 & 8 \\
6 & 3 & 5 & 9 & 0 & 4 & 7 & 2 & 8 & 1 \\
5 & 0 & 6 & 8 & 7 & 1 & 4 & 3 & 2 & 9
\end{array}
$$

Fig. 6. An SOLS(10).

Table 4

r	π										r	π										r	π									
37	0	1	2	8	5	6	7	4	3	9	55	0	1	2	3	4	5	7	6	9	8	70	0	1	2	3	4	5	6	8	7	9
41	0	1	8	7	2	5	3	6	4	9	56	0	1	2	3	4	6	5	8	9	7	71	0	1	2	3	5	7	9	6	8	4
42	0	1	2	4	5	6	9	8	7	3	57	0	1	2	3	4	5	9	6	8	7	72	0	1	2	3	4	5	6	9	8	7
43	0	1	2	4	9	5	3	6	7	8	58	0	1	2	3	4	5	7	9	6	8	73	0	1	2	3	5	4	8	7	6	9
44	0	1	4	8	6	3	9	7	2	5	59	0	1	2	3	4	5	7	8	9	6	74	0	1	2	3	4	8	9	6	5	7
45	0	1	3	2	4	6	5	7	8	9	60	0	1	2	3	4	5	8	9	6	7	75	0	1	2	3	7	4	9	6	8	5
46	0	1	2	3	6	5	9	4	7	8	61	0	1	2	3	4	5	6	9	7	8	76	0	1	2	3	4	7	6	5	8	9
47	0	1	2	3	9	4	7	5	6	8	62	0	1	2	3	4	6	8	5	7	9	77	0	1	2	7	8	5	4	6	9	3
48	0	1	2	3	6	4	7	5	9	8	63	0	1	2	3	4	5	8	6	7	9	78	0	1	2	5	6	4	7	3	8	9
49	0	1	2	3	9	7	5	8	4	6	64	0	1	2	3	4	6	7	8	5	9	79	0	1	3	9	4	6	8	5	7	2
50	0	1	2	3	4	9	5	6	7	8	65	0	1	2	3	4	6	8	9	7	5	80	0	1	2	3	9	5	6	7	8	4
51	0	1	2	3	6	8	9	4	7	5	66	0	1	2	3	4	5	6	7	9	8	81	0	1	7	3	6	5	2	4	8	9
52	0	1	2	3	4	5	9	6	7	8	67	0	1	2	3	4	7	5	9	6	8	82	1	2	4	6	5	9	7	3	8	0
53	0	1	2	3	4	6	5	9	8	7	68	0	1	2	3	4	5	9	7	8	6	83	8	0	4	6	2	9	3	1	5	7
54	0	1	2	3	4	7	9	8	5	6	69	0	1	2	3	4	5	7	9	8	6	84	5	6	4	0	9	1	8	7	3	2

Lemma 4.2. *There exists an r-SOLS*(10) *for every* $r \in [14, 97]$.

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p = m = 2$, $k = 2$, $l = 0$, $n = 5$ and $R_1 = \{5, 7, 10, 11, 13, 14, 15, 17, 19, 21, 25\}$, the input designs, q_1-SOLS(5) for $q_1 \in R_1$, are from Theorems 1.2 and 1.3, then we obtain an r-SOLS(10) for every $r = q_{11} + q_{12} \in 2 \otimes R_1 \supset [14, 40] \backslash \{37\}$.

Give a proper permutation π to the rows of a self-orthogonal Latin square L_1 in Fig. 6 we obtain an r-SOLS(10) for $r \in [41, 84]$, π and r are listed in Table 4.

Give one or two proper order-2-interchanges to the Latin square L_1 in Fig. 6 in turn, we obtain a new r-SOLS(10). We list r and the order-2-interchanges in Table 5.

Table 5

r	Order-2-interchanges		r	Order-2-interchanges	
85	0, 2; 0, 3	0, 3; 5, 7	89	0, 2; 0, 3	3, 6; 8, 9
86	0, 2; 0, 3	0, 7; 7, 9	91	0, 2; 0, 3	4, 6; 5, 9
87	0, 2; 0, 3	0, 6; 3, 9	93	0, 2; 0, 3	
88	0, 2; 0, 3	2, 3; 3, 9			

0	9	8	7	6	4	5	3	2	1
7	1	9	6	2	3	8	5	4	0
6	5	2	8	9	1	4	7	0	3
8	2	4	3	1	9	7	0	5	6
9	7	3	0	4	6	2	8	1	5
3	8	7	4	0	5	9	1	6	2
4	0	1	5	8	2	6	9	3	7
2	3	0	9	5	8	1	6	7	4
1	6	5	2	3	7	0	4	8	9
5	4	6	1	7	0	3	2	9	8

0	2	1	4	3	6	5	9	7	8
9	1	0	2	6	4	3	5	8	7
8	7	2	0	5	9	1	3	6	4
7	4	5	3	9	8	0	6	2	1
6	5	7	8	4	2	9	1	3	0
4	9	6	7	0	5	8	2	1	3
3	0	4	9	1	7	6	8	5	2
5	8	9	1	2	3	4	7	0	6
1	3	8	6	7	0	2	4	9	5
2	6	3	5	8	1	7	0	4	9

Fig. 7. r-SOLS(10) for $r \in \{95, 97\}$.

Table 6

r	π											r	π											r	π										
31	0	1	2	3	4	5	6	7	8	10	9	53	0	1	2	3	4	5	7	6	9	10	8	74	0	1	2	3	5	4	6	9	10	8	7
33	0	1	2	3	4	5	6	9	10	7	8	54	0	1	2	3	5	4	7	6	9	10	8	75	0	1	2	3	4	5	8	10	9	7	6
34	0	1	4	10	9	8	7	6	5	2	3	55	0	1	2	3	4	6	5	7	9	10	8	76	0	1	2	3	5	4	8	10	7	9	6
35	0	1	2	3	4	5	6	8	7	10	9	56	0	1	2	3	5	4	6	8	10	7	9	77	0	1	2	3	4	5	8	10	7	9	6
36	0	1	4	3	2	6	5	10	9	8	7	57	0	1	2	3	4	5	6	8	10	7	9	78	0	1	2	3	5	6	8	10	4	7	9
37	0	1	2	3	4	6	5	8	7	10	9	58	0	1	2	3	5	4	6	8	9	10	7	79	0	1	2	3	4	6	8	10	5	7	9
38	0	1	3	2	7	8	10	9	5	4	6	59	0	1	2	3	4	5	7	10	8	6	9	80	0	1	2	3	5	7	9	6	4	10	8
39	0	1	2	3	8	9	10	7	4	5	6	60	0	1	2	3	5	6	7	4	10	9	8	81	0	1	2	3	4	7	10	6	9	5	8
40	0	1	2	3	9	10	6	8	7	4	5	61	0	1	2	3	4	5	9	10	7	8	6	82	0	1	2	3	5	10	4	8	7	9	6
41	0	1	2	3	7	8	9	10	4	6	5	62	0	1	2	3	5	4	7	8	10	6	9	83	0	1	2	3	5	7	10	6	9	4	8
42	0	1	2	3	5	4	6	8	7	10	9	63	0	1	2	3	4	5	6	8	10	9	7	84	0	1	2	3	5	7	10	9	6	8	4
43	0	1	2	3	4	5	7	6	10	9	8	64	0	1	2	3	5	4	7	10	6	8	9	85	0	1	2	3	5	8	10	7	6	9	4
44	0	1	2	3	7	9	10	4	8	5	6	65	0	1	2	3	4	5	6	9	10	8	7	86	0	1	2	4	5	7	10	9	3	8	6
45	0	1	2	3	4	5	6	10	9	8	7	66	0	1	2	3	5	4	9	10	7	8	6	87	0	1	2	3	5	6	10	9	4	8	7
46	0	1	2	3	7	8	10	4	5	9	6	67	0	1	2	3	4	5	7	8	10	6	9	88	0	1	2	4	5	9	8	6	10	3	7
47	0	1	2	3	4	5	8	10	6	9	7	68	0	1	2	3	5	6	7	10	9	4	8	89	0	1	2	4	8	10	7	6	3	5	9
48	0	1	2	3	5	4	6	9	10	7	8	69	0	1	2	3	4	5	7	9	10	8	6	90	0	1	2	6	10	5	7	9	4	8	3
49	0	1	2	3	4	5	6	7	9	10	8	70	0	1	2	3	5	4	7	10	8	6	9	91	0	1	2	4	7	10	6	5	8	9	3
50	0	1	2	3	5	4	6	10	9	8	7	71	0	1	2	3	4	5	7	10	9	6	8	92	0	1	2	5	8	4	7	10	6	9	3
51	0	1	2	3	4	7	9	5	10	6	8	72	0	1	2	3	5	4	8	10	9	7	6	93	0	1	3	4	2	10	7	8	6	5	9
52	0	1	2	3	5	4	8	10	6	9	7	73	0	1	2	3	4	6	8	7	10	5	9	95	0	1	3	6	10	4	8	9	2	5	7

From [16] we have an FSOLS(2^5), and from Theorem 3.3 we have an ISOLS(10; 2, 2, 2, 2), an ISOLS(10; 2, 2, 2) and an ISOLS(10; 2, 2). Filling all the holes of the above squares with a symmetric Latin square of order 2 we can obtain r-SOLS(10) for $r = 90, 92, 94$ and 96, respectively.

r-SOLS(10) for $r \in \{95, 97\}$ are listed in Fig. 7. □

Lemma 4.3. *There exists an r-SOLS*(11) *for every* $r \in [15, 118]$.

