Characteristic Conditions of the Generation of $C_0$ Semigroups in a Hilbert Space

Dong-Hua Shi

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084 China
E-mail: dshi@math.tsinghua.edu.cn

and

De-Xing Feng

Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China
E-mail: dxfeng@iss03.iss.ac.cn

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In this paper new characteristic conditions, in terms of $A$ and the first order resolvent of $A$ and $A^*$, which assure that $A$ generates a $C_0$ semigroup in a Hilbert space are proposed and proved. The conditions can be used to investigate the well-posedness problem of non-dissipative systems. An example is also given to show how to use them.

Key Words: $C_0$ semigroups; generation conditions; Hilbert space; non-dissipative systems.

1. INTRODUCTION

In recent years there has been growing interest in the stabilization of flexible robot arms and other flexible structures. Among the many control methods developed, most of them used the control law that makes the corresponding closed loop system dissipative. More recently, a new kind of feedback control method which makes the closed loop system non-dissipative is particularly attractive due to its easy implementation and better performance in practice (see [6]). Since the closed loop system is a

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non-dissipative system in this case, it is very difficult to investigate the well-posedness problem for such a system, or it is hard to check whether the closed loop operator generates a $C_0$ semigroup. The difficulty is in that it is not easy to find the expression of $R^k(\lambda; \mathcal{A})$ and to estimate its norm for $k \geq 2$, and this is necessary to use the Hille–Yosida theorem. The goal of this paper is to give a new sufficient and necessary condition in terms of $\mathcal{A}$ and the first order resolvent of $\mathcal{A}$ and $\mathcal{A}^*$ which assures that $\mathcal{A}$ will generate a $C_0$ semigroup in a Hilbert space $\mathcal{H}$ and can be easily verified to non-dissipative systems in some cases.

The main result of this paper is the following Theorem 1.1:

**Theorem 1.1.** A linear operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is the infinitesimal generator of a $C_0$ semigroup $T(t)$ satisfying

$$\|T(t)\| \leq Me^{\sigma_0 t}, \quad \forall t \geq 0,$$

for some $M \geq 1$ and $\sigma_0 \in \mathbb{R}$, if and only if

1. $\mathcal{A}$ is a closed densely defined operator;
2. $\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, contains $\{\lambda \in \mathbb{C} | \Re \lambda > \sigma_0\}$, and for any $\lambda = \sigma + i\tau \in \mathbb{C}$ with $\sigma > \sigma_0$, the resolvent estimates

$$\sup_{\sigma > \sigma_0} (\sigma - \sigma_0) \int_{\mathbb{R}} \|R(\sigma + i\tau; \mathcal{A})x\|^2 d\tau < +\infty, \quad \forall x \in \mathcal{H},$$

and

$$\sup_{\sigma > \sigma_0} (\sigma - \sigma_0) \int_{\mathbb{R}} \|R(\sigma - i\tau; \mathcal{A}^*)y\|^2 d\tau < +\infty, \quad \forall y \in \mathcal{H},$$

are satisfied.

From Theorem 1.1 together with [5] the following corollaries are immediate.

**Corollary 1.2.** A linear operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is the infinitesimal generator of a $C_0$ semigroup $T(t)$, which satisfies (1.1) and is continuous for $t > 0$ in the uniform operator topology if and only if

1. Conditions (i) and (ii) in Theorem 1.1 hold;
2. $\sup_{x \in \mathcal{H}, \|x\|=1} \|R(\sigma + i\tau; \mathcal{A})x\|^2 d\tau \to 0, \quad a \to \infty, \sigma > \sigma_0$.

**Corollary 1.3.** A linear operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is the infinitesimal generator of a $C_0$ semigroup $T(t)$, which satisfies (1.1) and is compact if and only if

1. Conditions 1 and 2 in Corollary 1.2 hold;
2. $R(\lambda; \mathcal{A})$ is compact for $\lambda \in \rho(\mathcal{A})$. 
In terms of $\mathcal{A}$ or the first order resolvent $R(\lambda; \mathcal{A})$, Corollary 1.2 gives the characteristic conditions which assure the continuity for $t > 0$ of a $C_0$ semigroup $T(t)$ in the uniform operator topology, and the conditions of Corollary 1.3 assure that a $C_0$ semigroup $T(t)$ is compact.

In Section 2, the proof of Theorem 1.1 and a corollary obtained from the proof are given. In Section 3, an example is studied to show how to use the above conditions to investigate the well-posedness problem for non-dissipative system. Finally, in Section 4, whether Theorem 1.1 still holds in general Banach spaces is discussed further and another proof of the sufficiency of Theorem 1.1 is obtained in this section.

2. PROOF OF THEOREM 1.1

For the $C_0$ semigroup $T(t)$ in a Hilbert space $\mathcal{H}$ generated by $\mathcal{A}$, denote

$$\omega(\mathcal{A}) = \lim_{t \to \infty} \frac{\log \|T(t)\|}{t}, \quad s(\mathcal{A}) = \sup \{ \text{Re}\, \lambda | \lambda \in \sigma(\mathcal{A}) \},$$

where $\sigma(\mathcal{A})$ is the set of the spectrum of $\mathcal{A}$. First we prove two lemmas.

**Lemma 2.1.** Under the conditions (1.2) and (1.3), for every $x \in \mathcal{H}$ and $\sigma_1 > \sigma_0$, we have

$$\|R(\lambda; \mathcal{A})x\| \to 0, \quad \text{when} \quad \text{Re} \, \lambda \geq \sigma_1 \quad \text{and} \quad |\lambda| \to +\infty.$$  

**Proof.** Let $\lambda = \sigma + i\tau$. For $\sigma > \sigma_0$ and $x \in \mathcal{H}$, we define the mapping $S_\sigma: \mathcal{H} \to Y = L^2(\mathbb{R}; \mathcal{H})$ by

$$(S_\sigma x)(\tau) = \sqrt{\sigma - \sigma_0} R(\sigma + i\tau; \mathcal{A})x, \quad \forall \tau \in \mathbb{R}.$$  

It follows from (1.2) that $S_\sigma$ is a closed linear operator of $\mathcal{H}$ into $Y$ with $\mathcal{D}(S_\sigma) = \mathcal{H}$. So by the closed graph theorem $S_\sigma$ is bounded. Moreover, the condition (1.2) implies that

$$\sup \{ \|S_\sigma x\|_Y | \sigma > \sigma_0 \} < +\infty.$$  

Thus by using the uniform boundedness principle, there exists a positive constant $M_1$ independent of $\sigma$ such that

$$\left( \int_{\mathbb{R}} \|R(\sigma + i\tau; \mathcal{A})x\|^2 \, d\tau \right)^{1/2} \leq \frac{M_1 \|x\|}{\sqrt{\sigma - \sigma_0}}. \quad (2.1)$$

