JOURNAL OF DIFFERENTIAL EQUATIONS 49, 281–329 (1983)

Homotopy Invariants of Repeller–Attractor Pairs, II. Continuation of R–A Pairs*

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Received November 28, 1980; revised February 2, 1982

INTRODUCTION

This paper is the sequel to [14][1] and we use the same notation and definitions given there. The basic motivation for the results of this paper arises from the following geometrical considerations. Consider a continuous 1-parameter family of flows in the plane parametrized by $\lambda \in [0, 1]$. Suppose that for any given $\lambda \in [0, 1]$ there are two distinct points in the plane A_λ^* and A_λ which are non-degenerate hyperbolic critical points for the flow at the parameter value λ. For simplicity, in this Introduction, we assume that A_λ^* and A_λ do not vary with $\lambda \in [0, 1]$. Assume that the phase portraits of the flow in a neighborhood containing A_λ^* and A_λ are those given in Figs. 1a and 1b for $\lambda = 0$ and $\lambda = 1$, respectively. In particular there are no heteroclinic orbits from A_λ^* to A_λ for $\lambda = 0$ or $\lambda = 1$; however, the relative position of the local unstable and stable manifolds of A_λ^* and A_λ, respectively, does a flip-flop as λ changes from zero to one.

It then follows from a simple "shooting" argument or Wazewski's lemma (cf. [2]) that there exists $\bar{\lambda}$ between zero and one so that the phase portrait in a neighborhood of $A_{\bar{\lambda}}^*$ and $A_{\bar{\lambda}}$ is that of Fig. 1c; in particular there is a heteroclinic orbit from $A_{\bar{\lambda}}^*$ to $A_{\bar{\lambda}}$.

Heuristically, the existence of this heteroclinic orbit can be captured algebraically as follows. Think of the local unstable manifold to A_λ^* as the vector which it represents when considered as a relative homology class in an appropriate topological pair (N_1, N_3), where $N_1 \supset N_2 \supset N_3$ is an index triple for the repeller–attractor pair (A_λ^*, A_λ) as shown to exist in [14]. Assuming that the same triple $\langle N_1, N_2, N_3\rangle$ can be used for both $\lambda = 0$ and $\lambda = 1$ (which in general is not the case) schematically the situation is that of Fig. 2 where the vectors represented by the unstable manifolds to A_λ^*,

* Partially supported by NSF Grant MCS-8103459.

[1] See the Appendix for corrections to the proofs of [14, Theorem 3.2 and Proposition 4.4].

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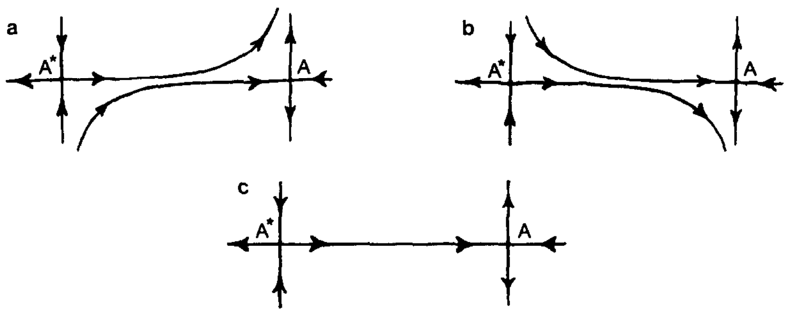

FIG. 1a. The flow at $\lambda = 0$.
FIG. 1b. The flow at $\lambda = 1$.
FIG. 1c. The flow at $\lambda = \bar{\lambda}$, $0 < \bar{\lambda} < 1$.

$\lambda = 0, 1$, have been superimposed on the same picture. The difference between these two vectors is clearly non-zero.

Now in [14, Sect. 4] it was noted that each of these vectors is the image by the splitting map of the same homology class. Hence from the fact that the difference is non-zero we conclude that the splitting map is not constant along the path of flows. It follows from Theorem 3.3 below that there is necessarily a heteroclinic orbit from the repeller to the attractor for some value $\bar{\lambda}$ between zero and one.

The above argument can be carried out in arbitrary finite dimension (in fact in infinite dimension if appropriate compactness conditions are present) for arbitrary repeller–attractor (R–A) pairs, not just critical points.

Stated more precisely, the basic aims of this paper are:

(1) to establish a framework which makes precise the notion of a continuous family of R–A pairs (for a continuous family of local semi-flows), which is done in Section 1;

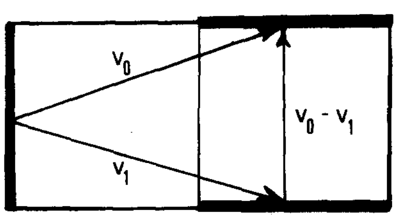

FIG. 2. A schematic of the difference between the local unstable manifolds to A^* at $\lambda = 0, 1$ represented as relative homology classes v_0, v_1 respectively in (N_1, N_3), where N_1 is the large rectangle and N_3 is the union of the three heavily drawn arcs in the boundary of N_1. The difference $v_0 - v_1$ is a relative class in (N_2, N_3), where N_2 is the union of the right-hand square and N_3. $N_1 \supset N_2 \supset N_3$ is an index triple for the R–A pair (A^*, A).

(2) to show that there are natural equivalences between the index sequences of any two R–A pairs along a path of R–A pairs; the proof of which occupies most of Section 2 and culminates in Theorem 2.20 and its corollaries;

(3) to show that if the splitting class for the index sequence of an R–A pair is defined at each point along a path of R–A pairs, then the equivalences mentioned in (2) restrict to natural transformations between the splitting classes, which is Theorem 2.24 below; hence, if the equivalences mentioned in (2) do not induce a natural transformation between the maps on homology induced by the splitting classes of two sequences defined at the endpoints of a path of R–A pairs, then at some point on the path the splitting class is not defined, implying there is a heteroclinic orbit from the repeller to the attractor. This last mentioned fact is Theorem 3.3 below.

Of course, to show that the equivalences of (2) do not induce a natural transformation, it suffices to show that the appropriate rectangle of maps is not commutative. In the simple example above, the equivalences of (2) are the appropriate identity maps because the path of R–A pairs is the constant path. Thus, the non-commutativity of the appropriate rectangle of maps follows immediately from the non-zero difference, illustrated in Fig. 2, between the maps induced on homology by the splitting classes at the endpoints of the path. A more sophisticated example arises in [16] where a bounded traveling wave to a system of non-linear reaction-diffusion equations occurs as a heteroclinic orbit between two steady states forming an R–A pair.

1. The Space of R–A Pairs Associated to a Product Parametrization

1.1. Definition of Product Parametrizations. A product parametrization of a local semi-flow $\Phi \subset \Gamma$ is a homeomorphism φ: $X \times \Lambda \to \Phi$ such that for each λ, $\varphi(X \times \{\lambda\})$ is a local semi-flow.

The restriction $\varphi \mid X \times \{\lambda\}$ will be denoted φ_λ and its image Φ_λ. More generally, if $K \subset \Lambda$, the restriction $\varphi \mid X \times K$ will be denoted φ_K and its image Φ_K; it is immediate from 1.2 below that Φ_K is a local semi-flow as Φ is.

Henceforth for any product parametrization $\varphi: X \times \Lambda \to \Phi$ it will be assumed that $\Phi \subset \Gamma_0$; i.e., Φ is Hausdorff. It follows that both X and Λ are Hausdorff. Also it will always be assumed that X is locally compact so that Φ_λ is locally compact for each $\lambda \in \Lambda$. Note that it is not assumed that Λ is locally compact so that Φ need not be locally compact; however, if $K \subset \Lambda$ is compact, then certainly Φ_K is locally compact since $X \times K$ is.

Before giving the basic motivating example for the definition, let us prove one proposition which says that the slices Φ_λ behave as one expects.

1.2. PROPOSITION. *Let $\varphi: X \times \Lambda \to \Phi$ be a product parametrization of a local semi-flow. Then for each $\lambda \in \Lambda$, Φ_λ is both positively and negatively invariant relative to Φ.*

Proof. If $\gamma \in \Phi_\lambda$ and $N \equiv \gamma \cdot [0, t] \subset \Phi$, set $N_\lambda = N \cap \Phi_\lambda$ and note that because Φ_λ is a local flow, $\sigma \mid N_\lambda(\gamma)$ cannot be less than t without contradicting its definition. This shows the relative positive invariance. The relative negative invariance follows by applying the same arguments to $\gamma \cdot -t$ in place of γ whenever $\gamma \cdot [-t, 0] \subset \Phi$, since $\Phi_\lambda \cap \Phi_\eta = \varnothing$ if $\lambda \neq \eta$. ∎

1.3. EXAMPLE. Motivation for the previous definition is provided by the class of differential equations depending on a parameter

$$\frac{dx}{dt} = f(x, \lambda), \tag{1}$$

where $x \in \mathbf{R}^n$, $\lambda \in [0, 1]$, and f is continuous and locally Lipshitz in x. Γ is taken to be the collection of curves into $\mathbf{R}^n \times \Lambda$ with closed graph and open domain endowed with the compact-open topology; the flow on Γ is the translation of domain flow; Γ_0 is the set of curves in Γ with zero in their domain; and φ assigns to each point (x, λ) the maximal integral curve of (1) with initial condition x. Details are given in [2, Chap. II, Sect. 3].

1.4. DEFINITION (The Space of Isolated Invariant Sets of a Product Parametrization) [2, Chap. IV, Sect. 2]. Given $\varphi: X \times \Lambda \to \Phi \subset \Gamma_0$ as in 1.1, let $\mathscr{S} = \mathscr{S}(\varphi)$ be the set of ordered pairs (λ, S), where $\lambda \in \Lambda$ and S is a Φ_λ-isolated invariant set. Sometimes notation will be abused and we shall refer to $S \in \mathscr{S}(\varphi)$ when we mean $(\lambda, S) \in \mathscr{S}(\varphi)$.

Given a compact subset N of X, let $\Lambda(N)$ be the set of $\lambda \in \Lambda$ such that $\varphi_\lambda(N)$ is an isolating Φ_λ-neighborhood. Define $\sigma_N: \Lambda(N) \to \mathscr{S}$ by $\sigma_N(\lambda) = (\lambda, S)$, where S is the maximal invariant set of $\varphi_\lambda(N)$.

It will be shown below that for each compact $N \subset X$, $\Lambda(N)$ is an open (possibly empty) subset of X. For the moment accept this, and let $\mathscr{B}$ be the collection of sets of the form $\sigma_N(U)$, where $N \subset X$ is compact and $U \subset \Lambda(N)$ is open. The collection $\mathscr{B}$ is a basis for a topology on $\mathscr{S}$. To show this the following lemma is needed. Note that it is immediate from this lemma that $\Lambda(N)$ is open in Λ for each compact $N \subset X$ (take $K = N$ in the lemma).

