JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 16, 73-83 (1966)

A Class of Continuous Linear Programming Problems

N. LEVINSON*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts

INTRODUCTION

Tyndall [1] treats rigorously a continuous linear programming problem and its dual. Here the problem will be generalized and the proofs shortened. References to relevant work of Bellman, Wolfe, Lehman, Koopmans, and Dorfman, Samuelson, and Solow will be found in [1]. A number of interesting examples and counter examples are also given in [1].

If $A(t)$ is a matrix for $0 \leqslant t \leqslant T$ with entries $a_{ij}(t)$ and if $\rho(t)$ is a scalar on $[0, T]$ such that every entry

$$a_{ij}(t) \leqslant \rho(t), \qquad t \in [0, T],$$

then the notation

$$A(t) \leqslant \rho(t) \qquad t \in [0, T]$$

will be used. The meaning of $\rho(t) \leqslant A(t)$ is now clear. If $\tilde{A}(t)$ is a matrix on $[0, T]$ with the same number of rows and columns as $A(t)$, then

$$A(t) \leqslant \tilde{A}(t)$$

means

$$a_{ij}(t) \leqslant \tilde{a}_{ij}(t)$$

for all entries. The transpose of A will be denoted by A'. Also

$$| A | = \sum_{i,j} | a_{ij} | .$$

Vectors are regarded as matrices of one column.

* Supported in part by the Office of Naval Research and the National Science Foundation, NSF GP-4364.

The abbreviation p.w. contin. will be used for piecewise continuous; that is, continuous except for a finite number of simple discontinuities. In Theorems 1 and 2

$B(t) \geqslant 0$ is an $n \times m$ matrix p.w. contin. on $[0, T]$

$\gamma(t)$ is an n vector p.w. contin. on $[0, T]$

$\alpha(t)$ is an m vector p.w. contin. on $[0, T]$

$K(t, s)$ is an $n \times m$ matrix p.w. contin. on $[0, T] \times [0, T]$

(By the p.w. contin. of $K(t, s)$ is meant continuous except for possible simple discontinuities on a finite number of differentiable arcs.)

A bounded measurable m vector $z(t) \geqslant 0$ on $[0, T]$ such that

$$B(t)\, z(t) \leqslant \gamma(t) + \int_0^t K(t, s)\, z(s)\, ds, \qquad 0 \leqslant t \leqslant T, \tag{1.1}$$

is said to be feasible for the Primal Problem. The Primal Problem is said to be feasible if there exists a feasible $z(t)$.

REMARK. If $\gamma(t) \geqslant 0$ then the Primal Problem is feasible since $z(t) = 0$ is feasible.

A feasible $\tilde{z}(t)$ is said to be extremal if

$$\int_0^T \tilde{z}(t) \cdot \alpha(t)\, dt \geqslant \int_0^T z(t) \cdot \alpha(t)\, dt \tag{1.2}$$

for all feasible $z(t)$.

The case treated in [1] is where $B(t)$ and $K(t, s)$ are both constant matrices.

THEOREM 1. *Let the Primal Problem (1.1) be feasible. Let there exist a p.w. contin. vector $\lambda(t) \geqslant 0$, $|\lambda(t)| \leqslant 1$ and a constant $b > 0$ such that*

$$\lambda'(t)\, B(t) \geqslant b > 0. \tag{1.3}$$

Then there exists an extremal $\tilde{z}(t)$.

A bounded measurable n-vector $w(t) \geqslant 0$ on $[0, T]$ such that

$$B'(t)\, w(t) \geqslant \alpha(t) + \int_t^T K'(s, t)\, w(s)\, ds, \qquad 0 \leqslant t \leqslant T, \tag{1.4}$$

is said to be feasible for the Dual Problem. The Dual Problem is said to be feasible if there exists a feasible $w(t)$. (The relationship of the two problems is shown in (4.21).)

A feasible $\tilde{w}(t)$ is said to be extremal if

$$\int_0^T \tilde{w}(t) \cdot \gamma(t)\, dt \leqslant \int_0^T w(t) \cdot \gamma(t)\, dt \tag{1.5}$$

for all feasible $w(t)$.