Similarly, there exists $M_2 > 0$ such that

$$\left( \int_{\mathbb{R}} \|R(\sigma - i\tau; \mathcal{A}^*)y\|^2 \, d\tau \right)^{1/2} \leq \frac{M_2 \|y\|}{\sqrt{\sigma - \sigma_0}}. \quad (2.2)$$
By the Schwarz inequality it follows from (2.1) and (2.2) that for $\sigma > \sigma_0$, 
\[
\left\| \int_{\tau_1}^{\tau_2} R^2(\sigma + i\tau; \mathcal{A}) x d\tau \right\|
\]
\[
= \sup_{y \in \mathcal{H}, \|y\| = 1} \left( \int_{\tau_1}^{\tau_2} R^2(\sigma + i\tau; \mathcal{A}) x d\tau, y \right) \mathcal{H}
\]
\[
= \sup_{y \in \mathcal{H}, \|y\| = 1} \int_{\tau_1}^{\tau_2} (R(\sigma + i\tau; \mathcal{A}) x, R(\sigma - i\tau; \mathcal{A}^*) y) \mathcal{H} d\tau
\]
\[
\leq \sup_{y \in \mathcal{H}, \|y\| = 1} \left( \int_{\tau_1}^{\tau_2} \|R(\sigma + i\tau; \mathcal{A}) x\|^2 d\tau \right)^{1/2}
\]
\[
\times \left( \int_{\tau_1}^{\tau_2} \|R(\sigma - i\tau; \mathcal{A}^*) y\|^2 d\tau \right)^{1/2}
\]
\[
\leq \frac{M_1 M_2 \|x\|}{\sigma - \sigma_0}, \quad \forall \tau_1, \tau_2 \in \mathbb{R}, x \in \mathcal{H},
\]
(2.3)
which means that $\int_{\tau} R^2(\sigma + i\tau; \mathcal{A}) x d\tau$ is convergent. Since
\[
R(\sigma + i\tau_1; \mathcal{A}) x = R(\sigma + i\tau_0; A) x - i \int_{\tau_0}^{\tau_1} R^2(\sigma + i\tau; \mathcal{A}) x d\tau,
\]
(2.4)
the limit
\[
\lim_{|\tau| \to +\infty} R(\sigma + i\tau; \mathcal{A}) x
\]
exists in the topology of $\mathcal{H}$. Using this fact together with (2.1), we obtain
\[
\lim_{|\tau| \to +\infty} R(\sigma + i\tau; \mathcal{A}) x = 0, \quad \forall x \in \mathcal{H}, \sigma > \sigma_0.
\]
(2.5)
It follows from (2.3) and (2.4) that
\[
\|R(\sigma + i\tau_1; \mathcal{A}) x\| \leq \|R(\sigma + i\tau_0; \mathcal{A}) x\| + \frac{M_1 M_2 \|x\|}{\sigma - \sigma_0}.
\]
(2.6)
Let $\tau_0 \to \infty$ in (2.6); we then get
\[
\|R(\sigma + i\tau; \mathcal{A})\| \leq \frac{M_1 M_2}{\sigma - \sigma_0}, \quad \sigma > \sigma_0.
\]
(2.7)
So if \( \sigma \geq \max\{\sigma_1, |\tau|\} \), then it follows from (2.7) that
\[
\|R(\lambda, \mathcal{A})x\| \leq \frac{2M_1M_2}{\sqrt{|\lambda|^2 - 4\sigma_0}}\|x\| \to 0, \quad \text{as } |\lambda| \to +\infty. \quad (2.8)
\]

On the other hand, by using the resolvent identity and (2.7) when \( \sigma_1 \leq \text{Re} \lambda \leq |\tau| \), we obtain
\[
\|R(\lambda, A)x\| \leq \|R(\sigma_1 + i\tau; \mathcal{A})x\|\]
\[
+ |\sigma - \sigma_1| \|R(\lambda; \mathcal{A})\| \|R(\sigma_1 + i\tau; \mathcal{A})x\|
\]
\[
< (1 + M_1M_2)\|R(\sigma_1 + i\tau; \mathcal{A})x\|. \quad (2.9)
\]

Note that if \( |\tau| \geq |\lambda|/\sqrt{2} \), we have \( |\tau| \to \infty \) as \( |\lambda| \to \infty \). Thus the desired assertion is derived from (2.8) and (2.9).

The following lemma is easily verified, therefore the proof is omitted.

**Lemma 2.2.** Let \( \lambda = \sigma + i\tau \) and \( \mu = \sigma_1 + i\tau_1 \). For every \( x \in \mathcal{A} \), we have
\[
\frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{e^{\lambda t}}{\lambda^k} x d\lambda = \begin{cases} x, & \sigma > 0, \ k = 1, \ t > 0, \\ 0, & \sigma < 0, \ k \geq 1, \ t > 0, \end{cases} \quad (2.10)
\]
\[
\frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{e^{\mu t}}{\mu^k(\mu - \lambda)} x d\mu = 0, \quad \sigma < \sigma_1 < 0, \ t > 0, \quad (2.11)
\]
\[
\frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{e^{\mu t}}{\mu^k(\mu - \lambda)} x d\mu = \frac{e^{\lambda t}}{\lambda^k} x, \quad \sigma < \sigma_1 < 0, \ t > 0, \quad (2.12)
\]
\[
\frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{1}{\lambda} R(\lambda; \mathcal{A}) x d\lambda = 0, \quad \sigma > 0, \quad (2.13)
\]
\[
\frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{1}{\lambda(\lambda - \sigma_1)} R(\lambda; \mathcal{A}) x d\lambda = \frac{R(\sigma_1; \mathcal{A}) x}{\sigma_1}, \quad \sigma_1 > \sigma > 0. \quad (2.14)
\]

**Proof of Theorem 1.1.** (Necessity) It is obvious that (i) holds by the Hille–Yosida theorem (see [2]). As for (ii), we need only prove (1.2), because according to the theory of adjoint semigroups (see [2]), \( T^*(t) \) is a
$C_0$ semigroup in $\mathcal{H}$ with the same properties. By a lemma in [1], we have

$$\|R(\sigma + i\tau; \mathcal{A})x\|^2 = \int_{-\infty}^{\infty} e^{-i\xi\tau} f(\xi) d\xi$$

where

$$f(\xi) = \int_{\max(0, -\xi)}^{\infty} e^{-\sigma(\xi + 2s)} (T(\xi + s)x, T(s)x)_{\mathcal{H}} ds.$$ 

Therefore, for $\xi \geq 0$, using (1.1), we have

$$|f(\xi)| \leq \int_{0}^{\infty} M^2 \|x\|^2 e^{-\sigma(\xi + 2s)} e^{\sigma(\xi + 2s)} ds$$

$$= \frac{M^2 \|x\|^2}{2(\sigma - \sigma_0)} e^{-(\sigma - \sigma_0)\xi} \leq \frac{M^2 \|x\|^2}{2(\sigma - \sigma_0)}, \quad (2.15)$$

and for $\xi < 0$ we have

$$|f(\xi)| \leq \int_{-\xi}^{\infty} M^2 \|x\|^2 e^{-\sigma(\xi + 2s)} e^{\sigma(\xi + 2s)} ds$$

$$= \frac{M^2 \|x\|^2}{2(\sigma - \sigma_0)} e^{(\sigma - \sigma_0)\xi} \leq \frac{M^2 \|x\|^2}{2(\sigma - \sigma_0)}, \quad (2.16)$$

