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A Noncommutative Central Limit Theorem for CCR-Algebras

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A noncommutative generalization of the central limit theorem for even completely positive mappings between two CCR-algebras is proved. Quasi-free completely positive mappings are found to be the generalizations of the gaussian distributed random variables.

I. INTRODUCTION

In probability theory the central limit theorem describes the behaviour of suitably normalised large sums of independent identical stochastic variables. In order to generalize this theorem we translate it in the language of C^*-algebras. Let $(X_k)_{k\in\mathbb{N}_0}$ be a sequence of independent real-valued stochastic variables which are copies of the same stochastic variable X. Suppose X has zero mean and finite variance σ. Denote by F_X the distribution-function of X. Consider the commutative C^*-algebra $\mathscr{A}$ of almost periodic functions on $\mathbb{R}$. We can associate to X a state ω_X of $\mathscr{A}$ defined by

$$\omega_X\colon \mathscr{A} \to \mathbb{C}$$

$$f \mapsto \int_{\mathbb{R}} f(x)\, dF_X(x).$$

Further let $\mathscr{A}^n = \otimes^n \mathscr{A}$ ($= C^*$-algebra of almost periodic functions on $\mathbb{R}^n$) and $\omega_X^n = \otimes^n \omega_X$. Define an embedding Φ^n of $\mathscr{A}$ into $\mathscr{A}^n$ by

$$\Phi^n\left(\sum_{k=1}^{N} \lambda_k \exp it_k x\right) = \sum_{k=1}^{N} \lambda_k \exp i\frac{t_k}{\sqrt{n}}(x_1 + \cdots + x_n).$$

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Then $\omega_X^n \circ \Phi^n = \omega_{S_n}$, where $S_n = 1/\sqrt{n} \sum_{k=1}^n X_k$ and hence the central limit theorem becomes

$$w^* - \lim_{n \to \infty} \omega_X^n \circ \Phi^n = \omega_G,$$

where G is the gaussian stochastic variable with distribution function $F_G(x) = 1/(\sqrt{2\pi\sigma}) \exp - x^2/2\sigma^2$.

Generalizations of this theorem have been made to the case where the algebra $\mathscr{A}$ is noncommutative. In [1, and 2] Hudson *et al.* proved a central limit theorem for even states of the CAR-algebra (i.e., the C^*-algebra of cannonical anticommutation relations). In [3] this result has further been generalized to the case of even completely positive mappings between two CAR-algebras (here also the range algebra is noncommutative).

In this paper we prove another noncommutative generalization of the central limit theorem for the case of even completely positive mappings between two CCR-algebras (i.e., C^*-algebras of cannonical commutation relations) built on nondegenerate symplectic spaces. We find a family of projections (labelled by states) from the set of even completely positive mappings onto the set of quasifree completely positive mappings. So the latter can be considered as the noncommutative generalization of the gaussian distributions.

The central limit construction goes along the same lines as for the CAR-case treated in [3]. However, completely different problems arise. Due to the nonseparability of the topology on the CCR-algebra, one needs natural regularity conditions, some of which play the analogous role as the condition of finite variance in the commutative case. Moreover the construction leads to a CCR-algebra built on a degenerate symplectic space which has nontrivial two-sided *-ideals, and so faithfulness of one of its representations, needed in the construction, is no longer automatically guaranteed.

In the special case of states we can also treat the situation of infinite variance. In fact we have here the following complete result: let ω be an even regular state of the CCR-algebra $\overline{\Delta(H, \sigma)}$ with GNS-triplet $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ and let $B_\omega(\phi)$ be the generator of the unitary group $\lambda \in \mathbb{R} \mapsto \pi_\omega(w(\lambda\phi))$ for $\phi \in H$, then the central limit of ω is a quasi-free state or the unique central state depending on whether Ω_ω belongs or does not belong to the domain of $B_\omega(\phi)$.

The central limit theorem derived in this paper might be applied to problems of mathematical physics. It provides a tool to approximate the dynamics of a system by its quasi-free projection, which could be used to shed a new light on the dynamical Hartree–Fock theory.

II. Preliminaries and Notation

1. *Completely Positive Mappings*

A linear mapping T from a C^*-algebra $\mathscr{A}$ in a C^*-algebra $\mathscr{B}$ is called completely positive if

$$\sum_{i,j=1}^{n} y_i^* \, T(x_i^* x_j) \, y_j \geqslant 0$$

for all choices of $x_i \in \mathscr{A}$, $y_i \in \mathscr{B}$, and $n \in \mathbb{N}_0$. An equivalent condition is that $T \otimes 1_n: \mathscr{A} \otimes M_n \to \mathscr{B} \otimes M_n$ (where M_n is the algebra of complex $n \times n$ matrices) is positive for any $n \in \mathbb{N}_0$.

In the case where $\mathscr{B} = \mathscr{B}(\mathscr{H})$(= bounded operators on a hilbert space $\mathscr{H}$) T admits a Stinespring decomposition $(\mathscr{H}_T, \sigma_T, V_T)$, i.e. there exists a hilbertspace $\mathscr{H}_T$, a representation σ_T of $\mathscr{A}$ on $\mathscr{H}_T$ and a bounded linear mapping $V_T: \mathscr{H} \to \mathscr{H}_T$ such that $\forall x \in \mathscr{A}: T(x) = V_T^* \sigma_T(x) V_T$ and $[\sigma_T(\mathscr{A}) V_T \mathscr{H}] = \mathscr{H}_T$; the triplet $(\mathscr{H}_T, \sigma_T, V_T)$ is unique up to unitary equivalence. In the sequel we will deal with CCR-algebras, which are unital. Therefore we will restrict ourselves to unity preserving completely positive (upcp) mappings T (i.e., T maps the unit of $\mathscr{A}$ in the unit of $\mathscr{B}$). In the case of a upcp mapping $T: \mathscr{A} \to \mathscr{B}(\mathscr{H})$ V_T is an isometry.

2. *CCR-Algebras*

Let (H, σ) be a symplectic space, i.e., H is a real vector space and σ is an application from $H \times H$ into $\mathbb{R}$ which is bilinear and antisymmetric. If there exists a $\phi \in H$ with $\phi \neq 0$ such that for all $\phi' \in H: \sigma(\phi, \phi') = 0$, then σ will be called degenerate; if on the other hand $\sigma(\phi, \phi') = 0$ for all $\phi' \in H$ implies $\phi = 0$, then σ is nondegenerate.

For every symplectic space (H, σ) there exists a unique (up to an isomorphism) involutive unital algebra $\Delta(H, \sigma)$ determined by an injective mapping $W: H \to \Delta(H, \sigma)$ satisfying

$$W(\phi_1)\, W(\phi_2) = W(\phi_1 + \phi_2)\, e^{-i\sigma(\phi_1, \phi_2)}, \qquad \phi_1, \phi_2 \in H,$$

$$W(\phi)^* = W(-\phi), \qquad \phi \in H,$$

$$\Delta(H, \sigma) \text{ is generated by } \{W(\phi) | \phi \in H\}.$$

If σ is nondegenerate, there exists a unique C^*-norm on $\Delta(H, \sigma)$; the completion $\overline{\Delta(H, \sigma)}$ with respect to this norm is a unital simple C^*-algebra (called the CCR-algebra over (H, σ)). If on the contrary σ is degenerate, there is no unique C^*-norm on $\Delta(H, \sigma)$. However there exists a C^*-norm $\|\cdot\|$ (called the minimal regular norm) such that for every other C^*-norm $\|\cdot\|_0$ on $\Delta(H, \sigma)$ one has: $\forall x \in \Delta(H, \sigma): \|x\|_0 \leqslant \|x\|$. The completion of $\Delta(H, \sigma)$ with respect to the minimal regular norm is denoted by $\overline{\Delta(H, \sigma)}$. Furthermore there exists a one to one correspondence between the set of closed two-sided $*$-ideals I of $\overline{\Delta(H,\sigma)}$ with $I \cap \Delta(H,\sigma) = \{0\}$ and the sets of C^*-norms $\|\cdot\|_I$ on $\Delta(H,\sigma)$ such that

$$\overline{\Delta(H,\sigma)}^I = \overline{\Delta(H,\sigma)}/I,$$

where $\overline{\Delta(H,\sigma)}^I$ denotes the completion of $\Delta(H,\sigma)$ with respect to the norm $\|\cdot\|_I$ corresponding to the ideal I.

The center of $\overline{\Delta(H,\sigma)}$ is the C^*-subalgebra $\overline{\Delta(H_0)}$ of $\overline{\Delta(H,\sigma)}$ generated by $\{W(\phi)|\phi \in H_0\}$, where $H_0 = \{\phi \in H | \forall \phi \in H: \sigma(\phi,\psi) = 0\}$. A closed two-sided $*$- ideal J of $\overline{\Delta(H,\sigma)}$ is trivial if and only if $J \cap \overline{\Delta(H_0)}$ is trivial. The CCR-algebra with a nondegenerate symplectic form is studied in [4], whereas the degenerate case has been treated in [5].

