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Continuity of Best Reciprocal Polynomial Approximation on $[0, \infty)$

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INTRODUCTION

Over the past 10 years considerable progress has been made in studying various questions concerning rational approximation on unbounded sets. To a large extent the starting point of this effort was the paper of Cody, Meinardus and Varga [6] and this has led to investigations of best approximation properties in various settings [1–4, 8–9] and studies of the error of best approximation [10–12].

In this paper we wish to study the best approximation properties of strong uniqueness and continuity of the best approximation operator for reciprocal polynomial approximation on $[0, \infty)$ of continuous positive functions tending to 0 as $x \to \infty$. Thus, we define

$$C_0^+[0, \infty) = \{f \in C[0, \infty): f(x) > 0, x \in [0, \infty) \text{ and } \lim_{x \to \infty} f(x) = 0\}. \quad (1)$$

and

$$R_n = \{1/p: p \in \Pi_n, p(x) > 0, x \in [0, \infty)\}, \qquad n \geq 1, \quad (2)$$

where Π_n denotes the class of all algebraic real polynomials of degree $\leq n$. Furthermore, define $\|f\| = \sup\{|f(x)|: x \in [0, \infty)\}$ in what follows. In this

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setting, it is known that best approximations exist and are unique [3, 4] and that the following characterization theorem holds:

THEOREM 1 [4]. *Let $f \in C_0^+[0, \infty)$, $f \notin R_n$ with $n \geq 1$. Then $1/p^*$ is the best approximation to f from R_n on $[0, \infty)$ iff*

(i) (*standard alternation*) *there exist* $\{x_i\}_{i=0}^{n+1}$, $0 \leq x_0 < x_1 < \cdots < x_{n+1}$, *such that* $|f(x_i) - 1/p^*(x_i)| = \|f - 1/p^*\|$, $i = 0,\ldots, n+1$, and

$$f(x_i) - \frac{1}{p^*(x_i)} = -\left(f(x_{i+1}) - \frac{1}{p^*(x_{i+1})}\right), \qquad i = 0,\ldots, n;$$

or

(ii) (*nonstandard alternation*) $\partial p^* \leq n - 1$ *and there exist* $\{x_i\}_{i=0}^{n}$, $0 \leqslant x_0 < x_1 < \cdots < x_n$ *such that* $f(x_i) - 1/p^*(x_i) = (-1)^{n-i} \|f - 1/p^*\|$.

In both cases the points $\{x_i\}$ are called extreme points. Also, we wish to note that for $n \geqslant 1$, p^* cannot be a constant. Indeed, since $f(x) > 0$ for all $x \in [0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then in order for the reciprocal of a constant, $1/c^*$, to be a best approximation to f, we must have that $c^* = 2/M$, where $M = \max_{x\geqslant 0} f(x)$. Since $f(x) \to 0$ as $x \to \infty$ we can find $x_0 > 0$ such that $f(x) < M$ for $x \geqslant x_0$. It is then easily seen that for $p^*(x) = \epsilon(x - x_0) + c^*$ with $\epsilon > 0$ and sufficiently small that $\|f - 1/p^*\| < \|f - 1/c^*\|$ by a straightforward continuity-compactness argument.

In addition, it has been shown in [3] that if $1/p^* \in R_n$ is the best approximation to $f \in C_0^+[0, \infty)$ from R_n with $\partial p^* = n$ then both strong uniqueness (i.e., $\|f - 1/p\| - \|f - 1/p^*\| \geqslant \gamma \| 1/p - 1/p^*\|$, $\gamma = \gamma(f) > 0$, for all $1/p \in R_n$) and Lipschitz continuity of the best approximation operator at f (i.e., $\| 1/p^* - 1/p_g \| \leqslant \beta \|f - g\|$, $\beta = \beta(f) > 0$, $g \in C_0^+[0, \infty)$ and $1/p_g$ the best approximation to g from R_n) hold. Furthermore, it was shown in [3] that for each f whose corresponding best approximation from R_n, $1/p^*$, satisfies $\partial p^* \leqslant n - 2$ the strong uniqueness theorem cannot hold. In this present paper we shall prove that if $f \in C_0^+[0, \infty)$ has $1/p^* \in R_n$ as its best approximation then (i) if $\partial p^* \leqslant n - 2$ (i.e., $1/p^*$ is deficient of order 2 or more) then the best approximation operator is discontinuous at f and (ii) if $\partial p^* = n - 1$ then the best approximation operator is continuous at f. It remains open as to whether or not a strong uniqueness theorem holds in the case that $\partial p^* = n - 1$.

