Journal of Mathematical Analysis and Applications **250**, 579–588 (2000)
doi:10.1006/jmaa.2000.6995, available online at http://www.idealibrary.com on IDEAL®

Hyers–Ulam–Rassias Stability of a Jensen Type Functional Equation

Tiberiu Trif

Facultatea de Matematică şi Informatică, Universitatea Babeş-Bolyai, Str. Kogălniceanu 1, 3400 Cluj-Napoca, Romania
E-mail: ttrif@math.ubbcluj.ro

Submitted by William F. Ames

Received October 26, 1999

In this paper we solve the Jensen type functional equation (1.1). Likewise, we investigate the Hyers–Ulam–Rassias stability of this equation.

Key Words: Hyers–Ulam–Rassias stability; functional equation; Jensen's functional equation.

1. INTRODUCTION

More than half a century ago S. M. Ulam [15] raised the following problem concerning the stability of homomorphisms: given a group G_1, a metric group G_2 with metric $d(\cdot, \cdot)$, and a positive real number ϵ, does there exist a positive real number δ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \le \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $d(f(x), h(x)) \le \epsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [5] under the assumption that G_1 and G_2 are Banach spaces. Th. M. Rassias [13] proved the following substantial generalization of the result of Hyers:

THEOREM 1.1. *Let X and Y be Banach spaces, let $\theta \in [0, \infty[$, and let $p \in [0, 1[$. If a function $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

0022-247X/00 $35.00

for all $x \in X$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Due to this fact, the Cauchy functional equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers–Ulam–Rassias stability property on (X, Y). This terminology is also applied to other functional equations (see [7] for more detailed definitions).

Later, many Rassias type theorems concerning the stability of different functional equations were obtained by numerous authors (see, for instance, [2, 3, 9–11]).

In this paper we deal with the Jensen type functional equation

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]. \tag{1.1}$$

Equation (1.1) has been considered for the first time by T. Popoviciu [12], in connection with the following inequality: if I is a nonempty interval and $f : I \to \mathbf{R}$ is a convex function, then it holds that

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \geq 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \tag{1.2}$$

for all $x, y, z \in I$. Today the inequality (1.2) is commonly known as the Popoviciu inequality.

In Section 2 of this paper we solve the functional equation (1.1). In Section 3, using ideas from the papers of Th. M. Rassias [13] and D. H. Hyers [5], the Hyers–Ulam–Rassias stability of Eq. (1.1) will be investigated.

2. SOLUTIONS OF EQ. (1.1)

It is interesting to note that the functional equation (1.1) is equivalent to the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y). \tag{2.1}$$

It is well known that if X and Y are real linear spaces, then a function $f : X \to Y$ satisfying $f(0) = 0$ is a solution of Eq. (2.1) if and only if it is additive.

THEOREM 2.1. *Let X and Y be real linear spaces. A function $f: X \to Y$ satisfies* (1.1) *for all $x, y, z \in X$ if and only if there exist an element $B \in Y$ and an additive mapping $A: X \to Y$ such that*

$$f(x) = A(x) + B \qquad \text{for all } x \in X.$$

Proof. Necessity. Set $B := f(0)$ and then define the mapping $A: X \to Y$ by $A(x) := f(x) - B$. Then $A(0) = 0$ and

$$3A\left(\frac{x+y+z}{3}\right) + A(x) + A(y) + A(z) = 2\left[A\left(\frac{x+y}{2}\right) + A\left(\frac{y+z}{2}\right) + A\left(\frac{z+x}{2}\right)\right] \tag{2.2}$$

for all $x, y, z \in X$. We claim that A is additive.

