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Orbifold index cobordism invariance

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ABSTRACT

We prove cobordism index invariance for pseudo-differential elliptic operators on closed orbifolds with K-theoretical methods.

0. Introduction

Orbifolds, which play an important role in mathematical physics, have been recently studied from many viewpoints, including groupoid and symplectic geometrical. Orbifold index theory has played an important role in these studies.

Firstly, Atiyah established in [1] many important results on the index of G-transversally elliptic operators, which are closely related to elliptic operators on orbifolds as we will explain later. In the late seventies Kawasaki gave several proofs, one of which making extensive use of Atiyah's results, of an index theorem for orbifolds [11–13]. I later proved a K-theoretical index theorem for orbifolds with operator algebraic means [7], and Berline and Vergne computed the index of transversally elliptic operators via heat equation methods in [4]. Their work was then deepened by Vergne who proved a general index theorem for orbifolds [17]. By using elliptic estimates, I established in [8] some spectral properties of the eigenvalues of the Laplacian on orbifolds, and in [9] I defined orbifold eta invariants and established an index theorem for orbifolds with boundary.

There have been many proofs of the cobordism invariance of the index of pseudo-differential elliptic operators on closed manifolds. (See for example [15,10,2].) Recently, Carvalho proved in [5] a K_G-theoretical (G compact Lie) index cobordism invariance theorem for pseudo-differential G-equivariant elliptic operators that are multiplication at infinity. Her methods are topological and rely heavily on key properties of Atiyah's K_G-functor. It has also recently come to our attention that Braveman proved the cobordism invariance of orbifold indices analytically (see [3]), and that Carvalho extended her cobordism invariance results to families (see [6]).

In this note we will prove the cobordism invariance of the index of pseudo-differential operators on orbifolds topologically by a generalization of Carvalho's method. Our main result is indeed the following theorem.

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Theorem 4.6. *Let Q_i be a closed orbifold which is the locally free quotient of an action of a compact Lie group on a smooth closed manifold, and let P_i be an elliptic pseudo-differential operator on Q_i with symbol p_i, $i = 1, 2$. Suppose that (Q_1, p_1) is orbifold symbol cobordant to (Q_2, p_2). Then $Ind(P_1) = Ind(P_2)$.*

To prove this result, instead than working directly on the orbifold, we work on its G-frame bundle. More in general, any closed effective orbifold Q arises as the locally free quotient of an action of a compact Lie group G on a closed G-manifold M, see e.g. [7]. (M is a G-frame bundle of Q, and in general G is a compact Lie group.) Therefore an elliptic pseudo-differential operator P on Q lifts to a G-transversally elliptic operator $\tilde{P}$ on M with G-transverse symbol class in $K_G^0(T_G^*(M))$. Atiyah's distributional index homomorphism $Ind: K_G^0(T_G(M)) \to \mathcal{D}'(G)$ calculates the index of P, and, consequently, the index of $\tilde{P}$; see Section 1 for details.

There are two main ingredients in our proof. One is the push-forward property of the transverse index Ind under G-embeddings established by Atiyah in [1], and the other is the K-theoretical proof provided by Carvalho in [5] for the invariance of the index of elliptic operators on manifolds under cobordism. Here we generalize Carvalho's approach to the context of transversally elliptic operators in the framework of [1]. So $Ind(\sigma) = 0$ if σ arises from a trivial symbol G-cobordism, see Theorem 4.3. By reinterpreting this at the orbifold level, we obtain our orbifold index cobordism invariance result, Theorem 4.6.

More in detail, the contents of this note are as follows. In Section 1 we recall the definition of Atiyah's distributional transverse index Ind. In Section 2, we define restriction and boundary maps, together with symbol G-cobordism. In Section 3 we detail properties of the index, restriction, and boundary maps with respect to G-equivariant embeddings. In Section 4 we will state and prove our main results.

In the sequel, all orbifolds and manifolds are assumed to be smooth, $Spin^c$, connected, and closed, unless otherwise specified. Moreover, G will denote a compact connected Lie group. By a G-manifold we mean a manifold with a smooth and proper action of G.

1. The distributional index homomorphism

In this section we will review the definition and some of the main properties of the distributional G-index homomorphism for G-transversally elliptic operators (cf. [1] for details).