MAIN RESULTS

In this section we state and prove our main results. The first result establishing the discontinuity of the best approximation operator is given in two parts. The first theorem will treat this problem for the case that either $1/p^*$ is deficient of order 3 or more, or $1/p^*$ is deficient of order 2 and $f - 1/p^*$ possesses a standard alternating sequence. In this case we can prove even stronger results concerning the discontinuous behavior of the best approximation operator. The second theorem will treat the discontinuity of the best approximation operator when $1/p^*$ is deficient of order 2 with only non-standard alternation holding for $f - 1/p^*$. Our final result will be to prove that the best approximation operator is continuous whenever $1/p^*$ is deficient of order 1.

THEOREM 2. *Let $f \in C_0^+[0, \infty)$, $f \notin R_n$ and $1/p^* \in R_n$ be the best approximation to f from R_n. Further, assume that $\partial p^* \leqslant n - 2$ and that if $\partial p^* = n - 2$ then $f - 1/p^*$ possesses a standard alternating set. Then, given $\epsilon > 0$ there exists $\delta > 0$, $\{1/p_k\}_{k=1}^{\infty} \subset R_n$ and $\{g_k\}_{k=1}^{\infty} \subset C_0^+[0, \infty)$ such that each g_k has $1/p_k$ as its best approximation from R_n, g_k converges uniformly to f on $[0, \infty)$ and $\delta \leqslant \| 1/p^* - 1/p_k \| \leqslant \epsilon$ for all k.*

Remark. This theorem establishes that not only is the best approximation operator discontinuous at f but, in fact, that it is also not possible for a local (relative to $1/p^*$) strong uniqueness result to hold.

Proof. Set $E = \| f - 1/p^* \| > 0$ and assume without loss of generality that $\epsilon \leqslant E/4$. Set $\delta = \epsilon/8$. Since we are assuming throughout this paper that $n \geqslant 1$, we have that $p^*(x)$ is not identically equal to a constant which implies that $\lim_{x\to\infty} p^*(x) = \infty$. Select $\beta > 0$ such that $f(x) \leqslant \epsilon$ and $p^*(x) > 4/\epsilon$ for all $x \geqslant \beta$. Set $e_k = (\epsilon/4 + 2/p^*(k))^{-1}$ and note that for $k \geqslant \beta$, $p^*(k) > e_k$. Define $p_k \in \Pi_n$ by

$$p_k(x) = e_k + (p^*(x) - e_k)\left[\left(\frac{x}{k} - 1\right)^2 + \frac{e_k}{p^*(k) - e_k}\right], \qquad k \geqslant \beta.$$

Since for all $x \geqslant \beta$, $p^*(x) > e_k$, we have that

$$(p^*(x) - e_k)\left[\left(\frac{x}{k} - 1\right)^2 + \frac{e_k}{p^*(k) - e_k}\right] > 0$$

implying that $p_k(x) > e_k > 4/3\epsilon$ for $x \geqslant \beta$.

Next, observe that $e_k \to 4/\epsilon$ as $k \to \infty$ and $(x/k - 1)^2$ converges uniformly to 1 on $[0, \beta]$ as $k \to \infty$. Thus, p_k converges uniformly to p^* on $[0, \beta]$ as $k \to \infty$. Now, let

$$\eta = \min\left(\min_{x\in[0,\beta]} f(x),\ \min_{x\in[0,\beta]} \frac{1}{p^*(x)},\ \epsilon\right)$$

and select $\mu \geqslant \beta$ such that for $k \geqslant \mu$, $\max_{x\in[0,\beta]} | 1/p_k(x) - 1/p^*(x)| \leqslant \eta/2$. This implies that $1/p_k(x) \geqslant 1/p^*(x) - \eta/2 \geqslant \eta/2 > 0$ for all $x \in [0, \beta]$ and $k \geqslant \mu$. Hence $1/p_k \in R_n$ for $k \geqslant \mu$ as $p_k(x) > 0$ for all $x \in [0, \infty)$.