THEOREM 2. *Let*

$$\gamma(t) \geqslant 0, \qquad K(t, s) \geqslant 0. \tag{1.6}$$

Let there exist $\delta > 0$ such that for each i, j, and t

$$\text{either} \qquad b_{ij}(t) = 0 \qquad \text{or else} \qquad b_{ij}(t) \geqslant \delta. \tag{1.7}$$

Also for each t and j let there exist $i_j = i_j(t)$ such that

$$b_{i_j j}(t) \geqslant \delta. \tag{1.8}$$

Then the Dual Problem is feasible and there exists an extremal $\tilde{w}(t)$. (If (1.8) is satisfied then so is (1.3) by taking $\lambda(t)$ with all entries equal to $1/n$. In case B is constant (1.7) and (1.8) are implied by the Hypotheses of [1].)

THEOREM 3. *If in addition to the hypothesis of Theorem 2, $B(t)$, $\alpha(t)$, $\gamma(t)$ and $K(t, s)$ are all continuous on $[0, T]$ and $[0, T] \times [0, T]$, respectively, then*

$$\int_0^T \tilde{z}(t) \cdot \alpha(t)\, dt = \int_0^T \tilde{w}(t) \cdot \gamma(t)\, dt. \tag{1.9}$$

Moreover if $z(t)$ is feasible for (1.1) and $w(t)$ for (1.4) and if

$$\int_0^T z(t) \cdot \alpha(t)\, dt = \int_0^T w(t) \cdot \gamma(t)\, dt, \tag{1.10}$$

then $z(t)$ and $w(t)$ are both extremal.

2. PRIMAL PROBLEM

The simplification in the present treatment is effected by treating the continuous problem directly in Theorems 1 and 2.

GRONWALL'S LEMMA. *Let the integrable scalar $g(t) \geqslant 0$ satisfy*

$$g(t) \leqslant c_1 + c_2 \int_0^t g(s)\, ds, \qquad 0 \leqslant t \leqslant T, \tag{2.1}$$

where $c_1 \geqslant 0$, $c_2 > 0$. Then

$$g(t) \leqslant c_1 e^{c_2 t}, \qquad 0 \leqslant t \leqslant T. \tag{2.2}$$

PROOF. Let

$$G(t) = \int_0^t g(s)\, ds.$$

Then from (2.1)

$$\frac{d}{dt}(e^{-c_2 t}\, G(t)) \leqslant c_1 e^{-c_2 t}.$$

Hence

$$G(t) \leqslant \frac{c_1}{c_2}(e^{c_2 t} - 1),$$

which in (2.1) gives (2.2).

PROOF OF THEOREM 1. From (1.1)

$$\lambda'(t)\, B(t)\, z(t) \leqslant \lambda'(t)\, \gamma(t) + \int_0^t \lambda'(t)\, K(t, s)\, z(s)\, ds.$$

If

$$|\, \gamma(t)\, | \leqslant c_1 \qquad |\, K(t, s)\, | \leqslant c, \tag{2.3}$$

then by (1.3)

$$b\, |\, z(t)\, | \leqslant c_1 + c \int_0^t |\, z(s)\, |\, ds.$$

By Gronwall's lemma

$$|\, z(t)\, | \leqslant \frac{c_1}{b} \exp\left(\frac{ct}{b}\right) \leqslant \frac{c_1}{b} \exp\left(\frac{cT}{b}\right) \tag{2.4}$$

For all feasible $z(t)$ let

$$\ell.u.b. \int_0^T z(t) \cdot \alpha(t)\, dt = M.$$

Then there exists a sequence of feasible $z^{(j)}(t), j \geqslant 1$, such that

$$\lim_{j\to\infty} \int_0^T z^{(j)}(t) \cdot \alpha(t)\, dt = M. \tag{2.5}$$

By (2.4)

$$|\, z^{(j)}(t)\, | \leqslant \frac{c_1}{b} \exp\left(\frac{cT}{b}\right). \tag{2.6}$$

Hence by weak convergence in $L^2(0, T)$, there exists $z^{(0)}(t)$ to which a subsequence of $z^{(j)}(t)$ converges weakly [2, p. 64] as $j \to \infty$. Calling this weakly convergent subsequence itself $z^{(j)}(t)$, the weak convergence implies in (2.5) that

$$\int_0^\infty z^{(0)}(t) \cdot \alpha(t)\, dt = M \tag{2.7}$$

and in (1.1) with $z = z^{(j)}(t)$, that

$$\limsup_{j\to\infty} B(t)\, z^{(j)}(t) \leqslant \gamma(t) + \int_0^t K(t,s)\, z^{(0)}(s)\, ds. \tag{2.8}$$

It remains to discuss the feasibility of $z^{(0)}(t)$ and to see if $z^{(0)}(t) \geqslant 0$.