A representation π of $\overline{\Delta(H,\sigma)}$ on a hilbert space $\mathscr{H}$ is called regular if for all $\phi \in H$ the unitary group $\lambda \in \mathbb{R} \mapsto \pi(W(\lambda\phi))$ is weakly or (equivalently) strongly continuous. By Stone's theorem we get a selfadjoint (unbounded) operator $B(\phi)$ on $\mathscr{H}$ generating the group, i.e., $\pi(W(\lambda\phi)) = e^{i\lambda B(\phi)}$.

Every positive functional ω on $\Delta(H,\sigma)$ with $\omega(\mathbb{1}) = 1$ extends to a state $\bar{\omega}$ of $\overline{\Delta(H,\sigma)}$. If no confusion can occur, we will use also the notation ω to denote $\bar{\omega}$. A state is called regular if its GNS representation is regular. The parity automorphism τ of $\Delta(H,\sigma)$ defined by $\tau(W(\phi)) = W(-\phi)$ can be extended to an automorphism of $\overline{\Delta(H,\sigma)}$ also denote by τ. A state ω of $\overline{\Delta(H,\sigma)}$ is said to be even if $\omega \circ \tau = \omega$.

A state ω is called an even quasi-free state if there exists a bilinear symmetric form $s: H \times H \to \mathbb{R}$ such that $|\sigma(\phi_1,\phi_2)|^2 \leqslant s(\phi_1,\phi_1)\, s(\phi_2,\phi_2)$ and $\omega(W(\phi)) = \exp -\frac{1}{2}s(\phi,\phi)$. For a general study of quasi-free states of CCR-algebras (with a nondegenerate σ) see [6].

Let T be a completely positive mapping from $\overline{\Delta(H,\sigma)}$ into $\overline{\Delta(H',\sigma')}$. Then T is called even if $T \circ \tau = \tau' \circ T$, where $\tau(\tau')$ is the parity automorphism of $\overline{\Delta(H,\sigma)}$ $(\overline{\Delta(H',\sigma')})$. By an immediate generalization of the arguments given in [7] we see that the mapping $T: \Delta(H,\sigma) \to \Delta(H',\sigma'): W(\phi) \to W(A\phi)\,\rho_A(W(\phi))$ with

A a linear operator from H into H',

ρ_A a state of $\overline{\Delta(H,\sigma_A)}$, where $\sigma_A(\cdot,\cdot) = \sigma'(\cdot,\cdot) - \sigma(A\cdot, A\cdot)$

extends to a upcp mapping $\bar{T}$ from $\overline{\Delta(H,\sigma)}$ into $\overline{\Delta(H',\sigma')}$. Following [7] we call $\bar{T}$ an even quasi-free completely positive (eqfcp) mapping if ρ_A is even quasifree state of $\overline{\Delta(H,\sigma_A)}$. If no confusion is possible, the notation T will also be used for $\bar{T}$.

III. A Central Limit Theorem

In this section we want to show how eqfcp mappings arise as the limit of an increasing sequence of large products of an even unity preserving completely positive (eufcp) mapping.

Let T be an eufcp mapping from a CCR-algebra $\bar{\Delta}=\overline{\Delta(H,\sigma)}$ into another CCR-algebra $\bar{\Delta}'=\overline{\Delta(H',\sigma')}$. We assume that σ and σ' are non-degenerate. For any $n \in \mathbb{N}_0$ we consider $\bar{\Delta}^n=\bar{\Delta}(\oplus_{k=1}^n (H,\sigma))$ and $\bar{\Delta}'^n=\bar{\Delta}(\oplus_{k=1}^n (H',\sigma'))$. It is known [4, 3.4.1] that $f_n: \bar{\Delta}^n \to \otimes_{k=1}^n \bar{\Delta}: W(\phi_1 \oplus \cdots \oplus \phi_n) \mapsto W(\phi_1) \otimes \cdots \otimes W(\phi_n)$ and $f'_n: \bar{\Delta}'^n \to \otimes_{k=1}^n \bar{\Delta}'$: $W(\psi_1 \oplus \cdots \oplus \psi_n) \mapsto W(\psi_1) \otimes \cdots \otimes W(\psi_n)$ define isomorphisms between $\bar{\Delta}^n$ and $\otimes_{k=1}^n \bar{\Delta}$ and $\bar{\Delta}'^n$ and $\otimes_{k=1}^n \bar{\Delta}'$, where the C^*-cross norm, used to complete the algebraic tensorproducts, is unique since $\bar{\Delta}$ and $\bar{\Delta}'$ are nuclear [8, Theorem 10.10]. By [9, Proposition IV.4.23] we can now define an eupcp mapping $T^n = f_n^{-1} \circ \otimes_{k=1}^n T \circ f_n$ from $\bar{\Delta}^n$ into $\bar{\Delta}'^n$.

Next we need to inject $\bar{\Delta}$ into $\bar{\Delta}^n$ by the homomorphism Φ^n defined by $W(\phi) \mapsto W((1/\sqrt{n}) \oplus_{k=1}^n \phi)$, $\phi \in H$. Composing Φ^n with T^n we arrive at an eupcp mapping from $\bar{\Delta}$ into $\bar{\Delta}'^n$. A central limit theorem should tell something about the limit $T^n \circ \Phi^n$ as $n \to \infty$.

In the case of states $\bar{\Delta}' = \mathbb{C}$ $(H' = \{0\})$ and so $\bar{\Delta}'^n = \mathbb{C}$. In the general case, however, the range space of $T^n \circ \Phi^n$ becomes arbitrarely large and in order to make some statement about $\lim_{n \to \infty} T^n \circ \Phi^n$ one is forced to "cut off" the range space in some way. Therefore we need an additional element in the construction.

Take an even state ω of $\bar{\Delta}'$ with GNS triplet $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$. The n-fold product state $\omega^n (= \otimes_{k=1}^n \omega \circ f'_n)$ of $\bar{\Delta}'^n$ can be realised as a vector state in the following representation:

$$\mathscr{H}_\omega^n = \bigotimes_{k=1}^n \mathscr{H}_\omega, \qquad \pi_\omega^n = \bigotimes_{k=1}^n \pi_\omega \circ f'_n, \qquad \Omega_\omega^n = \bigotimes_{k=1}^n \Omega_\omega,$$

then $\omega^n(x) = \langle \Omega_\omega^n | \pi_\omega^n(x)\, \Omega_\omega^n \rangle$, $x \in \bar{\Delta}'^n$.

As $\bar{\Delta}'^n$ is simple, π_ω^n is faithful and so we can without loss of generality consider the completely positive mapping $\pi_\omega^n \circ T^n \circ \Phi^n$ from $\bar{\Delta}$ into $\mathscr{B}(\mathscr{H}_\omega^n)$. By cutting off $T^n \circ \Phi^n$ we then mean that we restrict $\pi_\omega^n \circ T^n \circ \Phi^n(x)$, $x \in \bar{\Delta}^n$ to the invariant subspace $[\{\pi_\omega^n \circ T^n \circ \Phi^n(\bar{\Delta}),\ \pi_\omega^n \circ \Phi'^n(\bar{\Delta}')\}''\, \Omega_\omega^n]$ of $\mathscr{H}_\omega^n$, where Φ'^n is the injection $W(\psi) \mapsto W((1/\sqrt{n}) \otimes_{k=1}^n \psi)$, $\psi \in H'$ from $\bar{\Delta}'$ into $\bar{\Delta}'^n$.

So we will have to calculate (among others) limits of the type $\lim_{n \to \infty}$ $\langle \Omega_\omega^n | \pi_\omega^n(\Phi'^n(W(\psi))\, T^n \circ \Phi^n(W(\phi)))\, \Omega_\omega^n \rangle$ and since $\langle \Omega_\omega^n |\, \pi_\omega^n(\Phi'^n(W(\psi))\, T^n \circ \Phi^n(W(\phi))\, \Omega_\omega^n \rangle = \langle \Omega_\omega |\, \pi_\omega(W((\psi/\sqrt{n})))\, V_T^* \sigma_T(W((\psi/\sqrt{n})))\, V_T \Omega_\omega \rangle^n$, where $(\mathscr{H}_T, \sigma_T, V_T)$ is the Stinespring triplet for $\pi_\omega \circ T$, it is natural to require the following regularity conditions on ω and T:

(R.1) The representation π_ω is regular (which is equivalent with the continuity of the application $\lambda \in \mathbb{R} \mapsto \omega(W(\lambda\psi))$ for $\psi \in H'$)

(R.2) The representations σ_T is regular (which is equivalent with the continuity of the application $\lambda \in \mathbb{R} \mapsto \omega(xT(W(\lambda\phi))\,y)$ for x, $y \in \bar{\Delta}'$ and $\phi \in H$).