Now, since $p_k(k) = 2e_k$ we have that $1/p_k(k) - 1/p^*(k) = \epsilon/8 = \delta$. Thus, $\| 1/p_k - 1/p^* \| \geqslant \delta$ for all $k \geqslant \mu$. Also, for $k \geqslant \mu$ we have that $| 1/p_k(x) - 1/p^*(x)| \leqslant \eta/2 < \epsilon$ for $x \in [0, \beta]$. In addition, for $x \geqslant \beta$ we have that $1/p^*(x) < \epsilon/4$ and $1/p_k(x) \leqslant 3\epsilon/4$ for $k \geqslant \mu$. Hence $\| 1/p_k - 1/p^* \| \leqslant \epsilon$ as claimed.

Finally, define g_k, for $k \geqslant \mu$, by

$$\begin{aligned} g_k(x) &= f(x) + \frac{1}{p_k(x)} - \frac{1}{p^*(x)}, && x \in [0, \beta] \\ &= f(x), && x \geqslant \beta + \frac{1}{k} \\ &= \text{linear with endpoint values } f(\beta) + \frac{1}{p_k(\beta)} - \frac{1}{p^*(\beta)} \\ &\quad \text{and } f\left(\beta + \frac{1}{k}\right), && x \in \left(\beta, \beta + \frac{1}{k}\right). \end{aligned}$$

Clearly, $g_k \in C[0, \infty)$, $k \geqslant \mu$ and since $f(x) + 1/p_k(x) - 1/p^*(x) \geqslant f(x) - \eta/2 > 0$ for $x \in [0, \beta]$ we have that $g_k \in C_0^+[0, \infty)$ for $k \geqslant \mu$. Since $f(\beta) + 1/p_k(\beta) - 1/p^*(\beta) < f(\beta) + \eta/2 \leqslant f(\beta) + \epsilon/2 \leqslant f(\beta) + E/8 \leqslant \frac{3}{8}E$ and $f(\beta + 1/k) \leqslant E/4$ we have that $g_k(x) \leqslant \frac{3}{8}E$ for $x \in [\beta, \beta + 1/k]$ implying $g_k(x) \leqslant 3E/8$ for $x \geqslant \beta$. Also, $\| 1/p_k - 1/p^* \| < \epsilon \leqslant E/4$ implies that $1/p_k(x) < 1/p^*(x) + E/4 \leqslant E/2$ for $x \geqslant \beta$. From this it follows that $| g_k(x) - 1/p_k(x)| \leqslant \frac{7}{8}E$ for $x \geqslant \beta$. In addition, for $x \in [0, \beta]$ we have that $g_k(x) - 1/p_k(x) = f(x) - 1/p^*(x)$ and this implies that $g - 1/p_k$ exhibits the same alternating behavior as $f - 1/p^*$ on $[0, \infty)$. Thus, if $f - 1/p^*$ has a standard alternating set so does $g_k - 1/p_k$ implying that $1/p_k$ is the best approximation to g_k from R_n on $[0, \infty)$. If $f - 1/p^*$ possesses only a nonstandard alternating set then so does $g_k - 1/p_k$. Since in this case we must have that $\partial p_k \leqslant n - 3$, we must have that $\partial p_k \leqslant n - 1$ implying once again that $1/p_k$ is the best approximation to g_k from R_n on $[0, \infty)$. Since it is clear that g_k converges uniformly to f on $[0, \infty)$, the proof is completed by relabeling the sequences $\{1/p_k\}_{k=\mu}^{\infty}$ and $\{g_k\}_{k=\mu}^{\infty}$ as $\{1/p_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively. ∎