Hence $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and the Fourier transform of $f(\xi)$ is the nonnegative function $(2\pi)^{-1/2}\|R(\sigma + i\tau; \mathcal{A})x\|^2$. Using Lemma (21.50) in [4], it follows from (2.15) and (2.16) that

$$\frac{1}{2\pi} \int_{\mathbb{R}} \|R(\sigma + i\tau; \mathcal{A})x\|^2 d\tau = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\tau) d\tau \leq \|f\|_{L^\infty} \leq \frac{M^2 \|x\|^2}{2(\sigma - \sigma_0)}, \quad (2.17)$$

which means that (1.2) holds.
(Sufficiency) For a fixed $\sigma > \sigma_0$, we define the linear operator $T_\sigma(t)$ in $\mathcal{H}$:

$$T_\sigma(t)x = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R(\lambda; \mathcal{A}) x d\lambda$$

(2.18)

$$= \frac{e^{\sigma t}}{2\pi} \int_{\mathbb{R}} e^{i\tau t} R(\sigma + i\tau; \mathcal{A}) x d\tau, \quad t > 0, x \in \mathcal{H}.$$ (2.19)

Integration by parts yields that

$$\int_{\tau_1}^{\tau_2} e^{i\tau t} R(\sigma + i\tau; \mathcal{A}) x d\tau$$

$$= \frac{1}{t} \int_{\tau_1}^{\tau_2} e^{i\tau t} R^2(\sigma + i\tau; \mathcal{A}) x d\tau$$

$$+ \frac{1}{it} e^{i\tau_2} R(\sigma + i\tau_2; \mathcal{A}) x$$

$$- \frac{1}{it} e^{i\tau_1} R(\sigma + i\tau_1; \mathcal{A}) x,$$

$$t > 0, \tau_1, \tau_2 \in \mathbb{R}, x \in \mathcal{H}.$$ (2.20)

By using Schwarz's inequality, it follows from (2.1) and (2.2) that

$$\left( \int_{\tau_1}^{\tau_2} e^{i\tau t} R^2(\sigma + i\tau; \mathcal{A}) x d\tau, y \right)_{\mathcal{H}}$$

$$= \int_{\tau_1}^{\tau_2} \left( e^{i\tau t} R(\sigma + i\tau; \mathcal{A}) x, R(\sigma - i\tau; \mathcal{A}^*) y \right)_{\mathcal{H}} d\tau$$

$$\leq \left( \int_{\tau_1}^{\tau_2} \| R(\sigma + i\tau; \mathcal{A}) x \|^2 d\tau \right)^{1/2} \left( \int_{\tau_1}^{\tau_2} \| R(\sigma - i\tau; \mathcal{A}^*) y \|^2 d\tau \right)^{1/2}$$

$$\leq \frac{M_1 M_2}{(\sigma - \sigma_0)} \| x \| \| y \|, \quad \forall x, y \in \mathcal{H}.$$ (2.21)

Therefore, it follows from (2.20), (2.21) and Lemma 2.1 that the improper Riemann integral $\int_{\mathbb{R}} e^{i\tau t} R(\sigma + i\tau; \mathcal{A}) x d\tau$ converges in the topology of $\mathcal{H}$.
and satisfies
\[ \left\| \int e^{i\tau} R(\sigma + i\tau; \mathcal{A}) x \, d\tau \right\| \leq \frac{M_1 M_2}{t(\sigma - \sigma_0)} \|x\|. \]  
(2.22)

Hence \( T_\sigma(t) (t > 0) \) defined by (2.19) is a linear bounded operator in \( \mathcal{H} \).

Next, we proceed to verify that \( T_\sigma(t) \) is a \( C_0 \) semigroup.

1. First we show that \( T_\sigma(t) \) is independent of \( \sigma \), \( \theta \), \( \theta' \), and \( \eta \).

   For each \( \beta > 0 \), let \( \Gamma_\beta \) be the rectangular path with vertices at \( \sigma \pm i\beta \) and \( \sigma_1 \pm i\beta \). By Cauchy’s theorem,

\[ \int_{\Gamma_\beta} e^{i\lambda \mathcal{A}} R(\lambda; \mathcal{A}) x \, d\lambda = 0. \]  
(2.23)

Without loss of generality, we may assume that \( \sigma < \sigma_1 \). Using Lemma 2.1, we know that

\[ \left\| \int_{\sigma}^{\sigma_1} e^{i(s + i\beta)\mathcal{A}} R(s + i\beta; \mathcal{A}) x \, ds \right\| \leq e^{\sigma_1} \int_{\sigma}^{\sigma_1} \|R(s + i\beta; \mathcal{A}) x\| \, ds \to 0, \]

\[ \beta \to \infty. \]

Thus, by letting \( \beta \to \infty \) in (2.23), it follows from (2.19) that \( T_\sigma(t) = T_{\sigma_1}(t) \).

For every \( t > 0 \), it follows from (2.19) and (2.22) that

\[ \|T(t)\| \leq \inf_{\sigma > \sigma_0} \frac{M_1 M_2 e^{\sigma t}}{2\pi t(\sigma - \sigma_0)} = \frac{e^{\sigma_0 t} M_1 M_2}{2\pi} \min_{\sigma > \sigma_0} \frac{e^{(\sigma - \sigma_0) t}}{(\sigma - \sigma_0) t} = M e^{\sigma_0 t}, \]

(2.24)

where \( M = M_1 M_2 e^{t} / 2\pi \).

2. Then we prove that \( T(t + s) = T(t) T(s) \) for \( t, s > 0 \). Without loss of generality, we may assume that \( \sigma_0 < 0 \), otherwise we only need to replace \( \mathcal{A} \) by \( \mathcal{A} = \mathcal{A} - (\sigma_0 + 1) I \) and \( T(t) \) by \( \tilde{T}(t) = e^{-(\sigma_0 + 1)t} T(t) \). So we can take \( \sigma_0 < \sigma < \sigma_1 < 0 \). For \( x \in \mathcal{B}(\mathcal{A}^k) \), \( k = 1, 2, \ldots \), we have the equality

\[ R(\lambda; \mathcal{A}) x = \sum_{j=1}^{k} \frac{1}{\lambda^j} \mathcal{A}^{j-1} x + \frac{1}{\lambda^k} R(\lambda; \mathcal{A}) \mathcal{A}^k x. \]  
(2.25)
Using (2.25) with $k = 2$ and Lemma 2.2, for $x \in \mathcal{D}(\mathcal{A}^2)$ we have

$$T(t)x = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t} R(\lambda; \mathcal{A})x \, d\lambda$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{2} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{\lambda t}}{\lambda^j} \, d\lambda \lambda^{j-1}x \, d\lambda + \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{\lambda t}}{\lambda^2} R(\lambda; \mathcal{A})x \, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t} R(\lambda; \mathcal{A})x \, d\lambda. \quad (2.26)$$