1.5. LEMMA. *Let (N, K) be a compact pair in X (perhaps $N = K$) and* *suppose that* $\nu \in \Lambda(N) \cap \Lambda(K)$ *and that* $\varphi_\nu(N)$ *and* $\varphi_\nu(K)$ *have the same maximal invariant set. Then there exists* W *open about* ν *so that* $W \subset \Lambda(N) \cap \Lambda(K)$ *and for* $\lambda \in W$, $\varphi_\lambda(N)$ *and* $\varphi_\lambda(K)$ *have the same maximal invariant set.*

Proof. It suffices to show that for λ in a neighborhood of ν, the minimum of the entrance and exit time maps on $\Phi_\lambda(N)$ is finite at each point $\varphi_\lambda(x)$ for $x \in N \backslash \mathrm{int}(K)$. This follows from the upper semi-continuity of the two maps

$$\tilde{\sigma}^*, \tilde{\sigma}\colon N \times \Lambda \to [0, \infty]$$

defined by

$$\tilde{\sigma}^* = (\sigma^* \mid \tilde{N}) \circ \varphi, \qquad \tilde{\sigma} = (\sigma \mid \tilde{N}) \circ \varphi,$$

where $\tilde{N} = \varphi(N \times \Lambda)$. These maps are well-defined and upper semi-continuous by [13, Proposition 2.7] since $\{\varphi(N \times \lambda) \colon \lambda \in \Lambda\}$ is an upper semi-continuous decomposition of $\tilde{N}$ by compact sets. ∎

1.6. PROPOSITION. $\mathscr{B}$ *is a basis for a topology on* $\mathscr{S}$.

Proof. The proof of this proposition is contained implicitly in the proof of [2, Theorem IV.1.3]. It is trivial that $\mathscr{B}$ covers $\mathscr{S}(\varphi)$, and if $(\lambda, S) \in \sigma_{N_1}(U_1) \cap \sigma_{N_2}(U_2)$, then from Lemma 1.5 it follows that there is an open W about λ in the intersection of $\Lambda(N_1)$, $\Lambda(N_2)$, and $\Lambda(N_1 \cap N_2)$ so that

$$(\lambda, S) \in \sigma_{N_1 \cap N_2}(U \cap W) \subset \sigma_{N_1}(U_1) \cap \sigma_{N_2}(U_2),$$

where U is the intersection of U_1 and U_2. Conley uses [2, Lemma IV.1.2.A] instead of Lemma 1.5 to guarantee W above. ∎

Remark. When X is a compact, boundaryless manifold, Λ is the space of smooth (autonomous) vectorfields on X, and if φ assigns to (x, λ) the maximal integral curve of λ with initial condition x in a manner similar to that of Example 1.3, then $\mathscr{S}(\varphi)$, as topologized above, is a special case of Montgomery's definition, given in [17], of the space of isolated invariant sets associated to all the continuous flows on a fixed compact metric space.

1.7. THEOREM [2, IV.1.3]. *Define* $\pi\colon \mathscr{S}(\varphi) \to \Lambda$ *by* $\pi(\lambda, S) = \lambda$. *Then* π *is a surjective local homeomorphism*; *in particular, for* N *compact in* X, $\pi \mid \sigma_N(\Lambda(N))$ *is a homeomorphism with inverse* σ_N.

Proof. First, because the empty neighborhood isolates the empty invariant set, π is surjective; i.e., $\Lambda(\varnothing) = X$ so

$$\pi(\sigma_\varnothing(\Lambda(\varnothing))) = X.$$

The rest of the proof is given by Conley as cited above, or can easily be supplied by the reader. ∎

Remark. It is straightforward to show that the topology on $\mathscr{S}$ defined by the basis $\mathscr{B}$ coincides with the finest topology on $\mathscr{S}$ which makes each σ_N continuous where N ranges over compact subsets of X.

1.8. DEFINITION (The Space of Repeller–Attractor Pairs). The reader is referred to [23, pp. 29–30] for the usual definition, functorial construction, and universal properties which characterize the pullback of a map $p: M \to B$ by a map $q: B' \to B$, denoted $q^*(p): q^*M \to B'$, which is a morphism in the same category in which p and q are morphisms assuming this category admits products.

Letting $\Delta: \Lambda \to \Lambda \times \Lambda \times \Lambda$ be the generalized diagonal map, note that Δ is an embedding, and because the finite product of local homeomorphisms is a local homeomorphism, note too that

$$\bar{\pi} = \pi \times \pi \times \pi: \mathscr{S}(\varphi) \times \mathscr{S}(\varphi) \times \mathscr{S}(\varphi) \to \Lambda \times \Lambda \times \Lambda$$

is one. Set

$$\mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) = \Delta^*(\mathscr{S}(\varphi) \times \mathscr{S}(\varphi) \times \mathscr{S}(\varphi)),$$

so that we have the commutative diagram

$$\begin{array}{ccc} \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) & \xrightarrow{\bar{\Delta}} & \mathscr{S}(\varphi) \times \mathscr{S}(\varphi) \times \mathscr{S}(\varphi) \\ \Big\downarrow {\scriptstyle \Delta^*(\bar{\pi})} & & \Big\downarrow {\scriptstyle \bar{\pi}} \\ \Lambda & \xrightarrow{\Delta} & \Lambda \times \Lambda \times \Lambda. \end{array}$$

Clearly, $\mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi)$ can be naturally identified with the set of quadruples (λ, S_1, S_2, S_3), where $\lambda \in \Lambda$ and $(\lambda, S_i) \in \mathscr{S}(\varphi)$, $i = 1, 2, 3$, and we do so henceforth.

Having carried out the above procedure, define the repeller–attractor space of φ, $\mathscr{R}\mathscr{A}(\varphi)$, as all those points

$$(\lambda, S, A^*, A) \in \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi)$$

such that (A^*, A) is a repeller–attractor pair in S, and give $\mathscr{R}\mathscr{A}(\varphi)$ the relative topology. Let $\pi_2 = \Delta^*(\bar{\pi}) \mid \mathscr{R}\mathscr{A}(\varphi)$. We wish to show that π_2 is a surjective local homeomorphism. This will follow from Lemma 1.10 below due to Conley. The proof of 1.10 requires the following lemma also due to Conley which is a convenient criterion for finding attractor neighborhoods. Proofs of these lemmas are given in [2, Chap. II, Proposition 5.1.C and Proposition 5.3.C, respectively]. More detailed proofs are given in [12].

1.9. LEMMA [2, Chap. II, Sect. 5.1]. *Let $S \subset \Gamma_0$ be a compact Hausdorff invariant set. Suppose $U \subset S$ is compact and for some $\bar{t} > 0$, $U \cdot \bar{t} \subset \mathrm{int}_S(U)$. Then for some $t_0 \geqslant \bar{t}$,*

$$U \cdot [t_0, \infty) \subset \mathrm{int}_S(U).$$

Hence U is an attractor neighborhood with attractor $\omega(U)$.

1.10. LEMMA [2, Chap. II, Sect. 5.3]. *Let $S \subset \Gamma$ be a compact invariant set and (A^*, A) an R–A pair in S. Suppose there exist disjoint Γ-open neighborhoods $U^* \supset A^*$ and $U \supset A$. Then there exists a Γ-open neighborhood V of S and if $\tilde{S}$ is a compact invariant set contained in V then*

$$(\omega^*(U^* \cap \tilde{S}; \tilde{S}), \omega(U \cap \tilde{S}; \tilde{S}))$$

is an R–A pair in $\tilde{S}$, and

$$\omega^*(U^* \cap \tilde{S}; \tilde{S}) \subset U^*, \qquad \omega(U \cap \tilde{S}; \tilde{S}) \subset U.$$

Proof. Set $K = S \backslash U^*$. Then for some $\bar{t} > 0$, $K \cdot \bar{t} \subset U$. To exhibit such a $\bar{t}$, note that by [14, Proposition 1.4] there exists a positively invariant S-neighborhood of A, call it P, with $P \subset U$. Note that each point of K is moved along a flow line in finite positive time into the S-interior of P; hence the existence of $\bar{t}$ follows from the continuity of the flow induced on S and the compactness of K.

It follows that there is W_1 open in Γ and containing K so that $W_1 \cdot \bar{t} \subset U$. Set $V \equiv U^* \cup W_1$ and suppose $\tilde{S} \subset V$. Claim

$$\mathrm{cl}_{\tilde{S}}(U \cap \tilde{S}) \cdot \bar{t} \subset U \cap \tilde{S}. \tag{1}$$

To see this, note that as U is disjoint from U^* and as $\tilde{S} \subset V$, it follows that $U \cap \tilde{S} \subset W_1$; hence

$$(U \cap \tilde{S}) \cdot \bar{t} \subset U \cap \tilde{S} \tag{2}$$

since $W_1 \cdot \bar{t} \subset U$ and since $\tilde{S}$ is invariant, and in fact (1) holds because if γ is a limit point of $U \cap \tilde{S}$ in $\tilde{S} \subset U^* \cup W_1$, then $\gamma \notin U^*$ since U^* is open and disjoint from $U \cap \tilde{S}$; whence $\gamma \in W_1$ so that $\gamma \cdot \bar{t} \subset U \cap \tilde{S}$ also. By Lemma 1.9, (1) shows that $\mathrm{cl}_{\tilde{S}}(U \cap \tilde{S})$ is an attractor neighborhood with attractor $\tilde{A} \equiv \omega(U \cap \tilde{S}; \tilde{S})$, necessarily contained in U.

Let $\tilde{A}^*$ be the dual repeller of $\tilde{A}$ in $\tilde{S}$. By choice of $\bar{t}$ above, for $\gamma \in W_1 \cap \tilde{S}$, $\omega(\gamma) \subset \tilde{A}$ and note that points of $\tilde{A}^*$ are characterized by the condition that their ω-limit sets in $\tilde{S}$ be disjoint from $\tilde{A}$; hence as $\tilde{S} \subset V$, $\tilde{A}^* \subset U^*$. Then the invariance of $\tilde{A}^*$ yields that $\tilde{A}^* \subset \omega^*(U^* \cap \tilde{S}; \tilde{S})$. The reverse inclusion follows because $\tilde{A}^*$ is the largest invariant subset of $\tilde{S}$ with

ω-limit set disjoint from $\tilde{A}$ and because $\omega^*(U^* \cap \tilde{S}; \tilde{S})$ is disjoint from $\tilde{A}$ as follows from an argument similar to one given in [2, Chap. II, Sect. 5.1.A]. ∎

Remark. This proof follows the one given in [2] as cited with some expansion and correction. In particular: first, no detail on how to find $\bar{t}$ is given in [2]; second, instead of showing inclusion (1) above, Conley shows the inclusion which is stated by replacing U with W_1 in (2), from which the inclusion stated by replacing U with W_1 in (1) does not in general follow although this is what is needed to conclude that $\omega(W_1 \cap \tilde{S})$ is an attractor and equals $\omega(U \cap \tilde{S})$ as claimed in [2]; third, no remark in [2] is made to show that $\omega^*(U^* \cap \tilde{S}; \tilde{S}) \subset \tilde{A}^*$. Note that the above proof does not require that S be Hausdorff.