For the case that $\partial p^* = n - 2$ and $f - 1/p^*$ has only a nonstandard alternating sequence we have the slightly weaker theorem:

THEOREM 3. *Let $f \in C_0^+[0, \infty)$, $f \notin R_n$ and, $1/p^* \in R_n$ be its best approximation from R_n. Further, assume that $\partial p^* = n - 2$ and $f - 1/p^*$ possesses only a nonstandard alternating set. Then there exists $\{1/p_k\}_{k=1}^{\infty} \subset R_n$ and $\{g_k\}_{k=1}^{\infty} \subset C_0^+[0, \infty)$ such that for each k, $1/p_k$ is the best approximation to g_k from R_n on $[0, \infty)$, g_k converges uniformly to f on $[0, \infty)$ and $\| 1/p_k - 1/p^* \| \geqslant \frac{7}{8}E$, where $E = \|f - 1/p^* \| > 0$.*

Proof. Select $\beta > 0$ such that $p^*(x) \geqslant 8/E$, $f(x) \leqslant E/8$ and $p^*(x)$ is monotone increasing for $x \geqslant \beta$. For $k \geqslant \beta$, define

$$p_k(x, t) = t + (p^*(x) - t)\left[\left(\frac{x}{k} - 1\right)^2 + \frac{t}{p^*(k) - t}\right],$$
$$0 \leqslant t \leqslant \frac{1}{E}, \ x \geqslant 0.$$

Note that $p_k(x, t)$ is continuous on $[0, \infty) \times [0, 1/E]$. Define $h(t) = \min\{p_k(x, t): x \in [\beta, 2k]\}$ and observe that h is a continuous function of t, $0 \leqslant t \leqslant 1/E$. In addition, $h(0) = \min\{p^*(x)(x/k - 1)^2: x \in [\beta, 2k]\} = 0$ as $k \geqslant \beta$ and that $h(1/E) = \min\{1/E + (p^*(x) - 1/E)[(x/k - 1)^2 + 1/(Ep^*(k) - 1)]: x \in [\beta, 2k]\} > 1/E$ as $p^*(x) > 1/E$ on $[\beta, 2k]$. Select $e_k \in (0, 1/E)$ so that $h(e_k) = 1/E$. Thus, $p_k(x, e_k) \geqslant 1/E$ for $x \in [\beta, 2k]$. Select $e_k \in (0, 1/E)$ so that $h(e_k) = 1/E$. Thus, $p_k(x, e_k) \geqslant 1/E > 0$ for $x \in [\beta, 2k]$. Observe that $p_k(x, e_k)$ converges uniformly to $p^*(x)$ on $[0, \beta]$ as $k \to \infty$ since $0 < e_k \leqslant 1/E$, $e_k/(p^*(k) - e_k) \to 0$ and $(x/k - 1)^2$ converges uniformly to 1 on $[0, \beta]$ as $k \to \infty$.

Next, let

$$\eta = \min\left(\min_{x \in [0,\beta]} f(x), \min_{x \in [0,\beta]} \frac{1}{p^*(x)}, \frac{E}{4}\right) > 0.$$

Select $\mu \geqslant \beta$ such that $k \geqslant \mu$ implies that $\sqrt{k} \geqslant \beta$, $k > 1$, $\max\{|1/p_k(x, e_k) - 1/p^*(x)|: x \in [0, \beta]\} \leqslant \eta/2$. Thus, for $k \geqslant \mu$, $1/p_k(x, e_k) \geqslant \eta/2 > 0$, for all $x \in [0, \beta]$. This implies that for $k \geqslant \mu$, $1/p_k(x, e_k)$ is positive and converges uniformly to $1/p^*(x)$ on $[0, \beta]$. In addition, for $k \geqslant \mu$ and $x \in [\beta, \sqrt{k}]$ we have $p_k(x, e_k) \geqslant e_k + (p^*(x) - e_k)[(1/\sqrt{k} - 1)^2] \geqslant e_k + \frac{1}{2}(p^*(x) - e_k) \geqslant \frac{1}{2}p^*(x) \geqslant 4/E$ as $p^*(x) \geqslant 8/E$ for $x \geqslant \beta$. Since $\max\{1/p_k(x, e_k): x \in [\beta, 2k]\} = E$ we have that if $t_k \in [\beta, 2k]$ is such that $1/p_k(t_k, e_k) = E$ then $t_k > \sqrt{k}$ for $k \geqslant \mu$.

