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On Mittag-Leffler-type functions in fractional evolution processes

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Abstract

We review a variety of fractional evolution processes (so defined being governed by equations of fractional order), whose solutions turn out to be related to Mittag-Leffler-type functions. The chosen equations are the simplest of the fractional calculus and include the Abel integral equations of the second kind, which are relevant in typical inverse problems, and the fractional differential equations, which govern generalized relaxation and oscillation phenomena.

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1. Introduction

The purpose of this review is to outline the fundamental role of Mittag-Leffler-type functions in fractional evolution processes. By a fractional evolution process we mean a phenomenon governed by an integro-differential equation containing integrals and/or derivatives of fractional order in time. Particular attention is devoted to the technique of Laplace transforms for treating our equations in a way accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigor.

The plan of the paper is as follows. In the first part we recall the main properties of the Mittag-Leffler function (Section 2) and we introduce the linear operators of fractional integration and fractional differentiation in the framework of the Riemann–Liouville fractional calculus (Section 3). In the second part, by applying the technique of Laplace transforms, we then consider

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some linear integral and differential equations of fractional order, which are associated with evolution processes of physical interest. For these equations we derive their analytical solutions in terms of Mittag-Leffler-type functions. In Section 4 we treat Abel-type integral equations of the second kind which are related to a class of inverse problems. In Section 5 we consider ordinary differential equations of fractional order which are related to generalized processes of relaxation and oscillation, that we refer to as fractional relaxation and fractional oscillation, respectively.

The present review is essentially based on our original works, to which we shall refer in the following. For other related reviews of ours, see the invited lectures on applications of the *Fractional Calculus* that we have given in recent times in international *Advanced Courses* and *Workshops*, i.e., [14,15,20,35–38].

2. Mittag-Leffler-type functions

The Mittag-Leffler function is so named from the great Swedish mathematician who introduced it at the beginning of this century in a sequence of five notes [40–44]. In this section we shall consider the Mittag-Leffler function and some of the related functions which are relevant for their connection with fractional evolution processes. It is our purpose to provide a short reference-historical background and a review of the main properties of these functions with particular regard to their Laplace transforms.

2.1. Reference-historical background

We note that the Mittag-Leffler-type functions, being ignored in the common books on special functions, are unknown to the majority of scientists. Even in the 1991 Mathematics Subject Classification these functions cannot be found. However, in the new AMS classification foreseen for the year 2000, a place for them has been reserved: 33E12 ("Mittag-Leffler functions").

A description of the most important properties of these functions with relevant references can be found in the third volume of the Bateman Project [13], in the chapter XVIII of the *miscellaneous functions*. The treatise where great attention is devoted to them are those by Dzherbashyan [11,12]. We also recommend the classical treatise on complex functions by Sansone and Gerretsen [49] and the recent book on fractional calculus by Podlubny [46].

Since the times of Mittag-Leffler several scientists have recognized the importance of the Mittag-Leffler-type functions, providing interesting mathematical and physical applications, which unfortunately are not much known.

As pioneering works of mathematical nature in the field of fractional integral and differential equations, we like to quote those by Hille and Tamarkin [28] and Barret [2]. In 1930 Hille and Tamarkin have provided the solution of the Abel integral equation of the second kind in terms of a Mittag-Leffler function, whereas in 1954 Barret has expressed the general solution of the linear fractional differential equation with constant coefficients in terms of Mittag-Leffler functions.

As former applications in physics we like to quote the contributions by Cole in 1933 [8] in connection with nerve conduction, see also [10], and by Gross [26] in 1947 in connection with mechanical relaxation. Subsequently, in 1971, Caputo and Mainardi [7] have shown that Mittag-Leffler functions are present whenever derivatives of fractional order are introduced in the constitutive equations of a linear viscoelastic body. Since then, several other authors have pointed out the relevance of these functions for fractional viscoelastic models, see, e.g., [35].

Gorenflo and Yamamoto [25] have used asymptotic properties of Mittag-Leffler functions for a detailed analysis of the transition from the second kind to the first kind Abel integral equation, thereby considering the equation of second kind as a (singular) perturbation of that of first kind. Whereas in applications Mittag-Leffler functions usually occur with real arguments, Witte [53] has extensively exploited their properties in the complex domain for studying equations of fractional convection and fractional diffusion–convection and for developing difference schemes for their numerical treatment.

2.2. The Mittag-Leffler functions $E_\alpha(z)$, $E_{\alpha,\beta}(z)$

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane:

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \ z \in \mathbb{C}. \tag{2.1}$$

It turns out that $E_\alpha(z)$ is an *entire function* of order $\rho = 1/\alpha$ and type 1. This property is still valid but with $\rho = 1/\mathrm{Re}\{\alpha\}$, if $\alpha \in \mathbb{C}$ with *positive real part*, as formerly noted by Mittag-Leffler himself in [43].