LEMMA 2.1. *Let the uniformly bounded sequence of scalar measurable functions* $\{f_j(t)\}$, $j \geqslant 1$, *converge weakly on* $[0, T]$ *to* $f_0(t)$. *Let*

$$\limsup_{j\to\infty} f_j(t) = f_u(t). \tag{2.9}$$

Then, except on a set of zero measure,

$$f_0(t) \leqslant f_u(t), \qquad 0 \leqslant t \leqslant T.$$

PROOF OF LEMMA. For $n \geqslant 1$ let

$$f_0(t) \geqslant f_u(t) + \frac{1}{n} \tag{2.10}$$

on a set E_n of $[0, T]$. From (2.9)

$$\lim_{j\to\infty} [\sup_{i\geqslant j} f_i(t)] = f_u(t). \tag{2.11}$$

Using weak convergence, we obtain

$$\int_{E_n} f_0(t)\, dt = \lim_{j\to\infty} \int_{E_n} f_j(t)\, dt \leqslant \lim_{j\to\infty} \int_{E_n} \sup_{i\geqslant j} f_i(t)\, dt.$$

Using (2.11) and the dominated convergence theorem of Lebesgue, the above gives

$$\int_{E_n} f_0(t)\, dt \leqslant \int_{E_n} f_u(t)\, dt.$$

This and (2.10) give $m(E_n)/n \leqslant 0$ so that $m(E_n) = 0$. But $E_{n+1} \supset E_n$ so that

$$m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n) = 0,$$

which proves the lemma.

By using the lemma on each entry of $z^{(j)}(t)$ and by using (2.6) it follows that $z^{(0)}(t)$ is uniformly bounded from above except possibly on a set of zero measure where it can be assumed to be zero.

By using this lemma on each row of the vector $B(t)\, z^{(j)}(t)$ in (2.8) it follows that

$$B(t)\, z^{(0)}(t) \leqslant \gamma(t) + \int_0^t K(t, s)\, z^{(0)}(s)\, ds \tag{2.12}$$

except possibly on a set of zero measure, E_0. Since the left side of (2.8) is non-negative, so is the right side. Hence on E_0 replace $z^{(0)}(t)$ by the zero vector. This will not change the right side of (2.12) or (2.7). Calling the so modified $z^{(0)}(t)$, $\tilde{z}^{(0)}(t)$, (1.1) is satisfied by $\tilde{z}^{(0)}(t)$.

To show $z^{(0)}(t) \geqslant 0$ except on a set of zero measure note that if one defines $f_l(t) = \liminf f_j(t)$ in place of (2.9), one finds as in Lemma 2.1 that $f_0(t) \geqslant f_l(t)$ except possibly on a set of measure 0. By applying this to each row of $z^{(0)}(t)$, $z^{(0)}(t) \geqslant 0$ except possibly on a set of zero measure $\tilde{E}_0$, and hence so is $\tilde{z}^{(0)}(t) \geqslant 0$ except possibly on $\tilde{E}_0$. Take $\tilde{z}(t)$ as $\tilde{z}^{(0)}(t)$ off of $\tilde{E}_0$ and as the zero vector on $\tilde{E}_0$ and the theorem is proved.

3. Dual Problem

Let c be as in (2.3) and let $|\alpha(t)| \leqslant c_2$. Let

$$\rho(t) = \frac{c_2}{\delta} \exp\left[\frac{c(T-t)}{\delta}\right]. \tag{3.1}$$

Then

$$\delta\rho(t) = c_2 + c\int_t^T \rho(s)\, ds, \qquad 0 \leqslant t \leqslant T. \tag{3.2}$$

Take $\omega(t)$ as an n-vector with all entries $\rho(t)$. Then $\omega(t)$ is feasible. Indeed by (1.8) and (3.2)

$$B'(t)\, \omega(t) \geqslant \delta\rho(t) \geqslant \alpha(t) + \int_t^T K'(s, t)\, \omega(s)\, ds. \tag{3.3}$$

Hence the Dual Problem is feasible.

Since it is desired to minimize

$$\int_0^T w(t) \cdot \gamma(t)\, dt,$$

and since $\gamma(t) \geqslant 0$, it is advantageous to decrease each positive entry of $w(t)$ if possible.