Next, note that for $x \geqslant k \geqslant \mu$, $p_k(x, e_k)$ is a monotone increasing function of x and that $p_k(2k, e_k) = e_k + (p^*(2k) - e_k)(1 + e_k/(p^*(2k) - e_k)) \geqslant p^*(2k) \geqslant 8/E$. Thus, $1/p_k(x, e_k) \leqslant E/8$ for $x \geqslant 2k$. Summarizing, we have shown that $1/p_k(x, e_k) \leqslant E/4$ for $x \in [0, \sqrt{k}]$, $1/p_k(x, e_k) \leqslant E/8$ for $x \geqslant 2k$ and $1/p_k(x, e_k) \leqslant E$ for $x \in [\sqrt{k}, 2k]$ with $t_k \in [\sqrt{k}, 2k]$ a point at which the value E is attained.

Next, define α_k by $E - \alpha_k = \max\{(1/p_k(x, e_k) - f(x)): x \in [\beta, 2k]\}$. Since $f(x) \leqslant E/8$ for $x \geqslant \beta$ and $1/p_k(t_k, e_k) = E$ we have that $E - \alpha_k \geqslant E - f(t_k) \geqslant \frac{7}{8}E$ implying that $E/8 \geqslant f(t_k) \geqslant \alpha_k$. Let $y_k \in [\beta, 2k]$ be such that $1/p_k(y_k, e_k) - f(y_k) = E - \alpha_k$ for each $k \geqslant \mu$. Since $1/p_k(x, e_k) \leqslant E/8$ for $x \in [\beta, \sqrt{k}]$ we have that $y_k \in [\sqrt{k}, 2k]$. Also, since $f(t_k) \to 0$ as $k \to \infty$ (as $t_k \to \infty$) it follows that $\alpha_k \to 0$ as $k \to \infty$. Noting that $f(x) \leqslant E/8$ for $x \in [\beta, \infty)$ and that $1/p_k(x, e_k) \leqslant E/8$ for $x \geqslant 2k$ we have that $|f(x) - 1/p_k(x, e_k)| \leqslant E - \alpha_k$ for $x \in [\beta, \infty)$ and $k \geqslant \mu$. Also, since $1/p_k(t_k, e_k) = E$

and $1/p^*(t_k) \leqslant E/8$ we have that $\| 1/p_k - 1/p^* \| \geqslant | 1/p_k(t_k, e_k) - 1/p^*(t_k)| \geqslant 7E/8$.

Now define g_k by (for $k \geqslant \mu$)

$$\begin{aligned}
g_k(x) &= f(x) + \frac{1}{p_k(x, e_k)} - \frac{1}{p^*(x)}, && x \in [0, \beta], \ \left| f(x) - \frac{1}{p^*(x)} \right| \leqslant E - \alpha_k \\
&= E - \alpha_k + \frac{1}{p_k(x, e_k)}, && x \in [0, \beta], \quad f(x) - \frac{1}{p^*(x)} > E - \alpha_k \\
&= -E + \alpha_k + \frac{1}{p_k(x, e_k)}, && x \in [0, \beta], \quad f(x) - \frac{1}{p^*(x)} < -E - \alpha_k \\
&= f(x), && x \geqslant \beta + \frac{1}{k} \\
&= \text{linear on } \left[\beta, \beta + \frac{1}{k}\right] \text{ with endpoint values} \\
&\qquad f(\beta) + \frac{1}{p_k(\beta, e_k)} - \frac{1}{p^*(\beta)} \text{ and } f\left(\beta + \frac{1}{k}\right).
\end{aligned}$$