Therefore, in terms of the closeness of $\mathcal{A}$, for $x \in \mathcal{D}(\mathcal{A}^2)$ it follows from (2.26) and (2.25) that

$$T(s)T(t)x$$

$$= -\frac{1}{4\pi^2} \int_{\sigma - i\infty}^{\sigma + i\infty} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{e^{\lambda t_\mu}}{\lambda^2} R(\lambda; \mathcal{A})R(\mu; \mathcal{A})x \, d\mu \, d\lambda$$

$$= -\frac{1}{4\pi^2} \int_{\sigma - i\infty}^{\sigma + i\infty} d\lambda \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{e^{\lambda t_\mu}}{\lambda^2} \left( \frac{R(\lambda; \mathcal{A})x}{\mu - \lambda} \right) + R(\mu; \mathcal{A})x \right) \bigg|_{\lambda = \mu} \, d\mu$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{\lambda t}}{\lambda^2} R(\lambda; \mathcal{A})x \, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t_\mu} R(\mu; \mathcal{A})x \, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda (t+1)} R(\lambda; \mathcal{A})x \, d\lambda$$

$$= T(t+s)x,$$

where we have used the resolvent identity and Lemma 2.2. Finally, noting that $T(t)$ is bounded and the $\mathcal{D}(\mathcal{A}^4)$ is dense in $\mathcal{H}$ (because $\mathcal{D}(\mathcal{A}) = R(\sigma; \mathcal{A})\mathcal{H}$ and $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$, so $R(\sigma; \mathcal{A})$ maps a dense set of $\mathcal{H}$ to a dense set of $\mathcal{H}$, and hence $\mathcal{D}(\mathcal{A}^4) = R^4(\sigma; \mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$). Then the desired result follows.
(3) To obtain the strong continuity of $T(t)$, it is sufficient to prove that $\lim_{t \to 0^+} T(t) x = x$ for any $x \in D(\mathcal{A})$ because of the uniform boundedness of $T(t)$ for $t \in [0, 1]$. Now taking $\sigma > \max(0, \sigma_0)$, we have

$$T(t)x - x = \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} e^{\lambda t} R(\lambda; \mathcal{A}) x \, d\lambda - x$$

$$= \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} e^{\lambda t} \frac{x}{\lambda} \, d\lambda - x + \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} e^{\lambda t} R(\lambda; \mathcal{A}) x \, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} e^{\lambda t} \frac{x}{\lambda} R(\lambda; \mathcal{A}) x \, d\lambda,$$

where we have used Lemma 2.2 and (2.25) with $k = 1$. It follows from (1.2) that the integral

$$\int_{\sigma - \infty}^{\sigma + \infty} \frac{\|R(\lambda; \mathcal{A}) x\|}{|\lambda|} \, d\lambda$$

is convergent. Therefore, by Lemma 2.2, we have

$$\lim_{t \to 0^+} \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} e^{\lambda t} \frac{R(\lambda; \mathcal{A}) x \, d\lambda}{\lambda} = \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{1}{\lambda} R(\lambda; \mathcal{A}) x \, d\lambda = 0.$$

(2.28)

So the desired result follows from (2.27) and (2.28).

(4) Finally, we demonstrate that $\mathcal{A}$ is the infinitesimal generator of $T(t)$. Let $\mathcal{A}$ be the infinitesimal generator of $T(t)$. For $x \in D(\mathcal{A})$, taking $\sigma_1 > \sigma > \max(\sigma_0, \omega(\mathcal{A}), 0)$ and using (2.25) and Lemma 2.2, we have

$$R(\sigma_1; \mathcal{A}) x = \int_0^\infty e^{-\sigma_1 t} T(t) x \, dt$$

$$= \int_0^\infty e^{-\sigma_1 t} x \, dt + \int_0^\infty e^{-\sigma_1 t} \left( \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} e^{\lambda t} \frac{R(\mu; \mathcal{A}) x \, d\mu}{\mu} \right) \, dt$$

$$= \frac{x}{\sigma_1} + \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \left( \int_0^{\infty} e^{\mu t - \sigma_1 t} \, dt \right) \frac{R(\mu; \mathcal{A}) x}{\mu} \, d\mu$$

$$= \frac{x}{\sigma_1} + \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{1}{\mu(\mu - \sigma_1)} R(\mu; \mathcal{A}) x \, d\mu$$

$$= \frac{x}{\sigma_1} + \frac{R(\sigma_1; \mathcal{A}) x}{\sigma_1} = R(\sigma_1; \mathcal{A}) x, \quad \forall x \in D(\mathcal{A}).$$

(2.29)
Since \( \mathcal{D}(\mathcal{A}) \) is dense in \( \mathcal{H} \), (2.29) holds for all \( x \in \mathcal{H} \), which implies that \( \mathcal{A} = \mathcal{A}' \). Then the proof of Theorem 1.1 is complete.

By the proof of Theorem 1.1, the following result is immediate.

**Corollary 2.3.** Let \( \mathcal{A} \) be the infinitesimal generator of a \( C_0 \) semigroup \( T(t) \) in a Hilbert space \( \mathcal{H} \). Then we have

\[
\omega(\mathcal{A}) = \inf\{\sigma | \sigma > R(\sigma + i \cdot) x \in H_2(\sigma; \mathcal{H}), \forall x \in \mathcal{H}\},
\]

(2.30)

where \( H_2(\sigma; \mathcal{H}) \) is the Hardy space, for the precise meaning of which, see Section 6.4 in [4].

### 3. Application to the Vibrating String with Non-Dissipative Boundary Condition

In this section, we give an example to show how to apply the previous theorem to the well-posedness problem of a non-dissipative system. Consider a vibrating string system with a non-dissipative boundary condition,

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2}(x, t) - \frac{\partial^2 w}{\partial x^2}(x, t) &= 0, \quad \text{in } (0, 1) \times \mathbb{R}^+ \\
\frac{\partial w}{\partial t}(0, t) &= 0, \quad t > 0, \\
\frac{\partial w}{\partial x}(1, t) + \alpha w(1, t) &= 0, \quad t > 0,
\end{align*}
\]

(3.1)

where \( \alpha < 0 \) and \( \alpha \neq -1 \). It is obvious that (3.1) is a dissipative system when \( \alpha > 0 \). Now we incorporate (3.1) into certain function space. To this end, we define the product Hilbert space \( \mathcal{H} = V_0^1 \times L_2^0(0, 1) \), where

\[
V_0^1 = \{ \phi \in H^1(0, 1) | \phi(0) = 0 \}
\]

and \( H^k(0, 1) \) is the usual Sobolev space of order \( k \). The inner product in \( \mathcal{H} \) is defined as

\[
(Y_1, Y_2)_{\mathcal{H}} = \int_0^1 w_1^* w_2 + v_1 v_2 \, dx,
\]

where \( Y_k = [w_k, v_k]^T \in \mathcal{H}, k = 1, 2 \). We define a linear operator \( \mathcal{A} \) in \( \mathcal{H} \):

\[
\mathcal{A} \left[ \begin{array}{c} w \\ v \end{array} \right] = \left[ \begin{array}{c} v \\ w'' \end{array} \right], \quad \left[ \begin{array}{c} w \\ v \end{array} \right] \in \mathcal{D}(\mathcal{A}),
\]

\[
\mathcal{D}(\mathcal{A}) = \{ [w, v]^T \in \mathcal{H} | w \in H^2(0, 1), v \in V_0^1, w'(1) + \alpha v(1) = 0 \}.
\]
Then (3.1) can be written as the following linear evolution equation in $\mathcal{A}$:

$$\frac{dY}{dt} = \mathcal{A}Y. \quad (3.2)$$

A simple calculation shows that the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ is given by

$$\mathcal{A}^* = \begin{bmatrix} w \\ u \end{bmatrix} = -\begin{bmatrix} v \\ w' \end{bmatrix}, \quad \begin{bmatrix} w \\ u \end{bmatrix} \in \mathcal{D}(\mathcal{A}^*),$$

$$\mathcal{D}(\mathcal{A}^*) = \{ [w, v] \in \mathcal{A} | w \in H^2(0, 1), u \in V_0^1, w'(1) - \alpha v(1) = 0 \}.$$  