Proof. Let $L = (a_{ij})$ be a symmetric Latin square of order 11, where $a_{ij} = i + j \pmod{11}$ $(0 \leqslant i, j \leqslant 10)$. Give a permutation π to the rows of L we obtain a new r-SOLS(11) for r shown in Table 6.

$$L_1=\begin{array}{ccccccccccc}
a & 8 & 9 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0\\
9 & a & 7 & 8 & 5 & 6 & 3 & 4 & 1 & 0 & 2\\
8 & 6 & a & 9 & 7 & 4 & 5 & 2 & 0 & 3 & 1\\
6 & 9 & 8 & a & 4 & 3 & 1 & 0 & 7 & 2 & 5\\
7 & 4 & 6 & 5 & a & 0 & 2 & 1 & 9 & 8 & 3\\
4 & 7 & 5 & 2 & 1 & a & 0 & 9 & 3 & 6 & 8\\
5 & 2 & 4 & 0 & 3 & 1 & a & 6 & 8 & 7 & 9\\
2 & 5 & 3 & 1 & 0 & 9 & 7 & 8 & 4 & a & 6\\
3 & 0 & 1 & 6 & 9 & 2 & 8 & 5 & a & 4 & 7\\
0 & 1 & 2 & 3 & 8 & 7 & 6 & a & 5 & 9 & 4\\
1 & 3 & 0 & 4 & 2 & 8 & 9 & 7 & 6 & 5 & a
\end{array}
\qquad
L_2=\begin{array}{ccccccccccc}
0 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & a & 7 & 9\\
a & 1 & 0 & 2 & 5 & 3 & 7 & 4 & 9 & 8 & 6\\
9 & a & 2 & 7 & 0 & 4 & 8 & 1 & 6 & 3 & 5\\
8 & 7 & 5 & 3 & a & 0 & 9 & 6 & 4 & 2 & 1\\
7 & 9 & 8 & 6 & 4 & 2 & a & 0 & 5 & 1 & 3\\
4 & 6 & a & 1 & 8 & 5 & 2 & 3 & 7 & 9 & 0\\
3 & 8 & 9 & a & 7 & 1 & 6 & 2 & 0 & 5 & 4\\
6 & 5 & 3 & 9 & 2 & a & 4 & 7 & 1 & 0 & 8\\
5 & 4 & 7 & 0 & 1 & 9 & 3 & a & 8 & 6 & 2\\
1 & 3 & 4 & 5 & 6 & 8 & 0 & 9 & 2 & a & 7\\
2 & 0 & 6 & 8 & 9 & 7 & 1 & 5 & 3 & 4 & a
\end{array}$$

Fig. 8. A 13-SOLS(11) and a 118-SOLS(11).

Table 7

r	Square	Order-2-interchanges				r	Square	Order-2-interchanges		
15	L_1	0, 2; 5, 6				96	L_2	0, 8; 7, 8	0, 10; 0, 1	2, 9; 3, 10
17	L_1	0, 1; 4, 5				97	L_2	0, 8; 7, 8	0, 10; 0, 1	1, 7; 6, 7
18	L_1	1, 9; 0, 9				98	L_2	0, 8; 7, 8	0, 10; 0, 1	0, 8; 1, 3
19	L_1	0, 6; 1, 8				99	L_2	0, 8; 7, 8	0, 10; 0, 1	2, 10; 0, 4
20	L_1	0, 6; 1, 8	2, 7; 0, 7			100	L_2	0, 8; 7, 8	0, 10; 0, 1	1, 4; 0, 6
21	L_1	0, 6; 1, 8	0, 2; 5, 6			101	L_2	0, 8; 7, 8	0, 10; 0, 1	2, 3; 0, 6
22	L_1	0, 6; 1, 8	1, 9; 0, 9			102	L_2	0, 8; 7, 8	0, 10; 0, 1	0, 1; 1, 2
23	L_1	0, 6; 1, 8	0, 1; 4, 5			103	L_2	0, 8; 7, 8	0, 3; 1, 9	
24	L_1	0, 6; 1, 8	2, 7; 2, 9			104	L_2	0, 8; 7, 8	0, 10; 0, 1	
25	L_1	0, 6; 1, 8	0, 1; 6, 7			105	L_2	0, 8; 7, 8	1, 4; 0, 6	
26	L_1	0, 6; 1, 8	0, 1; 6, 7	0, 7; 7, 8		106	L_2	0, 8; 7, 8	4, 6; 1, 2	
27	L_1	0, 6; 1, 8	0, 1; 6, 7	2, 5; 3, 7		107	L_2	0, 8; 7, 8	2, 9; 1, 9	
28	L_1	0, 6; 1, 8	0, 1; 6, 7	1, 9; 0, 9		108	L_2	0, 8; 7, 8	2, 3; 0, 6	
29	L_1	0, 6; 1, 8	0, 1; 6, 7	0, 1; 4, 5		109	L_2	0, 8; 7, 8	4, 10; 2, 3	
30	L_1	0, 6; 1, 8	0, 1; 6, 7	2, 7; 2, 9		110	L_2	0, 3; 1, 9		
32	L_1	0, 6; 1, 8	0, 1; 6, 7	0, 1; 4, 5	1, 9; 0, 9	112	L_2	0, 5; 4, 7		
94	L_2	0, 8; 7, 8	0, 10; 0, 1	1, 7; 6, 7	4, 6; 1, 2	114	L_2	2, 9; 1, 9		

Give several proper order-2-interchanges to L_1 or L_2 in Fig. 8 in turn, we can get a new r-SOLS. We list the order-2-interchanges and the corresponding r in Table 7.

Give order-2-interchange $I_{(0,1;3,4)}$ to the 14-SOLS(11) in Fig. 9 we obtain a 16-SOLS(11).

From [16] we have an ISOLS(11; 2, 2, 2, 2, 2), from Theorem 3.3 we have an ISOLS(11; 2, 2, 2, 2), an ISOLS (11; 2, 2, 2) and an ISOLS(11; 2, 2). Filling the holes of all the above squares with a symmetric Latin square of order 2 we then obtain r-SOLS(11) for $r \in \{111, 113, 115, 117\}$.

116-SOLS(11) is shown in Fig. 9 and 118-SOLS(11) is in Fig. 8. □

Lemma 4.4. *There exists an r-SOLS(12) for every $r \in [16, 138]$.*

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p = m = 2$, $k = 2$, $l = 0$, $n = 6$ and $R_1 = [8, 31]$, the input designs, q_1-SOLS(6) for $q_1 \in R_1$, are from Theorems 1.3, then we obtain an r-SOLS(12) for every $r = q_{11} + q_{12} \in 2 \otimes R_1 = [16, 62]$.

Suppose that $L = (a_{ij})$ is a symmetric Latin square of order 12, where $a_{ij} = i + j \pmod{12}$ $(0 \leqslant i, j \leqslant 11)$. Give a proper permutation π to the rows of L we obtain an r-SOLS(12) for r shown in Table 8.

Give several proper order-2-interchanges to the SOLS(12) in Fig. 10 in turn, we obtain a new r-SOLS(12). We list the order-2-interchanges and the corresponding r in Table 9.

0	2	1	3	4	5	6	7	8	9	a
1	0	2	4	3	6	5	8	7	a	9
2	1	0	5	6	3	4	9	a	7	8
4	3	6	0	7	8	9	a	2	1	5
3	4	5	8	0	9	a	6	1	2	7
6	5	4	7	a	1	8	2	9	3	0
5	6	3	a	9	7	1	0	4	8	2
8	7	a	9	5	0	2	1	3	6	4
7	8	9	1	2	a	3	4	0	5	6
a	9	8	2	1	4	7	5	6	0	3
9	a	7	6	8	2	0	3	5	4	1

0	1	2	3	4	5	6	7	8	9	a
9	2	3	4	5	6	7	8	a	0	1
8	a	4	0	1	2	3	6	7	5	9
7	9	5	1	3	4	8	2	6	a	0
6	8	7	5	9	1	0	a	2	3	4
4	7	a	2	6	3	9	0	5	1	8
5	4	9	6	7	0	a	3	1	8	2
2	6	0	9	8	a	1	5	4	7	3
3	5	8	a	0	7	2	1	9	4	6
a	3	1	7	2	8	4	9	0	6	5
1	0	6	8	a	9	5	4	3	2	7

Fig. 9. A 14-SOLS(11) and a 116-SOLS(11).