Observe that $g_k(x) > 0$ for all $x \geqslant 0$. Indeed, for $x \in [0, \beta]$ with $| f(x) - 1/p^*(x)| \leqslant E - \alpha_k$ we have that $g_k = f(x) + 1/p_k(x, e_k) - 1/p^*(x) \geqslant f(x) - \eta/2 \geqslant \eta/2 > 0$. For $x \in [0, \beta]$ with $f(x) - 1/p^*(x) > E - \alpha_k$, $g_k(x) = E - \alpha_k + 1/p_k(x, e_k) \geqslant \frac{7}{8}E + 1/p_k(x, e_k) > 0$ and for $x \in [0, \beta]$ with $f(x) - 1/p^*(x) < -E + \alpha_k$, $g_k(x) = -E + \alpha_k + 1/p_k(x, e_k) > -E + \alpha_k + 1/p^*(x) + 1/p_k(x, e_k) - 1/p^*(x) > f(x) - \eta/2 \geqslant \eta/2 > 0$. Since $f(\beta) + 1/p_k(\beta, e_k) - 1/p^*(\beta) \geqslant f(\beta) - \eta/2 \geqslant \eta/2 > 0$ and $f(\beta + 1/k) > 0$ we have that $g_k(x) > 0$ on $[\beta, \beta + 1/k]$ and finally g_k is positive on $[\beta + 1/k, \infty)$ as f is. To see that $g_k(x)$ is continuous on $[0, \infty)$ one must only check on $[0, \beta]$ as for $x > \beta$ it is clearly continuous. However, on $[0, \beta]$, $g_k(x)$ is simply the truncation of $f(x) - 1/p^*(x)$ to the range $[-E + \alpha_k, E - \alpha_k]$ plus the continuous function $1/p_k(x, e_k)$ showing that $g_k \in C_0^+[0, \infty)$.

Next, let us consider $|g_k(x) - 1/p_k(x, e_k)|$. Note that by construction $| g_k(x) - 1/p_k(x, e_k)| \leqslant E - \alpha_k$ for $x \in [0, \beta]$ and that, if $\{x_i\}_{i=0}^n$ with $x_0 < x_1 < \cdots < x_n$ is a nonstandard alternating set for $f - 1/p^*$ then we must have that $x_n < \beta$ and

$$g_k(x_i) - 1/p_k(x_i, e_k) = \operatorname{sgn}(f(x_i) - 1/p^*(x_i))(E - \alpha_k) = (-1)^{n-i}(E - \alpha_k).$$

Next, on $[\beta, \beta + 1/k]$ we have that $f(\beta) \leqslant E/8$, $f(\beta + 1/k) \leqslant E/8$ and $| 1/p_k(\beta, e_k) - 1/p^*(\beta)| \leqslant E/8$ so that $g_k(\beta) \leqslant E/4$ and $g_k(\beta + 1/k) \leqslant E/8$. Thus, $g_k(x) \leqslant E/4$ on $[\beta, \beta + 1/k]$. Also, recall that $1/p_k(x, e_k) \leqslant E/4$ on $[0, \sqrt{k}]$ so that $| g_k(x) - 1/p_k(x, e_k)| \leqslant E/4$ on $[\beta, \beta + 1/k]$. Finally, we noted earlier that $|f(x) - 1/p_k(x, e_k)| \leqslant E - \alpha_k$ on $[\beta, \infty)$ so that $| g_k(x) -$

$|1/p_k(x, e_k)| \leqslant E - \alpha_k$ on $[0, \infty)$. Since there exists $y_k \in [\sqrt{k}, 2k]$ at which $f(y_k) - 1/p_k(y_k, e_k) = -(E - \alpha_k)$ we have that $g_k - 1/p_k$ possesses a standard alternating set at the points $x_0 < x_1 < \cdots < x_n < y_k$ and thus $1/p_k$ is the best approximation to g_k from R_n on $[0, \infty)$. Finally, it is a straightforward argument to prove that g_k converges uniformly to f. Thus, once again reindexing the sequence $\{1/p_k\}_{k=\mu}^{\infty}$ gives the desired result. ∎

Next, we wish to show that if $f \in C_0^+[0, \infty)$ has $1/p^*$ as its best approximation from R_n with $\partial p^* = n - 1$ then the best approximation operator is continuous. This we do in the following theorem.