Let X be a G-manifold and let $\mathcal{G}$ be the Lie algebra of G. To each $V \in \mathcal{G}$ associate the vector field V_G on X defined by

$$V_G(f) = \operatorname*{Lim}_{t\to 0} \frac{f(Exp(tV_x)) - f(x)}{t}, \quad \forall f \in C^\infty(X).$$

Definition 1.1. ([1]) Let X be a G-manifold. Define the G-invariant space $T_G^*(X) \subseteq T^*(X)$ by

$$T_G^*(X) = \{ v \in T^*(X) \mid v(V_G) = 0,\ \forall V \in \mathcal{G} \}.$$

Let X be a G-manifold and let D be a pseudo-differential operator acting on sections of the G-vector bundle E. D is said to be G-transversally elliptic G if the symbol of D is invertible on $T_G^*(X)$, except for the zero section. We will call such an operator a G-t.e.p.d. operator for short.

We will now recall how the transverse index of D is defined, see [1, Lecture 2]. If $\mathcal{G}$ denotes the Lie algebra of G, and $\mathcal{X}_j$, $j = 1, \dots, k$, the first order differential operators defined by the action of $\mathcal{G}$ on E, denote by Δ_G the following operator

$$\Delta_G = 1 - \sum_{j=1}^{k} \mathcal{X}_j^2.$$

Let λ be an eigenvalue of Δ_G, and denote by $\mathcal{C}^\infty(X, E)_\lambda$ the kernel of the operator $\Delta_G - \lambda$. Since D is G-invariant, D commutes with Δ_G, and induces an operator

$$D_\lambda : \mathcal{C}^\infty(X, E)_\lambda \to \mathcal{C}^\infty(X, E)_\lambda,$$

with index $Ind(D_\lambda)$. Define

$$Ind(D) = \sum_\lambda Ind(D_\lambda).$$

This sum converges in the sense of distributions, and is equal to the distributional index of D. If we denote by $\mathcal{D}'(G)$ the group of the G-invariant distributions on G, then the index of D is an element of $\mathcal{D}'(G)$ [1].

Let K_G^s, $s = 0, 1$, be Atiyah's equivariant K-theory functor. Then the symbol of D determines a class $\sigma_D \in K_G^0(T_G^*(X))$, in analogy G-elliptic operators [1]. $\sigma_D \in K_G^0(T_G^*(X))$ is called the G-transverse symbol class of D. We have

Theorem 1.2. *([1, Theorem 2.6]) Let X be a G-manifold and let D be a G-t.e.p.d. operator. Then the index of D depends only on the class $\sigma_D \in K_G^0(T_G^*(X))$. In particular, there exists an index homomorphism*

$$Ind: K_G^0\big(T_G^*(X)\big) \to \mathcal{D}'(G),$$

such that $Ind(\sigma_D) = Ind(D)$.

Ind can also be defined for non-compact G-manifolds via equivariant G-embeddings into compacts G-manifolds, for G compact [1].

If the action of G on X is locally free, and if D is the lift of a pseudo-differential elliptic operator on the quotient orbifold X/G, *Ind* computes the distributional orbifold index, and consequently the orbifold numerical index of the operator [13,7,17].

2. Boundary maps and symbol *G*-cobordism

We say that a G-manifold X is the G-boundary of a G-manifold with boundary W if $\partial(W) = X$, and X has a collared G-invariant neighborhood in W of type $X \times [0,1)$ with product G-action, which is assumed to be trivial on the second factor; we will write $X = \partial_G(W)$. We will also say that W has G-boundary X. Note that, $T_G^*(W)|_X = T_G^*(X) \times \mathbb{R}$.