Our problem is whether $\mathcal{A}$ generates a $C_0$ semigroup and satisfies the spectrum-determined assumption, i.e., $\omega(\mathcal{A}) = \sigma(\mathcal{A})$. It is easy to see that $(\mathcal{A}Y, Y)_{\mathcal{A}} = -\alpha|v'(1)|^2 \geq 0$ for $Y \in \mathcal{D}(\mathcal{A})$. So $\mathcal{A}$ is not dissipative, and it is not easy to verify Hille–Yosida’s conditions. By using Theorem 1.1, however, we can give an affirmative answer to the above problem. To do this, first for any $F = [f, g] \in \mathcal{A}$ and $\lambda = \sigma + i\tau$ we solve the resolvent equation $\lambda Y - \mathcal{A}Y = F$ with $Y = [w, v] \in \mathcal{D}(\mathcal{A})$, i.e.,

$$\lambda w - v = f,$$

$$\lambda v - w' = g,$$

$$w(0) = v(0) = 0,$$

$$w'(1) + \alpha v(1) = 0. \quad (3.3)$$

Denote

$$Q = \begin{cases} \{ \lambda \in \mathbb{C} | \lambda \neq \frac{1}{2} \ln \frac{\alpha - 1}{\alpha + 1} + ik\pi, k \in \mathbb{Z} \}, & \text{if } \alpha < -1, \\
\{ \lambda \in \mathbb{C} | \lambda \neq \frac{1}{2} \ln \frac{1 - \alpha}{1 + \alpha} + i(k + \frac{1}{2})\pi, k \in \mathbb{Z} \}, & \text{if } \alpha > -1, \end{cases}$$

where $\mathbb{Z}$ is the integer set. It is easy to see that for $\lambda \in Q$, (3.3) has the solution

$$w'(x) = w'_g(x) + w'_f(x), \quad v(x) = v_g(x) + v_f(x), \quad (3.4)$$
where, for simplicity, we have omitted the dependence of \( w \) and \( v \) on \( \lambda = \sigma + i\tau \), and

\[
\begin{align*}
    w^f_\lambda(x) &= \int_0^x \frac{\left[ \cosh \lambda(1 - s) + \alpha \sinh \lambda(1 - s) \right] g(s) ds}{\cosh \lambda + \alpha \sinh \lambda} \cosh \lambda x \\
    &\quad - \int_0^x \cosh \lambda(x - s) g(s) ds, \\
    w^f_\nu(x) &= \int_0^x \frac{\left[ \sinh \lambda(1 - s) + \alpha \cosh \lambda(1 - s) \right] f'(s) ds}{\cosh \lambda + \alpha \sinh \lambda} \cosh \lambda x \\
    &\quad - \int_0^x \sinh \lambda(x - s) f'(s) ds, \\
    v^f_\lambda(x) &= \int_0^x \frac{\left[ \cosh \lambda(1 - s) + \alpha \sinh \lambda(1 - s) \right] g(s) ds}{\cosh \lambda + \alpha \sinh \lambda} \sinh \lambda x \\
    &\quad - \int_0^x \sinh \lambda(x - s) g(s) ds, \\
    v^f_\nu(x) &= \int_0^x \frac{\left[ \sinh \lambda(1 - s) + \alpha \cosh \lambda(1 - s) \right] f'(s) ds}{\cosh \lambda + \alpha \sinh \lambda} \sinh \lambda x \\
    &\quad - \int_0^x \cosh \lambda(x - s) f'(s) ds.
\end{align*}
\] (3.5) (3.6) (3.7) (3.8)

It is well known that \( \{ \sqrt{2} \sin n\pi s, n = 1, 2, \ldots \} \) is an orthonormal basis of \( L^2[0, 1] \). Since \( f', g \in L^2[0, 1] \), we can expand \( f' \) and \( g \) in \( L^2[0, 1] \) as

\[
    f'(s) = \sum_{n=1}^\infty f_n \sin n\pi s, \quad g(s) = \sum_{n=1}^\infty g_n \sin n\pi s, \quad (3.9)
\]

where \( f_n \) and \( g_n \) are Fourier coefficients of \( f' \) and \( g \), respectively, satisfying

\[
    \sum_{n=1}^\infty |f_n|^2 = 2\|f'\|_2^2, \quad \sum_{n=1}^\infty |g_n|^2 = 2\|g\|_2^2. \quad (3.10)
\]

Substituting (3.9) into (3.5) and (3.7), and noting the equalities

\[
    \int_0^x \sinh \lambda(x - s) \sin n\pi s ds = \frac{n\pi \sinh \lambda x - \lambda \sin n\pi x}{n^2\pi^2 + \lambda^2},
\]

\[
    \int_0^x \cosh \lambda(x - s) \sin n\pi s ds = \frac{n\pi (\cosh \lambda x - \cos n\pi x)}{n^2\pi^2 + \lambda^2},
\]
we obtain

\[ w'_g(x) = w'_{g1}(x) + w'_{g2}(x), \quad v_g(x) = v_{g1}(x) + v_{g2}(x), \quad (3.11) \]

where

\[ w'_{g1}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \pi g_n}{\cosh \lambda + \alpha \sinh \lambda (n^2 \pi^2 + \lambda^2)} \cosh \lambda x, \quad (3.12) \]
\[ w'_{g2}(x) = \sum_{n=1}^{\infty} \frac{n \pi g_n}{n^2 \pi^2 + \lambda^2} \cos n \pi x, \quad (3.13) \]
\[ v_{g1}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \pi g_n}{\cosh \lambda + \alpha \sinh \lambda (n^2 \pi^2 + \lambda^2)} \sinh \lambda x, \quad (3.14) \]
\[ v_{g2}(x) = \sum_{n=1}^{\infty} \frac{\lambda g_n}{n^2 \pi^2 + \lambda^2} \sin n \pi x. \quad (3.15) \]

It follows from (3.4) that

\[ \int_{\mathbb{R}} ||Y||^2 \, d\tau = \int_{\mathbb{R}} (||w'||^2 + ||v||^2) \, d\tau \leq 2(I_g + I_f), \quad (3.16) \]

where

\[ I_g = \int_{\mathbb{R}} (||w'_{g1}||^2 + ||v_{g1}||^2) \, d\tau, \quad I_f = \int_{\mathbb{R}} (||w'_{g2}||^2 + ||v_{g2}||^2) \, d\tau. \]