(R.3) $\Omega_\omega \in \mathscr{D}(B_\omega(\psi))$ for $\psi \in H'$, where $\mathscr{D}(B_\omega(\psi))$ is the domain of the generator $B_\omega(\psi)$ of $\lambda \mapsto \pi_\omega(W(\lambda\psi))$.

(R.4) $V_T\Omega_\omega \in \mathscr{D}(B_T(\phi))$ for $\phi \in H$, where $\mathscr{D}(B_T(\phi))$ is the domain of the generator $B_T(\phi)$ of $\lambda \mapsto \sigma_T(W(\lambda\phi))$.

Now we are able to construct the objects we need to give a precise meaning to the "cutting off" procedure and the limit sketched above.

Let $H_1' = H' \times H$ and equip it with the usual real vector space structure,

$$(\psi_1, \phi_1) + (\psi_2, \phi_2) = (\psi_1 + \psi_2, \phi_1 + \phi_2),$$
$$\lambda(\psi_1, \phi_1) = (\lambda\psi_1, \lambda\phi_1), \qquad \phi_1, \phi_2 \in H, \psi_1, \psi_2 \in H', \lambda \in \mathbb{R}.$$

Define on H_1' a bilinear form $S_T^\omega: H_1' \times H_1' \to \mathbb{C}$ by

$$S_T^\omega((\psi_1, \phi_1), (\psi_2, \phi_2)) = \langle (B_\omega(\psi_1) + V_T^* B_T(\phi_1)\, V_T)\, \Omega_\omega | (B_\omega(\psi_2) + V_T^* B_T(\phi_2)\, V_T)\, \Omega_\omega \rangle.$$

Note that

$$S_T^\omega((\psi_2, \phi_2), (\psi_1, \phi_1)) = \overline{S_T^\omega((\psi_1, \phi_1), (\psi_2, \phi_2))},$$
$$S_T^\omega((\psi, \phi), (\psi, \phi)) \geqslant 0.$$

The kernel H_0 of S_T^ω is by positivity a linear subspace of H_1' so we can pass to the quotient space $H_1 = H_1'/H_0$. We also use the notation S_T^ω for the quotient bilinear form on H_1. The canonical surjection from H_1' onto H_1 will be denoted by Λ. Finally let σ_T^ω be the symplectic form on H_1 defined by

$$\sigma_T^\omega(\Lambda(\psi_1, \phi_1), \Lambda(\psi_2, \phi_2)) = \operatorname{Im} S_T^\omega(\Lambda(\psi_1, \phi_1), \Lambda(\psi_2, \phi_2)).$$

Remark that $|\sigma_T^\omega(\Lambda(\psi_1, \phi_1), \Lambda(\psi_2, \phi_2))|^2 \leqslant S_T^\omega(\Lambda(\psi_1, \phi_1), \Lambda(\psi_1, \phi_1))\, S_T^\omega(\Lambda(\psi_2, \phi_2), \Lambda(\psi_2, \phi_2))$. We end up with a symplectic space (H_1, σ_T^ω) which can in general be degenerate.

LEMMA III.1. *Let A_j (resp. B_j) ($j = 1,..., k$) be self-adjoint operators on a hilbert space $\mathscr{H}$ (resp. $\mathscr{H}'$) with domains $\mathscr{D}(A_j)$ (resp. $\mathscr{D}(B_j)$). Let Ω be a normalized vector of $\mathscr{H}$ and $V: \mathscr{H} \to \mathscr{H}'$ an isometry such that for all $j = 1,..., k$,*

$$\Omega \in \mathscr{D}(A_j), \qquad V\Omega \in \mathscr{D}(B_j)$$

and

$$\langle \Omega | A_j \Omega \rangle = \langle V\Omega | B_j V\Omega \rangle = 0.$$

Then $\lim_{n\to\infty} \langle \Omega | e^{iA_1/\sqrt{n}} V^* e^{iB_1/\sqrt{n}} V \cdots e^{iA_k/\sqrt{n}} V^* e^{iB_k/\sqrt{n}} V\Omega \rangle^n$ *exists and equals*

$$\exp -\tfrac{1}{2}\Bigg(\sum_{j=1}^{k} \|A_j \Omega\|^2 + \|B_j V\Omega\|^2 + 2\langle A_j \Omega | V^* B_j V\Omega \rangle + 2 \sum_{1 \leqslant i < j \leqslant k} \langle (A_i + V^* B_i V)\, \Omega | (A_j + V^* B_j V)\, \Omega \rangle \Bigg).$$

Proof. For shorthand notation we write

$$g_k(\lambda) = \langle \Omega | e^{i\lambda A_1} V^* e^{i\lambda B_1} V \cdots e^{i\lambda A_k} V^* e^{i\lambda B_k} V\Omega \rangle.$$

Clearly $\lim_{n\to\infty} g_k(1/\sqrt{n}) = 1$. It is then a matter of elementary analysis to check that

$$\exists N \in \mathbb{N}, \forall n > N: \qquad | g_k(1/\sqrt{n}) - 1 - \log g_k(1/\sqrt{n})| \leqslant | g_k(1/\sqrt{n}) - 1|^2$$

(where the branch cut for the logarithmic function is chosen along the negative real half axis). Because of $g_k(1/\sqrt{n})^n = \exp n \log g_k(1/\sqrt{n})$ it will therefore be sufficient to examine the limit $\lim_{\lambda\to 0} (1/\lambda^2)(g_k(\lambda) - 1)$.

One can easily see that in any unital associative algebra $\mathscr{A}$, $\forall k \in N_0$, $\forall x_j \in \mathscr{A}$ $(j = 1,..., k)$,

$$(x_1 x_2 \cdots x_k - 1) = \sum_{l=1}^{k} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_l \leqslant n} \times (x_{j_1} - 1)(x_{j_2} - 1) \cdots (x_{j_l} - 1).$$

By applying this for $x_j = e^{i\lambda A_j} V^* e^{i\lambda B_j} V$ $(j = 1,..., k)$ the proof of the lemma reduces to showing that:

(i)
$$\lim_{\lambda\to 0} \frac{1}{\lambda^2} \langle \Omega | (e^{i\lambda A_j} V^* e^{i\lambda B_j} V - 1)\, \Omega \rangle = -\tfrac{1}{2}(\|A_j \Omega\|^2 + \|B_j V\Omega\|^2 + 2\langle A_j \Omega | V^* B_j V\Omega \rangle),$$

for $j = 1,..., k$.

(ii)
$$\lim_{\lambda\to 0} \frac{1}{\lambda^2} \langle \Omega | (e^{i\lambda A_{j_1}} V^* e^{i\lambda B_{j_1}} V - 1) \times (e^{i\lambda A_{j_2}} V^* e^{i\lambda B_{j_2}} V - 1)\, \Omega \rangle = -\langle (A_{j_1} + V^* B_{j_1} V)\, \Omega | (A_{j_2} + V^* B_{j_2} V)\, \Omega \rangle,$$

for $j_1, j_2 = 1,..., k$

(iii)
$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\langle\Omega|(e^{i\lambda A_{j_1}}V^*e^{i\lambda B_{j_1}}V-1)\cdots(e^{i\lambda A_{j_l}}V^*e^{i\lambda B_{j_l}}V-1)\,\Omega\rangle = 0 \qquad \text{when} \quad l>2,$$

for $j_1,\ldots,j_l=1,\ldots,k$.

From the assumptions of the lemma we have that the function $\lambda\to f(\lambda)=\langle\Omega|(e^{i\lambda A_j}V^*e^{i\lambda B_j}V-1)\,\Omega\rangle$ is twice differentiable at $\lambda=0$ and $f(0)=f'(0)=0$. So (i) follows from a twofold application of de l'Hopital's rule.