Table 8

r	π												r	π											
63	0	1	2	3	4	6	7	10	5	8	9	11	94	0	1	2	3	4	7	10	8	9	6	5	11
64	0	1	2	3	4	5	7	10	8	6	9	11	95	0	1	2	3	5	7	10	4	6	9	8	11
65	0	1	2	3	4	6	9	8	5	10	7	11	96	0	1	2	3	5	6	10	9	4	8	7	11
66	0	1	2	3	4	6	7	8	10	9	5	11	97	0	1	2	3	5	8	10	9	7	6	4	11
67	0	1	2	3	4	5	10	8	9	7	6	11	98	0	1	2	4	5	8	3	10	9	7	6	11
68	0	1	2	3	4	6	5	8	10	9	7	11	99	0	1	2	3	5	10	6	9	4	8	7	11
69	0	1	2	3	4	6	7	5	10	8	9	11	100	0	1	2	3	7	6	10	5	8	4	9	11
70	0	1	2	3	4	5	6	8	10	9	7	11	101	0	1	2	3	5	10	8	6	4	9	7	11
71	0	1	2	3	4	5	7	10	9	8	6	11	102	0	1	2	4	6	10	9	7	5	3	8	11
72	0	1	2	3	4	5	6	9	10	8	7	11	103	0	1	2	4	6	9	10	8	5	3	7	11
73	0	1	2	3	4	5	7	8	10	6	9	11	104	0	1	2	5	7	6	10	3	9	4	8	11
74	0	1	2	3	4	5	9	10	8	7	6	11	105	0	1	2	4	6	9	7	10	5	3	8	11
75	0	1	2	3	4	5	7	9	10	8	6	11	106	0	1	2	5	10	6	3	9	4	8	7	11
76	0	1	2	3	4	6	7	10	8	9	5	11	107	0	1	2	4	10	8	5	3	9	7	6	11
77	0	1	2	3	4	5	7	10	9	6	8	11	108	0	1	2	4	10	8	6	3	9	7	5	11
78	0	1	2	3	4	6	7	10	5	9	8	11	109	0	1	2	4	10	8	7	5	3	9	6	11
79	0	1	2	3	4	6	9	5	10	8	7	11	110	0	1	3	5	7	10	8	6	4	2	9	11
80	0	1	2	3	4	6	8	7	10	5	9	11	111	0	1	3	7	6	10	5	4	8	2	9	11
81	0	1	2	3	5	4	6	9	10	8	7	11	112	0	1	2	6	7	10	5	9	4	8	3	11
82	0	1	2	3	4	7	9	8	6	10	5	11	113	0	2	4	8	10	6	1	3	5	7	9	11
83	0	1	2	3	4	6	9	8	10	7	5	11	114	0	3	6	4	9	1	10	2	7	5	8	11
84	0	1	2	3	4	7	6	10	8	5	9	11	115	0	2	4	6	8	10	5	1	3	7	9	11
85	0	1	2	3	4	5	8	10	9	7	6	11	116	0	3	4	7	10	2	9	6	1	5	8	11
86	0	1	2	3	4	6	9	7	10	8	5	11	117	0	3	6	2	1	4	7	10	9	5	8	11
87	0	1	2	3	4	6	8	10	5	7	9	11	118	0	2	10	6	1	9	5	7	8	4	3	11
88	0	1	2	3	4	5	8	10	7	9	6	11	120	0	6	10	2	4	8	3	7	9	1	5	11
89	0	1	2	3	4	6	7	10	9	5	8	11	122	1	3	8	9	2	4	7	10	0	5	6	11
90	0	1	2	3	4	6	8	10	7	9	5	11	123	0	4	6	10	2	3	8	9	1	5	7	11
91	0	1	2	3	4	6	10	9	7	8	5	11	124	1	3	8	2	4	9	7	5	10	0	6	11
92	0	1	2	3	4	7	9	6	10	5	8	11	126	1	5	9	10	3	4	8	0	2	6	7	11
93	0	1	2	3	4	6	10	9	5	8	7	11													

Fill the hole of an ISOLS(12, 3) from Theorem 3.2 with a symmetric Latin square of order 3 to get a 138-SOLS(12). This completes the proof. □

Lemma 4.5. *There exists an r-SOLS*(13) *for every* $r \in [17, 163]$.

Proof. Start with an SOLS(4), applying Construction 3.8 with $m = 4$, $n = 3$, $t = 0$, $R_1 = \{4, 9, 16\}$ and $R_2 = \{3, 9\}$, the input designs, q_1-SOLS(4) for $q_1 \in R_1$ and q_2-MOLS(3) for $q_2 \in R_2$, are from Theorems 1.1–1.3, then we obtain an r-SOLS(13) for every $r = \sum_{i=1}^{4} (q_{1i} - 1) + 2\sum_{j=1}^{6} q_{2j} + 1 \in 4 \otimes (R_1 - 1) + 6 \otimes (2R_2) + 1$.

0	2	7	9	6	b	a	3	5	8	1	4
3	1	8	6	a	7	2	b	9	4	5	0
5	b	2	0	8	1	4	9	3	6	7	a
a	4	1	3	0	9	8	5	7	2	b	6
9	0	5	8	4	6	b	1	a	3	2	7
1	8	9	4	7	5	0	a	2	b	6	3
7	a	b	2	9	3	6	4	0	5	8	1
b	6	3	a	2	8	5	7	4	1	0	9
2	7	6	b	1	4	9	0	8	a	3	5
6	3	a	7	5	0	1	8	b	9	4	2
4	9	0	5	b	2	3	6	1	7	a	8
8	5	4	1	3	a	7	2	6	0	9	b

Fig. 10. An SOLS(12).

Table 9

r	Order-2-interchanges				r	Order-2-interchanges	
119	0, 2; 4, 9	0, 7; 2, 7	3, 4; 1, 4	0, 2; 6, 11	130	0, 2; 4, 9	2, 5; 1, 9
121	0, 2; 4, 9	0, 7; 2, 7	0, 2; 6, 11		131	0, 2; 4, 9	2, 9; 2, 11
125	0, 2; 4, 9	0, 7; 2, 7	0, 10; 7, 9		132	0, 2; 4, 9	0, 11; 3, 10
127	0, 2; 4, 9	0, 7; 2, 7	1, 6; 3, 6		133	0, 2; 4, 9	5, 10; 7, 10
128	0, 2; 4, 9	0, 2; 6, 11			134	0, 2; 4, 9	1, 6; 3, 6
129	0, 2; 4, 9	0, 7; 2, 7			136	0, 7; 2, 7	
135	5, 10; 7, 10	2, 9; 2, 11	3, 9; 0, 11	4, 6; 1, 8	137	0, 2; 4, 9	

A computer search shows that $4\otimes(R_1-1)+6\otimes(2R_2)+1=\{49, 54, 59, 61, 64, 66, 69, 71, 73, 76, 78, 81, 83, 85, 88, 90, 93, 95, 97, 100, 102, 105, 107, 109, 112, 114, 117, 119, 121, 124, 126, 129, 131, 133, 136, 138, 141, 143, 145, 148, 150, 155, 157, 162, 169\}$.

Suppose that $L=(a_{ij})$ is a symmetric Latin square of order 13, where $a_{ij}=i+j \pmod{13}$ $(0\leqslant i, j\leqslant 12)$. Give a proper permutation π to the rows of L we obtain an r-SOLS(13) for r shown in Table 10.

Give several proper order-2-interchanges to the 15-SOLS(13), L_1, or SOLS(13), L_2, in Fig. 11 in turn, we obtain a new r-SOLS(13). We list the order-2-interchanges and the corresponding r in Table 11.

We denote the 16-SOLS(13) shown in Fig. 12 by L. Then $I_{(0,1;3,4)}(L)$ is an 18-SOLS(13), $I_{(6,9;2,8)}(I_{(1,9;2,11)}(L))$ is a 28-SOLS(13). 159-SOLS(13) is in Fig. 12. 160-SOLS(13) is from Lemma 2.3. Fill the hole of an ISOLS(13, 3) from Theorem 3.2 with a symmetric Latin square of order 3 to get a 163-SOLS(13). This completes the proof. □

5. Existence of r-SOLS(v) for $14\leqslant v\leqslant 27$

Lemma 5.1. *There exists an r-SOLS(14) for every $r\in[18, 190]$.*

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p=m=2$, $k=2$, $l=0$, $n=7$ and $R_1=[9, 45]\cup\{47, 49\}$, the input designs, q_1-SOLS(7) for $q_1\in R_1$, are from Theorems 1.2 and 1.3, then we obtain an r-SOLS(14) for every $r=q_{11}+q_{12}\in 2\otimes R_1\supset[18, 94]$.

Start with an SOLS(4) with a symmetric transversal off the main diagonal from Theorem 3.9, applying Construction 3.8 with $m=4, n=3, t=1, R_1=\{4, 9, 16\}, R_2=\{3, 9\}, R_3=\{4, 6, 8, 9, 12, 16\}$, the input designs, q_1-SOLS(4) for $q_1\in R_1$, q_2-MOLS(3) for $q_2\in R_2$, and q_3-MOLS(4) for $q_3\in R_3$, are from Theorems 1.1–1.3, then we obtain an r-SOLS(14) for every $r=\sum_{i=1}^{4}(q_{1i}-1)+2\sum_{j=1}^{4}q_{2j}+2\sum_{k=1}^{2}(q_{3k}-1)+2\in 4\otimes(R_1-1)+4\otimes(2R_2)+2\otimes(2(R_3-1))+2$.