Then for (1.2) to hold, it is enough to prove that

\[ I_g \leq \frac{c_1 ||g||^2}{\sigma}, \quad (3.17) \]
\[ I_f \leq \frac{c_2 ||f'||^2}{\sigma}. \quad (3.18) \]

Here and below \(c_i \ (i = 1, 2, \ldots)\) are all positive constants. First we prove (3.17). It follows from (3.11) that

\[ I_g \leq 2 \int_{\mathbb{R}} (||w'_{g1}||^2 + ||v_{g1}||^2) \, d\tau + 2 \int_{\mathbb{R}} ||w'_{g2}||^2 \, d\tau + 2 \int_{\mathbb{R}} ||v_{g2}||^2 \, d\tau. \quad (3.19) \]
We have
\[
\|w_{g_2}'\|_2^2 = \sum_{n,m} \frac{nm\pi^2\delta_{mn}}{2(n^2\pi^2 + \lambda^2)(m^2\pi^2 + \lambda^2)} g_n \overline{g}_m
\]
\[
= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n^2\pi^2|g_n|^2}{|n^2\pi^2 + \lambda^2|^2},
\] (3.20)
\[
\|v_{g_2}'\|_2^2 = \sum_{n,m} \frac{|\lambda|^2\delta_{mn}}{2(n^2\pi^2 + \lambda^2)(m^2\pi^2 + \lambda^2)} g_n \overline{g}_m
\]
\[
= \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\lambda|^2|g_n|^2}{|n^2\pi^2 + \lambda^2|^2},
\] (3.21)
and
\[
\|w_{g_1}'\|_2^2 + \|v_{g_1}'\|_2^2 = \frac{\sinh 2\sigma}{2\sigma|\cosh \lambda + \alpha \sinh \lambda|^2}
\]
\[
\times \sum_{n,m} \frac{(-1)^{n+1}(-1)^{m+1}nm\pi^2}{(n^2\pi^2 + \lambda^2)(m^2\pi^2 + \lambda^2)} g_n \overline{g}_m,
\] (3.22)
where \(\delta_{mn}\) is the Kronecker delta and \(\Sigma_{n,m}\) is understood as \(\lim_{N \to \infty} \sum_{n=1}^{N}\sum_{m=1}^{N}\). By the residue theorem, we have
\[
\int_{\mathbb{R}} \frac{1}{|n^2\pi^2 + \lambda^2|^2} d\tau = \frac{\pi}{2\sigma(n^2\pi^2 + \sigma^2)},
\] (3.23)
\[
\int_{\mathbb{R}} \frac{|\lambda|^2}{|n^2\pi^2 + \lambda^2|^2} d\tau = \frac{\pi}{2\sigma} + \frac{\sigma\pi}{2(n^2\pi^2 + \sigma^2)},
\] (3.24)
\[
\int_{\mathbb{R}} \frac{1}{(n^2\pi^2 + \lambda^2)(m^2\pi^2 + \lambda^2)} d\tau
\]
\[
= \frac{8\sigma\pi}{((n + m)^2\pi^2 + 4\sigma)((n - m)^2\pi^2 + 4\sigma^2)}.
\] (3.25)
It follows from (3.20), (3.23), and (3.10) that
\[
\int_{\mathbb{R}} \|w_{g_2}'\|_2^2 d\tau = \frac{1}{2} \sum_{n=1}^{\infty} n^2\pi^2|g_n|^2 \int_{\mathbb{R}} \frac{1}{|n^2\pi^2 + \lambda^2|^2} d\tau
\]
\[
= \sum_{n=1}^{\infty} \frac{n^2\pi^2|g_n|^2}{4\sigma(n^2\pi^2 + \sigma^2)} \leq \frac{\pi}{2\sigma} \|g\|_2^2.
\] (3.26)
Similarly, by using (3.21), (3.24), and (3.10), we have

\[
\int_{\mathbb{R}} \left\| v_{g,1} \right\|_{2}^{2} d\tau = \frac{1}{2} \sum_{n=1}^{\infty} |g_{n}|^{2} \int_{\mathbb{R}} \left| \frac{\lambda}{n^{2}\pi^{2} + \lambda^{2}} \right|^{2} d\tau
\]

\[
= \frac{\pi}{4\sigma} \sum_{n=1}^{\infty} \left( |g_{n}|^{2} + \frac{\sigma^{2}|g_{n}|^{2}}{n^{2}\pi^{2} + \sigma^{2}} \right)
\]

\[
\leq \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} |g_{n}|^{2} = \frac{\pi}{\sigma} \left\| g \right\|_{\frac{3}{2}}^{2}.
\]  

(3.27)

A simple calculation gives

\[
|\cosh \lambda + \alpha \sinh \lambda|^{2} = \frac{1}{4} \left[ (1 + \alpha)^{2}e^{2\sigma} + (1 - \alpha)^{2}e^{-2\sigma}
\right.
\]

\[
+ 2(1 - \alpha^{2})\cos 2\tau \right]
\]

\[
\geq \frac{1}{4} \left[ (1 + \alpha)^{2}e^{2\sigma} + (1 - \alpha)^{2}e^{-2\sigma} - 2|1 - \alpha^{2}| \right].
\]  

(3.28)

It can be easily seen from (3.28) that for every \( \varepsilon > 0 \) there exists a positive constant \( c_{\varepsilon} \) such that

\[
\frac{\sinh 2\sigma}{2|\cosh \lambda + \alpha \sinh \lambda|^{2}} \leq \frac{e^{2\sigma} - e^{-2\sigma}}{(1 + \alpha)^{2}e^{2\sigma} + (1 - \alpha)^{2}e^{-2\sigma} - 2|1 - \alpha^{2}|} \leq c_{\varepsilon},
\]  

(3.29)

provided that \( \sigma > \sigma_{0} + \varepsilon \), where \( \sigma_{0} = \frac{1}{2}\ln(1 - \alpha)/(1 + \alpha) > 0 \). Substituting (3.29) into (3.22), we have

\[
\int_{\mathbb{R}} \left\| v_{g,1}' \right\|_{2}^{2} + \left\| v_{g,1} \right\|_{2}^{2} d\tau \leq c_{\varepsilon} \int_{\mathbb{R}} \sum_{n,m} \frac{nm\pi^{2}(-1)^{n+m}}{(n^{2}\pi^{2} + \lambda^{2})(m^{2}\pi^{2} + \lambda^{2})} g_{n}g_{m} d\tau,
\]

\( \sigma > \sigma_{0} + \varepsilon \).