1.11. PROPOSITION. *$\pi_2: \mathscr{R}\mathscr{A}(\varphi) \to \Lambda$ is a surjective local homeomorphism.*

Proof. Because the empty invariant set has $(\varnothing, \varnothing)$ as an R–A pair it follows that π_2 is surjective.

Because we have that $\Delta^*(\bar{\pi}): \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \to \Lambda$ is a local homeomorphism being the pullback of one by the embedding Δ, it suffices to show that $\mathscr{R}\mathscr{A}(\varphi)$ is an open subset of

$$\mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi).$$

Let $(\lambda, S, A^*, A) \in \mathscr{R}\mathscr{A}(\varphi)$. Because A^* and A are disjoint compact subsets of the open Hausdorff Γ_0, there are disjoint open neighborhoods U_0^* and U_0 of A^* and A, respectively. Then choose N_0^* and N_0 compact subsets of X so that (λ, A^*) and (λ, A) are in the images of $\sigma_{N_0^*}$ and σ_{N_0}, respectively, and so that

$$\varphi_\lambda(N_0^*) \subset U_0^* \qquad \text{and} \qquad \varphi_\lambda(N_0) \subset U_0.$$

By the continuity of φ and the compactness of N_0^* and N_0, choose W_0 Λ-open, with $\lambda \in W_0 \subset \Lambda(N_0^*) \cap \Lambda(N_0)$, so that

$$\varphi(N_0^* \times W_0) \subset U_0^* \qquad \text{and} \qquad \varphi(N_0 \times W_0) \subset U_0.$$

Then $\varphi(\operatorname{int} N_0^* \times W_0)$ and $\varphi(\operatorname{int} N_0 \times W_0)$ are Φ-open subsets of U_0^* and U_0, respectively; hence

$$\varphi(\operatorname{int} N_0^* \times W_0) = \Phi \cap U_1^* \qquad \text{and} \qquad \varphi(\operatorname{int} N_0 \times W_0) = \Phi \cap U_1,$$

where U_1^* and U_1 are open in Γ and $U_1^* \subset U_0^*$ and $U_1 \subset U_0$. As $A^* \subset U_1^*$ and $A \subset U_1$, and as (A^*, A) is an R–A pair of S, choose an open $V \supset S$ as guaranteed by the previous Lemma 1.10.

Now choose $M \subset X$ compact so that $\varphi_\lambda(M)$ isolates S and $\varphi_\lambda(M) \subset V$, and choose W_1 open in Λ satisfying $\lambda \in W_1 \subset \Lambda(M)$, and if $\eta \in W_1$, then $\varphi_\eta(M) \subset V$. Then choose $N_1^* \subset \operatorname{int} N_0^*$ and $N_1 \subset \operatorname{int} N_0$, both compact subsets of M, so that $\varphi_\lambda(N_1^*)$ and $\varphi_\lambda(N_1)$ isolate A^* and A, respectively. As both $\varphi_\lambda(N_1^*)$ and $\varphi_\lambda(N_0^*)$ isolate A^*, and as both $\varphi_\lambda(N_1)$ and $\varphi_\lambda(N_0)$ isolate A, by 1.5 choose W_2 open in Λ,

$$\lambda \in W_2 \subset \Lambda(N_0^*) \cap \Lambda(N_1^*) \cap \Lambda(N_0) \cap \Lambda(N_1),$$

so that if $\eta \in W_2$, then both $\varphi_\eta(N_0^*)$ and $\varphi_\eta(N_1^*)$ have the same maximal invariant set, and also both $\varphi_\eta(N_0)$ and $\varphi_\eta(N_1)$ have the same maximal invariant set.

Set $W \equiv W_0 \cap W_1 \cap W_2$ and define Q by

$$Q \equiv W \times \sigma_M(W) \times \sigma_{N_1^*}(W) \times \sigma_{N_1}(W).$$

Note that $Q \cap \mathscr{S} \oplus \mathscr{S} \oplus \mathscr{S}$ is an open neighborhood of (λ, S, A^*, A) in $\mathscr{S} \oplus \mathscr{S} \oplus \mathscr{S}$. To finish we show that $Q \cap \mathscr{S} \oplus \mathscr{S} \oplus \mathscr{S} \subset \mathscr{R}\mathscr{A}$. Let $(\eta, S', A'', A') \in Q \cap \mathscr{S} \oplus \mathscr{S} \oplus \mathscr{S}$. Then $\varphi_\eta(M) \subset V$ and $\varphi_\eta(M)$ isolates S'; whence by choice of V,

$$(\omega^*(S' \cap U_1^*), \omega(S' \cap U_1))$$

is an R–A pair of S', and $\omega^*(S' \cap U_1^*) \subset S' \cap U_1^*$ and $\omega(S' \cap U_1) \subset S' \cap U_1$. Also, $S' \cap U_1^* \subset \Phi_\eta \cap U_1^* = \varphi_\eta(\operatorname{int} N_0^*)$ and $S' \cap U_1 \subset \Phi_\eta \cap U_1 = \varphi_\eta(\operatorname{int} N_0)$. In particular, $\omega^*(S' \cap U_1^*)$ is an invariant subset of $\varphi_\eta(N_0^*)$; but as $\eta \in W$, $\varphi_\eta(N_0^*)$ and $\varphi_\eta(N_1^*)$ have the same maximal invariant set which is A'' as $(\eta, A'') \in \sigma_{N_1^*}(W)$. It follows that $\omega^*(S' \cap U_1^*) \subset A''$.

On the other hand, as $N_1^* \subset M$, the maximal invariant subset of $\varphi_\eta(N_1^*)$ is a subset of the maximal invariant subset of $\varphi_\eta(M)$; i.e., $A'' \subset S'$; and as $\varphi_\eta(N_1^*) \subset U_1^* \cap \Phi_\eta$, in fact $A'' \subset S' \cap U_1^*$; whence $A'' \subset \omega^*(S' \cap U_1^*)$. Thus $A'' = \omega^*(S' \cap U_1^*)$, and similarly, $A = \omega(S' \cap U_1)$. It follows that $(\eta, S', A'', A') \in \mathscr{R}\mathscr{A}$. ∎

2. Continuation of $\mathscr{J}(S; A^*, A)$ along Paths

The aim of this section is to be able to relate and compare $\mathscr{J}(c(0))$ and $\mathscr{J}(c(1))$, where $c: I \to \mathscr{R}\mathscr{A}(\varphi)$ is a continuous path. The results of this section are mostly direct analogues of the results obtained in [2, Chap. IV] for $\mathscr{J}(S)$, the exception being the results on the splitting class μ, although the use of the language of category theory here, in particular Definition 2.1, is new, and hopefully this serves the ideal of clarification rather than the bane of obfuscation.

2.1. DEFINITION OF THE CATEGORY OF CONNECTED SIMPLE SYSTEMS IN $\mathscr{K}$. Let $\mathscr{K}$ be a category. Define a category $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{K})$ as follows:

(1) The objects are connected simple systems which are subcategories of $\mathscr{K}$;

(2) If $\mathscr{A}$ and $\mathscr{B}$ are connected simple systems which are subcategories of $\mathscr{K}$, a morphism from $\mathscr{A}$ to $\mathscr{B}$, called a map between connected simple systems, is a covariant functor F on the product category $\mathscr{A} \times \mathscr{B}$ to the morphism category of $\mathscr{K}$ satisfying:

(i) if (A, B) is an object of $\mathscr{A} \times \mathscr{B}$, $F(A, B)$ is a morphism in $\mathscr{K}$, $F(A, B)\colon A \to B$;

(ii) if $f\colon A \to A'$ and $g\colon B \to B'$ are morphisms in $\mathscr{A}$ and $\mathscr{B}$, respectively, $F(f, g)$ is the commutative diagram of morphisms in $\mathscr{K}$

$$\begin{tikzcd} A \arrow[r, "F(A,B)"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A' \arrow[r, "F(A',B')"] & B' \end{tikzcd}.$$

(3) If $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ are maps between connected simple systems in $\mathscr{K}$, their composite $G \circ F\colon \mathscr{A} \to \mathscr{C}$ is defined as follows:

(i) $G \circ F(A, C)$ is the composite $G(B, C) \circ F(A, B)$, where B is any object of $\mathscr{K}$. This definition is independent of the choice of B, for if B' is another object of $\mathscr{B}$, there is the diagram

$$\begin{tikzcd} A \arrow[r, "F(A,B)"] \arrow[dr, "F(A,B')"'] & B \arrow[r, "G(B,C)"] \arrow[d] & C \\ & B' \arrow[ur, "G(B',C)"'] & \end{tikzcd}$$

where the vertical arrow is the unique morphism in $\mathscr{B}$ from B to B', and it is commutative by virtue of the fact that F and G are maps between connected simple systems.

(ii) if $f\colon A \to A'$ and $h\colon C \to C'$ are morphisms in $\mathscr{A}$ and $\mathscr{C}$, respectively, then $G \circ F(f, h)$ is the juxtaposition of $F(f, g)$ and $G(g, h)$ where g: $B \to B'$ is any morphism in $\mathscr{B}$; i.e.,

$$\begin{tikzcd} A \arrow[r, "F(A,B)"] \arrow[d, "f"'] & B \arrow[r, "G(B,C)"] \arrow[d, "g"] & C \arrow[d, "h"] \\ A' \arrow[r, "F(A',B')"] & B' \arrow[r, "G(B',C')"] & C' \end{tikzcd}.$$

This definition is independent of the choice of g, for if $\bar{g}\colon \bar{B} \to \bar{B}'$ is another morphism in $\mathscr{B}$ there is the commutative diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[d] \arrow[ddr] & & B \arrow[d] \arrow[ddl] \arrow[ddr] & \\
A' \arrow[rr] \arrow[ddr] & & B' \arrow[ddl] \arrow[ddr] & \\
 & \bar{B} \arrow[rr] \arrow[d] & & C \arrow[d] \\
 & \bar{B}' \arrow[rr] & & C'
\end{tikzcd}$$

where all the arrows have been previously defined save for $B \to \bar{B}$ and $B' \to \bar{B}'$ which are the unique morphisms in $\mathscr{B}$ between the given object pairs. The diagram is commutative because F and G are maps between connected simple systems and because $\mathscr{B}$ is a connected simple system.