The G-equivariant operation of restriction to X induces restriction K_G-theory homomorphisms

$$\rho_{W,X}^s: K_G^s\big(T_G^*(W)\big) \to K_G^s\big(T_G^*(X) \times \mathbb{R}\big), \quad s = 0, 1.$$

By definition,

$$K_G^1\big(T_G^*(X) \times \mathbb{R}\big) \cong K_G^0\big(T_G^*(X) \times \mathbb{R}^2\big).$$

Hence, for $s = 1$,

$$\rho_{W,X}^1: K_G^1\big(T_G^*(W)\big) \to K_G^0\big(T_G^*(X) \times \mathbb{R}^2\big).$$

Theorem 2.1. *([5]) Let W be a G-manifold having as G-boundary the G-manifold X, i.e., $\partial_G(W) = X$. The symbol G-boundary map $\partial_X^W: K_G^1(T_G^*(W)) \to K_G^0(T_G^*(X))$ is defined by the following equality*

$$\partial_X^W = \big(\beta_{T_G^*(X)}^0\big)^{-1} \circ \rho_{W,X}^1,$$

where $\beta_{T_G^*(X)}^0: K_G^0(T_G^*(X)) \to K_G^0(T_G^*(X) \times \mathbb{R}^2)$ is the equivariant Bott isomorphism.

Definition 2.2. ([5]) Let $\Upsilon_i = (X_i, \sigma_i)$ with X_i a G-manifold and $\sigma_i \in K_G^0(T_G^*(X_i))$, $i = 1, 2$. Then we say that Υ_1 and Υ_2 are symbol G-cobordant if there exists a pair $\mathcal{W} = (W, \sigma)$, with W a G-manifold with boundary, and $\sigma \in K_G^1(T_G^*(W))$, such that the G-boundary of W is $X_1 \sqcup X_2$, and

$$\partial_X^W(\sigma) = -\sigma_1 \oplus \sigma_2.$$

If Υ_1 and Υ_2 are symbol G-cobordant, we will write $\Upsilon_1 \sim \Upsilon_2$. Note that $\sim$ is an equivalence relation.

3. Embeddings, index, and symbol *G*-cobordisms

We will now describe the behaviors of the index, restriction and boundary homomorphisms with respect to G-embeddings. We will require that all G-embeddings $\varphi: X \to Y$ of G-manifolds are G-equivariant, K-oriented, and admit open G-invariant tubular neighborhoods. If X and Y are G-manifolds with boundary, we also assume that a G-embedding $\varphi: X \to Y$ restricts to a G-embedding on ∂X. (This follows from the existence of tubular neighborhoods for G-manifolds with boundary.)

Let $\varphi: X \to Y$ be a G-embedding of X into Y. Then φ induces K_G-theory 'wrong way functoriality' shriek maps $\varphi_!^s: K_G^s(T_G^*(X)) \to K_G^s(T_G^*(Y))$, $s = 0, 1$, defined as below [1].

Let N be an open G-invariant tubular neighborhood of $\varphi(X)$ in Y and let

$$\tau_{X,N}^s: K_G^s\big(T_G^*(X)\big) \to K_G^s\big(T_G^*(N)\big), \quad s = 0, 1,$$

be the Thom homomorphism. Moreover, let

$$\kappa_{N,Y}^s: K_G^s\big(T_G^*(N)\big) \to K_G^s\big(T_G^*(Y)\big), \quad s = 0, 1,$$

be the K_G-theory maps induced by the open embedding $\kappa: T_G^*(N) \to T_G^*(Y)$.

Definition 3.1. Let $\varphi : X \to Y$ be a G-embedding of G-manifolds. Then the shriek map

$$\varphi_!^s : K_G^s\big(T_G^*(X)\big) \to K_G^s\big(T_G^*(Y)\big), \quad s = 0, 1,$$

is defined by the following equality

$$\varphi_!^s = \kappa_{N,Y}^s \circ \tau_{X,N}^s, \quad s = 0, 1.$$

The following deep theorem, which states the invariance of the transverse index under push-forwards, is proved by Atiyah in [1].

Theorem 3.2. *([1]) Let $\varphi : X \to Y$ be a G-embedding of G-manifolds. Then the diagram below is commutative*:

$$\begin{array}{ccc} K_G^0(T_G^*(X)) & \xrightarrow{\varphi_!^0} & K_G^0(T_G^*(Y)) \\ \downarrow{\scriptstyle Ind} & & \downarrow{\scriptstyle Ind} \\ \mathcal{D}'(G) & \xrightarrow{Id} & \mathcal{D}'(G) \end{array}$$

Moreover, if $\varphi : X \to Y$ and $\psi : Y \to Z$ are G-embeddings of G-manifolds, then

$$(\varphi \circ \psi)_!^s = \varphi_!^s \circ \psi_!^s, \quad s = 0, 1.$$

In the above theorem, Y can also be assumed to be non-compact because of the functoriality of the shriek maps, and the open G-embeddings index invariance [1].