Putting $y = x$ and $z = -2x$ in (2.2) yields

$$A(-2x) = 4A\left(-\frac{x}{2}\right) \qquad \text{for all } x \in X. \tag{2.3}$$

Replacing x by $-x$ in (2.3) we get

$$A(2x) = 4A\left(\frac{x}{2}\right) \qquad \text{for all } x \in X. \tag{2.4}$$

Replacing x by $2x$ in (2.4) we get

$$A(4x) = 4A(x) \qquad \text{for all } x \in X. \tag{2.5}$$

Putting $y = z = 0$ in (2.2) and taking account of (2.4) we obtain

$$3A\left(\frac{x}{3}\right) = A(2x) - A(x) \qquad \text{for all } x \in X. \tag{2.6}$$

Putting $y = x$ and $z = 0$ in (2.2) and taking account of (2.6) we obtain

$$A(4x) - A(2x) = 4A\left(\frac{x}{2}\right) \qquad \text{for all } x \in X. \tag{2.7}$$

From (2.4), (2.5), and (2.7) it follows that

$$A(2x) = 2A(x) \qquad \text{for all } x \in X. \tag{2.8}$$

Putting $y = x$ and $z = -x$ in (2.2) and taking account of (2.6) and (2.8) we get

$$A(-x) = -A(x) \qquad \text{for all } x \in X. \tag{2.9}$$

Finally, putting $z = -x - y$ in (2.2) and taking account of (2.8) and (2.9) we get

$$A(x+y) = A(x) + A(y) \qquad \text{for all } x, y \in X.$$

Therefore A is additive, as claimed, and $f(x) = A(x) + B$ for all $x \in X$.

Sufficiency. This is obvious. ∎

Remark. From Theorem 2.1 it follows that a continuous function $f : \mathbf{R} \to \mathbf{R}$ satisfies (1.1) for all $x, y, z \in \mathbf{R}$ if and only if it has the form $f(x) = ax + b$, with a and b arbitrary real constants. This result has been established by T. Popoviciu [12].

3. HYERS–ULAM–RASSIAS STABILITY OF EQ. (1.1)

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$Df(x,y,z) := 3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) - 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

for all $x, y, z \in X$.

THEOREM 3.1. *Assume that $\delta, \theta \in [0, \infty[$ and that $p \in]0, 1[$. If the function $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \le \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.1}$$

for all $x, y, z \in X$, then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - f(0) - A(x)\| \le \frac{\delta}{3} + \frac{\theta}{2^{1-p} - 1}\|x\|^p \tag{3.2}$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Proof. Let $g : X \to Y$ be the function defined by $g(x) := f(x) - f(0)$. Then $g(0) = 0$ and, since $Dg(x,y,z) = Df(x,y,z)$ for all $x, y, z \in X$, we have

$$\|Dg(x,y,z)\| \le \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.3}$$

for all $x, y, z \in X$.

Putting $y = x$ and $z = -2x$ in (3.3) we get

$$\left\| g(-2x) - 4g\left(-\frac{x}{2}\right)\right\| \le \delta + \theta(2 + 2^p)\|x\|^p \qquad \text{for all } x \in X.$$

Replacing x by $-2x$ in the above relation yields

$$\|g(4x) - 4g(x)\| \le \delta + \theta 2^{2p}(1 + 2^{1-p})\|x\|^p \tag{3.4}$$

for all $x \in X$.

Next we prove by induction on n that for all $x \in X$ it holds that

$$\|2^{-2n}g(2^{2n}x) - g(x)\| \le \delta \sum_{k=1}^{n} 2^{-2k} + \theta(1 + 2^{1-p})\|x\|^p \sum_{k=1}^{n} 2^{-2(1-p)k}. \tag{3.5}$$