Because for all $j=1,\ldots,k$ the functions $\lambda\to e^{i\lambda A_j}\Omega$ and $\lambda\to V^*e^{i\lambda B_j}V\Omega$ are normdifferentiable and $\|e^{i\lambda A_j}\|=1$, it follows that $\lambda\to e^{i\lambda A_j}V^*e^{i\lambda B_j}V\Omega$ is normdifferentiable, namely,

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}(e^{i\lambda A_j}V^*e^{i\lambda B_j}V-1)\,\Omega=i(A_j+V^*B_jV)\,\Omega.$$

Hence, if $\lambda\to x_\lambda$ is a uniformly bounded strongly continuous one parameter family of operators on $\mathscr{H}$, one has for all $j_1,\ldots,j_l=1,\ldots,k$,

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\langle\Omega|(e^{i\lambda A_{j_1}}V^*e^{i\lambda B_{j_1}}V-1)\,x_\lambda(e^{i\lambda A_{j_l}}V^*e^{i\lambda B_{j_l}}V-1)\,\Omega\rangle$$
$$=-\langle(A_{j_1}+V^*B_{j_1}V)\,\Omega|x_0(A_{j_l}+V^*B_{j_l}V)\,\Omega\rangle.$$

By taking $x_\lambda=1$ (resp. $x_\lambda=(e^{i\lambda A_{j_2}}V^*e^{i\lambda B_{j_2}}V-1)\cdots(e^{i\lambda A_{j_{l-1}}}V^*e^{i\lambda B_{j_{l-1}}}V-1)$) one finds (ii) (resp. (iii)). ∎

COROLLARY III.2. *Let* $T:\bar{\mathcal{A}}\to\bar{\mathcal{A}}'$ *be an* eupcp *mapping and* ω *an even state of* $\bar{\mathcal{A}}'$ *satisfying the regularity conditions* (R.1) → (R.4). *Then using the same notation as above,* $\forall k\in\mathbb{N}_0$, $\forall\phi_i\in H$, $\forall\psi_i\in H'$ $(i=1,\ldots,k)$,

$$\lim_{n\to\infty}\omega^n(\Phi'^n(W(\psi_1))\,T^n\circ\Phi^n(W(\phi_1))\cdots\Phi'^n(W(\psi_k))\,T^n\circ\Phi^n(W(\phi_k)))$$

exists and equals $\exp -\frac{1}{2}Q_k((\psi_1,\phi_1),\ldots,(\psi_k,\phi_k))$, *where* Q_k *is given by*

$$Q_1((\psi,\phi))=\langle B_\omega(\psi)\,\Omega_\omega|B_\omega(\psi)\,\Omega_\omega\rangle+\langle B_T(\phi)\,V_T\Omega_\omega|B_T(\phi)\,V_T\Omega_\omega\rangle$$
$$+\,2\langle B_\omega(\psi)\,\Omega_\omega|\,V_T^*\,B_T(\phi)\,V_T\Omega_\omega\rangle$$

$$Q_k((\psi_1,\phi_1),\ldots,(\psi_k,\phi_k))$$
$$=\sum_{i=1}^{k}Q_1((\psi_i,\phi_i))+2\sum_{1\leqslant i<j\leqslant k}S_T^\omega((\psi_i,\phi_i),(\psi_j,\phi_j)).$$

Proof. The result follows from the definition of ϕ^n, Φ'^n, T^n, and ω^n and by taking in Lemma III.1, $A_j = B_\omega(\phi_j)$, $B_j = B_T(\psi_j)$ $(j = 1,..., k)$, $\Omega = \Omega_\omega$, and $V = V_T$. ▮

Before proceeding we need

LEMMA III.3. *Let (H, σ) be a (possibly degenerate) symplectic space and let $s: H \times H \to C$ be a bilinear form on H satisfying*

$$|\sigma(\phi_1, \phi_2)|^2 \leqslant s(\phi_1, \phi_1)\, s(\phi_2, \phi_2) \qquad \textit{for all} \quad \phi_1, \phi_2 \in H,$$

$$s(\phi, \phi) \geqslant 0 \quad \textit{for all } \phi \in H \quad \textit{and} \quad s(\phi, \phi) = 0 \Rightarrow \phi = 0.$$

Then $\omega_s: \Delta(H, \sigma) \to \mathbb{C}: W(\phi) \mapsto \exp - \frac{1}{2}s(\phi, \phi)$ extends to a state of $\overline{\Delta(H, \sigma)}$ with a faithful GNS *representation.*

Proof. It is easily checked that ω_s is positive on $\Delta(H, \sigma)$ and hence extends to a state of $\overline{\Delta(H, \sigma)}$. Since the kernel K of the GNS representation of ω_s is a closed two sided *-ideal of $\overline{\Delta(H, \sigma)}$, we know from [5] that it will be sufficient to show that $K \cap \overline{\Delta(H_0)} = \{0\}$, where $\overline{\Delta(H_0)}$ is the C^*-subalgebra of $\overline{\Delta(H, \sigma)}$ generated by $\{W(\phi) | \phi \in H_0\}$ with $H_0 = \{\phi \in H | \forall \psi \in H: \sigma(\phi, \phi) = 0\}$. This will be proved by showing that the state ω_s^0 (= restriction of ω_s to $\overline{\Delta(H_0)}$) can be extended to a KMS-state of a larger algebra.

Let $\mathscr{H} = H_0 \oplus H_0$ and $S = s \oplus s$ a real scalar product on $\mathscr{H}$. The mapping $J: \mathscr{H} \to \mathscr{H}: (\phi, \psi) \mapsto (-\psi, \phi)$ defines a complex structre on $(\mathscr{H}, S)$. Further let σ be the nondegenerate symplectic form on $\mathscr{H}$ given by

$$\sigma(\xi, \eta) = \lambda S(J\xi, \eta), \qquad \xi, \eta \in \mathscr{H}$$

with $0 < \lambda < 1$. Clearly there exists a natural embedding of $\overline{\Delta(H_0)}$ into $\overline{\Delta(\mathscr{H}, \sigma)}$ and ω_s^0 extends to a state ω_S of $\overline{\Delta(\mathscr{H}, \sigma)}$ satisfying

$$\omega_S(W(\xi)) = \exp - \tfrac{1}{2}S(\xi, \xi).$$

Consider now the one parameter-group α_t of automorphism of $\overline{\Delta(\mathscr{H}, \sigma)}$ defined by

$$\alpha_t(W(\xi)) = W(e^{tJ}\xi).$$

Then ω_S is a β-KMS state with respect to α_t for $\beta = \log(1 + \lambda)/(1 - \lambda)$. Hence by [10, Theorem 13.3] and since $\overline{\Delta(\mathscr{H}, \sigma)}$ is simple, ω_S is faithful, so ω_s^0 is faithful and this implies $K \cap \overline{\Delta(H_0)} = \{0\}$. ▮

Now we are able to identify the limit we found in Corollary III.2 as an even quasi-free state on the CCR-algebra $\overline{\Delta(H_1, \sigma_T^\omega)}$. More precisely we have

LEMMA III.4. *We assume that ω and T obey the regularity condition* (R.1) → (R.4) *and use the same notation as above. Let $Q_k'((\psi_1, \phi_1),...,$*

$(\psi_k, \phi_k)) = Q_k((\psi_1, \phi_1),..., (\psi_k, \phi_k)) - \sum_{j=1}^{k} Q_1''(\psi_j, \phi_j))$ for $k \in \mathbb{N}_0$, $\phi_j \in H$, $\psi_j \in H'$ $(j = 1,..., k)$, where

$$Q_1''((\psi_j, \phi_j)) = R(\phi_j, \phi_j) + 2i\sigma_T^\omega(\Lambda(\psi_j, 0), \Lambda(0, \phi_j))$$

and R is the bilinear form on H_1 defined by

$$R(\phi, \phi') = \langle B_T(\phi)\, V_T\Omega_\omega \,|\, B_T(\phi')\, V_T\Omega_\omega\rangle - \langle V_T^* B_T(\phi)\, V_T\Omega_\omega \,|\, V_T^* B_T(\phi')\, V_T\Omega_\omega\rangle.$$

Then $Q_k'((\psi_1, \phi_1),..., (\psi_k, \phi_k))$ depends only on the classes $\Lambda(\psi_j, \phi_j)$ $(j = 1,..., k)$ and the mapping

$$W(\Lambda(\psi_1, \phi_1)) \cdots W(\Lambda(\psi_k, \phi_k)) \mapsto \exp - \tfrac{1}{2} Q_k'((\psi_1, \phi_1),..., (\psi_k, \phi_k))$$

defines a quasi-free state ω^∞ of $\overline{\Delta(H_1, \sigma_T^\omega)}$ with a faithful GNS *representation.*

Proof. Since $Q_k'((\psi_1, \phi_1),..., (\psi_k, \phi_k)) = 2 \sum_{1 \leqslant i < j \leqslant k} S_T^\omega((\psi_i, \phi_i), (\psi_j, \phi_j)) + \sum_{j=1}^{k} S_T^\omega((\psi_j, \phi_j), (\psi_j, \phi_j))$, the positivity of S_T^ω and the definition of Λ imply that $Q_k'((\psi_1, \phi_1),..., (\psi_k, \phi_k))$ depends only on $\Lambda(\psi_j, \phi_j)$ $(j = 1,..., k)$. Because $Q_k'((\psi_1, \phi_1),..., (\psi_k, \phi_k)) = Q_{k-1}'((\psi_1, \phi_1),..., (\psi_j + \psi_{j+1}, \phi_j + \phi_{j+1}),..., (\psi_k, \phi_k)) + 2i\sigma_T^\omega(\Lambda(\psi_j, \phi_j), \Lambda(\psi_{j+1}, \phi_{j+1}))$ for $1 \leqslant j \leqslant k-1$, the mapping

$$\omega^\infty\colon W(\Lambda(\psi_1, \phi_1)) \cdots W(\Lambda(\psi_k, \phi_k)) \mapsto \exp - \tfrac{1}{2} Q_k'((\psi_1, \phi_1),..., (\psi_k, \phi_k))$$

is well defined on $\Delta(H_1, \sigma_T^\omega)$ and ω^∞ is determined by

$$\omega^\infty(W(\Lambda(\psi, \phi))) = \exp - \tfrac{1}{2} S_T^\omega(\Lambda(\psi, \phi), \Lambda(\psi, \phi)).$$