The straight-forward verification that this composition is associative is omitted.

(4) If $\mathscr{A}$ is an object of $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H})$, $1_{\mathscr{A}}\colon \mathscr{A} \to \mathscr{A}$ is the map between connected simple systems defined by: $1_{\mathscr{A}}(A, A')$ is the unique morphism of $\mathscr{A}$ from A to A'. Let B be another object of $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H})$ and $F\colon \mathscr{A} \to \mathscr{B}$ a map. Then $F \circ 1_{\mathscr{A}} = F$. For if (A, B) is an object pair of $\mathscr{A} \times \mathscr{B}$ by (3) above

$$F \circ 1_{\mathscr{A}}(A, B) = F(A, B) \circ 1_{\mathscr{A}}(A, A) = F(A, B) \circ 1_A = F(A, B).$$

Similarly $1_{\mathscr{B}} \circ F(A, B) = F(A, B)$. Let $f\colon A \to A'$ and $g\colon B \to B'$ be morphisms in $\mathscr{A}$ and $\mathscr{B}$, respectively. Then $F \circ 1_{\mathscr{A}}(f, g) = F(f, g) \circ 1_{\mathscr{A}}(f, f) = F(f, g)$ since the diagram of $F \circ 1_{\mathscr{A}}(f, g)$ is

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "f"'] & A \arrow[r, "F(A,B)"] \arrow[d, "f"] & B \arrow[d, "g"] \\
A' \arrow[r, "1_{A'}"] & A' \arrow[r, "F(A',B')"] & B'
\end{tikzcd}$$

which collapses to the diagram of $F(f, g)$, i.e., the right-hand square in the above diagram. Similarly $1_{\mathscr{B}} \circ F(f, g) = F(f, g)$. Thus for each object $\mathscr{A}$ of $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H})$, $1_{\mathscr{A}}$ serves as an identity morphism.

This completes showing that $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H})$ is a well-defined category.

Remark 1. From the diagram of (2)(ii) above, as every morphism of $\mathscr{A}$ is an equivalence, it follows that $F(A', B') = g \circ F(A, B) \circ f^{-1}$; hence this equation serves to define a map between connected simple systems by having the right-hand side define the left.

Remark 2. It follows from Remark 1 that if $\mathscr{A}$ and $\mathscr{B}$ are connected simple systems in $\mathscr{H}$ and $F_1(A,B): A \to B$, $F_2(A',B'): A' \to B'$ are two morphisms in $\mathscr{H}$ between two object pairs (A,B), (A',B') of $\mathscr{A} \times \mathscr{B}$, then $F_1(A,B)$ and $F_2(A',B')$ define the same map between $\mathscr{A}$ and $\mathscr{B}$ if, and only if, the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "F_1(A,B)"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "F_2(A',B')"] & B' \end{tikzcd}$$

where the vertical arrows are the unique morphisms in $\mathscr{A}$ and $\mathscr{B}$, respectively.

Our main interest in $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H})$ is in those morphisms which are equivalences, and Remark 1 above is important in the proof of the following proposition which characterizes the equivalences.

2.2. Proposition. *A map between connected simple systems in $\mathscr{H}$ F: $\mathscr{A} \to \mathscr{B}$ is an equivalence in $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H})$ if, and only if, for each object pair (A,B) of $\mathscr{A} \times \mathscr{B}$ $F(A,B): A \to B$ is an equivalence in $\mathscr{H}$.*

Proof. If F is an equivalence with inverse $G: \mathscr{B} \to \mathscr{A}$, then for any object pair (A,B),

$$\begin{aligned} 1_A &= 1_{\mathscr{A}}(A,A) = G \circ F(A,A) = G(B,A) \circ F(A,B), \\ 1_B &= 1_{\mathscr{B}}(B,B) = F \circ G(B,B) = F(A,B) \circ G(B,A), \end{aligned} \tag{1}$$

which shows that $F(A,B)$ has inverse $G(B,A)$.

Conversely, suppose for each object pair (A,B), $F(A,B): A \to B$ has an inverse $G(B,A): B \to A$. Then by Remark 1 following 2.1 above, for any other pair (A',B'), $F(A',B') = gF(A,B)f^{-1}$, where $f: A \to A'$, $g: B \to B'$ are the unique morphisms of $\mathscr{A}$ and $\mathscr{B}$, respectively, from A to A' and B to B', respectively. Thus

$$G(B',A') = F(A',B')^{-1} = fF(A,B)^{-1}g^{-1} = fG(B,A)g^{-1};$$

hence the diagram

$$\begin{tikzcd} B \arrow[r, "G(B,A)"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ B' \arrow[r, "G(B',A')"] & A' \end{tikzcd}$$

is commutative.

It follows that there is a map $G\colon \mathscr{B} \to \mathscr{A}$ so that for each object pair (A, B) of $\mathscr{A} \times \mathscr{B}$ $G(B, A)$ is the inverse of $F(A, B)$; hence the equalities (1) above hold and it follows that G is an inverse for F as regards object pairs.

It remains to show that G is an inverse for F with respect to morphism pairs. Accordingly, suppose that $f\colon A \to A'$ and $\bar{f}\colon \bar{A} \to \bar{A}'$ are morphisms of $\mathscr{A}$. Then for any morphism $g\colon B \to B'$ of $\mathscr{B}$

$$\begin{aligned} G \circ F(f, \bar{f}) &= (1_{\mathscr{A}} \circ G) \circ F(f, \bar{f}) = (1_{\mathscr{A}} \circ G)(g, \bar{f}) \circ F(f, g) \\ &= 1_{\mathscr{A}}(f, \bar{f}) \circ G(g, f) \circ F(f, g) \end{aligned}$$

and this last composition is the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{F(A,B)} & B & \xrightarrow{G(B,A)} & A & \longrightarrow & \bar{A} \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ A' & \xrightarrow{F(A',B')} & B' & \xrightarrow{G(B',A')} & A' & \longrightarrow & \bar{A}' \end{array}$$

where $A \to \bar{A}$, $A' \to \bar{A}'$ are the unique morphisms in $\mathscr{A}$. As this diagram is commutative and since

$$G(B, A) \circ F(A, B) = 1_A$$

and

$$G(B', A') \circ F(A', B') = 1_{A'},$$

this diagram collapses to its right-hand square which is precisely $1_{\mathscr{A}}(f, \bar{f})$. Similarly if $g\colon B \to B'$ and $\bar{g}\colon \bar{B} \to \bar{B}'$ are morphisms of $\mathscr{B}$, then $F \circ G(g, \bar{g}) = 1_{\mathscr{B}}(g, \bar{g})$. ∎

Remark. The proposition coupled with Remark 1 following 2.1 shows that $F\colon \mathscr{A} \to \mathscr{B}$ is an equivalence if, and only if, for some pair (A, B),

$$F(A, B)\colon A \to B$$

is an equivalence.

Recall [cf. 18, I.7] that the fundamental groupoid of a space E, $\Pi(E)$, is the category whose objects are the points of E, and for $x, y \in E$, the set of morphisms from x to y, $\pi(x, y)$, is the set of homotopy classes of paths in E with initial point y and endpoint x. If x, y, $z \in E$ and $c_2 \in \pi(x, y)$, $c_1 \in \pi(y, z)$ the composition $c_1 * c_2$ is the usual multiplication of paths,

$$\begin{aligned} c_1 * c_2(t) &= c_1(2t), & 0 \leqslant t \leqslant \tfrac{1}{2}, \\ &= c_2(2t - 1), & \tfrac{1}{2} \leqslant t \leqslant 1. \end{aligned}$$

With this and Definition 2.1 we can restate our goal, namely, to define a contravariant functor on the fundamental groupoid, $\Pi(\mathscr{R}\mathscr{A}(\varphi))$, of $\mathscr{R}\mathscr{A}(\varphi)$ which assigns to an object $(S_\lambda; A_\lambda^*, A_\lambda) \in \mathscr{R}\mathscr{A}(\varphi)$ the connected simple system $\mathscr{I}(S_\lambda; A_\lambda^*, A_\lambda^\bullet)$ and which assigns to the homotopy class of a path $c: I \to \mathscr{R}\mathscr{A}(\varphi)$ a map from $\mathscr{I}(c(0))$ to $\mathscr{I}(c(1))$. As any path can be traveled backwards (i.e., every morphism in $\Pi(\mathscr{R}\mathscr{A}(\varphi))$ is an equivalence), this map must be an equivalence. To enable us to define these maps several lemmas are needed. Several of these as noted are proved in [2] and we give only the statements here.

In the remainder of Section 2, $\varphi: X \times \Lambda \to \Gamma_0$ will always denote a product parametrization of a local semi-flow Φ, and recall our standing assumption that for each $\lambda \in \Lambda$, Φ_λ is locally compact. Also recall that as a consequence of 1.2, for every $K \subset \Lambda$, Φ_K is a local semi-flow. Finally recall that for any map $\pi: E \to B$, a map $\sigma: K \to E$ for $K \subset B$ is called a section of π over K (or more loosely, a section of E when π is understood from context) if, and only if, $\pi \circ \sigma = 1_K$.

2.3. Lemma [2, IV.2.1.B]. *Let $K \subset \Lambda$ be compact. Then $S \subset \Phi_K$ is an isolated invariant set relative to Φ_K, if and only if, for some continuous section $\sigma: K \to \mathscr{S}(\varphi)$ of $\pi: \mathscr{S}(\varphi) \to \Lambda$,*

$$S = \bigcup \{\sigma(\lambda): \lambda \in K\}.$$

Remark. Note the abuse of notation $S = \bigcup \{\sigma(\lambda): \lambda \in K\}$, for actually $\sigma(\lambda) = (\lambda, S_\lambda)$, S_λ being Φ_λ-isolated, and S is the union over K of these S_λ.

When S is isolated the section σ is defined by

$$\sigma(\lambda) = (\lambda, S \cap \Phi_\lambda),$$

and note that 1.2 guarantees that $S \cap \Phi_\lambda$ is invariant—it is trivial to verify that it is Φ_λ-isolated since S is Φ_K-isolated and K is compact.

Note that Conley tacitly uses Proposition 1.2 throughout [2, Chap. IV], in particular for the proof of 2.3 above, in the instance already mentioned, and also, given the section σ, in proving that S is Φ_K-isolated. Namely, having constructed a compact Φ_K-neighborhood N of S so that $N \cap \Phi_\lambda$ is an isolating Φ_λ-neighborhood of $\sigma(\lambda)$ for each $\lambda \in K$, if $\gamma \cdot \mathbf{R} \subset N$ then $\gamma \cdot \mathbf{R} \subset N \cap \Phi_\lambda$ for some λ. Hence $\gamma \cdot \mathbf{R} \subset \sigma(\lambda) \subset S$, which shows that N isolates S.