The Mittag-Leffler function provides a simple generalization of the exponential function because of the substitution of $n! = \Gamma(n+1)$ with $(\alpha n)! = \Gamma(\alpha n + 1)$. Particular cases of (2.1), from which elementary functions are recovered, are

$$E_2(+z^2) = \cosh z, \qquad E_2(-z^2) = \cos z, \quad z \in \mathbb{C} \tag{2.2}$$

and

$$E_{1/2}(\pm z^{1/2}) = \mathrm{e}^z[1 + \mathrm{erf}\,(\pm z^{1/2})] = \mathrm{e}^z\,\mathrm{erfc}(\mp z^{1/2}), \quad z \in \mathbb{C}, \tag{2.3}$$

where erf (erfc) denotes the (complementary) error function defined as

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2}\,\mathrm{d}u, \qquad \mathrm{erfc}(z) := 1 - \mathrm{erf}(z), \ z \in \mathbb{C}. \tag{2.4}$$

In (2.4) by $z^{1/2}$ we mean the principal value of the square root of z in the complex plane cut along the negative real semi-axis. With this choice $\pm z^{1/2}$ turns out to be positive/negative for $z \in \mathbb{R}^+$. A straightforward generalization of the Mittag-Leffler function, originally due to Agarwal in 1953 based on a note by Humbert, see [1,29,30], is obtained by replacing the additive constant 1 in the argument of the Gamma function in (2.1) by an arbitrary complex parameter β. For the generalized Mittag-Leffler function we agree to use the notation

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{C}, \ z \in \mathbb{C}. \tag{2.5}$$

Particular simple cases are

$$E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}, \qquad E_{2,2}(z) = \frac{\sinh(z^{1/2})}{z^{1/2}}. \tag{2.6}$$

We note that $E_{\alpha,\beta}(z)$ is still an entire function of order $\rho = 1/\alpha$ and type 1. For lack of space we prefer to continue with the Mittag-Leffler functions in one parameter: since here we shall limit ourselves to consider evolution processes characterized by a single fractional order, the more general functions with two parameters turn out to be redundant. However, we find it convenient to introduce other functions depending on a single parameter which turn out to be related by simple relations to the original Mittag-Leffler functions, and to consider them as belonging to the class of Mittag-Leffler-type functions.

In a broad sense the class of Mittag-Leffler-type functions is much larger than that considered here, since several generalizations have been introduced by mathematicians to take into account more parameters and variables. In addition to the books [11,46], we would like to cite a few relevant papers, namely [16,18,27,31,50,51].

2.3. The Mittag-Leffler integral representation and asymptotic expansions

Many of the most important properties of $E_\alpha(z)$ follow from Mittag-Leffler's *integral representation*

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{\mathrm{Ha}} \frac{\zeta^{\alpha-1}e^{\zeta}}{\zeta^\alpha - z}\,d\zeta, \quad \alpha > 0,\ z \in \mathbb{C}, \tag{2.7}$$

where the path of integration Ha (the *Hankel path*) is a loop which starts and ends at $-\infty$ and encircles the circular disk $|\zeta| \leqslant |z|^{1/\alpha}$ in the positive sense: $-\pi \leqslant \arg\zeta \leqslant \pi$ on Ha. To prove (2.7), expand the integrand in powers of ζ, integrate term-by-term, and use Hankel's integral for the reciprocal of the Gamma function.

The integrand in (2.7) has a branch-point at $\zeta = 0$. The complex ζ-plane is cut along the negative real semi-axis, and in the cut plane the integrand is single-valued: the principal branch of ζ^α is taken in the cut plane. The integrand has poles at the points $\zeta_m = z^{1/\alpha}\,e^{2\pi i m/\alpha}$, m integer, but only those of the poles lie in the cut plane for which $-\alpha\pi < \arg z + 2\pi m < \alpha\pi$. Thus, the number of the poles inside Ha is either $[\alpha]$ or $[\alpha+1]$, according to the value of $\arg z$.

The most interesting properties of the Mittag-Leffler function are associated with its asymptotic developments as $z \to \infty$ in various sectors of the complex plane. These properties can be summarized as follows. For the case $0 < \alpha < 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha}\exp(z^{1/\alpha}) - \sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \quad |\arg z| < \alpha\pi/2, \tag{2.8}$$

$$E_\alpha(z) \sim -\sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \quad \alpha\pi/2 < \arg z < 2\pi - \alpha\pi/2. \tag{2.9}$$

For the case $\alpha \geqslant 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha}\sum_{m}\exp(z^{1/\alpha}e^{2\pi i m/\alpha}) - \sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \tag{2.10}$$

where m takes all integer values such that $-\alpha\pi/2 < \arg z + 2\pi m < \alpha\pi/2$, and $\arg z$ can assume any value between $-\pi$ and $+\pi$ inclusive. From the asymptotic properties (2.8)–(2.10) and the definition of the order of an entire function, we infer that the Mittag–Leffler function is an *entire function of order* $1/\alpha$ for $\alpha > 0$; in a certain sense each $E_\alpha(z)$ is the simplest entire function of its order, see [47]. The Mittag-Leffler function also furnishes examples and counter-examples for the growth and other properties of entire functions of finite order (see [4]).

2.4. The Laplace transform pairs related to the Mittag-Leffler functions

The Mittag-Leffler functions are connected to the Laplace integral through the equation

$$\int_0^\infty \mathrm{e}^{-u} E_\alpha(u^\alpha z)\,\mathrm{d}u = \frac{1}{1-z}, \quad \alpha > 0. \tag{2.11}$$

The integral at the LHS was evaluated by Mittag-Leffler who showed that the region of its convergence contains the unit circle and is bounded by the line $\mathrm{Re}\, z^{1/\alpha} = 1$. Putting in (2.11) $u = st$ and $u^\alpha z = -\lambda t^\alpha$ with $t \geqslant 0$ and $\lambda \in \mathbb{C}$, and using the sign $\div$ for the juxtaposition of a function depending on t with its Laplace transform depending on s, we get the following Laplace transform pairs:

$$e_\alpha(t;\lambda) := E_\alpha(-\lambda t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha + \lambda}, \quad \mathrm{Re}\, s > |\lambda|^{1/\alpha}. \tag{2.12}$$

Later we shall show the key role of the Mittag-Leffler-type functions $e_\alpha(t;\lambda)$ in treating certain fractional integral and differential equations. In particular, we shall discuss their peculiar characters for $0 < \alpha < 1$ and for $1 < \alpha < 2$ related to *fractional relaxation* and *fractional oscillation* processes, respectively.