By the preceding lemma we know that ω^∞ extends to a quasi-free state of $\overline{\Delta(H_1, \sigma_T^\omega)}$ with a faithful GNS-representation. ∎

THEOREM III.5. *With the same conditions and notations as in Corollary* III.2 *and Lemma* III.4 *there exists a unique* eqfcp *mapping T^∞ from $\bar{\Delta}$ into $\overline{\Delta(H_1, \sigma_T^\omega)}$ such that*

$$\begin{aligned}
\lim_{n\to\infty} \omega^n(&\Phi'^n(W(\psi_1))\, T^n \circ \Phi^n(W(\phi_1)) \cdots \Phi'^n(W(\psi_k)) \\
&\times T^n \circ \Phi^n(W(\phi_k))\, T^n \circ \Phi^n(x) \\
&\times \Phi'^n(W(\xi_1))\, T^n \circ \Phi^n(W(\eta_1)) \cdots \Phi'^n(W(\xi_l))\, T^n \circ \Phi^n(W(\eta_l))) \\
= \exp &\left[-\tfrac{1}{2} \sum_{j=1}^{k} Q_1''(\psi_j, \phi_j) + \sum_{j=1}^{l} Q_1''(\xi_j, \eta_j))\right] \\
&\times \omega^\infty(W(\Lambda(\psi_1, \phi_1)) \cdots W(\Lambda(\psi_k, \phi_k)) \\
&\times T^\infty(x)\, W(\Lambda(\xi_1, \eta_1)) \cdots W(\Lambda(\xi_l, \eta_l))) \qquad (1)
\end{aligned}$$

for all $k, l \in \mathbb{N}$, $\psi_j, \xi_j \in H'$, $\phi_j, \eta_j \in H$ and $x \in \bar{\Delta}$. This mapping T^∞ satisfies

$$T^\infty(W(\phi)) = W(\Lambda(0, \phi))\,\rho(W(\phi)), \quad \text{where } \rho(W(\phi)) = \exp -\tfrac{1}{2}R(\phi, \phi).$$

Proof. Let L be the mapping $H \to H_1 : \phi \mapsto \Lambda(0, \phi)$ and σ_L the symplectic form on H defined by $\sigma_L(\phi_1, \phi_2) = \sigma(\phi_1, \phi_2) - \sigma_T^\omega(L\phi_1, L\phi_2)$. Note that $\sigma_L(\phi_1, \phi_2) = \operatorname{Im} R(\phi_1, \phi_2)$. From the complete positivity of T and $T(1) = 1$ it follows that $R'(x_1, x_2) = \langle x_1 | x_2 \rangle - \langle V_T^* x_1 | V_T^* x_2 \rangle$ is a positive sesquilinear form on $\mathscr{H}_T$. Remark that $R'(x_1, x_2) = R(\phi_1, \phi_2)$ when $x_j = B_T(\phi_j)\, V_T \Omega_\omega (j = 1, 2)$. Applying the Cauchy–Schwartz inequality for R' yields

$$|\sigma_L(\phi_1, \phi_2)|^2 \leqslant |R(\phi_1, \phi_2)|^2 \leqslant R(\phi_1, \phi_1)\, R(\phi_2, \phi_2).$$

Hence $\rho(W(\phi)) = \exp -\frac{1}{2}R(\phi, \phi)$ defines an even quasi-free state of $\overline{\Delta(H, \sigma_L)}$ and the mapping $W(\phi) \mapsto W(\Lambda(0, \phi))\,\rho(W(\phi))$ extends to an eqfcp mapping T from $\bar{\Delta}$ into $\overline{\Delta(H_1, \sigma_T^\omega)}$ [7]. It follows immediately from Corollary III.2 and Lemma III.4 that this mapping satisfies (1) for $x = W(\chi)$ and by linearity and uniform continuity (1) holds for all $x \in \Delta$. Finally uniqueness of T^∞ follows easily from the faithfulness of the GNS representation of ω^∞. ∎

Starting from an eupcp mapping $T: \bar{\Delta} \to \bar{\Delta}'$ we have found an eqfcp mapping $T^\infty: \bar{\Delta} \to \overline{\Delta(H_1, \sigma_T^\omega)}$ as a "central limit." If we want to end up with an eqfcp mapping from $\bar{\Delta}$ into $\bar{\Delta}'$ we still need an eqfcp mapping from $\overline{\Delta(H_1, \sigma_T^\omega)}$ into $\bar{\Delta}'$ in order to compose it with T^∞. It will be shown in Proposition III.6 that, since $\sigma'(\psi_1, \psi_2) = \sigma_T^\omega(\Lambda(\psi_1, 0), \Lambda(\psi_2, 0))$ there exists a natural embedding Γ of $\bar{\Delta}'$ into $\overline{\Delta(H_1, \sigma_T^\omega)}$ given by $W(\psi) \mapsto W(\Lambda(\psi, 0))$. Therefore we look for a conditional expectation from $\overline{\Delta(H_1, \sigma_T^\omega)}$ onto $\Gamma(\bar{\Delta}')$.

We recall the definition of a conditional expectation: let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{B}$ a unital C^*-subalgebra in $\mathscr{A}$, then a mapping $E: \mathscr{A} \to \mathscr{B}$ is called a conditional expectation if it satisfies:

(i) E is projection from $\mathscr{A}$ onto $\mathscr{B}$,

(ii) $E(xy) = xE(y)$; $x \in \mathscr{B}$, $y \in \mathscr{A}$,

(iii) E is completely positive.

Now we will show that the requirement of the existence of a conditional expectation $E: \overline{\Delta(H_1, \sigma_T^\omega)} \to \Gamma(\bar{\Delta}')$ implies a supplementary regularity condition on ω and T. First we prove

PROPOSITION III.6. (i) *Let (K, σ_1) be symplectic spaces such that there is a symplectic embedding $\gamma: (K_1, \sigma_1) \to (K, \sigma)$ (i.e., γ is linear and $\forall \phi, \psi \in K_1$: $\sigma_1(\phi, \psi) = \sigma(\gamma(\phi), \gamma(\psi))$).*

Then for every C^*-norm $\|\cdot\|_0$ on $\Delta(K,\sigma)$, there exists a unique C^*-norm $\|\cdot\|_1$ on $\Delta(K_1,\sigma_1)$ such that the mapping

$$\begin{aligned} \Gamma\colon \Delta(K_1,\sigma_1) &\to \Delta(K,\sigma) \\ W(\psi) &\mapsto W(\gamma(\psi)) \end{aligned} \tag{2}$$

extends to an embedding $\Gamma\colon \overline{\Delta(K_1,\sigma_1)}^{\,1} \to \overline{\Delta(K,\sigma)}^{\,0}$.

(ii) *Let (K_1,σ_1) and (K_2,σ_2) be symplectic spaces and $(K,\sigma) = (K_1 \oplus K_2, \sigma_1 \oplus \sigma_2)$ Denote $\Delta = \Delta(K,\sigma)$, $\Delta_1 = \Delta(K_1,\sigma_1)$, $\Delta_2 = \Delta(K_2,\sigma_2)$. Then for every C^*-norm $\|\cdot\|_0$ on Δ there exists C^*-norms $\|\cdot\|_1$ (resp. $\|\cdot\|_2$) on Δ_1 (resp. Δ_2) and a C^*-cross norm α* on *$\bar{\Delta}_1^1 \otimes \bar{\Delta}_2^2$ such that $\bar{\Delta}^0 \simeq \bar{\Delta}_1^1 \otimes_\alpha \bar{\Delta}_2^2$.*

Proof. (i) Clearly the mapping given by (2) is an injective homomorphism from $\Delta(K_1,\sigma_1)$ into $\Delta(K,\sigma)$. Defining $\|x\|_1 = \|\Gamma(x)\|_0$, $x \in \Delta(K_1,\sigma_1)$ one gets the statement.