Note that if $\sigma: K \to \mathscr{R}\mathscr{A}(\varphi) \subset \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi)$ is a section of π_2: $\mathscr{R}\mathscr{A}(\varphi) \to \Lambda$ over $K \subset \Lambda$, then σ defines three sections $\sigma_0, \sigma_1, \sigma_2: K \to \mathscr{S}(\varphi)$ by projecting σ onto the factors—in the notation of 1.8, $\sigma_i = p_{i+1} \circ \bar{\Delta} \circ \sigma$, where $p_{i+1}: \mathscr{S} \times \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ is a projection onto the $(i+1)$st factor, $i = 0, 1, 2$.

2.4. COROLLARY. *Let $K \subset \Lambda$ be compact. Then $S \subset \Phi_K$ is an isolated invariant set relative to Φ_K and (A^*, A) is an R–A pair of S if, and only if, for some continuous section $\sigma: K \to \mathscr{R}\mathscr{A}(\varphi)$ of $\pi_2: \mathscr{R}\mathscr{A}(\varphi) \to \Lambda$,*

$$S = \bigcup \{\sigma_0(\lambda): \lambda \in K\}$$

and

$$A^* = \bigcup \{\sigma_1(\lambda): \lambda \in K\}$$

and

$$A = \bigcup \{\sigma_2(\lambda): \lambda \in K\}.$$

Proof. Suppose S is Φ_K-isolated and (A^*, A) is an R–A pair of S. Then both A^* and A are also Φ_K-isolated. By 2.3 there are three sections σ_0, σ_1, $\sigma_2: K \to \mathscr{S}(\varphi)$ giving S, A^*, and A as in the statement of the corollary. Hence defining

$$\sigma: K \to \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \subset \Lambda \times \mathscr{S}(\varphi) \times \mathscr{S}(\varphi) \times \mathscr{S}(\varphi)$$

by

$$\sigma(\lambda) = (\lambda, \sigma_0(\lambda), \sigma_1(\lambda), \sigma_2(\lambda)),$$

σ is a section of $\Delta^*(\bar{\pi}): \mathscr{S} \oplus \mathscr{S} \oplus \mathscr{S} \to \Lambda$ over K. Writing for $\lambda \in K$, $S_\lambda = S \cap \Phi_\lambda$, $A_\lambda = A \cap \Phi_\lambda$, $A^*_\lambda = A^* \cap \Phi_\lambda$ by the remark following 2.3, $\sigma_0(\lambda) = (\lambda, S_\lambda)$, $\sigma_1(\lambda) = (\lambda, A^*_\lambda)$, $\sigma_2(\lambda) = (\lambda, A_\lambda)$, and because for each $\lambda \in K$, S_λ is closed in S, it follows from the definition of an R–A pair [2, Chap. II, Sect. 5] that (A^*_λ, A_λ) is an R–A pair of S_λ for $\lambda \in K$.

For the converse, note that by 2.3, S, A^*, and A are isolated invariant sets relative to Φ_K, and A^* and A are disjoint because (A^*_λ, A_λ) is an R–A pair of S_λ for $\lambda \in K$. Let N be an isolating Φ_K-neighborhood of A. Suppose $\gamma \in N \cap S$ and $\gamma \cdot \mathbf{R}^- \subset N$. Then for some unique $\lambda \in K$, $\gamma \in S_\lambda$ and $\omega^*(\gamma) \subset N \cap \Phi_\lambda$, but $N \cap \Phi_\lambda$ isolates A_λ so that

$$\omega^*(\gamma) \subset A_\lambda = \{\eta \in S_\lambda: \omega^*(\eta) \cap A^*_\lambda = \varnothing\};$$

hence $\gamma \in A_\lambda \subset A$. It follows that if $\gamma \in \partial_S(N \cap S)$ then $\gamma \cdot \mathbf{R}^- \not\subset N$. Then by [14, Lemma 1.4] $(N \cap S)^+$ is an attractor neighborhood, and as N isolates $A \subset S$, the attractor is A. Because the dual repeller of A is

$$\begin{aligned}\{\gamma \in S: \omega(\gamma) \cap A = \varnothing\} &= \bigcup_{\lambda \in K} \{\gamma \in S_\lambda: \omega(\gamma) \cap A_\lambda = \varnothing\} \\ &= \bigcup \{A^*_\lambda: \lambda \in K\} = A^*,\end{aligned}$$

(A^*, A) is an R–A pair of S. ∎

2.5. Definition and Proposition (Maps between Connection Indices over a Compact Pair in Λ). Let $K \subset \Lambda$ be compact, let S be an isolated invariant set relative to Φ_K, and let (A^*, A) be an R–A pair of S. Suppose $N_1 \supset N_2 \supset N_3$ is a nested index triple for (A^*, A).

For each compact $K' \subset K$, define

$$S_{K'} \equiv S \cap \Phi_{K'}, \qquad A^*_{K'} \equiv A^* \cap \Phi_{K'}, \qquad A_{K'} \equiv A \cap \Phi_{K'},$$

and

$$N_{i,K'} \equiv N_i \cap \Phi_{K'}.$$

Then for each compact $K' \subset K$,

(i) $N_{1,K'} \supset N_{2,K'} \supset N_{3,K'}$ is a nested index triple for the R–A pair $(A^*_{K'}, A_{K'})$ of $S_{K'}$.

(ii) The commutative diagram of functorial inclusion induced maps

$$\begin{array}{ccccc} N_{2,K'}/N_{3,K'} & \longrightarrow & N_{1,K'}/N_{3,K'} & \longrightarrow & N_{1,K'}/N_{2,K'} \\ \downarrow & & \downarrow & & \downarrow \\ N_2/N_3 & \longrightarrow & N_1/N_3 & \longrightarrow & N_1/N_2 \end{array} \tag{1}$$

induces a map between connected simple systems

$$F(K'; S; A^*, A)\colon \mathscr{I}(S_{K'}; A^*_{K'}, A_{K'}) \to \mathscr{I}(S, A^*, A),$$

and the definition of $F(K'; S; A^*, A)$ is independent of the choice of index triple used to define it.

(iii) If (K', K'') is a compact pair in K, then

$$F(K''; S; A^*, A) = F(K'; S; A^*, A) \circ F(K''; S_{K'}; A^*_{K'}, A_{K'}).$$

(iv) If $S = A^* \cup A$, so that the splitting class μ of $\mathscr{I}(S; A^*, A)$ of [14, Definition 4.3] is defined, then also the splitting class $\mu^{K'}$ of $\mathscr{I}(S_{K'}; A^*_{K'}, A_{K'})$ is defined, and the splittings μ and $\mu^{K'}$ are natural relative to $F(K'; S; A^*, A)$; i.e.,

$$\begin{aligned} &S^i F(K', S)(N_{1,K'}/N_{3,K'}, N_1/N_3) \circ S^i \mu^{K'} \\ &\quad = S^i \mu \circ S^i F(K', A^*)(N_{1,K'}/N_{2,K'}, N_1/N_2) \end{aligned}$$

$(i = 0, 1, 2, \ldots)$, where $F(K', S)$ and $F(K', A^*)$ are, respectively, "slices" of $F(K'; S; A^*, A)$ mapping $\mathscr{I}(S_{K'})$ to $\mathscr{I}(S)$ and $\mathscr{I}(A^*_{K'})$ to $\mathscr{I}(A^*)$ (see 2.6 below for the explicit definition) and where $S^i F$ and $S^i \mu$ denote the ith iterated suspensions.

Proof. Note that by 2.4 there is a section of $\mathscr{R}\mathscr{A}(\varphi)$ $\sigma\colon K \to \mathscr{R}\mathscr{A}(\varphi)$ whose three components unioned over K give S, A^*, and A, respectively. Then as $\sigma \mid K'$ is a section over K', again by 2.4, $(A^*_{K'}, A_{K'})$ is an R–A pair of $S_{K'}$. Then (i) follows immediately from 1.2 since $N_1 \supset N_2 \supset N_3$ is an index triple for (A^*, A). Note that it is important that K' be compact otherwise $\Phi_{K'}$ need not be closed relative to Φ_K, so $S_{K'}$ need not be closed relative to S and so not compact—analogous remarks hold for $A^*_{K'}$, $A_{K'}$, and $N_{i,K}$, $i = 1, 2, 3$. By the functorial construction of the long coexact sequence of an index triple given in [14, Theorem 3.2], the diagram (1) embeds in an infinite homotopy commutative ladder between the long coexact sequences extending the top and bottom rows of (1), and by Remark 1 after 2.1 this defines a map between connected simple systems $F(K'; S; A^*, A)$: $\mathscr{I}(S_{K'}; A^*_{K'}, A_{K'}) \to \mathscr{I}(S; A^*, A)$.

To see that the definition of $F(K'; S; A^*, A)$ is independent of the nested index triple used in defining it, let $N_1' \supset N_2' \supset N_3'$ be another index triple for (A^*, A) relative to Φ_K. By Remark 2 after 2.1 and the argument of the preceding paragraph, it suffices to show that there is a homotopy commutative (three-dimensional) diagram

$$\begin{tikzcd}
& N_{2,K'}/N_{3,K'} \arrow[rr] \arrow[dl] \arrow[dd] & & N_{1,K'}/N_{3,K'} \arrow[rr] \arrow[dl] \arrow[dd] & & N_{1,K'}/N_{2,K'} \arrow[dl] \arrow[dd] \\
N'_{2,K'}/N'_{3,K'} \arrow[rr] \arrow[dd] & & N'_{1,K'}/N'_{3,K'} \arrow[rr] \arrow[dd] & & N'_{1,K'}/N'_{3,K'} \arrow[dd] & \\
& N_2/N_3 \arrow[rr] \arrow[dl] & & N_1/N_3 \arrow[rr] \arrow[dl] & & N_1/N_2 \arrow[dl] \\
N'_2/N'_3 \arrow[rr] & & N'_1/N'_3 \arrow[rr] & & N'_1/N'_2 &
\end{tikzcd} \tag{2}$$

where the top and bottom faces of the prism induce the unique morphisms in $\mathscr{I}(S_{K'}; A^*_{K'}, A_{K'})$ and $\mathscr{I}(S; A^*, A)$, respectively, and where all vertical arrows are functorial inclusion induced maps. By an argument analogous to that given in Section 2 in the Appendix to this paper, which corrects the proof given in [14, Theorem 3.2] of the naturality of the long coexact sequences of $\mathscr{I}(S; A^*, A)$ relative to the Morse indices, it can be assumed that $N_i' \subset N_i$, $i = 1, 2, 3$. Then all the arrows of (2) can be assumed to be induced by inclusions of pairs and it is immediate that (2) is commutative.