Lemma 3.1. *Let $w(t)$ be feasible. Then there exists $v(t)$ also feasible and*

$$0 \leqslant v(t) \leqslant w(t); \qquad v(t) \leqslant \rho(t). \tag{3.4}$$

PROOF OF LEMMA 3.1. From (1.4)

$$\sum_i b_{ij}(t)\, w_i(t) \geqslant \alpha_j(t) + \sum_i \int_t^T k_{ij}(s, t)\, w_i(s)\, ds. \tag{3.5}$$

If

$$v_i(t) = \begin{cases} w_i(t), & w_i(t) \leqslant \rho(t) \\ \rho(t), & w_i(t) > \rho(t), \end{cases}$$

then, since $K \geqslant 0$,

$$\sum_i b_{ij}(t)\, w_i(t) \geqslant \alpha_j(t) + \sum_i \int_t^T k_{ij}(s, t)\, v_i(s)\, ds. \tag{3.6}$$

Let $\sum = \sum' + \sum''$ where $\sum'$ extends over the i of $\sum$ for which $w_i(t) \leqslant \rho(t)$ and $\sum''$ over those for which $w_i(t) > \rho(t)$. It $\sum''$ is zero for a given t and j, $w_i(t) = v_i(t)$ in (3.6) and the lemma is proved. If $\sum''$ is not zero,

$$\sum_i b_{ij}(t)\, v_i(t) \geqslant {\sum_i}'' b_{ij}(t)\, \rho(t) \geqslant \delta\rho(t)$$

by (1.7). By using the right side of (3.3) the proof of the lemma is complete.

Hence it can be assumed that every feasible $w(t)$ satisfies

$$0 \leqslant w(t) \leqslant \rho(t)$$

and so by (3.1) is uniformly bounded. Weak convergence can now be used much as in the proof of Theorem 1 to complete the proof that $\tilde{w}(t)$ exists. It should be noted however that in the analogue to (2.8) here lim inf is treated. Also on the set of measure 0 on which $w^{(0)}(t)$, the weak limit, is modified, it is replaced by $\omega(t)$ as defined below (3.2).

4. PROOF OF THEOREM 3. Now B, α, γ and K are all continuous. Let $h = T/N$ for large N. Let

$$B_{(j)} = B(jh), \qquad \alpha_{(j)} = \alpha(jh), \qquad \gamma_{(j)} = \gamma(jh), \qquad K_{(i,j)} = K(ih, jh) \tag{4.1}$$

Let $\zeta_{(j)}$ be m-vectors and consider the discrete problem

$$\begin{aligned} B_{(1)}\zeta_{(1)} &\leqslant \gamma_{(1)} + K_{(1,1)}\zeta_{(1)}h \\ B_{(2)}\zeta_{(2)} &\leqslant \gamma_{(2)} + K_{(2,1)}\zeta_{(1)}h + K_{(2,2)}\zeta_{(2)}h \\ &\vdots \\ B_{(N)}\zeta_{(N)} &\leqslant \gamma_{(N)} + \sum_j K_{(N,j)}\zeta_{(j)}h \end{aligned} \tag{4.2}$$

with $\zeta_{(j)} \geqslant 0$ and chosen to maximize

$$\sum_j \zeta_{(j)} \cdot \alpha_{(j)} h \tag{4.3}$$

if possible. If

$$\zeta = \begin{pmatrix} \zeta_{(1)} \\ \zeta_{(2)} \\ \vdots \\ \zeta_{(N)} \end{pmatrix},$$

this is a classical linear programming problem. It is feasible since $\zeta_{(j)} = 0$ is feasible.

Multiplying the jth equation of (4.2) by $\lambda'_{(j)} = \lambda'(jh)$,

$$b \mid \zeta_{(j)} \mid \leqslant c_1 + hc \sum_{i \leqslant j} \mid \zeta_{(i)} \mid$$

or

$$(b - ch) \mid \zeta_{(1)} \mid \leqslant c_1$$

$$(b - ch) \mid \zeta_{(j)} \mid \leqslant c_1 + hc \sum_{i \leqslant j-1} \mid \zeta_i \mid, \qquad j = 2, \cdots, N.$$

If $h < b/(2c)$ is small enough it follows easily by induction and $\exp(\epsilon) - 1 \geqslant \epsilon$ that

$$\mid \zeta_{(j)} \mid \leqslant \frac{2c_1}{b} \exp\left(\frac{4chj}{b}\right) \leqslant \frac{2c_1}{b} \exp\left(\frac{4cT}{b}\right). \tag{4.4}$$