Dividing both sides of (3.4) by 2^2 ensures the validity of (3.5) for $n = 1$. Now, assume that the inequality (3.5) holds true for some positive integer n. Replacing x in (3.4) by $2^{2n}x$ and then dividing both sides of (3.4) by $2^{2(n+1)}$ yields

$$\begin{aligned}\|2^{-2(n+1)}g(2^{2(n+1)}x) - 2^{-2n}g(2^{2n}x)\| \\ \le \delta 2^{-2(n+1)} + \theta(1 + 2^{1-p})\|x\|^p 2^{-2(1-p)(n+1)},\end{aligned}$$

hence

$$\begin{aligned}&\|2^{-2(n+1)}g(2^{2(n+1)}x) - g(x)\| \\ &\quad \le \|2^{-2(n+1)}g(2^{2(n+1)}x) - 2^{-2n}g(2^{2n}x)\| + \|2^{-2n}g(2^{2n}x) - g(x)\| \\ &\quad \le \delta \sum_{k=1}^{n+1} 2^{-2k} + \theta(1 + 2^{1-p})\|x\|^p \sum_{k=1}^{n+1} 2^{-2(1-p)k}\end{aligned}$$

for all $x \in X$. This completes the proof of the inequality (3.5).

Let x be any point in X. By virtue of (3.5) we have

$$\begin{aligned}&\|2^{-2n}g(2^{2n}x) - 2^{-2m}g(2^{2m}x)\| \\ &\quad = 2^{-2m}\|2^{-2(n-m)}g(2^{2(n-m)}2^{2m}x) - g(2^{2m}x)\| \\ &\quad \le 2^{-2m}\left(\delta \sum_{k=1}^{n-m} 2^{-2k} + \theta(1 + 2^{1-p})2^{2mp}\|x\|^p \sum_{k=1}^{n-m} 2^{-2(1-p)k}\right) \\ &\quad \le 2^{-2m}\left(\frac{\delta}{3} + \theta(1 + 2^{1-p})2^{2mp}\|x\|^p \frac{1}{2^{2(1-p)} - 1}\right) \\ &\quad = 2^{-2m}\left(\frac{\delta}{3} + \frac{\theta}{2^{1-p} - 1}2^{2mp}\|x\|^p\right)\end{aligned}$$

for all positive integers m and n with $m < n$. Since

$$\lim_{m\to\infty} 2^{-2m}\left(\frac{\delta}{3} + \frac{\theta}{2^{1-p}-1}2^{2mp}\|x\|^p\right) = 0,$$

it follows that $(2^{-2n}g(2^{2n}x))_{n\in\mathbf{N}}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define the mapping $A: X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^{-2n}g(2^{2n}x).$$

Let x, y, and z be any points in X. We have

$$\begin{aligned}\|DA(x,y,z)\| &= \lim_{n\to\infty} 2^{-2n}\left\|Dg\left(2^{2n}x, 2^{2n}y, 2^{2n}z\right)\right\| \\ &\le \lim_{n\to\infty} 2^{-2n}\left(\delta + \theta 2^{2np}\left(\|x\|^p + \|y\|^p + \|z\|^p\right)\right) \\ &= 0.\end{aligned}$$

Hence A satisfies (2.2) for all $x, y, z \in X$. Since $A(0) = 0$, it follows that A is additive. Moreover, by passing to the limit in (3.5) when $n \to \infty$, it follows that

$$\|g(x) - A(x)\| \le \frac{\delta}{3} + \frac{\theta}{2^{1-p}-1}\|x\|^p \qquad \text{for all } x \in X.$$

This inequality implies the validity of (3.2).

Now, let $\tilde{A}: X \to Y$ be another additive mapping satisfying

$$\left\|f(x) - f(0) - \tilde{A}(x)\right\| \le \frac{\delta}{3} + \frac{\theta}{2^{1-p}-1}\|x\|^p$$

for all $x \in X$. Then we have

$$\begin{aligned}\left\|A(x) - \tilde{A}(x)\right\| &= 2^{-n}\left\|A(2^n x) - \tilde{A}(2^n x)\right\| \\ &\le 2^{-n}\Big(\|A(2^n x) - f(2^n x) + f(0)\| \\ &\qquad + \left\|f(2^n x) - f(0) - \tilde{A}(2^n x)\right\|\Big) \\ &\le 2^{-n}\left(\frac{2\delta}{3} + \frac{2\theta}{2^{1-p}-1}2^{np}\|x\|^p\right)\end{aligned}$$

for all $x \in X$ and all positive integers n. Since

$$\lim_{n\to\infty} 2^{-n}\left(\frac{2\delta}{3} + \frac{2\theta}{2^{1-p}-1}2^{np}\|x\|^p\right) = 0,$$

we can conclude that $A(x) = \tilde{A}(x)$ for all $x \in X$. This proves the uniqueness of A.