Because the functorial inclusion induced diagram

$$\begin{tikzcd}
N_{2,K''}/N_{3,K''} \arrow[r] \arrow[d] & N_{1,K''}/N_{3,K''} \arrow[r] \arrow[d] & N_{1,K''}/N_{2,K''} \arrow[d] \\
N_2/N_3 \arrow[r] & N_1/N_3 \arrow[r] & N_1/N_2
\end{tikzcd}$$

factors into the commutative diagram of functorial inclusion induced maps

$$\begin{array}{ccccc} N_{2,K''}/N_{3,K''} & \longrightarrow & N_{1,K''}/N_{3,K''} & \longrightarrow & N_{1,K''}/N_{2,K''} \\ \downarrow & & \downarrow & & \downarrow \\ N_{2,K'}/N_{3,K'} & \longrightarrow & N_{1,K'}/N_{3,K'} & \longrightarrow & N_{1,K'}/N_{2,K'} \\ \downarrow & & \downarrow & & \downarrow \\ N_2/N_3 & \longrightarrow & N_1/N_3 & \longrightarrow & N_1/N_2 \end{array}$$

it follows that $F(K''; S; A^*, A) = F(K'; S; A^*, A) \circ F(K''; S_{K'}; A^*_{K'}, A_{K'})$; the tedious but straight-forward details are omitted.

For (iv), if $S = A^* \cup A$ then by [14, Sect. 1–2] for some $t > 0$ and N_1-open U, $A \subset U \subset N_2$, $\mu_{t,U}$ is defined, and regarding $N_{1,K'}/N_{2,K'}$ as embedded in N_1/N_2, by 1.2 the restriction $\mu_{t,U} \mid N_{1,K'}/N_{2,K'}$ can be regarded as a map into $N_{1,K'}/N_{3,K'}$ embedded in N_1/N_3. In particular the factor

$$(N_1 \backslash U)^t/(N_2 \backslash U) \to N_1/N_3^{-t}$$

of $\mu_{t,U}$ restricts to

$$(N_{1,K'} \backslash U')^t/(N_{2,K'} \backslash U') \to N_{1,K'}/N_{3,K'}^{-t},$$

where $U' = U \cap \Phi_{K'}$ so that $A_{K'} \subset U' \subset N_{2,K'}$ and U' is $N_{1,K'}$-open. Thus by [14, Sect. 4.1–2] $\mu^{K'}_{t,U'}$ is defined and clearly

$$\begin{aligned} &F(K'; S)(N_{1,K'}/N_{3,K'}, N_1/N_3) \circ \mu^{K'}_{t,U'} \\ &\quad = \mu_{t,U} \circ F(K'; A^*)(N_{1,K'}/N_{2,K'}, N_1/N_2), \end{aligned}$$

where

$$F(K', S)(N_{1,K'}/N_{3,K'}, N_1/N_3) \colon N_{1,K'}/N_{3,K'} \to N_1/N_3$$

and

$$F(K', A^*)(N_{1,K'}/N_{2,K'}, N_1/N_2) \colon N_{1,K'}/N_{2,K'} \to N_1/N_2$$

are the inclusion induced maps. The functorial construction of the long coexact sequences then yields the desired naturality for the non-trivial suspensions S^i, $i > 0$. ∎

Remark. When $K' = \{\lambda\}$ we write $F(\lambda; S; A^*, A)$ instead of $F(\{\lambda\}; S; A^*, A)$ and also $N_{1,\lambda} \supset N_{2,\lambda} \supset N_{3,\lambda}$, etc. Also, whenever clear from the context we omit the arguments when denoting maps between objects of connected simple systems.

2.6. Definition and Corollary (Maps between Morse Indices over a Compact Pair in Λ). Let $K \subset \Lambda$ be compact, and let S be an isolated invariant set relative to Φ_K. Suppose $\langle N_1, N_2 \rangle$ is an index pair for S relative to the isolating neighborhood N.

For each compact $K' \subset K$, define

$$N_{K'} \equiv N \cap \Phi_{K'}, \qquad N_{i,K'} \equiv N_i \cap \Phi_{K'} \quad (i = 1, 2),$$

and

$$S_{K'} \equiv S \cap \Phi_{K'}.$$

Then for each compact $K' \subset K$,

(i) $\langle N_{1,K'}, N_{2,K'} \rangle$ is an index pair for $S_{K'}$ relative to $N_{K'}$.

(ii) The functorial inclusion induced map

$$N_{1,K'}/N_{2,K'} \to N_1/N_2 \tag{1}$$

defines a map between connected simple systems

$$F(K', S)\colon \mathscr{I}(S_{K'}) \to \mathscr{I}(S)$$

with $F(K', S)(N_{1,K'}/N_{2,K'}, N_1/N_2)$ the inclusion induced map of (1) and the definition of $F(K', S)$ is independent of the index pair of S used to define it.

(iii) If (K', K'') is a compact pair in K, then

$$F(K'', S) = F(K', S) \circ F(K'', S_{K'}).$$

Proof. As S is a repeller relative to itself, $(S, \varnothing)$ is an R–A pair of S and $N_1 \supset N_2 \supset N_2$ a nested index triple of $(S, \varnothing)$ for any index pair $N_1 \supset N_2$ of S, and those "slices" of the diagrams in 2.5 corresponding to the indices of the repellers give a proof of the corollary. Alternatively, the reader can mimic the argument replacing index triple by index pair throughout with the resulting simplification in diagrams (the number of lattice points decreases by two-thirds). ∎

Remark. If $\lambda \in K$ we write $F(\lambda, S)$ instead of $F(\{\lambda\}, S)$. In [2, Chap. IV, Sects. 2.2A–B] Conley defined what amounts to $F(\lambda, S)$ though he did not make use of the language of category theory. He also shows the independence from the index pair, but his proof is different.

To continue our saga (and along paths) we now show that whenever $K \subset \Lambda$ is compact "small enough," and contractible in itself, $F(K'; S; A^*, A)$ defined in 2.5 above is an equivalence between connected simple systems. By the remark after 2.2, it suffices to show that each of the vertical arrows in diagram (1) of 2.5 above is an equivalence. The desired result will follow from the following special case proved by Conley.

2.7. THEOREM [2, Chap. IV, Sect. 2.2C]. *Let* $\sigma\colon W \to \mathscr{S}(\varphi)$ *be a section of* $\pi\colon \mathscr{S}(\varphi) \to \Lambda$ *over* W. *Then for each* $\lambda \in W$ *there exists a neighborhood* W_λ *about* λ *relative to* W *with the following equivalence map property: if* $K \subset W$ *is compact and contractible in itself and*

$$S \equiv \bigcup \{\sigma(\eta)\colon \eta \in K\},$$

then for each $v \in K$,

$$F(v; S)\colon \mathscr{I}(S_v) \to \mathscr{I}(S)$$

is an equivalence of connected simple systems.

2.8. COROLLARY. *Let* $\sigma\colon W \to \mathscr{R}\mathscr{A}(\varphi)$ *be a section of* $\pi_2\colon \mathscr{R}\mathscr{A}(\varphi) \to \Lambda$ *over* $W \subset \Lambda$. *Then for each* $\lambda \in W$ *there exists a* W-*neighborhood* W_λ *about* λ *with the following equivalence map property: if* $K \subset W_\lambda$ *is compact and contractible in itself and*

$$S \equiv \bigcup \{\sigma_0(\eta)\colon \eta \in K\},$$

$$A^* \equiv \bigcup \{\sigma_1(\eta)\colon \eta \in K\},$$

and

$$A \equiv \bigcup \{\sigma_2(\eta)\colon \eta \in K\}.$$

where σ_i *is defined as in* 2.4, $i = 0, 1, 2$, *then for each non-empty compact and contractible in itself set* $K' \subset K$,

$$F(K'; S; A^*, A)\colon \mathscr{I}(S_{K'}; A^*_{K'}, A_{K'}) \to \mathscr{I}(S; A^*, A)$$

is an equivalence of connected simple systems.

Proof. Let $N_1 \supset N_2 \supset N_3$ be a nested index triple for the R–A pair (A^*, A) of S relative to Φ_K, where K is a compact subset of W. We first will consider the special case of K' being a singleton, $K' = \{v\}$. Then with K' so restricted, referring to diagram (1) of the proof of 2.5 above, each of the vertical arrows, namely,

$$N_{2,v}/N_{3,v} \to N_2/N_3, \qquad N_{1,v}/N_{3,v} \to N_1/N_3, \qquad N_{1,v}/N_{2,v} \to N_1/N_3,$$

respectively, defines in the manner of 2.6 maps $F(v; A)\colon \mathscr{I}(A_v) \to \mathscr{I}(A)$, $F(v; S)\colon \mathscr{I}(S_v) \to \mathscr{I}(S)$, and $F(v, A^*)\colon \mathscr{I}(A^*_v) \to \mathscr{I}(A^*)$.

Now by 2.7, for each of the sections $\sigma_i\colon W \to \mathscr{S}(\varphi)$ $(i = 0, 1, 2)$, for each $\lambda \in W$, there are W-neighborhoods $W_{i,\lambda}$ $(i = 0, 1, 2)$ about λ so that if $K \subset W_{i,\lambda}$ is compact and contractible in itself, then in the notation of the preceding paragraph, respectively as $i = 0, 1, 2$, $F(v; S)$, $F(v; A^*)$, $F(v, A)$ is an equivalence for each $v \in K$. Set $W_\lambda = W_{0,\lambda} \cap W_{1,\lambda} \cap W_{2,\lambda}$. Then each of $F(v; S)$, $F(v; A^*)$, $F(v; A)$ is an equivalence for $v \in K \subset W_\lambda$, K compact and contractible in itself; hence by 2.2 each of the vertical arrows in diagram (1) of 2.5 is a homotopy equivalence; hence by the functorial construction of the long coexact Puppe-sequence, the infinite homotopy commutative ladder induced by diagram (1) of 2.5

$$\begin{array}{ccccccccc} N_{2,v}/N_{3,v} & \longrightarrow & N_{1,v}/N_{3,v} & \longrightarrow & N_{1,v}/N_{2,v} & \xrightarrow{c_1} & S(N_{2,v}/N_{3,v}) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ N_2/N_3 & \longrightarrow & N_1/N_3 & \longrightarrow & N_1/N_2 & \longrightarrow & S(N_2/N_3) & \longrightarrow & \cdots \end{array}$$

is an equivalence in the category of long coexact sequences of pointed spaces and homotopy commutative infinite ladders. By the remark after 2.2 and by 2.5, the above ladder defines an equivalence

$$F(v; S; A^*, A) \colon \mathscr{I}(S_v; A_v^*, A_v) \to \mathscr{I}(S; A^*, A).$$

For the general case of non-empty compact and contractible in itself K' with $K' \subset K \subset W_\lambda$, choose $v \in K'$; then by 2.5(iii) with $K'' = \{v\}$

$$F(v; S; A^*, A) = F(K'; S; A^*, A) \circ F(v; S_{K'}; A_{K'}^*, A_{K'}),$$

and by the result of the preceding paragraph $F(v; S; A^*, A)$ and $F(v; S_{K'}; A_{K'}^*, A_{K'})$ are equivalences whence $F(K'; S; A^*, A)$ is too. ∎

2.9. Definition (Partition of an Arc and Grids for $I \times I$). Let α be an arc and n a positive integer.

(1) A partition of α with $n+1$ partition points is an ordered n-tuple of elements of α, $(\lambda_0, \lambda_1, \ldots, \lambda_n)$, so that for some homeomorphism c: $[0, 1] \to \alpha$,

$$0 = c^{-1}(\lambda_0) < c^{-1}(\lambda_1) < \cdots < c^{-1}(\lambda_n) = 1,$$

and we define

$$[\lambda_{i-1}, \lambda_i] \equiv c[c^{-1}(\lambda_{i-1}), c^{-1}(\lambda_i)]$$

and call $[\lambda_{i-1}, \lambda_i]$ an interval of the partition for $i = 1, \ldots, n$. If $(\lambda_0, \ldots, \lambda_n)$ is a partition of α we call λ_0 the origin and λ_n the end of α.