Table 10

r	π													r	π												
37	0	1	2	3	4	5	6	7	8	9	10	12	11	84	0	1	2	3	4	6	5	7	10	11	9	12	8
39	0	1	2	3	4	5	6	7	8	11	12	9	10	86	0	1	2	3	4	6	5	7	9	11	12	10	8
40	0	1	4	12	11	10	9	8	7	6	5	2	3	87	0	1	2	3	4	5	6	8	9	11	12	10	7
41	0	1	2	3	4	5	6	7	8	10	9	12	11	89	0	1	2	3	4	5	6	9	11	12	8	7	10
42	0	1	5	4	3	2	7	6	12	11	10	9	8	91	0	1	2	3	4	5	6	8	11	12	10	9	7
43	0	1	2	3	10	11	12	6	7	8	9	5	4	92	0	1	2	3	4	6	5	7	11	12	9	10	8
44	0	1	8	12	11	10	9	6	7	4	5	2	3	94	0	1	2	3	4	6	5	7	11	10	12	9	8
45	0	1	2	3	4	5	6	8	7	10	9	12	11	96	0	1	2	3	4	6	5	7	10	12	11	9	8
46	0	1	3	2	9	8	12	11	10	4	5	7	6	98	0	1	2	3	4	6	5	8	12	11	7	10	9
47	0	1	2	3	4	6	5	8	7	10	9	12	11	99	0	1	2	3	4	5	6	8	9	12	11	7	10
48	0	1	4	3	2	12	11	8	7	10	9	6	5	101	0	1	2	3	4	5	6	8	12	11	9	10	7
50	0	1	2	3	5	4	7	6	9	8	12	11	10	103	0	1	2	3	4	5	6	8	12	11	7	10	9
51	0	1	2	3	4	6	5	7	8	10	9	12	11	104	0	1	2	3	4	6	7	9	11	5	10	12	8
52	0	1	2	3	5	12	7	8	9	10	4	11	6	106	0	1	2	3	4	6	5	9	12	10	11	8	7
53	0	1	2	3	4	5	6	7	9	8	12	11	10	108	0	1	2	3	4	6	7	9	11	5	12	10	8
55	0	1	2	3	4	5	6	7	8	12	11	10	9	110	0	1	2	3	4	6	8	10	12	5	9	11	7
56	0	1	2	4	3	5	8	7	6	12	11	10	9	111	0	1	2	3	4	5	6	9	12	10	11	8	7
57	0	1	2	3	4	5	6	7	10	12	8	11	9	113	0	1	2	3	4	5	8	11	9	12	7	6	10
58	0	1	2	3	4	6	5	10	11	12	7	8	9	115	0	1	2	3	4	5	10	12	7	11	9	8	6
60	0	1	2	3	4	6	5	8	7	12	11	10	9	116	0	1	2	3	4	6	11	7	10	12	9	5	8
62	0	1	2	3	4	6	5	12	11	10	9	8	7	118	0	1	2	3	4	6	12	7	11	10	5	9	8
63	0	1	2	3	4	5	6	7	9	8	11	12	10	120	0	1	2	3	5	6	10	9	12	8	4	11	7
65	0	1	2	3	4	5	6	8	7	9	11	12	10	122	0	1	2	3	5	7	10	12	9	6	8	4	11
67	0	1	2	3	4	5	6	7	8	10	12	9	11	123	0	1	2	3	5	12	6	11	10	4	9	8	7
68	0	1	2	3	4	6	5	8	7	10	11	12	9	125	0	1	2	4	5	7	9	12	3	6	8	10	11
70	0	1	2	3	4	6	5	8	7	10	12	9	11	127	0	1	2	4	5	8	12	11	6	3	9	10	7
72	0	1	2	3	4	6	5	7	9	10	11	12	8	128	0	1	2	4	5	10	8	12	11	9	3	7	6
74	0	1	2	3	4	6	5	7	9	10	12	8	11	130	0	1	2	4	8	10	3	6	9	12	11	5	7
75	0	1	2	3	4	5	6	8	7	10	12	11	9	132	0	1	3	7	9	11	2	4	8	10	12	6	5
77	0	1	2	3	4	5	6	7	8	10	12	11	9	135	0	1	3	4	9	6	12	10	2	7	5	11	8
79	0	1	2	3	4	5	6	7	8	11	12	10	9	137	0	1	2	4	5	6	12	11	10	3	9	8	7
80	0	1	2	3	4	6	5	7	10	11	12	8	9	151	0	1	3	5	7	12	11	9	2	4	6	8	10
82	0	1	2	3	4	6	5	7	10	12	11	8	9														

$$L_1 = \begin{array}{ccccccccccccc}
b & c & a & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
c & b & 9 & a & 7 & 8 & 5 & 6 & 3 & 4 & 1 & 0 & 2 \\
a & 8 & c & b & 9 & 6 & 7 & 4 & 5 & 2 & 0 & 3 & 1 \\
8 & a & b & c & 6 & 9 & 4 & 7 & 1 & 0 & 5 & 2 & 3 \\
9 & 6 & 8 & 7 & c & 3 & 2 & 1 & 0 & b & a & 5 & 4 \\
6 & 9 & 7 & 8 & 2 & c & 1 & 0 & b & a & 3 & 4 & 5 \\
7 & 4 & 6 & 5 & 3 & 0 & c & 2 & a & 1 & b & 9 & 8 \\
4 & 7 & 5 & 6 & 0 & 1 & 3 & c & 2 & 9 & 8 & b & a \\
5 & 2 & 4 & 0 & 1 & b & a & 3 & c & 8 & 7 & 6 & 9 \\
2 & 5 & 3 & 1 & b & a & 0 & 8 & 9 & c & 4 & 7 & 6 \\
3 & 0 & 1 & 4 & a & 2 & b & 9 & 6 & 5 & c & 8 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 8 & a & 7 & 6 & 9 & c & b \\
1 & 3 & 0 & 2 & 5 & 4 & 9 & b & 8 & 7 & 6 & a & c
\end{array}
\qquad
L_2 = \begin{array}{ccccccccccccc}
0 & a & 8 & c & 2 & 9 & 7 & b & 6 & 1 & 3 & 5 & 4 \\
8 & 1 & 9 & 6 & 7 & c & a & 0 & b & 2 & 5 & 4 & 3 \\
c & 8 & 2 & a & b & 6 & 9 & 1 & 0 & 7 & 4 & 3 & 5 \\
5 & c & 4 & 3 & a & 1 & b & 9 & 2 & 6 & 7 & 8 & 0 \\
a & 9 & 3 & 5 & 4 & b & 2 & c & 7 & 8 & 0 & 1 & 6 \\
3 & 4 & b & 9 & 1 & 5 & c & 6 & a & 0 & 8 & 7 & 2 \\
b & 3 & 5 & 8 & 9 & 2 & 6 & a & c & 4 & 1 & 0 & 7 \\
4 & b & c & 0 & 3 & a & 1 & 7 & 9 & 5 & 2 & 6 & 8 \\
9 & 5 & a & b & c & 7 & 0 & 4 & 8 & 3 & 6 & 2 & 1 \\
2 & 0 & 1 & 7 & 6 & 8 & 5 & 3 & 4 & 9 & b & c & a \\
6 & 7 & 0 & 2 & 5 & 3 & 4 & 8 & 1 & c & a & 9 & b \\
1 & 6 & 7 & 4 & 8 & 0 & 3 & 2 & 5 & a & c & b & 9 \\
7 & 2 & 6 & 1 & 0 & 4 & 8 & 5 & 3 & b & 9 & a & c
\end{array}$$

Fig. 11. A 15-SOLS(13) and an SOLS(13).

A computer search shows that $4 \otimes (R_1 - 1) + 4 \otimes (2R_2) + 2 \otimes (2(R_3 - 1)) + 2 \supset [95, 168] \cup [170, 175] \cup \{178, 179, 180, 182, 186\}$.

Fig. 13 is an SOLS(14), where a, b, c and d denote 10, 11, 12 and 13, respectively. Give several proper order-2-interchanges to the SOLS(14) in turn, we obtain a new r-SOLS(14). We list the order-2-interchanges and the corresponding r in Table 12.