(ii) One easily sees that

$$\begin{aligned} \pi\colon \Delta_1 \otimes \Delta_2 &\to \Delta \\ W(\phi) \otimes W(\psi) &\mapsto W(\phi \otimes \psi) \end{aligned}$$

defines an isomorphism. Let $\|x\|_1 = \|\pi(x \otimes 1\|_0, x \in \Delta_1$, $\|y\|_2 = \|\pi(1 \otimes y)\|_2, y \in \Delta_2$ then π can be extended to an injective homomorphism $\bar{\pi}\colon \bar{\Delta}_1^1 \otimes \bar{\Delta}_2^2 \to \bar{\Delta}^0$. Now we define a C^*-norm $\|\ \|_\alpha$ on the algebraic tensor product by

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\alpha = \left\| \bar{\pi} \left(\sum_{i=1}^n x_i \otimes y_i \right) \right\|_0, \qquad x_i \in \bar{\Delta}_1^1, y_i \in \bar{\Delta}_2^2.$$

It is known [10, Chap. IV.4] that every C^*-norm on the algebraic tensor product of two C^*-algebras is a cross-norm (i.e., $\|x \otimes y\|_\alpha = \|x\|_1\, \|y\|_2$). By construction of $\|\ \|_\alpha$, $\bar{\pi}$ can be extended to an injective homomorphism $\bar{\pi}\colon \bar{\Delta}_1^1 \otimes \bar{\Delta}_2^2 \to \bar{\Delta}^0$ which is also surjective since the range of $\bar{\pi}$ is closed and contains a dense subset (namely, $\pi(\Delta_1 \otimes \Delta_2) = \Delta$). ∎

Remark that in the case where σ_1 is nondegenerate $\bar{\Delta}^0 \simeq \bar{\Delta}_1 \otimes_{\min} \bar{\Delta}_2^2 = \bar{\Delta}_1 \otimes_{\max} \bar{\Delta}_2^2$ since there exists only one C^*-norm on Δ_1 and $\bar{\Delta}_1$ is nuclear.

LEMMA III.7. *Let (K,σ) be a nondegenerate symplectic space. If $x \in \overline{\Delta(K,\sigma)} \backslash \{0\}$ and there exists a linear functional $f\colon H \to \mathbb{R}$ such that:*

$$\forall \psi \in K, \qquad x W(\psi) = W(\psi)\, x e^{if(\psi)} \tag{3}$$

then there exists a unique $\phi \in K$ such that

$$f(\psi) = -2\sigma(\phi, \psi)$$

and $x = \lambda W(\phi)$ for some $\lambda \in \mathbb{C}$.

Proof. By taking the adjoint of (3) and using the linearity of f one has

$$\forall \psi \in K, \qquad W(\psi)\, x^* = x^* W(\psi)\, e^{if(\psi)}.$$

Hence $W(\psi)\, x^*x = x^*xW(\psi)$ for all $\psi \in K$. Thus x^*x belongs to the center of $\overline{\Delta(H, \sigma)}$ which is known to be trivial as σ is nondegenerate [5]. So $x^*x = \mu\mathbb{1}$ for some $\mu \in \mathbb{R}_0^+$ ($\mu = \|x\|^2$). Analogously one finds that $xx^* = \mu\mathbb{1}$. Let $y = x/\sqrt{\mu}$, then y is unitary and

$$\forall \psi \in K, \qquad e^{if(\psi)}\, \mathbb{1} = yW(\psi)\, y^* W(-\psi).$$

Now consider the vector space σ_K of linear functionals on K,

$$\sigma_K = \{\sigma_\phi : K \to \mathbb{R} : \psi \mapsto \sigma(\phi, \psi) \mid \phi \in K\}.$$

Because σ is nondegenerate, σ_K separates K and induces a Hausdorff topology on K (called the σ_K-topology) defined by $\psi_\alpha \to^{\sigma_K} \psi$ iff $\forall \phi \in K$, $\sigma_\phi(\psi_\alpha) \to \sigma_\phi(\psi)$. We want to show that f is continuous for this topology on K.

Take a net $(\psi_\alpha)_{\alpha \in I}$ in K such that ψ_α converges to a $\psi \in K$ in the σ_K-topology. Choose a sequence $y_n = \sum_{k=1}^{N_n} \lambda_{n,k} W(\phi_{n,k}) \in \Delta(K, \sigma)$ such that y_n converges to y in the norm topology. Then we can write

$$\begin{aligned}
|e^{if(\psi_\alpha)} - e^{if(\psi)}| &= \| yW(\psi_\alpha)\, y^* W(-\psi_\alpha) - yW(\psi)\, y^* W(-\psi)\| \\
&\leqslant \| yW(\psi_\alpha)\, y^* W(-\psi_\alpha) - y_n W(\psi_\alpha)\, y_n^* W(-\psi_\alpha)\| \\
&\quad + \| y_n W(\psi_\alpha)\, y_n^* W(-\psi_\alpha) - y_n W(\psi)\, y_n^* W(-\psi)\| \\
&\quad + \| y_n W(\psi)\, y_n^* W(-\psi) - yW(\psi)\, y^* W(-\psi)\| \\
&\leqslant 4M \| y - y_n\| + M \left\| \sum_{k=1}^{N_n} \lambda_{n,k} W(\phi_{n,k}) \right. \\
&\quad \left. \times \left(e^{2i\sigma(\psi_\alpha, \phi_{n,k})} - e^{2i\sigma(\psi, \phi_{n,k})}\right) \right\|,
\end{aligned}$$

where $M = \max \| y_n \|$. Clearly this implies $\lim_\alpha \; e^{if(\psi_\alpha)} = e^{if(\psi)}$ and by linearity of f: $\lim_\alpha f(\psi_\alpha) = f(\psi)$.

So f is a σ_K-continuous functional on K and since the topological dual of K in the σ_K-topology is σ_K, we have $f \in \sigma_K$ hence

$$\exists! \phi \in K, \forall \psi \in K, \qquad f(\psi) = -2\sigma(\phi, \psi).$$

As $W(-\phi)\, W(\phi) = W(\psi)\, W(-\phi)\, e^{2i\sigma(\phi, \psi)}$ and $xW(\psi) = W(\psi)\, xe^{-2i\sigma(\phi, \psi)}$ it follows that $W(\psi)\, xW(-\phi) = xW(-\phi)\, W(\psi)$ for all $\psi \in H$. Hence $xW(-\phi)$ belongs to the trivial center of $\overline{\Delta(H, \sigma)}$ and so $xW(-\phi) = \lambda\mathbb{1}$ or $x = \lambda W(\phi)$. ∎

PROPOSITION III.8. *Let* (K, σ) *and* (K_1, σ_1) *be symplectic spaces such that there exists a symplectic embedding* $\gamma: (K_1, \sigma_1) \to (K, \sigma)$. *Suppose* σ_1 *is nondegenerate. Let* $\|\cdot\|_0$ *be a* C^**-norm on* $\Delta(K, \sigma)$ *and denote* $\bar{\Delta}^0 = \overline{\Delta(K, \sigma)}^0$, $\bar{\Delta}_1 = \overline{\Delta(K_1, \sigma_1)}$ *and* Γ *is the embedding of* $\bar{\Delta}_1$ *into* $\bar{\Delta}^0$ (*see Proposition* III.6)

(i) *If there exists a conditional expectation* $E = \bar{\Delta}^0 \to \bar{\Delta}_1$, *then there exists a unique projection operator* $P: K \to \gamma(K_1)$ (*i.e., a linear operator satisfying* $\sigma(\gamma(\psi), P\phi) = \sigma(\gamma(\psi), \phi)$ *for all* $\psi \in K_1$ *and* $\phi \in K$, *a unique state* ω *of* $\bar{\Delta}_2^0$ ($= C^*$*-subalgebra in* $\bar{\Delta}^0$ *generated by* $\{W(\psi) | \psi \in K_2 = (1-P)K\}$) *such that*

$$\forall \phi \in K, \qquad E(W(\phi)) = W(P\phi)\, \omega(W((1-P)\phi)). \tag{4}$$

(ii) *Conversely, if* $P: K \to \gamma(K_1)$ *is a projection operator* (*in the above sense*) *and* ω *is a state of* $\bar{\Delta}_2^0$ *then* (4) *defines a conditional expectation* $E: \bar{\Delta}^0 \to \Gamma(\bar{\Delta}_1)$.

Proof. (i) Let E be a conditional expectation: $\bar{\Delta}^0 \to \Gamma(\bar{\Delta}_1)$. Take $\phi \in K$, then for all $\psi \in K_1$,

$$\begin{aligned} E(W(\phi))\, W(\gamma(\psi)) &= E(W(\phi)\, W(\gamma(\psi))) \\ &= E(W(\gamma(\psi)\, W(\phi))\, e^{-2i\sigma(\phi, \gamma(\psi))} \\ &= W(\gamma(\psi))\, E(W(\phi))\, e^{-2i\sigma(\phi, \gamma(\psi))}. \end{aligned}$$

From Lemma III.7 we know

$$\exists! \phi' \in \gamma(K_1), \forall \psi \in K_1, \qquad \sigma(\phi', \gamma(\psi)) = \sigma(\phi, \gamma(\psi))$$

therefore we define $P\phi = \phi'$. Moreover one has $E(W(\phi)) = W(P\phi)\, \lambda(\phi)$. On the other hand, since $\sigma(P\psi, (1-P)\phi) = 0$ for all $\phi, \psi \in K$, we have

$$\begin{aligned} \forall \phi \in K, \qquad E(W(\phi)) &= E(W(P\phi + (1-P)\phi)) \\ &= E(W(P\phi)\, W((1-P)\phi)) \\ &= W(P\phi)\, E(W(1-P)\phi)) \end{aligned}$$

hence $\lambda(\phi)\, \mathbb{1} = E(W((1-P)\phi))$. Taking any state ω_1 of $\Gamma(\bar{\Delta}_1)$ and defining $\omega = \omega_1 \circ E|_{\bar{\Delta}_2^0}$ one finds (4).