The classical dual to (4.2), (4.3) is

$$B'_{(1)}\omega_{(1)} \geqslant \alpha_{(1)} + \sum_{N \geqslant i \geqslant 1} K'(i, 1)\, \omega_{(i)} h$$

$$\vdots$$

$$B'_{(N)}\omega_{(N)} \geqslant \alpha_{(N)} + K'_{(N,N)}\omega_{(N)} h \tag{4.5}$$

with $\omega_{(j)} \geqslant 0$ and $\omega_{(j)}$ chosen to minimize

$$\sum \omega_{(j)} \cdot \gamma_{(j)} h \tag{4.6}$$

if possible. Let

$$\sigma_j = \frac{2c_2}{\delta} \exp\left[\frac{4c(N-j)\,h}{\delta}\right]. \tag{4.7}$$

Then

$$c_2 + ch \sum_{j \leqslant i \leqslant N} \sigma_i \leqslant c_2 + \frac{2chc_2}{\delta} e^{4c(N-j)h/\delta} \sum_{i \geqslant 0} e^{-4cih/\delta},$$

and for small h

$$\sum_{i \geqslant 0} e^{-4cih/\delta} = \frac{1}{1 - e^{-4ch/\delta}} < \frac{\delta}{3ch},$$

so that

$$c_2 + ch \sum_{j \leqslant i \leqslant N} \sigma_i \leqslant c_2 + \frac{2}{3} c_2 e^{4c(N-j)h/\delta} < \delta\sigma_j . \tag{4.8}$$

By using (4.8) it follows that if $\omega_{(j)}$ is taken with all its n entries equal to σ_j, then (4.5) is feasible. Also much as in Lemma 3.1 it can always be assumed for feasible $\omega_{(j)}$ that

$$0 \leqslant \omega_{(j)} \leqslant \sigma_j \leqslant \frac{2c_2}{\delta} \exp\left[\frac{4cT}{\delta}\right]. \tag{4.9}$$

Since (4.2) and its dual (4.5) are both feasible, it follows from the classical theory that there exists extremal $\tilde{\zeta}_{(j)}$ and $\tilde{\omega}_{(j)}$ such that

$$\sum \tilde{\zeta}_j \cdot \alpha_{(j)} h = \sum \tilde{\omega}_j \cdot \gamma_{(j)} h, \tag{4.10}$$

where by (4.4) and (4.9)

$$0 \leqslant \tilde{\zeta}_{(j)} \leqslant \frac{2c_1}{b} e^{4cT/b} \tag{4.11}$$

$$0 \leqslant \tilde{\omega}_{(j)} \leqslant \frac{2c_2}{b} e^{4cT/\delta} \tag{4.12}$$

and hence are uniformly bounded independent of h.

For $1 \leqslant j \leqslant N$ let

$$z_h(t) = \tilde{\zeta}_{(j)}, \qquad (j-1)\,h \leqslant t < jh. \tag{4.13}$$

Then since B, α, γ and K are uniformly continuous it follows from (4.11), (4.13), and the jth row of (4.2) that, given $\epsilon > 0$, for small enough h

$$B(t)\, z_h(t) \leqslant \gamma(t) + \epsilon + \int_0^t K(t, s)\, z_h(s)\, ds. \tag{4.14}$$

This is (1.1) with γ replaced by $\gamma + \epsilon$. Hence there is an extremal $\tilde{z}_\epsilon(t)$ for which (4.14) holds with z_h replaced by $\tilde{z}_\epsilon$.

Also

$$\int_0^T \tilde{z}_\epsilon(t) \cdot \alpha(t)\, dt \geqslant \int_0^T z_h(t) \cdot \alpha(t)\, dt \tag{4.15}$$

since $z_h(t)$ is feasible by (4.14). In place of (2.4)

$$|\, \tilde{z}_\epsilon(t) \,| \leqslant \frac{c_1 + n\epsilon}{b} \exp\left(\frac{cT}{b}\right). \tag{4.16}$$

For the given $\epsilon > 0$ and small enough h, (4.13) implies

$$\int_0^T z_h(t) \cdot \alpha(t)\, dt \geqslant \sum \tilde{\zeta}_{(j)} \cdot \alpha_{(j)} h - \epsilon.$$