Table 11

r	Square	Order-2-interchanges				
17	L_1	0, 2; 5, 6				
19	L_1	0, 1; 2, 3				
20	L_1	0, 11; 0, 12				
21	L_1	10, 12; 6, 7				
22	L_1	0, 11; 0, 12	0, 2; 5, 6			
23	L_1	10, 12; 6, 7	0, 1; 2, 3			
24	L_1	0, 11; 0, 12	0, 1; 2, 3			
25	L_1	10, 12; 6, 7	0, 1; 4, 5			
26	L_1	0, 11; 0, 12	0, 4; 7, 11			
27	L_1	0, 11; 0, 12	0, 11; 1, 11			
29	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 1; 2, 3		
30	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 11; 0, 12		
31	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 1; 6, 7		
32	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 11; 1, 11		
33	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 5; 7, 12		
34	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 11; 1, 11	0, 1; 2, 3	
35	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 5; 7, 12	0, 1; 2, 3	
36	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 5; 7, 12	8, 12; 0, 4	
38	L_1	10, 12; 6, 7	1, 7; 5, 10	0, 5; 7, 12	0, 11; 1, 11	
134	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7	5, 7; 7, 11	0, 6; 4, 5
139	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7	5, 7; 7, 11	9, 10; 9, 11
140	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7	5, 7; 7, 11	0, 2; 1, 3
142	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7	5, 7; 7, 11	
144	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7	9, 11; 9, 12	
146	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7	9, 10; 9, 11	
147	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7	0, 2; 1, 3	
149	L_2	1, 3; 1, 5	1, 2; 0, 1	4, 8; 4, 7		
152	L_2	1, 3; 1, 5	1, 2; 0, 1	3, 10; 7, 11		
153	L_2	1, 3; 1, 5	1, 2; 0, 1	9, 10; 9, 11		
154	L_2	1, 3; 1, 5	0, 6; 4, 5			
156	L_2	1, 3; 1, 5	1, 2; 0, 1			
158	L_2	1, 3; 1, 5	3, 9; 5, 11			
161	L_2	0, 6; 4, 5				

0	2	1	3	4	5	6	7	8	9	a	b	c
1	0	2	4	3	6	5	8	7	a	9	c	b
2	1	0	5	6	3	4	9	a	b	c	7	8
4	3	6	0	2	1	7	5	9	c	b	8	a
3	4	5	1	0	2	9	b	c	6	8	a	7
6	5	4	2	1	0	c	a	b	8	7	3	9
5	6	3	8	a	b	0	c	2	7	1	9	4
8	7	a	6	c	9	b	0	1	3	2	4	5
7	8	9	a	b	c	1	2	0	4	3	5	6
a	9	c	b	5	7	8	4	3	1	6	2	0
9	a	b	c	7	8	2	1	4	5	0	6	3
c	b	8	7	9	4	a	3	6	0	5	1	2
b	c	7	9	8	a	3	6	5	2	4	0	1

0	a	8	c	2	9	7	b	6	1	4	3	5
8	1	9	6	7	c	a	0	b	2	3	5	4
c	8	2	a	b	6	9	1	0	7	5	4	3
5	c	4	3	a	1	b	9	2	6	7	8	0
a	9	3	5	4	b	2	c	7	8	0	1	6
3	4	b	9	1	5	c	6	a	0	2	7	8
b	3	5	8	9	2	6	a	c	4	1	0	7
4	b	c	0	3	a	1	7	9	5	8	6	2
9	5	a	b	c	7	0	4	8	3	6	2	1
2	0	1	7	6	8	5	3	4	9	b	c	a
6	7	0	2	5	3	4	8	1	c	a	9	b
1	6	7	4	8	0	3	2	5	a	c	b	9
7	2	6	1	0	4	8	5	3	b	9	a	c

Fig. 12. A 16-SOLS(13) and a 159-SOLS(13).

From [1, Lemma 2.7] we have an ISOLS$(14; 4, 2)$. Filling the hole of side four with a 9-SOLS(4) from Theorem 1.3, and the hole of side two with a symmetric Latin square of order two we obtain a 187-SOLS(14).

From Theorem 3.2 we have an ISOLS$(14, 3)$. Filling the hole of side three with a symmetric Latin square of order three we obtain a 190-SOLS(14). This completes the proof. □

0	6	d	7	c	3	8	a	9	b	5	4	2	1
a	1	7	c	5	b	2	4	d	3	9	6	8	0
8	b	2	9	7	d	a	6	c	1	4	5	0	3
d	7	a	3	6	4	9	1	b	c	8	0	5	2
9	c	0	b	4	6	3	2	a	d	7	8	1	5
6	8	1	a	d	5	c	b	7	2	0	3	9	4
c	9	8	d	b	0	6	5	3	a	2	1	4	7
5	d	c	8	a	2	b	7	4	0	1	9	3	6
b	5	3	0	1	a	d	c	8	4	6	2	7	9
4	a	b	1	2	c	0	d	5	9	3	7	6	8
7	0	6	2	9	8	4	3	1	5	a	c	d	b
1	2	9	4	3	7	5	8	0	6	d	b	a	c
3	4	5	6	0	1	7	9	2	8	b	d	c	a
2	3	4	5	8	9	1	0	6	7	c	a	b	d

Fig. 13. An SOLS(14).

Table 12

r	Order-2-interchanges				r	Order-2-interchanges	
169	1, 3; 3, 9	0, 7; 5, 12	2, 4; 4, 10	0, 9; 4, 5	184	1, 3; 3, 9	1, 8; 1, 4
176	1, 3; 3, 9	0, 7; 5, 12	0, 9; 4, 5		185	1, 3; 3, 9	1, 12; 9, 12
177	1, 3; 3, 9	0, 7; 5, 12	0, 2; 2, 5		188	0, 1; 5, 9	
181	1, 3; 3, 9	0, 1; 6, 12			189	1, 3; 3, 9	
183	1, 3; 3, 9	1, 5; 6, 9					

Lemma 5.2. *There exists an r-SOLS*(15) *for every* $r \in [19, 219]$.

Proof. Start with a symmetric Latin square of order 3, applying Construction 3.6 with $p = m = 3$, $k = 3$, $l = 0$, $n = 5$ and $R_1 = \{5, 7, 10, 11, 13, 14, 15, 17, 19, 21, 25\}$, the input designs, q_1-SOLS(5) for $q_1 \in R_1$, are from Theorems 1.2 and 1.3, we can obtain an r-SOLS(15) for every $r = \sum_{i=1}^{3} q_{1i} \in 3 \otimes R_1 \supset [19, 61]$.

Start with an SOLS(4) with two symmetric transversal off the main diagonal from Theorem 3.9, applying Construction 3.8 with $m=4$, $n=3$, $t=2$, $R_1=\{4, 9, 16\}$, $R_2=\{3, 9\}$, $R_3=\{4, 6, 8, 9, 12, 16\}$, the input designs, q_1-SOLS(4) for $q_1 \in R_1$, q_2-MOLS(3) for $q_2 \in R_2$, and q_3-MOLS(4) for $q_3 \in R_3$ are from Theorems 1.1–1.3, then we obtain an r-SOLS(15) for every $r = \sum_{i=1}^{4} (q_{1i} - 1) + 2\sum_{j=1}^{2} q_{2j} + 2\sum_{k=1}^{4} (q_{3k} - 1) + 3 \in 4 \otimes (R_1 - 1) + 2 \otimes (2R_2) + 4 \otimes (2(R_3 - 1)) + 3$.

A computer search shows that $4 \otimes (R_1 - 1) + 2 \otimes (2R_2) + 4 \otimes (2(R_3 - 1)) + 3 \supset [63, 193] \cup [195, 200] \cup \{203, 204, 205, 207, 211, 212, 219\}$.

Suppose that $L = (a_{ij})$ is a symmetric Latin square of order 15, where $a_{ij} = i + j \pmod{15}$ $(0 \leqslant i, j \leqslant 14)$. Give the permutation $\pi = (0\ 1\ 2\ 3\ 4\ 5\ 7\ 6\ 9\ 8\ 11\ 10\ 14\ 13\ 12)$ to the rows of L we obtain a 62-SOLS(15).

Fig. 14 is an SOLS(15), where a, b, c, d and e denote 10, 11, 12, 13 and 14, respectively. Give several proper order-2-interchanges to the SOLS(15) in turn, we obtain a new r-SOLS(15). We list the order-2-interchanges and the corresponding r in Table 13.

216-SOLS(15) is from Lemma 2.3. This completes the proof. □

Lemma 5.3. *There exists an r-SOLS*(16) *for every* $r \in [20, 250] \cup \{252\}$.

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p = m = 2$, $k = 2$, $l = 0$, $n = 8$ and $R_1 = [10, 62]$, the input designs, q_1-SOLS(8) for $q_1 \in R_1$, are from Theorem 1.4, then we obtain an r-SOLS(16) for every $r = q_{11} + q_{12} \in 2 \otimes R_1 = [20, 124]$.

Start with an SOLS(4), applying Corollary 3.7 with $m = n = 4$, $R_1 = \{4, 9, 16\}$, $R_2 = \{4, 6, 8, 9, 12, 16\}$, the input designs, q_1-SOLS(4) for $q_1 \in R_1$ and q_2-MOLS(4) for $q_2 \in R_2$, are from Theorems 1.1–1.3, then we obtain an r-SOLS(16) for every $r = \sum_{i=1}^{4} p_i + 2\sum_{j=1}^{6} q_j \in 4 \otimes R_1 + 6 \otimes (2R_2)$, where $p_i \in R_1$, $q_j \in R_2$.