(ii) Conversely let $P: K \to \gamma(K_1)$ be a projection and ω a state of $\bar{\Delta}_2^0$. It follows from Proposition III.6 that there exists a isomorphism $\pi: \bar{\Delta}^0 \to \Gamma(\bar{\Delta}_1) \otimes \bar{\Delta}_2^0$ (where the C^*-cross norm on the tensor product is unique). One easily sees that E defined by (4) satisfies $E = \pi^{-1} \circ (Id \otimes \omega) \circ \pi$, where Id is the identical automorphism of $\Gamma(\bar{\Delta}_1)$. Hence by [10, Corollary 4.25] it follows that E is completely positive. The other conditions for having a conditional expectation are trivially fulfilled. ∎

It follows now from this last proposition that we have to require an additional regularity condition on ω and T in order to project the eqfcp mapping $T^{\infty}: \bar{\Delta} \to \overline{\Delta(H_1, \sigma_1^{\omega})}$, which was found in Theorem III.5, onto an eqfcp mapping $T_{\infty}: \bar{\Delta} \to \bar{\Delta}'$ by composing it with an even quasi-free conditional expectation, namely,

(R.5) For all $\phi \in H$, the functional

$$H' \to \mathbb{R}$$
$$\psi \mapsto \operatorname{Im}\langle B_{\omega}(\psi)\, \Omega_{\omega} | V_T^* B_T(\phi)\, V_T \Omega_{\omega}\rangle$$

is continuous for the $\sigma'_{H'}$-topology on H' (the $\sigma'_{H'}$-topology was defined in the proof of Lemma III.7). Indeed, since $\operatorname{Im} \langle B_{\omega}(\psi)\, \Omega_{\omega} | V_T^* B_T(\phi)\, V_T \Omega_{\omega}\rangle = \sigma_1^{\omega}(A(\psi, 0), A(0, \phi))$, condition (R.5) implies

$$\forall \phi \in H, \exists! \psi' \in H'; \forall \psi \in H', \qquad \sigma_T^{\omega}(A(\psi, 0), A(0, \phi)) = \sigma'(\psi, \psi')$$

(because the dual of H' in the $\sigma'_{H'}$-topology is $\sigma'_{H'}$) and hence yields a unique projection operator. $P: H_1 \to \gamma(H')$ defined by

$$\forall \phi \in H, \forall \psi \in H', \quad \sigma_T^{\omega}(A(\psi, 0), A(0, \phi)) = \sigma_T^{\omega}(A(\psi, 0), PA(0, \phi)).$$

Conversely the existence of such a projection operator trivially implies (R.5).

Assume now (R.5) and consider the conditional expectation

$$E_{\rho_1}: \overline{\Delta(H_2, \sigma_T^{\omega})} \to \Gamma(\bar{\Delta}')$$
$$W(A(\psi, \phi)) \mapsto W(PA(\psi, \phi))\, \rho_1(W((1-P)\, A(0, \phi)),$$

where ρ_1 is an even quasi-free state of $\overline{\Delta(H_1, \sigma_T^{\omega})}$. Defining $T_{\infty} = \Gamma^{-1} \circ E_{\rho_1} \circ T^{\infty}$ we end up with an eqfcp mapping from $\bar{\Delta}$ into $\bar{\Delta}'$ satisfying

$$T_{\infty}(W(\phi)) = W(A\phi)\, \rho(W(\phi))\, \rho_1(W(A(-A\phi, \phi)),$$

where $A: H \to H'$ is defined by $A\phi = \gamma^{-1} PA(0, \phi)$ or equivalently $\sigma'(\psi, A\phi) = \operatorname{Im} \langle B_{\omega}(\psi)\, \Omega_{\omega} | V_T^* B_T(\phi)\, V_T \Omega_{\omega}\rangle$ for all $\psi \in H'$, $\phi \in H$, and ρ was defined in Theorem III.4.

Finally remark that in general the quasi-free projection T_{∞} of T will depend on the choice of ω and ρ_1. If T happens to be quasi-free, then $T = T_{\infty}$ for all choices of ω and ρ_1.

IV. The Case of States

A special class of eupcp mappings are the even states $\rho: \overline{\Delta(H, \sigma)} \to \mathbb{C}$. Since $\mathbb{C} \simeq \overline{\Delta(\{0\}, 0)}$, we can perfectly well project an even state onto an even quasi-free one by applying the construction of Section III. However, we immediately see that, because $\mathbb{C}^n \simeq C$, the "cutting off" procedure is trivial in this case. The ingredients ω and E_{ρ_1} are trivially redundant and so the quasi-free projection of a state will only depend on the state itself. Moreover the regularity conditions concerning ω and E_{ρ_1} (namely, (R.1), (R.3), and (R.5)) are trivially satisfied. Therefore, Theorem III.5, when translated to the case of states, simplifies as follows:

Let (H, σ) be a nondegerate symplectice space and ρ an even state of $\overline{\Delta(H, \sigma)}$ with GNS triplet $(\mathscr{H}, \pi, \Omega)$ Assume:

(i) ρ is regular (i.e., the application $\lambda \mapsto \rho(W(\lambda\phi))$ is continuous for all $\phi \in H$(R.2).

(ii) $\Omega \in \mathscr{D}(B(\phi))$ for $\phi \in H$, where $\mathscr{D}(B(\phi))$ is the domain of the generator $B(\phi)$ of the strongly continuous unitary group $\lambda \mapsto \pi(W(\lambda\phi))$ (R.4).

Then, $\forall x \in \overline{\Delta(H, \sigma)}$, $\lim_{n \to \infty} \rho^n \circ \Phi^n(x)$ exists equals $\rho_\infty(x)$, where ρ_∞ is the even quasi-free state on $\overline{\Delta(H, \sigma)}$ determined by $\rho_\infty(W(\phi)) = \exp -\frac{1}{2} < B(\phi)\,\Omega \,|\, B(\phi)\,\Omega\rangle$.

But we can say more. In fact, the condition of the even parity of the state can be reduced to its weakest possible form and the regularity condition (ii) can completely be dropped. Then we have the following complete result:

Theorem IV.1. *Let*

→ (H, σ) *be a nondegenerator symplectic space,*

→ ρ *a regular state of* $\overline{\Delta(H, \sigma)}$ *with* GNS *triplet* $(\mathscr{H}, \pi, \Omega)$,

→ $B(\phi)$ *the generator of* $\lambda \to \pi(W(\lambda\phi))$.

(α) *If* $\Omega \in \mathscr{D}(B(\phi))$ *and* $\langle \Omega \,|\, B(\phi)\,\Omega\rangle = 0$ *for all* $\phi \in H$ *then* $\forall x \in \overline{\Delta(H, \sigma)}$, $\lim_{n \to \infty} \rho^n \circ \Phi^n(x)$ *exists and equals* $\rho_\infty(x)$, *where* ρ_∞ *is the even quasi-free state determined by*

$$\rho_\infty(W(\phi)) = \exp - \tfrac{1}{2}\langle B(\phi)\,\Omega \,|\, B(\phi)\Omega\rangle.$$

(β) *If* $\Omega \notin \mathscr{D}(B(\phi))$ *for all* $\phi \in H\backslash\{0\}$ *then* $\forall x \in \overline{\Delta(H, \sigma)}$, $\lim_{n \to \infty} \rho^n \circ \Phi^n(x)$ *exists and equals* $\omega_c(x)$, *where* ω_c *is the unique central state of* $\overline{\Delta(H, \sigma)}$.

Proof. By linearity and a continuity argument it is sufficient to prove the existence of the limit, $\lim_{n\to\infty} \rho^n \circ \Phi^n(x)$ for $x = W(\phi)$ with $\phi \in H\backslash\{0\}$. Since $B(\phi)$ is self-adjoint it has a spectral decomposition

$$B(\phi) = \int_{\mathbb{R}} x \, dE_x$$

and

$$\mathscr{D}(B(\phi)) = \{\xi \in \mathscr{H} \mid \int_{\mathbb{R}} x^2 d\langle \xi | E_x \xi \rangle < \infty\}.$$

Let $d\mu(x) = d\langle \Omega | E_x \Omega \rangle$. Denote by $\hat{\mu}$ the Fourier transform of the measure μ

$$\hat{\mu}(\lambda) = \int_{\mathbb{R}} e^{i\lambda x} \, d\mu(x)$$

then

$$\rho^n \circ \Phi^n(W(\phi)) = \rho(W(\phi/\sqrt{n}))^n = (\hat{\mu}(1/\sqrt{n}))^n$$

(α) If $\Omega \in \mathscr{D}(B(\phi))$ and $\langle \Omega | B(\phi) \Omega \rangle = 0$ one has $\int_{\mathbb{R}} x^2 \, d\mu(x) < \infty$ and $\int_{\mathbb{R}} x \, d\mu(x) = 0$. In this case $\hat{\mu}$ is twice differentiable and $(d\hat{\mu}/d\lambda)(0) = 0$. Hence, choosing the branch cut for the logarithmic function in the complex plane along the negative real half axis, we find by a twofold application of de l'Hopital's rule

$$\lim_{\lambda 0} \frac{1}{\lambda^2} \log \hat{\mu}(\lambda) = -\frac{1}{2} \int_{\mathbb{R}} x^2 \, d\mu(x) = -\frac{1}{2} \langle B(\phi) \, \Omega | B(\phi) \, \Omega \rangle$$

which proves case (α).