The proof of the last assertion in the theorem goes through in the same way as that of the theorem in [13]. ∎

The proof of the next theorem (containing the case $p = 0$), being similar to that of Theorem 3.1, is omitted.

THEOREM 3.2. *Let $\delta \in [0, \infty[$ and let $f : X \to Y$ be a function satisfying*

$$\|Df(x, y, z)\| \leq \delta \qquad \textit{for all } x, y, z \in X.$$

Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - f(0) - A(x)\| \leq \frac{\delta}{3} \qquad \textit{for all } x \in X.$$

If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

THEOREM 3.3. *Let $\theta \in [0, \infty[$, $p \in]1, \infty[$, and let $f : X \to Y$ be a function satisfying $f(0) = 0$ and*

$$\|Df(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.6}$$

for all $x, y, z \in X$. Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^{p-1}}{2^{p-1} - 1}\theta\|x\|^p \qquad \textit{for all } x \in X.$$

If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Proof. Putting $y = x$ and $z = -2x$ in (3.6) we see, as in the proof of Theorem 3.1, that

$$\left\| f(-2x) - 4f\left(-\frac{x}{2}\right)\right\| \leq \theta(2 + 2^p)\|x\|^p \qquad \text{for all } x \in X.$$

Replacing x by $-\frac{x}{2}$ in the above relation yields

$$\|2^2 f(2^{-2}x) - f(x)\| \leq \theta(1 + 2^{p-1})2^{1-p}\|x\|^p \qquad \text{for all } x \in X. \tag{3.7}$$

Starting from (3.7) it is easy to prove that

$$\|2^{2n} f(2^{-2n}x) - f(x)\| \leq \theta 2^{p-1}(1 + 2^{p-1})\|x\|^p \sum_{k=1}^{n} 2^{-2(p-1)k}$$

for all $x \in X$ and all positive integers n.

The rest of the proof is similar to the corresponding part of the proof of Theorem 3.1. ∎

Th. M. Rassias and P. Šemrl [14] have constructed a continuous function $f:\mathbf{R}\to\mathbf{R}$ to show that the functional inequality

$$\|f(x+y)-f(x)-f(y)\|\le\theta(\|x\|+\|y\|)$$

does not have Hyers–Ulam–Rassias stability property. In what follows we prove that their function serves as a counterexample to Theorem 3.3 for the case $p=1$.

THEOREM 3.4. *The continuous function* $f:\mathbf{R}\to\mathbf{R}$ *defined by* $f(x):=x\log_2(1+|x|)$ *satisfies the inequality*

$$|Df(x,y,z)|\le\tfrac{20}{3}(|x|+|y|+|z|) \tag{3.8}$$

for all $x,y,z\in\mathbf{R}$, *but*

$$\sup\left\{\left|\frac{f(x)-A(x)}{x}\right|\,\middle|\,x\in\mathbf{R}\setminus\{0\}\right\}=\infty$$

for each additive mapping $A:\mathbf{R}\to\mathbf{R}$.