0	c	a	d	9	6	1	b	e	2	8	7	3	5	4
9	1	d	7	c	2	0	a	6	b	e	8	5	4	3
c	0	2	6	7	1	9	e	a	8	b	d	4	3	5
8	9	b	3	5	0	d	4	1	c	7	e	2	a	6
5	b	8	a	4	d	2	6	3	e	c	9	1	0	7
2	4	c	9	3	5	e	d	0	7	1	6	a	b	8
a	5	3	2	b	7	6	8	d	0	4	1	e	c	9
4	e	9	1	0	a	c	7	5	3	6	2	d	8	b
3	d	4	e	1	c	b	2	8	6	0	5	9	7	a
e	3	1	4	6	b	a	5	7	9	d	0	8	2	c
b	8	5	0	e	9	3	1	2	4	a	c	7	6	d
d	a	e	5	2	8	7	c	4	1	3	b	6	9	0
6	7	0	8	d	3	4	9	b	5	2	a	c	e	1
1	6	7	b	8	e	5	3	c	a	9	4	0	d	2
7	2	6	c	a	4	8	0	9	d	5	3	b	1	e

Fig. 14. An SOLS(15).

Table 13

r	Order-2-interchanges				r	Order-2-interchanges	
194	0, 2; 0, 1	0, 8; 11, 13	0, 13; 4, 10	1, 3; 7, 13	210	0, 2; 0, 1	0, 8; 11, 13
201	0, 2; 0, 1	0, 8; 11, 13	0, 13; 4, 10	1, 2; 0, 6	213	0, 6; 1, 13	7, 13; 6, 8
202	0, 2; 0, 1	0, 8; 11, 13	0, 13; 4, 10		214	0, 2; 0, 1	0, 1; 0, 4
206	0, 2; 0, 1	0, 8; 11, 13	0, 1; 0, 4		215	0, 2; 0, 1	1, 2; 0, 6
208	0, 2; 0, 1	0, 8; 11, 13	0, 1; 1, 6		217	0, 6; 1, 13	
209	0, 2; 0, 1	0, 8; 11, 13	8, 11; 6, 11		218	0, 2; 0, 1	

Table 14

r	Order-2-interchanges			
231	0, 1; 4, 6	0, 4; 1; 9	0, 1; 5, 7	2, 7; 7, 13
239	0, 1; 4, 6	0, 4; 1; 9	2, 7; 7, 13	
243	0, 1; 4, 6	0, 4; 0; 8		

A computer search shows that $4 \otimes R_1 + 6 \otimes (2R_2) \supset [125, 230] \cup [232, 237] \cup \{240, 241, 242, 244, 248, 249\}$.

Filling the hole of side five of an ISOLS(16, 5) from Theorem 3.2 with an s-SOLS(5) for $s \in \{7, 14, 15, 19, 21\}$ from Theorem 1.3, we then obtain an r-SOLS(16) for $r \in \{238, 245, 246, 250, 252\}$.

Start with an SOLS(4), applying Inflation Construction, replace each cell of the SOLS(4) with the same SOLS(4) labelled by the element in that cell. We then get an SOLS(16) denoted by L. Give several proper order-2-interchanges to L in turn we get a new r-SOLS. We list the order-2-interchanges and the corresponding r in Table 14.

247-SOLS(16) is from Lemma 2.3. This completes the proof. □

Lemma 5.4. *There exists an r-SOLS*(17) *for every* $r \in [21, 283] \cup \{285\}$.

Proof. Applying Construction 3.14 with $n=5$, $k=2$, $r_1 \in M_6 = \{6\} \cup [8, 32]$, $P_1 = \{(0, 1), (1, 0), (2, 2), (3, 3), (4, 4)\}$, the input designs, r_1-MOLS(6) is from Theorem 1.1, 5-SOLS(5) with DOP set P_1, 6-SOLS(6) with x in the right bottom corner and DOP set $P_1 \cup \{(x, x)\}$, and the r_2-SOLS(6) with x in the right bottom corner and DOP set containing $P_1 \cup \{(x, x)\}$ for $r_2 \in \{6, 9, 11\}$, are from Figs. 15 and 16, we then obtain an r-SOLS(17) for every $r = 2r_1 + r_2 - 3 + 2 \in 2M_6 + \{6, 9, 11\} - 1 \supset [21, 68]$.

3	4	1	0	2
4	2	3	1	0
0	3	4	2	1
1	0	2	3	4
2	1	0	4	3

x	4	3	2	1	0
4	x	2	1	0	3
3	2	x	0	4	1
2	0	1	x	3	4
0	1	4	3	x	2
1	3	0	4	2	x

Fig. 15. A 5-SOLS(5) and a 6-SOLS(6).

x	1	3	4	2	0
0	x	4	2	3	1
3	4	0	1	x	2
4	2	x	0	1	3
2	3	1	x	0	4
1	0	2	3	4	x

4	x	1	2	3	0
x	3	4	0	2	1
3	4	x	1	0	2
2	1	0	4	x	3
0	2	3	x	1	4
1	0	2	3	4	x

Fig. 16. A 9-SOLS(6) and an 11-SOLS(6).

Applying Construction 3.8 with $m=n=4$, $t=0$, $R_1=\{5, 7, 10, 11, 13, 14, 15, 17, 19, 21, 25\}$, $R_2=\{4, 6, 8, 9, 12, 16\}$ we have an r-SOLS(17) for every $r = \sum_{i=1}^{4}(q_{1i}-1) + 2\sum_{j=1}^{6} q_{2j} + 1 \in 4 \otimes (R_1 - 1) + 6 \otimes (2R_2) + 1$, where $q_{1i} \in R_1$ and $q_{2j} \in R_2$.

A computer search shows that $1 + 4 \otimes (R_1 - 1) + 6 \otimes (2R_2) \supset [69, 275] \cup \{277, 278, 279, 281, 283, 285\}$.

276-SOLS(17) and 280-SOLS(17) are from Lemma 2.3. Filling the hole of an ISOLS(17, 4) from Theorem 3.2 with a 9-SOLS(4) we obtain a 282-SOLS(17). This completes the proof. □

Lemma 5.5. *There exists an r-SOLS*(18) *for every* $r \in [22, 318] \cup \{320\}$.

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p = m = 2$, $k = 2$, $l = 0$, $n = 9$ and $R_1 = [11, 79]$, the input designs, q_1-SOLS(9) for $q_1 \in R_1$, are from Lemmas 3.11, 4.1, and 3.5, then we obtain an r-SOLS(16) for every $r = q_{11} + q_{12} \in 2 \otimes R_1 = [22, 158]$.

Applying Construction 3.8 with $m=n=4$, $t=1$, $R_1=\{5, 7, 10, 11, 13, 14, 15, 17, 19, 21, 25\}$, $R_2=\{4, 6, 8, 9, 12, 16\}$, $R_3 = R_1$ we have an r-SOLS(18) for every $r = \sum_{i=1}^{4}(q_{1i}-1) + 2\sum_{j=1}^{4} q_{2j} + 2\sum_{k=1}^{2}(q_{3k}-1) + 2 \in 4 \otimes (R_1 - 1) + 4 \otimes (2R_2) + 2 \otimes (2(R_1 - 1)) + 2$, where $q_{1i}, q_{3k} \in R_1$ and $q_{2j} \in R_2$.

A computer search shows that $4 \otimes (R_1 - 1) + 4 \otimes (2R_2) + 2 \otimes (2(R_1 - 1)) + 2 \supset [159, 308] \cup \{310, 311, 312, 314, 316, 318\}$.

Filling the hole of an ISOLS(18, 5) from Theorem 3.2 with an s-SOLS(5) for $s \in \{10, 14, 21\}$ we obtain an r-SOLS(18) for $r \in \{309, 313, 320\}$. 315-SOLS(18) is from Lemma 2.3. Filling the hole of ISOLS(18, 4) from Theorem 3.2 with a 9-SOLS(4) we obtain a 317-SOLS(18). This completes the proof. □

Lemma 5.6. *There exists an r-SOLS*(19) *for every* $r \in [23, 357]$.

Proof. The first square in Fig. 17 is a 6-SOLS(6) with DOP set $Q = \{(0,0), (1,2), (2,1), (3,3), (4,5), (5,4)\}$. The second square in Fig. 17 is a 12-SOLS(7) with x in the right bottom corner and DOP set $Q \cup \{(0, 3), (3, 0), (4, 4), (5, x), (x, 5), (x, x)\}$. The third square in Fig. 17 is a 14-SOLS(7) with x in the right bottom corner and DOP set $Q \cup \{(0, 1), (1, 0), (0, 2), (2, 0), (2, 2), (4, 4), (5, 5), (x, x)\}$.

Applying Construction 3.14 with $n = 6$, $k = 1$, $r_1 \in M_7 = \{7\} \cup [9, 47]$, $P_1 = P = \{(i, i) : 0 \leqslant i \leqslant 5\}$ or $P_1 = Q = \{(0, 0), (1, 2), (2, 1), (3, 3), (4, 5), (5, 4)\}$, $r_2 = 7$ when $P_1 = P$, $r_2 \in \{12, 14\}$ when $P_1 = Q$, the input designs, r_1-MOLS(7) is from Theorem 1.1, 6-SOLS(6) with DOP set $P_1 = P$ and 7-SOLS(7) with DOP set $P \cup \{(x, x)\}$ are from symmetric Latin squares with entries $a_{ij} = i + j \pmod{v}$ $(i,j \in \mathbb{Z}_v)$, 6-SOLS(6) with DOP set $P_1 = Q$ and r_2-SOLS(7) with DOP set containing $Q \cup \{(x, x)\}$ for $r_2 \in \{12, 14\}$ are from Fig. 17, we then obtain an r-SOLS(19) for every $r = 2r_1 + r_2 - 3 + 1 \in 2M_7 + \{7, 12, 14\} - 2 \supset [23, 100]$.