2.5. Other formulas: summation and integration

For completeness hereafter we exhibit some formulas related to summation and integration of ordinary Mittag-Leffler functions (in one parameter α), referring the interested reader to Dzherbashian [11], Podlubny [46] for their proof and for their generalizations to two parameters.

Concerning summation we outline

$$E_\alpha(z) = \frac{1}{p} \sum_{h=0}^{p-1} E_{\alpha/p}(z^{1/p}\, \mathrm{e}^{\mathrm{i}2\pi h/p}), \quad p \in \mathbb{N} \tag{2.13}$$

from which we derive the *duplication formula*

$$E_\alpha(z) = \tfrac{1}{2}[E_{\alpha/2}(+z^{1/2}) + E_{\alpha/2}(-z^{1/2})]. \tag{2.14}$$

Concerning integration we outline another interesting *duplication formula*

$$\int_0^\infty \mathrm{e}^{-x^2/(4t)} E_\alpha(x^\alpha)\,\mathrm{d}x = \sqrt{\pi t}\, E_{\alpha/2}(t^{\alpha/2}), \quad t > 0. \tag{2.15}$$

3. Essentials of Riemann–Liouville fractional calculus

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term *fractional* is a misnomer, but it is retained following the prevailing use. This section is mostly based on the recent review by Gorenflo and Mainardi, [20]. For more details on the classical treatments of fractional calculus the reader is referred to specialized books, e.g., [24,39,45,46,48].

3.1. The fractional integral J^α

According to the Riemann–Liouville approach to fractional calculus, the notion of fractional integral of order $\alpha (\alpha > 0)$ is a natural consequence of the well-known formula (usually attributed to Cauchy), that reduces the calculation of the n-fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0,\ n \in \mathbb{N}. \tag{3.1}$$

From this definition we notice that $f_n(t)$ vanishes at $t=0$ with its derivatives of order $1, 2, \ldots, n-1$. For convention we require that $f(t)$ and henceforth $f_n(t)$ be a *causal* function, i.e., identically vanishing for $t < 0$.

In a natural way one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that $(n-1)! = \Gamma(n)$, and introducing the arbitrary *positive* real number α, one defines the *fractional integral of order* $\alpha > 0$:

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0,\ \alpha \in \mathbb{R}^+. \tag{3.2}$$

For complementation we define $J^0 := I$ (Identity operator), i.e., we mean $J^0 f(t) = f(t)$. Furthermore, by $J^\alpha f(0^+)$ we mean the limit (if it exists) of $J^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite. We note the *semigroup property* $J^\alpha J^\beta = J^{\alpha+\beta}, \alpha, \beta \geqslant 0$, which implies the *commutative property* $J^\beta J^\alpha = J^\alpha J^\beta$, and the effect of our operators J^α on the power functions

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \quad \alpha \geqslant 0,\ \gamma > -1,\ t > 0. \tag{3.3}$$

These properties are of course a natural generalization of those known when the order is a positive integer.

We note the following rule for the Laplace transform of the fractional integral:

$$J^\alpha f(t) \div \frac{\tilde{f}(s)}{s^\alpha}, \quad \alpha \geqslant 0 \tag{3.4}$$

which generalizes the case of an n-fold repeated integral (3.1).

It may be convenient to introduce the causal function

$$\Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0, \tag{3.5}$$

where the suffix $+$ just means that the function vanishes for $t < 0$. Being $\alpha > 0$, this function turns out to be *locally* absolutely integrable in $\mathbb{R}^+$. Let us now recall the notion of *Laplace convolution*, i.e., the convolution integral with two causal functions, $f(t) * g(t) := \int_0^t f(t-\tau) g(\tau)\,\mathrm{d}\tau = g(t) * f(t)$. Then we note from (3.2) and (3.5) that the fractional integral of order $\alpha > 0$ is the Laplace convolution of $\Phi_\alpha(t)$ and $f(t)$, i.e.,

$$J^\alpha f(t) = \Phi_\alpha(t) * f(t), \quad \alpha > 0. \tag{3.6}$$

3.2. The fractional derivatives D^α, D_*^α

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integrals and preserve the well-known properties of the ordinary derivative of integer order. Denoting by D^n the operator of ordinary differentiation of order n where n is a positive integer, we first note that $D^n J^n = I$, $J^n D^n \neq I$, i.e., D^n is left-inverse (but not right-inverse) to the corresponding integral operator J^n. In fact, we easily recognize from (3.1) that

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}, \quad t > 0. \tag{3.7}$$

As a consequence we expect that D^α is defined as left-inverse to J^α. For this purpose, introducing the positive integer m such that $m-1 < \alpha \leqslant m$, one defines the *fractional derivative of order* $\alpha > 0$ as $D^\alpha := D^m J^{m-\alpha}$. In fact $D^\alpha J^\alpha = D^m J^{m-\alpha} J^\alpha = D^m J^m = I$. When $\alpha = m$ we recover the standard derivative of integer order, whereas when α is not integer we get

$$D^\alpha f(t) := \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau\right], \quad m-1 < \alpha < m. \tag{3.8}$$