THEOREM 4. *Let $f \in C_0^+[0, \infty) \sim R_n$ and let $1/p^*$ be its best approximation from R_n on $[0, \infty)$ with $\partial p^* = n - 1$. Then, the best approximation operator is continuous at f.*

Proof. Let $\{g_k\}_{k=1}^{\infty} \subset C_0^+[0, \infty)$ with $g_k \to f$ uniformly on $[0, \infty)$. Further, let $1/p_k \in R_n$ be the best approximation to g_k on $[0, \infty)$ for each k. Then, we must prove that $\| 1/p_k - 1/p^* \| \to 0$ as $k \to \infty$. Let us first note that $\| g_k - 1/p_k \| \leqslant \| g_k - 1/p^* \|$ implying that $\lim_{k\to\infty} \sup \| g_k - 1/p_k \| \leqslant \lim_{k\to\infty} \sup \| g_k - 1/p^* \| = \| f - 1/p^* \| = E$. Also, $E = \| f - 1/p^* \| \leqslant \| f - 1/p_k \| \leqslant \| f - g_k \| + \| g_k - 1/p_k \|$ implying that $E = \lim_{k\to\infty} \inf(E - \| f - g_k \|) \leqslant \lim_{k\to\infty} \inf \| g_k - 1/p_k \|$. Combining these results gives that $\lim_{k\to\infty} \| g_k - 1/p_k \| = E$. In addition, since $E \leqslant \| f - 1/p_k \| \leqslant \| f - g_k \| + \| g_k - 1/p_k \|$ we also have that $\lim_{k\to\infty} \| f - 1/p_k \| = E$.

Next, fix $y \in [0, \infty)$ such that $f(y) = \max\{f(x): x \in [0, \infty)\}$. Then since a constant cannot be a best approximation to f from R_n on $[0, \infty)$ we must have that $2E < f(y)$. Select $\delta > 0$ such that for $x \in I = [y - \delta, y - \delta] \cap [0, \infty)$ we must have $f(x) \geqslant \frac{1}{2}(2E + f(y)) > 2E$. Choose β such that $k \geqslant \beta$ implies that $\| f - 1/p_k \| \leqslant \frac{3}{2}E$. Then for $k \geqslant \beta$ and $x \in I$, we have that

$$0 < m = 2E - \frac{3}{2}E \leqslant f(x) - \frac{3}{2}E \leqslant \frac{1}{p_k(x)}$$

$$\leqslant f(x) + \frac{3}{2}E \leqslant \| f \| + \frac{3}{2}E = M.$$

In addition, observe that the inequality $1/p_k(x) \leqslant M$ kolds for all $x \in [0, \infty)$ and $k \geqslant \beta$. Let $\{p_\nu\}$ be a subsequence of $\{p_k\}$. Then, since $1/M \leqslant p_\nu(x) \leqslant 1/m$ for all $x \in I$, there exists a subsequence $\{p_\mu\}$ of $\{p_\nu\}$ such that p_μ converges uniformly to some $\bar{p} \in \Pi_n$ on I. This implies that the coefficients of p_μ converge to the coefficients of $\bar{p}$ which in turn implies that for each $x \in [0, \infty)$, $p_\mu(x) \to \bar{p}(x)$. Thus, we must have $1/M \leqslant \bar{p}(x) \leqslant 1/m$ on I and $1/M \leqslant \bar{p}(x)$ on $[0, \infty)$. This last inequality shows that $1/p \in R_n$. Furthermore, for $x \in [0, \infty)$ fixed, $| f(x) - 1/\bar{p}(x) | = \lim_{\mu\to\infty} | f(x) - 1/p_\mu(x) | \leqslant \lim_{\mu\to\infty} \| f - 1/p_\mu \| = E$. Thus, $\| f - 1/\bar{p} \| \leqslant E$ implying that $\bar{p} = p^*$ by the uniqueness of best approximations from R_n. Since this is true for any subsequence $\{p_\nu\}$ of $\{p_k\}$ we must have that this is also true for the full sequence $\{p_k\}$. That is, that p_k converges uniformly to p^* on I and pointwise on $[0, \infty)$. To complete this argument we must prove that $1/p_k$ converges uniformly to $1/p^*$ on $[0, \infty)$. From the above discussion we have that $1/p_k$ converges pointwise to $1/p^*$ on $[0, \infty)$ and, in fact, on any fixed closed interval $[0, \alpha]$, $\alpha > 0$, $1/p_k$ converges uniformly to $1/p^*$ (due to the coefficient convergence).