0	1	2	3	4	5
2	0	1	5	3	4
1	2	0	4	5	3
3	4	5	0	1	2
5	3	4	2	0	1
4	5	3	1	2	0

x	2	0	3	5	1	4
1	4	5	x	3	2	0
0	x	4	5	2	3	1
3	5	x	4	1	0	2
4	0	1	2	x	5	3
2	1	3	0	4	x	5
5	3	2	1	0	4	x

0	1	5	2	3	x	4
2	0	x	4	5	1	3
4	x	2	3	1	5	0
1	4	3	5	x	0	2
3	5	0	x	4	2	1
x	2	4	1	0	3	5
5	3	1	0	2	4	x

Fig. 17. 6-SOLS(6) with associated 12-SOLS(7) and 14-SOLS(7).

Applying Construction 3.8 with $m=n=4, t=2, R_1=\{5, 7, 10, 11, 13, 14, 15, 17, 19, 21, 25\}, R_2=\{4, 6, 8, 9, 12, 16\}$, $R_3 = R_1$ we have an r-SOLS(19) for every $r = \sum_{i=1}^{4}(q_{1i} - 1) + 2\sum_{j=1}^{2} q_{2j} + 2\sum_{k=1}^{4}(q_{3k} - 1) + 3 \in 4 \otimes (R_1 - 1) + 2 \otimes (2R_2) + 4 \otimes (2(R_1 - 1)) + 3$, where $q_{1i}, q_{3k} \in R_1$ and $q_{2j} \in R_2$.

A computer search shows that $4 \otimes (R_1 - 1) + 2 \otimes (2R_2) + 4 \otimes (2(R_1 - 1)) + 3 \supset [101, 341]$.

Filling the hole of an ISOLS(19, 6) from Theorem 3.2 with an s-SOLS(6) for $s \in [17, 31]$ we obtain an r-SOLS(19) for every $r = 325 + s \in [342, 356]$. Filling the hole of ISOLS(19, 5) from Theorem 3.2 with a 21-SOLS(5) from Theorem 1.3 we obtain a 357-SOLS(19). This completes the proof. □

Lemma 5.7. *There exists an r-SOLS*(20) *for every* $r \in [24, 394] \cup \{396\}$.

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p=m=2,\ k=2,\ l=0,\ n=10$ and $R_1 = [12, 98]$, the input designs, q_1-SOLS(10) for $q_1 \in R_1$, are from Lemmas 3.11, 4.2, and 3.5, then we can obtain an r-SOLS(20) for every $r = q_{11} + q_{12} \in 2 \otimes R_1 = [24, 196]$.

Applying Corollary 3.7 with $m = 4$, $n = 5$, $R_1 = \{5, 7, 10, 11, 13, 14, 15, 17, 19, 21, 25\}$, $R_2 = R_1 \cup \{12, 16, 18\}$, the input designs, q_1-SOLS(5) for $q_1 \in R_1$ and q_2-MOLS(5) for $q_2 \in R_2$, are from Theorems 1.1–1.3, then we obtain an r-SOLS(20) for every $r = \sum_{i=1}^{4} q_{1i} + 2\sum_{j=1}^{6} q_{2j} \in 4 \otimes R_1 + 6 \otimes (2R_2)$.

A computer search shows that $4 \otimes R_1 + 6 \otimes (2R_2) \supset [197, 386] \cup \{388, 389, 390, 392, 394, 396\}$.

387-SOLS(20) and 391-SOLS(20) are from Lemma 2.3. Filling the hole of an ISOLS(20, 4) from Theorem 3.2 with a 9-SOLS(4) we obtain a 393-SOLS(20). This completes the proof. □

Lemma 5.8. *There exists an r-SOLS*(21) *for every* $r \in [25, 437]$.

Proof. Start with a symmetric Latin square of order 3, applying Construction 3.6 with $p = m = 3,\ k = 3,\ l = 0,$ $n = 7,\ R_1 = \{7\} \cup [9, 45]$, the input designs, q_1-SOLS(n) for $q_1 \in R_1$, are from Theorems 1.2 and 1.3, then we can obtain an r-SOLS(21) for every $r = \sum_{i=1}^{3} q_{1i} \in 3 \otimes R_1 \supset [25, 135]$.

Applying Construction 3.8 with $m=4, n=5, t=0, R_1=\{6\} \cup [8, 31], R_2=\{5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 25\}$ we have an r-SOLS(21) for every $r = \sum_{i=1}^{4}(q_{1i} - 1) + 2\sum_{j=1}^{6} q_{2j} + 1 \in 4 \otimes (R_1 - 1) + 6 \otimes (2R_2) + 1$, where $q_{1i} \in R_1$ and $q_{2j} \in R_2$.

A computer search shows that $4 \otimes (R_1 - 1) + 6 \otimes (2R_2) + 1 \supset [136, 421]$.

Filling the hole of an ISOLS(21, 6) from Theorem 3.2 with an s-SOLS(6) for $s \in [17, 31]$ we obtain an r-SOLS(21) for $r \in [422, 436]$. Filling the hole of ISOLS(21, 5) from Theorem 3.2 with a 21-SOLS(5) from Theorem 1.3 we obtain a 437-SOLS(21). This completes the proof. □

Lemma 5.9. *There exists an r-SOLS*(22) *for every* $r \in [26, 480]$.

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p = m = 2, k = 2, l = 0, n = 11$ and $R_1 = [13, 119]$, the input designs, q_1-SOLS(n) for $q_1 \in R_1$, are from Lemmas 3.11, 4.3, and 3.5, then we can obtain an r-SOLS(22) for every $r = q_{11} + q_{12} \in 2 \otimes R_1 = [26, 238]$.

Applying Construction 3.8 with $m=4, n=5, t=1, R_1=[8, 31], R_2=\{5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 25\}, R_3=[8, 32]$, the input designs, q_1-SOLS(6) for $q_1 \in R_1$, q_2-MOLS(5) for $q_2 \in R_2$, and q_3-MOLS(6) for $q_3 \in R_3$ are

0	1	2	3	4	5	6
2	0	3	4	5	6	1
1	4	0	5	6	2	3
4	3	6	0	2	1	5
3	6	5	1	0	4	2
6	5	1	2	3	0	4
5	2	4	6	1	3	0

x	0	1	2	3	4	5	6
0	x	2	1	4	3	6	5
2	1	x	0	5	6	3	4
1	2	0	x	6	5	4	3
4	3	6	5	x	0	1	2
3	4	5	6	0	x	2	1
6	5	4	3	2	1	x	0
5	6	3	4	1	2	0	x

0	x	1	2	3	4	5	6
x	1	2	0	4	3	6	5
2	0	x	1	5	6	3	4
1	2	0	x	6	5	4	3
4	3	6	5	x	0	1	2
3	4	5	6	0	x	2	1
6	5	4	3	2	1	x	0
5	6	3	4	1	2	0	x

Fig. 18. A 7-SOLS(7), an 8-SOLS(8), and a 13-SOLS(8).

from Theorems 1.1–1.3, we then obtain an r-SOLS(22) for every $r=\sum_{i=1}^{4}(q_{1i}-1)+2\sum_{j=1}^{4}q_{2j}+2\sum_{k=1}^{2}(q_{3k}-1)+2 \in 4\otimes(R_1-1)+4\otimes(2R_2)+2\otimes(2(R_3-1))+2$, where $q_{1i}\in R_1$, $q_{2j}\in R_2$ and $q_{3k}\in R_3$.

A computer search shows that $4\otimes(R_1-1)+4\otimes(2R_2)+2\otimes(2(R_3-1))+2\supset[239,446]$.

Filling the hole of an ISOLS(22, 7) from Theorem 3.2 with an s-SOLS(7) for $s\in[12,45]$ we obtain an r-SOLS(22) for $r\in[447,480]$. This completes the proof. □

Lemma 5.10. *There exists an r-SOLS(23) for every* $r\in[27,525]$.

Proof. Applying Construction 3.14 with $n=7, k=2, r_1\in M_8=\{8\}\cup[10,62], r_2\in\{8,11,13\}, P_1=\{(0,0),(1,2),(2,1),(3,4),(4,3),(5,6),(6,5)\}$, the input designs, r_1-MOLS(8) for $r_1\in M_8$ are from Theorem 1.1, 7-SOLS(7) with different ordered pair set P_1, 8-SOLS(8) with x in the right bottom corner and DOP set $P_1\cup\{(x,x)\}$, and 13-SOLS(8) with x in the right bottom corner and DOP set containing $P_1\cup\{(x,x)\}$, are from Fig. 18, 11-SOLS(8) with x in the right bottom corner and DOP set containing $P_1\cup\{(x,x)\}$, are from Lemma 3.10, we then obtain an r-SOLS(23) for every $r=2r_1+r_2-3+2\in 2M_8+\{8,11,13\}-1\supset[27,132]$.