Similarly to (3.3), the effect of the operator D^α on the power functions turns out to be

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \alpha \geqslant 0,\ \gamma > -1,\ t > 0. \tag{3.9}$$

Note the remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) \equiv 1$ if $\alpha \notin \mathbb{N}$. In fact, (3.9) with $\gamma = 0$ teaches us that

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \geqslant 0,\ t > 0. \tag{3.10}$$

This, of course, is $\equiv 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function in the points $0, -1, -2, \ldots$. We now observe that an alternative definition of the fractional derivative, originally introduced by Caputo [5,6] in the late 1960s and adopted by Caputo and Mainardi [7] in the framework of the theory of *Linear Viscoelasticity*, is $D_*^\alpha := J^{m-\alpha} D^m$ where $m-1 < \alpha \leqslant m$, $m \in \mathbb{N}$. When $\alpha = m$ we recover $D_*^m = D^m$, whereas when α is not integer we get

$$D_*^\alpha f(t) := \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau, \quad m-1 < \alpha < m. \tag{3.11}$$

This definition is of course more restrictive than (3.8) in which it requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha f(t) := D^m J^{m-\alpha} f(t) \neq J^{m-\alpha} D^m f(t) := D_*^\alpha f(t), \tag{3.12}$$

unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. For this purpose let us first note for $m-1 < \alpha \leqslant m$ and $t > 0$,

$$J^\alpha D_*^\alpha f(t) = J^\alpha J^{m-\alpha} D^m f(t) = J^m D^m f(t),$$

hence by (3.7)

$$J^{\alpha} D_*^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}. \tag{3.13}$$

Then, by $D^{\alpha} J^{\alpha} = I$ and (3.9), we derive for $t > 0$

$$D^{\alpha} f(t) = D_*^{\alpha} f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+), \quad m-1 < \alpha \leqslant m \tag{3.14}$$

and

$$D^{\alpha} \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0^+) \right) = D_*^{\alpha} f(t), \quad m-1 < \alpha \leqslant m. \tag{3.15}$$

The alternative definition (3.11) for the fractional derivative thus incorporates the initial values of the function and of its integer derivatives of lower order.

The subtraction of the Taylor polynomial of degree $m-1$ at $t = 0^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero can be easily recognized, i.e.,

$$D_*^{\alpha} 1 \equiv 0, \quad \alpha > 0. \tag{3.16}$$

We now explore the most relevant differences between the two fractional derivatives (3.8) and (3.11). We agree to denote (3.11) as the *Caputo fractional derivative* to distinguish it from the standard Riemann–Liouville fractional derivative (3.8). We observe, again by looking at (3.9), that $D^{\alpha} t^{\alpha-1} \equiv 0$.

From above we thus draw the following conclusions about functions which for $t > 0$ admit the same fractional derivative of order α, with $m-1 < \alpha \leqslant m$,

$$D^{\alpha} f(t) = D^{\alpha} g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j t^{\alpha-j}, \tag{3.17}$$

$$D_*^{\alpha} f(t) = D_*^{\alpha} g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j t^{m-j}. \tag{3.18}$$

In these formulas the coefficients c_j are arbitrary constants. For the two definitions we also note a difference with respect to the *formal* limit as $\alpha \to (m-1)^+$. From (3.8) and (3.11) we obtain, respectively,

$$\alpha \to (m-1)^+ \Rightarrow D^{\alpha} f(t) \to D^m J f(t) = D^{m-1} f(t), \tag{3.19}$$

$$\alpha \to (m-1)^+ \Rightarrow D_*^{\alpha} f(t) \to J D^m f(t) = D^{m-1} f(t) - f^{(m-1)}(0^+). \tag{3.20}$$

We now consider the *Laplace transform* of the two fractional derivatives. For the standard fractional derivative D^{α} the Laplace transform, assumed to exist, requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, \ldots, m-1$. In our notation the corresponding rule reads, see, e.g., [45,46],

$$D^{\alpha} f(t) \div s^{\alpha} \tilde{f}(s) - \sum_{k=0}^{m-1} D^k J^{(m-\alpha)} f(0^+) s^{m-1-k}, \quad m-1 < \alpha \leqslant m. \tag{3.21}$$

The Laplace transform of the Caputo fractional derivative requires the knowledge of the (bounded) initial values of the function and of its derivatives of integer order $k = 1, 2, \ldots, m-1$, in analogy with the case when $\alpha = m$. In fact, we can prove the corresponding Caputo rule, see, e.g., [6],

$$D_*^{\alpha} f(t) \div s^{\alpha} \tilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \leqslant m. \tag{3.22}$$

Gorenflo and Mainardi [20] and Podlubny [46] have recognized and pointed out the major utility of the Caputo fractional derivative in the treatment of differential equations of fractional order for *physical applications*. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order, even if the governing evolution equation is a generic integro-differential equation and therefore, in particular, a fractional differential equation.