(2) Suppose Q is a cover of α by subsets with non-void interior. Call a partition of α subordinate to Q if, and only if, for each interval of the partition there exists an element of Q having the interval as a subset of its interior.

(3) Let $P=(s_0,\ldots,s_m)$ and $P'=(t_0,\ldots,t_n)$ be partitions of I with $s_0=0=t_0$ and $s_m=1=t_n$. Then $P\times P'$ is called a grid for $I\times I$. The point (s_i,t_j) is called the (i,j)th grid point, the set $I\times\{t_j\}$ is called the jth horizontal arc of the grid, and the set $\{s_i\}\times I$ is called the ith vertical arc of the grid ($i=0,\ldots,m$; $j=1,\ldots,n$). The set $[s_{i-1},s_i]\times[t_{j-1},t_j]$ is called the (i,j)th cell of the grid, the set $[s_{i-1},s_i]\times\{t_r\}$ is called the bottom or top arc of the cell as $r=j-1$ or $r=j$, and the set $\{s_r\}\times[t_{j-1},t_j]$ is called the left or right arc of the cell as $r=i-1$ or $r=i$ ($i=1,\ldots,m$; $j=1,\ldots,n$).

(4) Let $P\times P'$ be a grid for $I\times I$ and Q a cover of $I\times I$ by sets with non-void interior. Then the grid $P\times P'$ is called subordinate to the cover Q if, and only if, for each cell of the grid there exists an element of Q having the cell as a subset of its interior.

2.10. Lemma (Independence of Partition). *Let $W\subset\Lambda$ and let σ: $W\to\mathscr{R}\mathscr{A}(\varphi)$ be a section of $\mathscr{R}\mathscr{A}(\varphi)$ over W. Suppose $\{W_\lambda:\lambda\in W\}$ is a W-open cover of W satisfying the equivalence map property of* 2.8, *and suppose $\alpha\subset W$ is an arc and for some $\eta\in W$, $\alpha\subset W_\eta$. Then for every partition $(\lambda_0,\ldots,\lambda_n)$ of α, setting $\alpha_i\equiv[\lambda_{i-1},\lambda_i]$ ($i=1,\ldots,n$),*

$$\begin{aligned}&F(\lambda_n;S_\alpha;A_\alpha^*,A_\alpha)^{-1}\circ F(\lambda_0;S_\alpha;A_\alpha^*,A_\alpha)\\&\quad=F_{n,n}^{-1}\circ F_{n-1,n}\circ F_{n-1,n-1}^{-1}\circ F_{n-2,n-1}\circ\cdots\circ F_{1,1}^{-1}\circ F_{0,1},\end{aligned}$$

where $(S_\alpha;A_\alpha^*,A_\alpha)\equiv\bigcup\{\sigma(\lambda):\lambda\in\alpha\}$ and

$$F_{i,j}\equiv F(\lambda_i;S_{\alpha_j};A_{\alpha_j}^*,A_{\alpha_j})\qquad(i=j-1,j;\ j=1,\ldots,n).$$

Proof. Note that by 2.4, S_α is an isolated invariant set relative to Φ_α and (A_α^*,A_α) is an R–A pair of S_α. Also since α and α_i ($i=1,\ldots,n$) are compact and contractible in themselves defining $F_{i,\alpha}$ and F_j by

$$\begin{aligned}F_{i,\alpha}&\equiv F(\lambda_i;S_\alpha;A_\alpha^*,A_\alpha)\qquad\text{for}\quad i=0,\ldots,n,\\F_j&\equiv F(\alpha_j;S_\alpha;A_\alpha^*,A_\alpha)\qquad\text{for}\quad j=1,\ldots,n,\end{aligned}$$

we have that $F_{i,\alpha}$, F_j, and $F_{i,j}$ are equivalences for all indices for which they are defined. Then by Proposition 2.5(iii)

$$F_{i,j}=F_j^{-1}\circ F_{i,\alpha}\qquad(i=j,j-1;\ j=1,\ldots,n).\tag{1}$$

By induction the proof immediately reduces to the case $n=2$. By (1)

$$\begin{aligned}&F_{2,2}^{-1}\circ F_{1,2}\circ F_{1,1}^{-1}\circ F_{0,1}\\&\quad=(F_{2,\alpha}^{-1}\circ F_2)(F_2^{-1}\circ F_{1,\alpha})(F_{1,\alpha}^{-1}\circ F_1)(F_1^{-1}\circ F_{0,\alpha})\\&\quad=F_{2,\alpha}^{-1}\circ F_{0,\alpha}.\quad\blacksquare\end{aligned}$$

2.11. PROPOSITION (Maps between Connection Indices over an Arc). *Let $\alpha \subset \Lambda$ be an arc and suppose $\sigma\colon \alpha \to \mathscr{RA}(\varphi)$ is a section over α of $\pi_2\colon \mathscr{RA}(\varphi) \to \Lambda$. Then:*

(1) *There exists a cover Q of α by open subsets satisfying the equivalence map property of 2.8 and a partition of α subordinate to this cover.*

(2) *If Q is a cover of α by open subsets satisfying the equivalence map property of 2.8, and if $P = (\lambda_0, \ldots, \lambda_n)$ is a partition of α subordinate to Q, defining $\alpha_i \equiv [\lambda_{i-1}, \lambda_i]$ and*

$$(S_{\alpha_i}; A^*_{\alpha_i}, A_{\alpha_i}) \equiv \bigcup \{\sigma(\lambda)\colon \lambda \in \alpha_i\} \qquad (i = 1, \ldots, n),$$

and

$$F_{i,j} \equiv F(\lambda_i; S_{\alpha_j}; A^*_{\alpha_j}, A_{\alpha_j}),$$

the composition $\bar{F}(\sigma, P)$ defined by

$$\bar{F}(\sigma, P) \equiv F^{-1}_{n,n} \circ F_{n-1,n} \circ F^{-1}_{n-1,n-1} \circ F_{n-1,n-1} \circ \cdots \circ F^{-1}_{1,1} \circ F_{0,1}$$

is an equivalence from $\mathscr{I}(\sigma(\lambda_0))$ to $\mathscr{I}(\sigma(\lambda_n))$ with inverse $\bar{F}(\sigma, \bar{P})$, where

$$\bar{P} = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_0).$$

(3) *If Q and Q' are (perhaps equal) covers of α as in (1) with corresponding subordinate partitions of α P and P', then if P and P' define the same orientation of α,*

$$\bar{F}(\sigma, P) = \bar{F}(\sigma, P').$$

(4) *Suppose $\alpha' \subset \Lambda$ is also an arc and $\sigma'\colon \alpha' \to \mathscr{RA}(\varphi)$ is a section of π_2 over α' so that $\alpha \cup \alpha'$ is an arc with partition $(\lambda_0, \lambda_1, \lambda_2)$ satisfying*

(i) $\alpha = [\lambda_0, \lambda_1]$ *and* $\alpha' = [\lambda_1, \lambda_2]$,

(ii) $\sigma(\lambda_1) = \sigma'(\lambda_1)$;

then $\sigma \cup \sigma'$ is a section of π_2 over $\alpha \cup \alpha'$, and if Q, Q', Q'' are open covers of α, α', and $\alpha \cup \alpha'$, respectively, satisfying the equivalence map property of 2.8 and P, P', and P'' are partitions of α, α', and $\alpha \cup \alpha'$, respectively, subordinate to Q, Q', and Q'', respectively, then

$$\bar{F}(\sigma \cup \sigma', P'') = \bar{F}(\sigma', P') \circ \bar{F}(\sigma, P).$$

(5) *If for each $\lambda \in \alpha$, $S_\lambda = A^*_\lambda \cup A_\lambda$, then for each $\lambda \in \alpha$, the splitting*

class μ^{λ} *of* $\mathscr{I}(S_{\lambda}; A_{\lambda}^{*}, A_{\lambda})$ *is defined, and for any open cover* Q *and partition* $P=(\lambda_0, \ldots, \lambda_n)$ *of* α *as in* (1), *the equivalence*

$$\bar{F}(\sigma, P)\colon \mathscr{I}(\sigma(\lambda_0)) \to \mathscr{I}(\sigma(\lambda_n))$$

is natural relative to the splitting classes μ^{λ_i} *of* $\mathscr{I}(\sigma(\lambda_i))$, $i=0$ *and* $i=n$.

Proof. Statement (1) follows by taking a uniform partition of a parameter interval for α with mesh fine enough so that its image by the parametrization is subordinate to a cover with the equivalence map property.

Statement (2) is immediate from the definitions, statement (3) follows by taking a common refinement of P and P' and applying Lemma 2.10, statement (4) follows from (3) and the definitions after taking a common refinement of $(\lambda_0, \lambda_1, \lambda_2)$ and P'', and statement (5) is immediate from 2.5(iv). ∎

We are almost ready to make $\mathscr{I}$ into a functor from the fundamental groupoid of $\mathscr{RA}(\varphi)$ to $\mathscr{CSS}(\mathscr{HLC})$, where $\mathscr{HLC}$ is the homotopy category of long coexact sequences of pointed spaces and homotopy commutative ladders between them.