This with (4.15) gives

$$\int_0^T \tilde{z}_\epsilon(t) \cdot \alpha(t)\, dt + \epsilon \geqslant \sum \tilde{\zeta}_{(j)} \cdot \alpha_{(j)} h. \tag{4.17}$$

Proceeding similarly with the Dual Problem by defining $w_h(t)$ much as in (4.13), etc., one has with $\alpha(t)$ in (1.4) replaced by $\alpha(t) - \epsilon$ an extremal $\tilde{w}_\epsilon(t)$ such that

$$\int_0^T \tilde{w}_\epsilon(t) \cdot \gamma(t)\, dt \leqslant \sum \tilde{\omega}_{(j)} \cdot \gamma_{(j)} h + \epsilon.$$

By combining this with (4.17) and (4.10),

$$\int_0^T \tilde{w}_\epsilon(t) \cdot \gamma(t)\, dt \leqslant \int_0^T \tilde{z}_\epsilon(t) \cdot \alpha(t)\, dt + 2\epsilon. \tag{4.18}$$

As in (4.15) let $\tilde{z}_\epsilon(t)$ be extremal for (1.1) with γ replaced by $\gamma + \epsilon$ and define

$$M(\epsilon) = \int_0^T \tilde{z}_\epsilon(t) \cdot \alpha(t)\, dt.$$

Given $\eta > 0$, $\tilde{z}_\eta(t)$ is feasible for $\epsilon > \eta$ and so it follows that $M(\epsilon)$ is monotone nondecreasing. Hence letting $\epsilon \to +0$, $M(+0)$ exists and

$$M = M(0) \leqslant M(+0). \tag{4.19}$$

On the other hand, by using (4.16), it follows much as in the proof of Theorem 1 that as $\epsilon \to +0$, a subsequence $\epsilon_j \to +0$ can be extracted for which $\tilde{z}_{\epsilon_j}(t)$ converges weakly, except possibly on an easily treated set of measure zero, to a feasible $z(t)$ for $\epsilon = 0$. This shows that $M(0) \geqslant M(+0)$ and so by (4.19), $M(0) = M(0+)$. Hence given $\eta > 0$, for sufficiently small ϵ

$$\int_0^T \tilde{z}_\epsilon(t) \cdot \alpha(t)\, dt < \int_0^T \tilde{z}(t) \cdot \alpha(t)\, dt + \eta.$$

A similar argument for the dual problem shows that

$$\int_0^T \tilde{w}_\epsilon(t) \cdot \gamma(t)\, dt > \int_0^T \tilde{w}(t) \cdot \gamma(t)\, dt - \eta.$$

The two above inequalities in (4.18) give

$$\int_0^T \tilde{w}(t) \cdot \gamma(t)\, dt < \int_0^T \tilde{z}(t) \cdot \alpha(t)\, dt + 2\epsilon + 2\eta$$

for any given $\epsilon > 0$ and $\eta > 0$. Thus

$$\int_0^T \tilde{w}(t) \cdot \gamma(t)\, dt \leqslant \int_0^T \tilde{z}(t) \cdot \alpha(t)\, dt. \tag{4.20}$$

On the other hand if $z(t)$ is feasible for (1.1) and $w(t)$ for (1.4), taking the scalar product of $w(t)$ with (1.1) and $z(t)$ with (1.4) and integrating each on $[0, T]$ leads easily to

$$\int_0^T w(t) \cdot \gamma(t)\, dt \geqslant \int_0^T z(t) \cdot \alpha(t)\, dt. \tag{4.21}$$

Using (4.21) with $w = \tilde{w}$, $z = \tilde{z}$ together with (4.20) proves (1.9).

Given (1.10),

$$\int_0^T w(t) \cdot \gamma(t)\, dt = \int_0^T z(t) \cdot \alpha(t)\, dt \leqslant \int_0^T \tilde{z}(t) \cdot \alpha(t)\, dt,$$

which with $z = \tilde{z}$ in (4.21) gives

$$\int_0^T w(t) \cdot \gamma(t)\, dt = \int_0^T \tilde{z}(t) \cdot \alpha(t)\, dt.$$

This with (1.9) shows w to be extremal. A similar argument applies to z and completes the proof of Theorem 3.

References

1. W. F. Tyndall. A duality theorem for a class of continuous linear programming problems, *J. Soc. Ind. Appl. Math.* **13** (1965), 644-666.
2. F. Riesz and B. Sz.-Nagy. "Lecons D'Analyse Fonctionnelle," Gauthier-Villars, Paris, 1955.