(3.30)
On the other hand, using (3.25) we get

\[
\int_{\mathbb{R}} \sum_{n,m} \frac{nm\pi^2 (-1)^{n+m}}{(n^2 \pi^2 + \lambda^2)(m^2 \pi^2 + \lambda^2)} g_n \bar{g}_m \, d\tau
\]

\[
= \lim_{N \to \infty} \sum_{n,m=1}^N \frac{nm\pi^2 (-1)^{n+m}}{(n^2 \pi^2 + \lambda^2)(m^2 \pi^2 + \lambda^2)} g_n \bar{g}_m \int_{\mathbb{R}} \frac{1}{(n^2 \pi^2 + \lambda^2)(m^2 \pi^2 + \lambda^2)} \, d\tau
\]

\[
= 8\sigma \pi \sum_{n,m} \frac{nm\pi^2 (-1)^{n+m} g_n \bar{g}_m}{(m + n)^2 \pi^2 + 4\sigma^2}\left(\frac{(n - m)^2 \pi^2 + 4\sigma^2}{(n + m)^2 \pi^2 + 4\sigma^2}\right)
\]

\[
= 2\sigma \pi \left(\sum_{n,m} \frac{(-1)^{n+m} g_n \bar{g}_m}{(n - m)^2 \pi^2 + 4\sigma^2} - \frac{(-1)^{n+m} g_n \bar{g}_m}{(n + m)^2 \pi^2 + 4\sigma^2}\right) . \quad (3.31)
\]

By the Cauchy inequality, we have

\[
\sum_{n,m} \frac{|g_n| |\bar{g}_m|}{(m + n)^2 \pi^2 + 4\sigma^2} \leq \left(\sum_{m,n} \frac{|g_n|^2}{(m + n)^2 \pi^2 + 4\sigma^2}\right)^{1/2}
\times \left(\sum_{m,n} \frac{|\bar{g}_m|^2}{(m + n)^2 \pi^2 + 4\sigma^2}\right)^{1/2}
\]

\[
= \sum_{m,n} \frac{|g_n|^2}{(m + n)^2 \pi^2 + 4\sigma^2}
\]

\[
\leq \sum_{n=1}^{\infty} |g_n|^2 \sum_{m=1}^{\infty} \frac{1}{m^2 \pi^2 + 4\sigma^2} \leq \frac{c_4}{\sigma} \|g\|_2^2, \quad (3.32)
\]

Similarly, we get

\[
\sum_{m,n} \frac{|g_n| |\bar{g}_m|}{(m - n)^2 \pi^2 + 4\sigma^2} \leq \sum_{n=1}^{\infty} |g_n|^2 \sum_{m=-\infty}^{+\infty} \frac{1}{m^2 \pi^2 + 4\sigma^2} = \frac{c_5}{\sigma} \|g\|_2^2 . \quad (3.33)
\]

Substituting (3.32) and (3.33) into (3.31), and then into (3.30), we know

\[
\int_{\mathbb{R}} \left(\|w_{g_t}\|_2^2 + \|v_{g_t}\|_2^2\right) \, d\tau \leq \frac{2\pi c_4 (c_4 + c_5)}{\sigma} \|g\|_2^2, \quad \sigma > \sigma_0 + \epsilon . \quad (3.34)
\]
Therefore, by combining (3.19), (3.26), (3.27), and (3.34), we see that (3.17) holds. We now turn to showing (3.18). We have
\[
\begin{align*}
  w'_f(x) &= w'_{f1}(x) + w'_{f2}(x) + w'_{f3}(x), \\
  v_f(x) &= v_{f1}(x) + v_{f2}(x) + v_{f3}(x),
\end{align*}
\]
where
\[
\begin{align*}
  w'_{f1}(x) &= \sum_{n=1}^{\infty} \frac{n \pi f_n (\sinh \lambda(1-x) + \alpha \cosh \lambda(1-x))}{(\cos \lambda + \alpha \sin \lambda)(n^2 \pi^2 + \lambda^2)}, \\
  w'_{f2}(x) &= \sum_{n=1}^{\infty} \frac{\alpha n \pi (-1)^{n+1} f_n}{(\cos \lambda + \alpha \sin \lambda)(n^2 \pi^2 + \lambda^2)} \cosh \lambda x, \\
  w'_{f3}(x) &= \sum_{n=1}^{\infty} \frac{\lambda f_n \sin n \pi x}{n^2 \pi^2 + \lambda^2}, \\
  v_{f1}(x) &= -\sum_{n=1}^{\infty} \frac{n \pi f_n (\cosh \lambda(1-x) + \alpha \sinh \lambda(1-x))}{(\cos \lambda + \alpha \sin \lambda)(n^2 \pi^2 + \lambda^2)}, \\
  v_{f2}(x) &= \sum_{n=1}^{\infty} \frac{\alpha n \pi (-1)^{n+1} f_n}{(\cos \lambda + \alpha \sin \lambda)(n^2 \pi^2 + \lambda^2)} \sinh \lambda x, \\
  v_{f3}(x) &= \sum_{n=1}^{\infty} \frac{n \pi f_n \cos n \pi x}{n^2 \pi^2 + \lambda^2}.
\end{align*}
\]
It follows from (3.36)–(3.41) that if we replace \( g_n \) in (3.12)–(3.15) by \( f_n \), we get \((1/\alpha)w'_{f2}, v_{f3}, (1/\alpha)v'_{f2}, \) and \( w'_{f3} \) respectively. Therefore, by the same argument as that for \( w'_{g1}, w'_{g2}, v'_{g1}, \) and \( v'_{g2} \), we have
\[
\int_{\mathbb{R}} \left( \|w'_{f2}\|_2^2 + \|v'_{f2}\|_2^2 + \|w'_{f3}\|_2^2 + \|v'_{f3}\|_2^2 \right) \, dt \leq \frac{\epsilon_n}{\sigma} \|f\|_2^2.
\]
Finally, it is easy to verify that
\[
\begin{align*}
  \|w'_{f1}\|_2^2 + \|v'_{f1}\|_2^2 &= \frac{(\alpha^2 + 1) \sinh 2 \sigma + 2 \alpha (\cosh 2 \sigma - 1)}{2 \sigma \cosh \lambda + \alpha \sinh \lambda^2} \\
  &\times \sum_{n,m} \frac{nm \pi^2}{(n^2 \pi^2 + \lambda^2)(m^2 \pi^2 + \lambda^2)} g_n \bar{g}_m.
\end{align*}
\]
Therefore, the same reasoning used to obtain (3.34) leads to
\[ \int_{\mathbb{R}} \left( \| w_{,1} \|^2 + \| v_{,1} \|^2 \right) \, d\tau \leq \frac{c_7}{\sigma} \| f' \|^2. \] (3.44)

Combining (3.44) and (3.42) implies that (3.18) holds.

Finally, for any \( F_* \in \mathcal{H} \), if there exists a \( Y_* \in \mathcal{D}(\mathcal{A}^*) \) such that \( \lambda Y_* - \mathcal{A}^* Y_* = F_* \) for \( \sigma > \sigma_0 + \epsilon \), where \( F_* = [f_*, g_*]^T \) and \( Y_* = [w_*, v_*]^T \), then it is easy to see that
\[ w_{,1} = -w_{,1} + w_{,1}, \quad v_* = v_{,1} - v_{,1}. \]

In terms of the same argument as that used to prove (1.2), we obtain (1.3).

To sum up, we have proved the following result.