Applying Construction 3.8 with $m=4, n=5, t=2, R_1=[8,31], R_2=\{5,7,10,11,12,13,14,15,16,17,18,19,21,25\}$, $R_3=[8,32]$ we obtain an r-SOLS(23) for every $r=\sum_{i=1}^{4}(q_{1i}-1)+2\sum_{j=1}^{2}q_{2j}+2\sum_{k=1}^{4}(q_{3k}-1)+3\in 4\otimes(R_1-1)+2\otimes(2R_2)+4\otimes(2(R_3-1))+3$, where $q_{1i}\in R_1$, $q_{2j}\in R_2$ and $q_{3k}\in R_3$.

A computer search shows that $4\otimes(R_1-1)+2\otimes(2R_2)+4\otimes(2(R_3-1))+3\supset[133,471]$.

From Theorem 3.4 we have an FSOLS($5^4 3^1$). Fill the four holes of side 5 with s-SOLS(5) for $s\in S_5=\{5,7,10,11,13,14,15,17,19,21,25\}$. Fill the hole of side 3 with a 3-SOLS(3). We then obtain an r-SOLS(23) for every $r\in 420+4\otimes S_5+3\supset[472,513]$.

Filling the hole of an ISOLS(23, 7) from Theorem 3.2 with an s-SOLS(7) for $s\in[34,45]$ we obtain an r-SOLS(23) for every $r\in 480+[34,45]=[514,525]$. This completes the proof. □

Lemma 5.11. *There exists an r-SOLS(24) for every* $r\in[28,572]$.

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p=m=2, k=2, l=0, n=12$ and $R_1=[14,138]$, the input designs, q_1-SOLS(n) for $q_1\in R_1$, are from Lemmas 3.11 and 4.4, then we can obtain an r-SOLS(24) for every $r=q_{11}+q_{12}\in 2\otimes R_1=[28,276]$.

Applying Corollary 3.7 with $m=4$, $n=6$, $R_1=[8,31]$, $R_2=[8,32]\cup\{34\}$, the input designs, q_1-SOLS(6) for $q_1\in R_1$ and q_2-MOLS(6) for $q_2\in R_2$, are from Theorems 1.1–1.3, then we obtain an r-SOLS(24) for every $r=\sum_{i=1}^{4}q_{1i}+2\sum_{j=1}^{6}q_{2j}\in 4\otimes R_1+6\otimes(2R_2)$.

It is easy to get that $4\otimes R_1+6\otimes(2R_2)=[128,532]$.

From Theorem 3.4 we have an FSOLS($5^4 4^1$). Fill the four holes of side 5 with s-SOLS(5) for $s\in\{14,15\}$. Fill the hole of side 4 with an SOLS(4). We then obtain an r-SOLS(24) for every $r\in 460+4\otimes\{14,15\}+16\supset[533,535]$.

Filling the hole of an ISOLS(24, 7) from Theorem 3.2 with an s-SOLS(7) for $s\in[9,45]$ we obtain an r-SOLS(24) for $r\in[536,572]$. □

Lemma 5.12. *There exists an r-SOLS(25) for every* $r\in[29,623]$.

Proof. Start with a symmetric Latin square of order 5, applying Construction 3.6 with $p=m=5, k=5, l=0, n=5$, $R_1=\{5,7,10,11,13,14,15,17,19,21,25\}$, the input designs, q_1-SOLS(5) for $q_1\in R_1$, are from Theorem 1.3, then we can obtain an r-SOLS(25) for every $r=\sum_{i=1}^{5}q_{1i}\in 5\otimes R_1\supset[29,111]$.

Applying Construction 3.8 with $m=4$, $n=6$, $t=0$, $R_1=\{7\}\cup[9,45]\cup\{47,49\}$, $R_2=\{6\}\cup[8,32]\cup\{34\}$ we have an r-SOLS(25) for every $r=\sum_{i=1}^{4}(q_{1i}-1)+2\sum_{j=1}^{6}q_{2j}+1\in 4\otimes(R_1-1)+6\otimes(2R_2)+1$, where $q_{1i}\in R_1$ and $q_{2j}\in R_2$.

A computer search shows that $4\otimes(R_1-1)+6\otimes(2R_2)+1\supset[112,597]$.

Filling the hole of an ISOLS(25, 8) from Theorem 3.2 with an s-SOLS(8) for $s\in[37,62]$ we obtain an r-SOLS(25) for every $r\in[598,623]$. □

Lemma 5.13. *There exists an r-SOLS(26) for every* $r\in[30,672]$.

Proof. Start with a symmetric Latin square of order 2, applying Construction 3.6 with $p=m=2$, $k=2$, $l=0$, $n=13$ and $R_1=[15,163]$, the input designs, q_1-SOLS(n) for $q_1\in R_1$, are from Lemmas 3.11 and 4.5, then we can obtain an r-SOLS(26) for every $r=q_{11}+q_{12}\in 2\otimes R_1=[30,326]$.

Applying Construction 3.8 with $m=4$, $n=6$, $t=1$, $R_1=[9,45]\cup\{47,49\}$, $R_2=[8,32]\cup\{34\}$, $R_3=R_1$ we have an r-SOLS(26) for every $r=\sum_{i=1}^{4}(q_{1i}-1)+2\sum_{j=1}^{4}q_{2j}+2\sum_{k=1}^{2}(q_{3k}-1)+2\in 4\otimes(R_1-1)+4\otimes(2R_2)+2\otimes(2(R_1-1))+2$, where $q_{1i},q_{3k}\in R_1$ and $q_{2j}\in R_2$.

It is easy to get that $4\otimes(R_1-1)+4\otimes(2R_2)+2\otimes(2(R_1-1))+2\supset[327,654]$.

Filling the hole of an ISOLS(26, 7) from Theorem 3.2 with an s-SOLS(7) for $s\in[28,45]$ we obtain an r-SOLS(26) for every $r\in[655,672]$. □

Lemma 5.14. *There exists an r-SOLS(27) for every* $r\in[31,727]$.

Proof. Start with a symmetric Latin square of order 3, applying Construction 3.6 with $p=m=3$, $k=3$, $l=0$, $n=9$, $R_1=\{9\}\cup[11,79]$, the input designs, q_1-SOLS(n) for $q_1\in R_1$, are from Theorem 1.2 and Lemmas 3.11, 4.1 and 3.5, then we can obtain an r-SOLS(27) for every $r=\sum_{i=1}^{3}q_{1i}\in 3\otimes R_1\supset[31,237]$.

Applying Construction 3.8 with $m=4$, $n=6$, $t=2$, $R_1=[9,45]\cup\{47\}$, $R_2=[8,32]$, $R_3=[9,47]$ we have an r-SOLS(27) for every $r=\sum_{i=1}^{4}(q_{1i}-1)+2\sum_{j=1}^{2}q_{2j}+2\sum_{k=1}^{4}(q_{3k}-1)+3\in 4\otimes(R_1-1)+2\otimes(2R_2)+4\otimes(2(R_3-1))+3$, where $q_{1i}\in R_1$, $q_{2j}\in R_2$ and $q_{3k}\in R_3$.

It is easy to get that $4\otimes(R_1-1)+2\otimes(2R_2)+4\otimes(2(R_3-1))+3\supset[238,677]$.

Filling the hole of an ISOLS(27, 8) from Theorem 3.2 with an s-SOLS(8) for $s\in[13,62]$ we obtain an r-SOLS(27) for $r\in[678,727]$. □

6. Concluding remarks

We are now in a position to give the main results of this paper.

Theorem 6.1. *For any integer* $9\leqslant v\leqslant 27$, *there exists an r-SOLS(v) if* $v\leqslant r\leqslant v^2$ *and* $r\notin\{v+1,v^2-1\}$ *with the possible exceptions of v and r shown in Table 15.*

Proof. Combining Theorem 1.2, Lemmas 3.5, 3.11, 4.1–4.5, and 5.1–5.14. □

The following is an updated theorem about the existence of r-self-orthogonal Latin squares.

Theorem 6.2. *For any integer* $v\geqslant 2$, *there exists an r-SOLS(v) if and only if* $v\leqslant r\leqslant v^2$ *and* $r\notin\{v+1,v^2-1\}$ *except the genuine and possible exceptions listed in Table 16.*

Table 15

Order v	Possible exceptions of r
12, 13, 14, 15	v^2-5, v^2-4, v^2-3
16, 17, 18, 20	v^2-5, v^2-3
19, 21, 22, 23, 24, 26	v^2-3

Table 16
Exceptions of r-SOLS(v) for $r \in [v, v^2] \setminus \{v+1, v^2-1\}$

Order v	Genuine exceptions of r	Possible exceptions of r
2	4	
3	5, 6, 7, 9	
4	6, 7, 8, 10, 11, 12, 13, 14	
5	8, 9, 12, 16, 18, 20, 22, 23	
6	32, 33, 34, 36	
7	46	
12,13,14,15		v^2-5, v^2-4, v^2-3
16,17,18,20		v^2-5, v^2-3
19,21,22,23,24,26		v^2-3

Proof. The necessity comes from Theorem 1.1. The sufficiency comes from Theorems 1.3–1.5, and Theorem 6.1. □

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