4. Abel integral equation of the second kind

The Abel integral equations of the first and second kind are the most simple integral equations of fractional order. Denoting by $u(t)$ the unknown function and $f(t)$ a given function, they can be written as

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = f(t), \quad \alpha > 0, \tag{4.1}$$

$$u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = f(t), \quad \alpha > 0, \quad \lambda \in \mathbb{C}. \tag{4.2}$$

Using (3.2) and (3.6) the integral term in (4.1) and (4.2) can be expressed as a fractional integral, namely

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = J^{\alpha} u(t) = \Phi_{\alpha}(t) * u(t). \tag{4.3}$$

For $0 < \alpha < 1$ the kernel function $\Phi_{\alpha}(t)$ turns out to be *weakly singular*.

It is well known that the above integral equations are so named from the great Norwegian mathematician, Niels Henrik Abel, who in 1823–26 was led to his famous equation (4.1), with $0 < \alpha < 1$, by the mechanical problem of the *tautochrone*. The interested reader is referred to Craig and Brown [9], Gorenflo and Vessella [24] and Gorenflo [14,15] for historical notes and detailed analysis with applications concerning the general class of Abel-type integral equations.

Here we limit ourselves to consider the Abel integral equation of the second kind, see (4.2), showing that its solution can be related in different ways to the Mittag-Leffler-type functions $e_{\alpha}(t; \lambda) := E_{\alpha}(-\lambda t^{\alpha})$ (see (2.12)). We shall use the method of the Laplace transforms, that makes easier and more comprehensible the treatment and leads to (2.12) in a straightforward way.

Applying the Laplace transform to (4.2) we obtain

$$\left[1 + \frac{\lambda}{s^{\alpha}}\right] \tilde{u}(s) = \tilde{f}(s) \;\Rightarrow\; \tilde{u}(s) = \frac{s^{\alpha}}{s^{\alpha} + \lambda} \tilde{f}(s). \tag{4.4}$$

It is quite natural to choose two different ways to get the inverse Laplace transforms from (4.4), according to the standard rules. Writing (4.4) as

$$\tilde{u}(s) = s\left[\frac{s^{\alpha-1}}{s^{\alpha}+\lambda}\tilde{f}(s)\right], \tag{4.4a}$$

we obtain

$$u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t-\tau)e_{\alpha}(\tau;\lambda)\,\mathrm{d}\tau. \tag{4.5a}$$

If we write (4.4) as

$$\tilde{u}(s) = \frac{s^{\alpha-1}}{s^{\alpha}+\lambda}[s\tilde{f}(s) - f(0^+)] + f(0^+)\frac{s^{\alpha-1}}{s^{\alpha}+\lambda}, \tag{4.4b}$$

we obtain

$$u(t) = \int_0^t f'(t-\tau)e_{\alpha}(\tau;\lambda)\,\mathrm{d}\tau + f(0^+)e_{\alpha}(t;\lambda). \tag{4.5b}$$

We also note that, $e_{\alpha}(t;\lambda)$ being a function differentiable with respect to t with $e_{\alpha}(0^+;\lambda)=E_{\alpha}(0^+)=1$, there exists another possibility to re-write (4.4), namely

$$\tilde{u}(s) = \left[s\frac{s^{\alpha-1}}{s^{\alpha}+\lambda} - 1\right]\tilde{f}(s) + \tilde{f}(s). \tag{4.4c}$$

Then we obtain

$$u(t) = \int_0^t f(t-\tau)e'_{\alpha}(\tau;\lambda)\,\mathrm{d}\tau + f(t) \tag{4.5c}$$

in agreement with the expression of the solution found by Hille and Tamarkin [28]. We see that the way (b) is more restrictive than the ways (a) and (c) since it requires that $f(t)$ be differentiable with $\mathscr{L}$-transformable derivative.

5. Fractional differential equations

We now analyze the most simple differential equations of fractional order, including those which, by means of fractional derivatives, generalize the well-known ordinary differential equations related to relaxation and oscillation phenomena. Generally speaking, we consider the following differential equation of fractional order $\alpha > 0$:

$$D_*^{\alpha}u(t) = D^{\alpha}\left(u(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}u^{(k)}(0^+)\right) = -u(t) + q(t), \quad t > 0, \tag{5.1}$$

where $u=u(t)$ is the field variable and $q(t)$ is a given function. Here m is a positive integer uniquely defined by $m-1 < \alpha \leqslant m$, which provides the number of the prescribed initial values $u^{(k)}(0^+) = c_k$, $k = 0,1,2,\ldots,m-1$. Implicit in the form of (5.1) is our desire to obtain solutions $u(t)$ for which the $u^{(k)}(t)$ are continuous for positive t and right-continuous at the origin $t=0$ for $k=0,1,2,\ldots,m-1$. In particular, the cases of *fractional relaxation* and *fractional oscillation* are obtained for $0<\alpha<1$ and $1<\alpha<2$, respectively.