Proof. It was proved in [14] that the function f satisfies

$$|f(x+y)-f(x)-f(y)|\le|x|+|y| \tag{3.9}$$

for all $x,y\in\mathbf{R}$. We claim that f satisfies also the inequality

$$|f(x+y+z)-f(x)-f(y)-f(z)|\le\tfrac{5}{3}(|x|+|y|+|z|) \tag{3.10}$$

for all $x,y,z\in\mathbf{R}$. Indeed, taking into account the inequality (3.9) we get

$$\begin{aligned}&|f(x+y+z)-f(x)-f(y)-f(z)|\\&\quad\le|f(x+y+z)-f(x+y)-f(z)|+|f(x+y)-f(x)-f(y)|\\&\quad\le|x+y|+|z|+|x|+|y|\\&\quad\le 2|x|+2|y|+|z|.\end{aligned}$$

Analogously we obtain

$$|f(x+y+z)-f(x)-f(y)-f(z)|\le 2|y|+2|z|+|x|$$

and

$$|f(x+y+z)-f(x)-f(y)-f(z)|\le 2|z|+2|x|+|y|.$$

Summing the last three inequalities implies the validity of (3.10). Since

$$\begin{aligned} Df(x,y,z) = 3f\left(\frac{x+y+z}{3}\right) - f(x+y+z) \\ -[f(x+y+z) - f(x) - f(y) - f(z)] \\ + 2\left[f(x+y+z) - f\left(\frac{x+y}{2}\right) - f\left(\frac{y+z}{2}\right) - f\left(\frac{z+x}{2}\right)\right], \end{aligned}$$

by virtue of (3.10) we have

$$\begin{aligned} |Df(x,y,z)| &\le \tfrac{5}{3}|x+y+z| + \tfrac{5}{3}(|x| + |y| + |z|) \\ &\quad + \tfrac{5}{3}(|x+y| + |y+z| + |z+x|) \\ &\le \tfrac{20}{3}(|x| + |y| + |z|) \end{aligned}$$

for all $x, y, z \in \mathbf{R}$. Hence (3.8) holds true.

Now, let $A : \mathbf{R} \to \mathbf{R}$ be any additive mapping. If A is continuous at a point, then there is a real number c such that $A(x) = cx$ for all $x \in \mathbf{R}$. In this case we have $|f(x) - A(x)|/|x| \to \infty$ as $x \to \infty$. On the other hand, if A is nowhere continuous, then the range of $x \in \mathbf{R}\setminus\{0\} \mapsto |f(x) - A(x)|/|x|$ is also unbounded, because the graph of A is everywhere dense in $\mathbf{R}^2$. ∎

REFERENCES

1. J. Aczél and J. Dhombres, "Functional Equations in Several Variables," Cambridge Univ. Press, Cambridge, UK, 1989.
2. P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.* **27** (1984), 76–86.
3. S. Czerwik, On the stability of the quadratic mappings in normed spaces, *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 59–62.
4. G. L. Forti, Hyers–Ulam stability of functional equations in several variables, *Aequationes Math.* **50** (1995), 143–190.
5. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. U.S.A.* **27** (1941), 222–224.
6. D. H. Hyers, G. Isac, and Th. M. Rassias, "Stability of Functional Equations in Several Variables," Birkhäuser, Boston/Basel/Berlin, 1998.
7. D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.* **44** (1992), 125–153.
8. S.-M. Jung, Hyers–Ulam–Rassias stability of functional equations, *Dynam. Systems Appl.* **6** (1997), 541–566.
9. S.-M. Jung, Hyers–Ulam–Rassias stability of Jensen's equation and its application, *Proc. Amer. Math. Soc.* **126** (1998), 3137–3143.
10. S.-M. Jung, On the Hyers–Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.* **222** (1998), 126–137.

11. S.-M. Jung, On the Hyers–Ulam–Rassias stability of a quadratic functional equation, *J. Math. Anal. Appl.* **232** (1999), 384–393.
12. T. Popoviciu, Sur certaines inégalités qui caractérisent les fonctions convexes, *An. Ştiint. Univ. Al. I. Cuza Iaşi Secţ. Ia Mat.* **11** (1965), 155–164.
13. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.
14. Th. M. Rassias and P. Šemrl, On the behavior of mappings that do not satisfy Hyers–Ulam stability, *Proc. Amer. Math. Soc.* **114** (1992), 989–993.
15. S. M. Ulam, "Problems in Modern Mathematics," Chap. VI, Wiley, New York, 1964.