(β) If $\Omega \notin \mathscr{D}(B(\phi))$ one has $\lim_{R\to\infty} \int_{|x| \leqslant R} x^2 \, d\mu(x) = +\infty$. In this case we have to show that $\lim_{\lambda\to 0} (1/\lambda^2) \log |\hat{\mu}(\lambda)|^2 = -\infty$. By concavity of the logarithmic function one has

$$\begin{aligned}
\frac{1}{\lambda^2} \log |\hat{\mu}(\lambda)|^2 &\leqslant \frac{1}{\lambda^2} (|\hat{\mu}(\lambda)|^2 - 1) \\
&= \frac{1}{\lambda^2} \left(\left(\int_{\mathbb{R}} \cos \lambda x \, d\mu(x) \right)^2 + \left(\int_{\mathbb{R}} \sin \lambda x \, d\mu(x) \right)^2 - 1 \right) \\
&= \frac{1}{\lambda^2} \left(\int_{\mathbb{R}} (\cos \lambda x - 1) \, d\mu(x) \int_{\mathbb{R}} (\cos \lambda x + 1) \, d\mu(x) \right. \\
&\quad \left. + \left(\int_{\mathbb{R}} \sin \lambda x \, d\mu(x) \right)^2 \right).
\end{aligned}$$

Remark that $\int_{\mathbb{R}}(1-\cos\lambda x)\,d\mu(x)\neq 0$ for λ small enough, and by the dominated convergence theorem $\lim_{\lambda\to 0}\int_{\mathbb{R}}(\cos\lambda x+1)\,d\mu(x)=2$. Hence it suffices to prove that

(i) $\lim_{\lambda\to 0}(1/\lambda^2)\int_{\mathbb{R}}(1-\cos\lambda x)\,d\mu(x)=+\infty$,

(ii) $\lim_{\lambda\to 0}(\int_{\mathbb{R}}\sin\lambda x\,d\mu(x))^2/\int_{\mathbb{R}}(1-\cos\lambda x)\,d\mu(x)=0$.

Since $1-\cos y\geqslant(2/\pi^2)y^2$ for $y\in[-\pi,\pi]$ (i) follows easily from

$$\frac{1}{\lambda^2}\int_{\mathbb{R}}(1-\cos\lambda x)\,d\mu(x)\geqslant\frac{1}{\lambda^2}\int_{|x|\leqslant\pi/|\lambda|}(1-\cos\lambda x)\,d\mu(x)$$
$$\geqslant\frac{2}{\pi^2}\int_{|x|\leqslant\pi/|\lambda|}x^2\,d\mu(x).$$

In order to check (ii) we distinsuish between two possible cases: either

(a) $\lim_{R\to+\infty}\int_{|x|\leqslant R}|x|\,d\mu(x)$ is finite

or

(b) $\lim_{R\to+\infty}\int_{|x|\leqslant R}|x|\,d\mu(x)=+\infty$.

In case (a) the validity of (ii) follows from (i) and the trivial estimate

$$\left|\int_{\mathbb{R}}\sin\lambda x\,d\mu(x)\right|\leqslant|\lambda|\int_{\mathbb{R}}|x|\,d\mu(x).$$

In case (b) we need a more subtle argument. First note that for all $r, s\in\mathbb{R}$ and $u, v\in\mathbb{R}_0^+$,

$$\frac{(r+s)^2}{u+v}\leqslant 2\frac{r^2}{u}+2\frac{s^2}{v}.$$

We apply this for

$$r=\int_{|x|\leqslant\pi/|\lambda|}\sin\lambda x\,d\mu(x),\qquad s=\int_{|x|>\pi/|\lambda|}\sin\lambda x\,d\mu(x),$$

$$u=\int_{|x|\leqslant\pi/|\lambda|}(1-\cos\lambda x)\,d\mu(x),\quad v=\int_{|x|>\pi/|\lambda|}(1-\cos\lambda x)\,d\mu(x).$$

Clearly s^2/v tends to zero in the limit $\lambda\to 0$. Indeed using the Cauchy–Schwartz inequality and $\sin^2 y\leqslant 2(1-\cos y)$ for all y, one has

$$0\leqslant\frac{s^2}{v}\leqslant\frac{\int_{|x|>\pi/|\lambda|}\sin^2\lambda x\,d\mu(x)}{\int_{|x|>\pi/|\lambda|}(1-\cos\lambda x)\,d\mu(x)}\int_{|x|>\pi/|\lambda|}d\mu(x)$$

$$\leqslant 2\int_{|x|>\pi/|\lambda|}d\mu(x).$$

As the measure μ has finite mass, this upper bound can be made arbitrarily small.

It remains to verify that $\lim_{\lambda\to 0}(r^2/u)=0$. One can easily see that

$$0\leqslant\frac{(\int_{|x|\leqslant\pi/|\lambda|}\sin\lambda x\,d\mu(x))^2}{\int_{|x|\leqslant\pi/|\lambda|}(1-\cos\lambda x)\,d\mu(x)}\leqslant\frac{(\int_0^{\pi/|\lambda|}\sin\lambda x\,d\nu(x))^2}{\int_0^{\pi/|\lambda|}(1-\cos\lambda x)\,d\nu(x)}$$
$$\leqslant\frac{\pi^2}{2}\frac{(\int_0^{\pi/|\lambda|}x\,d\nu(x))^2}{\int_0^{\pi/|\lambda|}x^2\,d\nu(x)},$$

where ν is the measure on $\mathbb{R}^+$ given by $d\nu(x)=d\mu(x)+d\mu(-x)$. Notice that (b) is equivalent with $\lim_{R\to+\infty}\int_0^R x\,d\nu(x)=+\infty$. So there exists an increasing sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^+$ such that $x_0=0$ and $\int_{x_k}^{x_{k+1}}x\,d\nu(x)=1$ for all k. Clearly $\lim_{k\to\infty}x_k=+\infty$ and for all $k\in\mathbb{N}_0$ and $R\in[0,x_k]$ one has

$$\frac{k}{x_k}=\frac{1}{x_k}\int_0^{x_k}x\,d\nu(x)$$
$$=\frac{1}{x_k}\int_0^R x\,d\nu(x)+\frac{1}{x_k}\int_R^{x_k}x\,d\nu(x)$$
$$\leqslant\frac{R}{x_k}+\int_{x\geqslant R}d\nu(x).$$

As ν has finite mass, this upper bound can be made arbitrarily small by choosing k and R large enough, so $\lim_{k\to\infty}(k/x_k)=0$.

Now consider an arbitrary $\varepsilon>0$; then there exists a $k_0\in\mathbb{N}_0$ such that $x_k\geqslant(3/\varepsilon)k$ for all $k\geqslant k_0$. Take $|\lambda|$ small enough such that $x_l\leqslant\pi/|\lambda|\leqslant x_{l+1}$ for some $l>k_0+1$. Then

$$\int_0^{\pi/|\lambda|}x\,d\nu(x)\leqslant\int_0^{x_{l+1}}x\,d\nu(x)=l+1$$

and

$$\int_0^{\pi/|\lambda|}x^2\,d\nu(x)\geqslant\int_0^{x_l}x^2\,d\nu(x)=\sum_{j=0}^{l-1}\int_{x_j}^{x_{j+1}}x^2\,d\nu(x)$$
$$\geqslant\sum_{j=0}^{l-1}x_j\geqslant\frac{3}{\varepsilon}\sum_{j=k_0}^{l-1}j$$
$$=\frac{3}{2\varepsilon}(l(l-1)-k_0(k_0+1)).$$

Thus

$$0\leqslant\frac{(\int_0^{\pi/|\lambda|}x\,d\nu(x))^2}{\int_0^{\pi/|\lambda|}x^2\,d\nu(x)}\leqslant\frac{2\varepsilon}{3}\frac{(l+1)^2}{l(l-1)-k_0(k_0+1)}$$

and again this upper bound can be made smaller than ε by choosing $|\lambda|$ small (or equivalently l large) enough. ∎

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