The following results, detailing the behavior of the transverse symbol with respect to G-embeddings and push-forwards, are generalizations of analogous results in [5]. We omit their proofs.

Lemma 3.3. *Let $\varphi : X \to Y$ be a G-embedding of G-manifolds. Then the diagram below is commutative*:

$$\begin{array}{ccc} K_G^0(T_G^*(X)) & \xrightarrow{\varphi_!^0} & K_G^0(T_G^*(Y)) \\ \downarrow{\scriptstyle \beta_X^0} & & \downarrow{\scriptstyle \beta_Y^0} \\ K_G^0(T_G^*(X) \times \mathbb{R}^2) & \xrightarrow{\tilde{\varphi}_!^0} & K_G^0(T_G^*(Y) \times \mathbb{R}^2) \end{array}$$

Here

$$\beta_Z^0 : K_G^0\big(T_G^*(Z)\big) \to K_G^0\big(T_G^*(Z) \times \mathbb{R}^2\big), \quad Z = X, Y,$$

is the Bott periodicity isomorphism and $\tilde{\varphi}$ is the lift of φ to $X \times \mathbb{R}^2$. (*The action of G on $\mathbb{R}^2$ is trivial.*)

Lemma 3.4. *Let $\varphi : X \to Y$ be a G-embedding of G-manifolds with boundary. Then the diagram below is commutative*:

$$\begin{array}{ccc} K_G^1(T_G^*(X)) & \xrightarrow{\varphi_!^1} & K_G^1(T_G^*(Y)) \\ \downarrow{\scriptstyle \rho_{X,X_0}^1} & & \downarrow{\scriptstyle \rho_{Y,Y_0}^1} \\ K_G^0(T_G^*(X_0) \times \mathbb{R}^2) & \xrightarrow{\tilde{\varphi}_{0!}^0} & K_G^0(T_G^*(Y_0) \times \mathbb{R}^2) \end{array}$$

where $\partial Z = Z_0$, $Z = X, Y$, $\varphi_0 = \varphi|_{X_0}$, and $\tilde{\varphi}_0$ is the lift of φ_0 to $X_0 \times \mathbb{R}^2$. (*Notation as in Section* 2 *and Lemma* 3.3.)

Proposition 3.5. *Let $\varphi : X \to Y$ be a G-embedding of G-manifolds with boundary. Then the diagram below is commutative*:

$$\begin{array}{ccc} K_G^1(T_G^*(X)) & \xrightarrow{\varphi_!^1} & K_G^1(T_G^*(Y)) \\ \downarrow{\scriptstyle \partial_{X_0}^X} & & \downarrow{\scriptstyle \partial_{Y_0}^Y} \\ K_G^0(T_G^*(X_0)) & \xrightarrow{\varphi_{0!}^0} & K_G^0(T_G^*(Y_0)) \end{array}$$

where $\varphi_0 = \varphi|_{X_0}$, $\partial Z = Z_0$, and the boundary map $\partial_{Z_0}^Z$, is as in Theorem 2.1. ($Z = X, Y$.)

4. The main result

In this final section, we will prove our main results, Theorems 4.3 and 4.6. But first two lemmas.

Let M be a G-manifold and $\sigma \in K_G(T^*_G(M))$. Recall that $\Upsilon = (M,\sigma) \sim 0$ (that is, Υ is symbol G-cobordant to zero) if there exist a G-manifold with boundary W with $X = \partial_G(W)$, and $\omega \in K^0_G(T^*_G(W))$ such that $\partial^W_X(\omega) = \sigma$.

Lemma 4.1. *Let M be a G-manifold and $\sigma \in K_G(T^*_G(M))$, with $\Upsilon = (M,\sigma) \sim 0$ via the G-cobordism W. If $\varphi : W \to Y$ is a G-embedding of W into the G-manifold with boundary Y, with restriction to the boundary given by φ_0, then*

$$\partial^Y_{\partial Y}\bigl(\varphi^1_!(\omega)\bigr) = \varphi^0_{0!}(\sigma),$$

which implies $\Upsilon_Y = (\partial Y, \varphi^0_{0!}(\sigma)) \sim 0$.