In order to establish this final fact, we must examine the coefficient convergence in more detail. Thus, let $p^*(x) = a_{n-1}^* x^{n-1} + \cdots + a_0^*$ with $a_{n-1}^* > 0$ (here we are using our hypothesis that $\partial p^* = n - 1$ and $1/p^* \in R_n$) and let $p_k(x) = a_n^k x^n + \cdots + a_0^k$, where we know that the leading nonzero coefficient of p_k must be positive. In addition, we have that $a_j^k \to a_j^*$ as $k \to \infty$ for $j = 0, 1,\ldots, n$, where $a_n^* = 0$. Thus, there exists $\gamma \geq \beta$ such that $k \geq \gamma$ implies that $a_{n-1}^k \geq a_{n-1}^*/2 > 0$ and $|a_j^k - a_j^*| \leq 1$ for $j = 0,\ldots, n-2$. Thus, given $\epsilon > 0$ there exists $\delta > 0$ such that $\hat{p}(x) = (a_{n-1}^*/2)\, x^{n-1} + (a_{n-2}^* - 1)\, x^{n-2} + \cdots + (a_0^* - 1) \geq 2/\epsilon$. Since $p_k(x) \geq \hat{p}(x)$ for $k \geq \gamma$ and $p^*(x) > \hat{p}(x)$ for all $x \geq \delta$ we have that

$$\left| \frac{1}{p_k(x)} - \frac{1}{p^*(x)} \right| \leq \left| \frac{1}{p_k(x)} \right| + \left| \frac{1}{p^*(x)} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $k \geq \gamma$ and $x \geq \delta$. On $[0, \delta]$ we have that $1/p_k$ converges uniformly to to $1/p^*$. Thus, we may select $K \geq \gamma$ such that $k \geq K$ implies $|1/p_k(x) - 1/p^*(x)| < \epsilon$ for all $x \in [0, \delta]$. Hence, for $k \geq K$ we have that $\| 1/p_k - 1/p^* \| < \epsilon$ implying the desired result. ∎

Concluding Remarks

Observe that the question of whether or not a strong uniqueness result holds for the case that $f \in C_0^+[0, \infty)$ with its best approximation $1/p^*$ from R_n satisfying $\partial p^* = n - 1$ remains open. Likewise, the question of Lipschitz continuity of the best approximation operator remains open in this case.

A second item of interest is that in ordinary rational approximation on a finite interval, nonstandard (i.e., fewer) alternation due to degeneracy of the best approximation may be unimportant as the set of f with degenerate best approximations is nowhere dense [5, 7]. If the corresponding result that $\{f$: the best approximation $1/p^* \in R_n$ has $\partial p^* < n\}$ was nowhere dense then we could expect to be able to usually employ the simpler theory of [8] for this problem. However, the continuity result for degree $n - 1$ implies that every f with nonstandard alternation and best approximation $1/p^* \in R_n$ with $\partial p^* = n - 1$ has all g sufficiently close with nonstandard alternation and best approximations of degree $n - 1$. In this regard, an interesting question is to characterize those f for which nonstandard alternation will occur. Some initial results in this direction have been obtained by the second author and D. Leeming.

Note added in proof. D. Schmidt has proved that strong uniqueness holds when $\partial p^* = n + 1$.

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