5.1. Generalities

We note that when α is the integer m Eq. (3.5) reduces to an ordinary differential equation whose solution can be expressed in terms of m linearly independent solutions of the *homogeneous* equation and of one particular solution of the *inhomogeneous* equation. We summarize the well-known result as follows:

$$u(t) = \sum_{k=0}^{m-1} c_k u_k(t) + \int_0^t q(t-\tau) u_\delta(\tau)\,\mathrm{d}\tau, \tag{5.2}$$

$$u_k(t) = J^k u_0(t), \ \ u_k^{(h)}(0^+) = \delta_{kh}, \ \ h,k = 0,1,\ldots,m-1, \tag{5.3}$$

$$u_\delta(t) = -u_0'(t). \tag{5.4}$$

Thus, the m functions $u_k(t)$ represent the *fundamental solutions* of the differential equation of order m, namely those linearly independent solutions of the *homogeneous* equation which satisfy the initial conditions in (5.3). The function $u_\delta(t)$, with which the free term $q(t)$ appears convoluted, represents the so-called *impulse-response solution*, namely the particular solution of the *inhomogeneous* equation with all $c_k = 0$, $k = 0,1,\ldots,m-1$, and with $q(t) = \delta(t)$. In the cases of standard relaxation $u'(t) = -u(t)$ and standard oscillation $u''(t) = -u(t)$ we recognize that $u_0(t) = \mathrm{e}^{-t} = u_\delta(t)$ and $u_0(t) = \cos t$, $u_1(t) = J\ u_0(t) = \sin t = \cos(t - \pi/2) = u_\delta(t)$, respectively.

Let us now solve (5.1) by the method of Laplace transforms. For this purpose we can use directly the Caputo rule (3.21) or, alternatively, reduce (5.1) with the prescribed initial conditions as an equivalent (fractional) integral equation and then treat the integral equation by the Laplace transform method. We obtain

$$\tilde{u}(s) = \sum_{k=0}^{m-1} c_k \frac{s^{\alpha-k-1}}{s^\alpha + 1} + \frac{1}{s^\alpha + 1}\tilde{q}(s). \tag{5.5}$$

Introducing the Mittag-Leffler-type functions, see (2.12),

$$e_\alpha(t) \equiv e_\alpha(t;1) := E_\alpha(-t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha + 1} \tag{5.6}$$

and

$$u_k(t) := J^k e_\alpha(t) \div \frac{s^{\alpha-k-1}}{s^\alpha + 1}, \quad k = 0,1,\ldots,m-1, \tag{5.7}$$

we find, from inversion of the Laplace transforms in (5.5),

$$u(t) = \sum_{k=0}^{m-1} c_k u_k(t) - \int_0^t q(t-\tau) u_0'(\tau)\,\mathrm{d}\tau. \tag{5.8}$$

For finding the last term in the RHS of (5.8), we have used the well-known rule for the Laplace transform of the derivative noting that $u_0(0^+) = e_\alpha(0^+) = 1$ and

$$\frac{1}{s^\alpha + 1} = -\left(s\frac{s^{\alpha-1}}{s^\alpha + 1} - 1 \right) \div -u_0'(t) = -e_\alpha'(t). \tag{5.9}$$

In particular formula (5.8) encompasses the solutions for $\alpha = 1$ and 2.

When α is not integer, namely for $m-1 < \alpha < m$, we note that $m-1$ represents the integer part of α (usually denoted by $[\alpha]$) and m the number of initial conditions necessary and sufficient to ensure the uniqueness of the solution $u(t)$. Thus, the m functions $u_k(t) = J^k e_\alpha(t)$ with $k = 0, 1, \ldots, m-1$ represent those particular solutions of the *homogeneous* equation which satisfy the initial conditions

$$u_k^{(h)}(0^+) = \delta_{kh}, \quad h, k = 0, 1, \ldots, m-1 \tag{5.10}$$

and therefore they represent the *fundamental solutions* of the fractional equation (5.1), in analogy with the case $\alpha = m$. Furthermore, the function $u_\delta(t) = -e'_\alpha(t)$ represents the *impulse-response solution*.

The reader is invited to verify that solution (5.8) has continuous derivatives $u^{(k)}(t)$ for $k = 0, 1, 2, \ldots, m-1$, which fulfill the m initial conditions $u^{(k)}(0^+) = c_k$. However, the so-called *impulse-response solution* of our equation (5.1), $u_\delta(t)$, is expected to be not so regular as solution (5.8). In fact, from (5.5), (5.8) and (5.9), one obtains

$$u_\delta(t) = -u'_0(t) \div \frac{1}{s^\alpha + 1}, \tag{5.11}$$

and therefore, using the limit theorem for Laplace transforms, one can recognize that, being $m-1 < \alpha < m$,

$$u_\delta^{(h)}(0^+) = 0, \quad h = 0, 1, \ldots, m-2, \quad u_\delta^{(m-1)}(0^+) = \infty. \tag{5.12}$$

5.2. Fractional relaxation and fractional oscillations

Hereafter, we are going to compute and exhibit the *fundamental solutions* and the *impulse-response solution* separately for the cases (a) $0 < \alpha < 1$ and (b) $1 < \alpha < 2$, pointing out the comparison with the corresponding solutions obtained when $\alpha = 1$ and 2. We prefer to derive the relevant properties of the basic functions $e_\alpha(t)$ directly from their representation as a Laplace inverse integral

$$e_\alpha(t) = \frac{1}{2\pi i} \int_{\mathrm{Br}} e^{st} \frac{s^{\alpha-1}}{s^\alpha + 1}\, ds, \tag{5.13}$$

in detail for $0 < \alpha \leqslant 2$, without detouring on the general theory of Mittag-Leffler functions in the complex plane. In (5.13) Br denotes the Bromwich path, i.e., a line $\mathrm{Re}\{s\} = \sigma$ with a value $\sigma \geqslant 1$, and $\mathrm{Im}\{s\}$ running from $-\infty$ to $+\infty$.