**Theorem 3.1.** The generator \( \mathcal{A} \) defined above generates a \( C_0 \) semigroup \( T(t) \). Then for any initial data \( Y_0 \in \mathcal{H} \), (3.2) has a unique weak solution \( Y(t) \) such that \( Y(\cdot) \in C([0, \infty), \mathcal{H}) \). Moreover, if \( Y_0 \in \mathcal{D}(\mathcal{A}) \), then \( Y(t) \) is the unique strong solution to (2.1) such that \( Y(\cdot) \in C([0, \infty), \mathcal{H}) \cap C([0, \infty), \mathcal{D}(\mathcal{A})) \). For every \( \epsilon > 0 \), there exists a positive constant \( M_\epsilon \) such that
\[ \| Y(t) \| \leq M_\epsilon e^{(\sigma_0 + \epsilon)t}, \]
where \( \sigma_0 = \frac{1}{4} \ln(1 - \alpha)/(1 + \alpha) = s(\mathcal{A}) \).

Although the method used here to prove Theorem 3.1 is, perhaps, not the simplest, the method of Theorem 3.1 is straightforward and can be applied to study the well-posedness problem for a large class of non-dissipative systems, such as the beam with non-dissipative boundary condition. We shall not discuss this problem here due to space limitations.

### 4. FURTHER DISCUSSION

In the above example the condition (1.3) is naturally satisfied when (1.2) holds. Then we may naturally ask whether the condition (1.2) implies (1.3). We guess that the answer is negative. Another problem is whether Theorem 1.1 can be extended to general Banach spaces. We have

**Theorem 4.1.** Let \( X \) be a Banach space and \( X^* \) be the dual of \( X \). Assume that \( \mathcal{A} \) is a closed densely defined operator in \( X \) such that \( \rho(\mathcal{A}) \cup \{ \lambda = \sigma + i\tau \in \mathbb{C} \mid \sigma > \sigma_0 \} \), and that for any \( \lambda = \sigma + i\tau \in \mathbb{C} \) with \( \sigma > \sigma_0 \),
\[ \sup_{\sigma > \sigma_0} (\sigma - \sigma_0) \int_{\mathbb{R}} \| R(\sigma + i\tau; \mathcal{A}) x \|^2 \, d\tau < +\infty, \quad \forall x \in X, \] (4.1)
\[ \sup_{\sigma > \sigma_0} (\sigma - \sigma_0) \int_{\mathbb{R}} \| R(\sigma - i\tau; \mathcal{A}^*) y^* \|^2 \, d\tau < +\infty, \quad \forall y^* \in X^*. \] (4.2)

Then \( \mathcal{A} \) generates a \( C_0 \) semigroup in \( X \).
Proof. The proof is similar to that of Theorem 12.6.1 in [4]. We denote the dual product $\langle x, y^* \rangle$ for $y^* \in X^*$, $x \in X$. Without loss of generality, we may assume that $\sigma_0 = 0$. It follows from (4.1) and (4.2) that $\langle R^2(\lambda; \mathcal{A})x, y^* \rangle \in H_\lambda(\sigma; X)$ and that there exists an $M_0 > 0$ such that

$$
\int_{-\infty}^{+\infty} \left| \langle R^2(\sigma + i\tau; \mathcal{A})x, y^* \rangle \right| d\tau \leq \frac{M_0}{\sigma} \|x\| \|y^*\|. \quad (4.3)
$$

By using Theorem 6.4.1 in [4], we have

$$
d\frac{d}{d\lambda} \langle R^2(\lambda; \mathcal{A})x, y^* \rangle = \frac{(-1)^j j!}{2\pi} \int_{-\infty}^{+\infty} \frac{\langle R^2(\sigma + i\tau; \mathcal{A})x, y^* \rangle}{(\lambda - \sigma - i\tau)^{j+1}} d\tau,
$$

$$
j = 1, 2, \ldots . \quad (4.4)
$$

Using (4.4) with $j = n - 2$, we get

$$
\langle R^{(n-1)}(\lambda; \mathcal{A})x, y^* \rangle
$$

$$
= - \frac{d^{n-2}}{d\lambda^{n-2}} \langle R^2(\lambda; \mathcal{A})x, y^* \rangle
$$

$$
= \frac{(-1)^{n-1}(n-2)!}{2\pi} \int_{-\infty}^{+\infty} \frac{\langle R^2(\sigma + i\tau; \mathcal{A})x, y^* \rangle}{(\lambda - \sigma - i\tau)^{n-1}} d\tau. \quad (4.5)
$$

Therefore, for $\lambda > \sigma > 0$, it follows from (4.5) and (4.3) that

$$
\|R^{(n-1)}(\lambda; \mathcal{A})x\| = \sup_{y^* \in X^*, \|y^*\| = 1} \langle R^{(n-1)}(\lambda; \mathcal{A})x, y^* \rangle
$$

$$
\leq \frac{(n-2)!}{2\pi} (\lambda - \sigma)^{(1-n)}
$$

$$
\times \sup_{y^* \in X^*, \|y^*\| = 1} \int_{-\infty}^{+\infty} \left| \langle R^2(\sigma + i\tau; \mathcal{A})x, y^* \rangle \right| d\tau
$$

$$
\leq M_0 (n-2)! (\lambda - \sigma)^{1-n} \sigma^{-1} \|x\|, \quad \forall n \geq 2. \quad (4.6)
$$

Taking $\sigma = \frac{1}{n}$ in (4.6), we have

$$
\|R^{(n-1)}(\lambda; \mathcal{A})\| \leq M_0 (n - 1)! \lambda^{-n}, \quad \forall n \geq 2, \lambda > 0. \quad (4.7)
$$
Furthermore, it follows from (4.1) that
\[ R(\lambda; \mathcal{A})x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{R(\sigma + i\tau; \mathcal{A})x}{\lambda - \sigma - i\tau} \, d\tau. \]  
(4.8)

Therefore, for \( \lambda > \sigma = \lambda/2 > 0 \), by (4.8) we have
\[
\|R(\lambda; \mathcal{A})x\| \leq \frac{1}{2\pi} \left( \int_{-\infty}^{+\infty} \|R(\sigma + i\tau; \mathcal{A})x\|^2 \, d\tau \right)^{1/2} \times \left( \int_{-\infty}^{+\infty} \left( (\lambda - \sigma)^2 + \tau^2 \right)^{-1} \, d\tau \right)^{1/2} \\
\leq M_3 \sigma^{-1/2} (\lambda - \sigma)^{-1/2} \left( \int_{-\infty}^{+\infty} \frac{1}{1 + \eta^2} \, d\eta \right)^{1/2} \\
\leq M_4 \lambda^{-1},
\]  
(4.9)

where \( M_3 \) and \( M_4 \) are positive constants. Thus, by the Hille–Yosida theorem, the desired result follows from (4.7) and (4.9).

We have used the Hille–Yosida theorem to get another proof of sufficiency of Theorem 1.1, but the proof given in Section 2 offers more useful information. Whether the conditions (4.1) and (4.2) are necessary for \( \mathcal{A} \) to generate a \( C_0 \) semigroup in the Banach space \( X \) remains a topic for further research.

REFERENCES