Proof. Apply the results of Section 3. □

Lemma 4.2. *We have*

$$K^i_G\bigl(T^*_G(E)\bigr) = 0, \quad i = 0,1,$$

where E is equal to the space $[0,1) \times \mathbb{R}^t$, $t > 0$.

Proof. Firstly, $K^i_G(T^*(E)) = 0$, $i = 0,1$, since this space is G-contractible [5]. Next, $K^i_G(T^*_G(E))$, $i = 0,1$, can be decomposed as the direct sum $K^i_G(T^*_G(E)) = \bigoplus_j K^i_G(T^*_G(E(j) - E(j+1)))$, where $E(j) = \{x \in E\colon \dim G_x \geqslant j\}$, $i = 0,1$, see [1, Theorem 8.4]. Also note that a decomposition similar to the above can be proved for $K^i_G(T^*(E))$, $i = 0,1$, with similar methods. Of course, in this latter case, each of the factors is the zero module.

Since on each of the spaces appearing in the above decompositions the action of the group can be assumed to have only finite stabilizers, by the Bott periodicity theorem it follows that each of the factors in the two decompositions are isomorphic in pairs, from which the result follows. Note that the Bott periodicity theorem can be applied to this case where the action is non-trivial by [16, Remark 2.8.7]. □

Theorem 4.3. *Suppose that M is a G-manifold, and let D be a G-p.d.t.e. operator on M with G-transverse symbol class $\sigma \in K^0_G(T^*_G(M))$. If $\Upsilon = (M,\sigma) \sim 0$, then $Ind(D) = 0$.*

Proof. Let $\mathcal{W} = (W,\omega)$, for some $\omega \in K^1_G(T^*_G(W))$, be a symbol G-cobordism between Υ and 0. Let $\varphi : W \to E$ be a G-embedding of W into $E = [0,1) \times \mathbb{R}^t$ (see e.g. [14]). If we denote by φ_0 the restriction of φ to the boundary, then by Lemma 4.1,

$$\partial^F_{\partial F}\bigl(\varphi^1_!(\omega)\bigr) = \varphi^0_{0!}(\sigma).$$

By Lemma 4.2 and the results in Section 3, $\varphi^1_!(\omega) = 0$, which implies $Ind(D) = 0$. □

As a corollary, we have

Corollary 4.4. *Suppose that M_i is a G-manifold, and let D_i be a G-p.d.t.e. on M_i with G-transverse symbol class $\sigma_i \in K^0_G(T^*_G(M_i))$, $i = 1,2$. Assume that $\Upsilon_1 = (M_1,\sigma_1) \sim \Upsilon_2 = (M_2,\sigma_2)$. Then $Ind(D_1) = Ind(D_2)$.*

Proof. We have that

$$(M_1 \sqcup M_2, -\sigma_1 \oplus \sigma_2) \sim 0.$$

Now $(-\sigma_1 \oplus \sigma_2)$ is the G-transverse symbol of $D^*_1 \oplus D_2$. Then the claim follows from Theorem 4.3. □

We can now prove the invariance under cobordism of the orbifold index.

Definition 4.5. Let Q_i be an orbifold. Assume that Q_i arises as the locally free quotient of the G-manifold M_i, $i = 1,2$. Let P_i be an elliptic pseudo-differential operator on Q_i with symbol p_i, and let $\tilde{P}_i$ be its lift to M_i with symbol $\tilde{p}_i$, $i = 1,2$. Then $\tilde{P}_i$ is a G-p.d.t.e. on M_i with G-transverse symbol class $\sigma_{\tilde{P}_i} \in K^0_G(T^*_G(M_i))$, $i = 1,2$. We say that (Q_1,p_1) is orbifold symbol cobordant to (Q_2,p_2) if $\Upsilon_1 = (M_1,\sigma_{\tilde{P}_1}) \sim \Upsilon_2 = (M_2,\sigma_{\tilde{P}_2})$.

Theorem 4.6. *Let Q_i be an orbifold and let P_i be an elliptic pseudo-differential operator on Q_i with symbol p_i, $i = 1,2$. Suppose that (Q_1,p_1) is orbifold symbol cobordant to (Q_2,p_2). Then $Ind(P_1) = Ind(P_2)$.*

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