For α not integer the power function s^α is uniquely defined as $s^\alpha = |s|^\alpha e^{i \arg s}$, with $-\pi < \arg s < \pi$, that is in the complex s-plane cut along the negative real semi-axis. The essential step consists in decomposing $e_\alpha(t)$ into two parts according to $e_\alpha(t) = f_\alpha(t) + g_\alpha(t)$, as indicated below. In case (a) the function $f_\alpha(t)$, in case (b) the function $-f_\alpha(t)$ is *completely monotone*; in both cases $f_\alpha(t)$ tends to zero as t tends to infinity, from above in case (a), from below in case (b). The other part, $g_\alpha(t)$, is identically vanishing in case (a), but of *oscillatory* character with exponentially decreasing amplitude in case (b). In order to obtain the desired decomposition of e_α we bend the Bromwich path of integration Br into the equivalent Hankel path $\mathrm{Ha}(1^+)$, a loop which starts from $-\infty$ along the lower side of the negative real semi-axis, encircles the circular disc $|s| = 1$ in the positive sense and ends at $-\infty$ along the upper side of the negative real semi-axis. One obtains

$$e_\alpha(t) = f_\alpha(t) + g_\alpha(t), \quad t \geqslant 0 \tag{5.14}$$

with

$$f_\alpha(t) := \frac{1}{2\pi i}\int_{\mathrm{Ha}(\varepsilon)} e^{st}\,\frac{s^{\alpha-1}}{s^\alpha+1}\,ds, \tag{5.15}$$

where now the Hankel path $\mathrm{Ha}(\varepsilon)$ denotes a loop constituted by a small circle $|s| = \varepsilon$ with $\varepsilon \to 0$ and by the two borders of the cut negative real semi-axis, and

$$g_\alpha(t) := \sum_h e^{s'_h t}\,\mathrm{Res}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s'_h} = \frac{1}{\alpha}\sum_h e^{s'_h t}, \tag{5.16}$$

where s'_h are the relevant poles of $s^{\alpha-1}/(s^\alpha+1)$. In fact the poles turn out to be $s_h = \exp[i(2h+1)\pi/\alpha]$ with unitary modulus; they are all simple but relevant are only those situated in the main Riemann sheet, i.e., the poles s'_h with argument such that $-\pi < \arg s'_h < \pi$.

If $0 < \alpha < 1$, there are no such poles, since for all integers h we have $|\arg s_h| = |2h+1|\,\pi/\alpha > \pi$; as a consequence,

$$g_\alpha(t) \equiv 0; \text{ hence, } e_\alpha(t) = f_\alpha(t), \quad \text{if } 0 < \alpha < 1. \tag{5.17}$$

If $1 < \alpha < 2$, then there exist precisely two relevant poles, namely $s'_0 = \exp(i\pi/\alpha)$ and $s'_{-1} = \exp(-i\pi/\alpha) = \overline{s'_0}$, which are located in the left half-plane. Then one obtains

$$g_\alpha(t) = \frac{2}{\alpha}\,e^{t\cos(\pi/\alpha)}\cos\left[t\sin\left(\frac{\pi}{\alpha}\right)\right] \quad \text{if } 1 < \alpha < 2. \tag{5.18}$$

We note that this function exhibits oscillations with circular frequency $\omega(\alpha) = \sin(\pi/\alpha)$ and with an amplitude decaying exponentially with rate $\lambda(\alpha) = |\cos(\pi/\alpha)|$.

It is now an exercise in complex analysis to show that the contribution from the Hankel path $\mathrm{Ha}(\varepsilon)$ as $\varepsilon \to 0$ is provided by

$$f_\alpha(t) := \int_0^\infty e^{-rt} K_\alpha(r)\,dr \tag{5.19}$$

with

$$K_\alpha(r) = -\frac{1}{\pi}\,\mathrm{Im}\left\{\left.\frac{s^{\alpha-1}}{s^\alpha+1}\right|_{s=r\,e^{i\pi}}\right\} = \frac{1}{\pi}\,\frac{r^{\alpha-1}\sin(\alpha\pi)}{r^{2\alpha}+2r^\alpha\cos(\alpha\pi)+1}. \tag{5.20}$$

This function $K_\alpha(r)$ vanishes identically if α is an integer, it is positive for all r if $0 < \alpha < 1$, negative for all r if $1 < \alpha < 2$. In fact in (5.20) the denominator is, for α not integer, always positive being $> (r^\alpha - 1)^2 \geqslant 0$. Hence $f_\alpha(t)$ has the aforementioned monotonicity properties, decreasing towards zero in case (a), and increasing towards zero in case (b). We also note that, in order to satisfy the initial condition $e_\alpha(0^+) = 1$, we find $\int_0^\infty K_\alpha(r)\,dr = 1$ if $0 < \alpha < 1$, $\int_0^\infty K_\alpha(r)\,dr = 1 - 2/\alpha$ if $1 < \alpha < 2$. In addition to the basic fundamental solutions, $u_0(t) = e_\alpha(t)$ we need to compute the impulse-response solutions $u_\delta(t) = -D^1 e_\alpha(t)$ for cases (a) and (b) and, only in case (b), the second fundamental solution $u_1(t) = J^1 e_\alpha(t)$. For this purpose we note that in general it turns out that

$$J^k f_\alpha(t) = \int_0^\infty e^{-rt} K_{\alpha,k}(r)\,dr \tag{5.21}$$

with

$$K_{\alpha,k}(r) := (-1)^k r^{-k} K_\alpha(r) = \frac{(-1)^k}{\pi}\,\frac{r^{\alpha-1-k}\sin(\alpha\pi)}{r^{2\alpha}+2r^\alpha\cos(\alpha\pi)+1}, \tag{5.22}$$

where $K_\alpha(r) = K_{\alpha,0}(r)$, and

$$J^k g_\alpha(t) = \frac{2}{\alpha}\, \mathrm{e}^{t\cos(\pi/\alpha)} \cos\left[t \sin\left(\frac{\pi}{\alpha}\right) - k\frac{\pi}{\alpha}\right]. \tag{5.23}$$

This can be done in direct analogy to the computation of the functions $e_\alpha(t)$, the Laplace transform of $J^k e_\alpha(t)$ being given by (5.7). For the impulse-response solution we note that the effect of the differential operator D^1 is the same as that of the virtual operator J^{-1}.

In conclusion, we can resume the solutions for the fractional relaxation and oscillation equations as follows:

(a) $0 < \alpha < 1$,

$$u(t) = c_0\, u_0(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, \mathrm{d}\tau, \tag{5.24a}$$

where $u_0(t) = f_\alpha(t)$, $u_\delta(t) = J^1 f_\alpha(t)$, with $u_0(0^+) = 1$, $u_\delta(0^+) = +\infty$;

(b) $1 < \alpha < 2$,

$$u(t) = c_0\, u_0(t) + c_1\, u_1(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, \mathrm{d}\tau, \tag{5.24b}$$

where $u_0(t) = f_\alpha(t) + g_\alpha(t)$, $u_1(t) = J^1 f_\alpha(t) + J^1 g_\alpha(t)$, $u_\delta(t) = J^{-1} f_\alpha(t) + J^{-1} g_\alpha(t)$, with $u_0(0^+) = 1$, $u_0'(0^+) = 0$, $u_1(0^+) = 0$, $u_1'(0^+) = 1$, $u_\delta(0^+) = 0$, $u_\delta'(0^+) = +\infty$.

We now desire to point out that in both cases (a) and (b) (in which α is just not integer), i.e., for *fractional relaxation* and *fractional oscillation*, all the fundamental and impulse-response solutions exhibit an *algebraic decay* as $t \to \infty$, as discussed below. This algebraic decay is the most important effect of the noninteger derivative in our equations, which dramatically differs from the *exponential decay* present in the standard relaxation and damped-oscillation phenomena.

Let us start with the asymptotic behaviour of $u_0(t)$. To this purpose we first derive an asymptotic series for the function $f_\alpha(t)$, valid for $t \to \infty$. Using the identity

$$\frac{1}{s^\alpha + 1} = 1 - s^\alpha + s^{2\alpha} - s^{3\alpha} + \cdots + (-1)^{N-1} s^{(N-1)\alpha} + (-1)^N \frac{s^{N\alpha}}{s^\alpha + 1}$$

in formula (5.15) and the Hankel representation of the reciprocal Gamma function, we (formally) obtain the asymptotic expansion (for α noninteger)

$$f_\alpha(t) = \sum_{n=1}^{N} (-1)^{n-1} \frac{t^{-n\alpha}}{\Gamma(1-n\alpha)} + O(t^{-(N+1)\alpha}) \quad \text{as } t \to \infty. \tag{5.25}$$

The validity of this asymptotic expansion can be established rigorously using the (generalized) Watson lemma (see [4]). We also can start from the spectral representation (5.19) and (5.20) and expand the spectral function for small r. Then the (standard) Watson lemma yields (5.25). We note that this asymptotic expansion coincides with that for $u_0(t) = e_\alpha(t)$, having assumed $0 < \alpha < 2$ ($\alpha \neq 1$). In fact the contribution of $g_\alpha(t)$ is identically zero if $0 < \alpha < 1$ and exponentially small as $t \to \infty$ if $1 < \alpha < 2$.

The asymptotic expansions of the solutions $u_1(t)$ and $u_\delta(t)$ are obtained from (5.25) integrating or differentiating term by term with respect to t. In particular, taking the leading term in (5.25), we obtain the asymptotic representations

$$u_0(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad u_1(t) \sim \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \quad u_\delta(t) \sim -\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}, \quad t \to \infty. \tag{5.26}$$

In [19] we have carried out a detailed analytical and numerical analysis about the zeros of the fundamental solutions in case (b) of fractional oscillations, providing illuminating plots and tables. In this respect we have improved some results by Wiman [52] about the asymptotic position of the zeros.

6. Conclusions

We have reviewed some simple and basic evolution equations of fractional order to outline the key role of the Mittag-Leffler-type functions. In the examples treated here these functions reduce to pure exponentials (of real or complex argument) when the order reduces to a positive integer. But, when the order is not an integer, the Mittag-Leffler-type functions exhibit a *power-law* asymptotic behaviour which shows how the described phenomena can be related to certain *scaling laws*. Scaling concepts are nowadays met in different disciplines, including physical, biological and economical sciences, so we expect that the Mittag-Leffler-type functions will increase their importance along with the Fractional Calculus which is their natural mathematical framework.

Here we have restricted our attention to processes depending only on time. If space coordinates are present, the governing equations contain partial derivatives which can be of fractional order as well. Particular importance is generally attributed to diffusion equations where fractional derivatives in time or in space are present. In some papers of ours (see, e.g., [17,21–23,32–35]), we have treated time-fractional diffusion equations where the Mittag-Leffler-type functions are still found but as Fourier or Laplace transforms of the fundamental solutions, which are now expressed in terms of Wright-type functions.

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