2.12. Definition (The Prototype of $\mathscr{I}\colon \Pi(\mathscr{RA}(\varphi)) \to \mathscr{CSS}(\mathscr{HLC})$). Let $\alpha \subset \Lambda$ be an arc with origin λ_0 and end λ_1 and let $\sigma\colon \alpha \to \mathscr{RA}(\varphi)$ be a section of π_2 over α. Define $\bar{F}(\sigma, [\lambda_0, \lambda_1])$ by

$$\bar{F}(\sigma, [\lambda_0, \lambda_1]) \equiv \bar{F}(\sigma, P),$$

where $P=(\eta_0, \ldots, \eta_n)$ is a partition of α subordinate to an open cover Q of α with the equivalence map property as guaranteed by 2.11(1) and with $\eta_0=\lambda_0$ and $\eta_n=\lambda_1$ (reverse the partition P if necessary).

By 2.11(3) this definition is independent of the choice of cover satisfying the equivalence map property and subordinate partition.

To help motivate what follows we outline how we define $\mathscr{I}$ to be a functor on the fundamental groupoid of $\mathscr{RA}(\varphi)$. We have already defined $\mathscr{I}$ on points of $\mathscr{RA}(\varphi)$; so suppose $c\colon I \to \mathscr{RA}(\varphi)$ is a path. We will "reparametrize" the local semi-flow over the image of $\pi_2 \circ c$, by pulling back φ along the path. In the new parametrization, the graph of c, gr(c), is a section of the space of R–A pairs over an oriented arc $\alpha=[\lambda_0, \lambda_1]$, and essentially, $\mathscr{I}(c)$ is defined by $\mathscr{I}(c)=\bar{F}(\mathrm{gr}(c), [\lambda_0, \lambda_1])$. The idea of pulling back along the path to get a section is more or less a standard technique.

2.13. Definition of the Pullback of a Product Parametrization. Let Y be a locally compact Hausdorff space, Γ' a space admitting a flow, denoted $(\gamma', t) \to \gamma' * t$, and Γ_0' an open Hausdorff subset of Γ'.

For $\psi: Y \times \Lambda' \to \Psi \subset \Gamma_0'$ a product parametrization of a local semi-flow, and

$$f: \Lambda'' \to \Lambda'$$

a continuous map on a Hausdorff space, make the following definitions:

(1) Define a flow on $\Gamma' \times \Lambda''$ by $(\gamma', \lambda'') * t \equiv (\gamma' * t, \lambda'')$.

(2) Define $f^*\Psi \equiv \Psi_{f(\Lambda'')} \times \Lambda''$, and note $f^*\Psi$ is a local semi-flow in $\Gamma_0' \times \Lambda''$ which is an open Hausdorff subset of $\Gamma' \times \Lambda''$.

(3) Define $f^*\psi: Y \times \Lambda'' \to f^*\Psi$ by

$$f^*\psi(y, \lambda'') = (\psi(y, f(\lambda''), \lambda'')),$$

and note that $f^*\psi$ is a product parametrization of $f^*\Psi$. $f^*\psi$ is called the pullback of ψ by f.

2.14. PROPOSITION. *Let Λ' and Λ'' be Hausdorff and*

$$\begin{tikzcd} & & \mathscr{E}(\varphi) \arrow[d, "\pi_2"] \\ \Lambda'' \arrow[r, "g"'] \arrow[urr, "G"] & \Lambda' \arrow[r, "h"'] \arrow[ur, "H"'] & \Lambda \end{tikzcd}$$

a commutative diagram of continuous maps where $\mathscr{E}$ is $\mathscr{R}\mathscr{A}$ or $\mathscr{S}$. Then:

(1) *There is a natural homeomorphism, $h^*\mathscr{E}(\varphi) \simeq \mathscr{E}(h^*\varphi)$, where $h^*\mathscr{E}$ is the pullback of $\mathscr{E}$ as in* 1.8.

(2) $\mathscr{E}((h \circ g)^* \varphi) \simeq \mathscr{E}(g^*(h^*\varphi))$.

(3) *Under the identification of* (1),

$$\operatorname{gr}(H): \Lambda' \to \mathscr{E}(h^*\varphi)$$

is a section of

$$h^*\pi_2: \mathscr{E}(h^*\varphi) \to \Lambda',$$

and

$$\operatorname{gr}(G): \Lambda' \to \mathscr{E}((h \circ g)^* \varphi)$$

is a section of

$$g^*h^*\pi_2: \mathscr{E}((h \circ g)^* \varphi) \to \Lambda'',$$

where $\mathrm{gr}(H)$ *and* $gr(G)$ *are the graph maps of* H *and* G;

$$\mathrm{gr}(H)(\lambda') = (\lambda', H(\lambda')), \qquad \mathrm{gr}(G)(\lambda'') = (\lambda'', G(\lambda'')),$$

and there is a commutative diagram

$$\begin{tikzcd}
\mathscr{E}((h \circ g)^* \varphi) \arrow[r, "\bar{g}"] & \mathscr{E}(h^*\varphi) \arrow[r, "\bar{h}"] & \mathscr{E}(\varphi) \arrow[d, "\pi_2"] \\
\Lambda'' \arrow[u, "\mathrm{gr}(G)"] \arrow[r, "g"] \arrow[ur, "G"] & \Lambda' \arrow[u, "\mathrm{gr}(H)"] \arrow[r, "h"] \arrow[ur, "H"] & \Lambda
\end{tikzcd}$$

where $\bar{h}$ *and* $\bar{g}$ *are defined as in* 1.8 *under the identification of* (1) *above.*

(4) *For each* $z \in \mathscr{E}(h^*\varphi)$ *and* $z' \in \mathscr{E}((h \circ g)^* \varphi)$, $\bar{h}$ *and* $\bar{g}$ *induce natural equivalences of connected simple systems*

$$\bar{h}_*: \mathscr{H}(z) \simeq \mathscr{H}(\bar{h}(z)), \qquad \bar{g}_*: \mathscr{H}(z') \simeq \mathscr{H}(\bar{g}(z')),$$

where $\mathscr{H}$ *is* $\mathscr{I}$ *or* $\mathscr{H}$ *is* $\mathscr{J}$ *depending as* $\mathscr{E}$ *is* $\mathscr{S}$ *or* $\mathscr{R}\mathscr{A}$, *and*

$$(\bar{h} \circ \bar{g})_* = \bar{h}_* \circ \bar{g}_*;$$

in particular $\bar{h}_*$ *and* $\bar{g}_*$ *commute with the maps of* 2.5 *or* 2.6 *as appropriate.*

Proof. The points of $h^*\mathscr{R}\mathscr{A}(\varphi)$ are precisely those 4-tuples $(\lambda', S_\lambda, A_\lambda^*, A_\lambda)$ with $\lambda' \in \Lambda'$, $(S_\lambda, A_\lambda^*, A_\lambda) \in \mathscr{R}\mathscr{A}(\varphi)$ and $h(\lambda') = \lambda$. On the other hand, if $\lambda' \in \Lambda'$, then S' is $(h^*\Phi)_{\lambda'}$-isolated if, and only if, $S' = S \times \{\lambda'\}$, where S is $\Phi_{h(\lambda')}$-isolated, and (A^*, A) is an R–A pair of S if, and only if, $(A^* \times \{\lambda'\}, A \times \{\lambda'\})$ is an R–A pair of S'. It follows that the correspondence

$$(\lambda', S_\lambda, A_\lambda^*, A_\lambda) \leftrightarrow (S_\lambda \times \{\lambda'\}, A_\lambda^* \times \{\lambda'\}, A_\lambda \times \{\lambda'\})$$

establishes a natural bijection between $h^*\mathscr{R}\mathscr{A}(\varphi)$ and $\mathscr{R}\mathscr{A}(h^*\varphi)$, and $(\lambda', S_\lambda) \leftrightarrow (S_\lambda \times \{\lambda'\})$ one between $h^*\mathscr{S}(\varphi)$ and $\mathscr{S}(h^*\varphi)$. Because for any product parametrization ψ, $\mathscr{R}\mathscr{A}(\psi)$ is an open subset of $\mathscr{S}(\psi) \oplus \mathscr{S}(\psi) \oplus \mathscr{S}(\psi)$, it follows that it suffices to show the second correspondence above bicontinuous, for then the first is the restriction of

$$\begin{aligned} h^*(\mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi) \oplus \mathscr{S}(\varphi)) &\simeq h^*\mathscr{S}(\varphi) \oplus h^*\mathscr{S}(\varphi) \oplus h^*\mathscr{S}(\varphi) \\ &\simeq \mathscr{S}(h^*\varphi) \oplus \mathscr{S}(h^*\varphi) \oplus \mathscr{S}(h^*\varphi). \end{aligned}$$

Accordingly if, $N \subset X$ is compact, and $U \subset \Lambda(N)$ open and $V \subset \Lambda'$ open, then $h^{-1}(U) \subset \Lambda'(N)$ is open and $V \times \sigma_N(U) \cap h^*\mathscr{S}(\varphi)$ corresponds to $\sigma'_N(V \cap h^{-1}(U))$, a basic open set in $\mathscr{S}(h^*\varphi)$; and if $U' \subset \Lambda'(N)$ is open, then $\sigma'_N(U')$ in $\mathscr{S}(h^*\varphi)$ corresponds to $U' \times \sigma_N(\Lambda(N)) \cap h^*(\mathscr{S}(\varphi))$ which is open in $h^*(\mathscr{S}(\varphi))$, which establishes that the bijection is bicontinuous.

To see (2), apply (1):

$$\mathscr{E}((h \circ g)^* \varphi) \simeq (h \circ g)^* \mathscr{E}(\varphi) \simeq g^* h^* \mathscr{E}(\varphi)$$
$$\simeq g^* \mathscr{E}(h^* \varphi) \simeq \mathscr{E}(g^*(h^* \varphi)).$$

For (3), certainly $\mathrm{gr}(H): \Lambda' \to \Lambda' \times \mathscr{E}(\varphi)$ is continuous and as $h = \pi_2 \circ H$, it follows that $\mathrm{gr}(H)$ has image in $h^*\Phi$ whence $\mathrm{gr}(H)$ is certainly a section of $h^*\mathscr{E}(\varphi)$. Similarly, $\mathrm{gr}(G)$ is a section of $(h \circ g)^* \mathscr{E}(\varphi)$. The commutativity of the diagram follows from an elementary chase left to the reader.

For (4), if $\lambda'' \in \Lambda''$, then $N' \times \{\lambda''\} \subset (g^*(h^*\Phi))_{\lambda''}$ isolates if, and only if, $N' \subset (h^*\Phi)_{g(\lambda'')}$ isolates if, and only if, for some $N \subset \Phi_{hg(\lambda'')}$, N isolates and $N' = N \times \{g(\lambda'')\}$; and similarly for index pairs and triples; whence (4) follows. ▮

2.15. Definition of $\mathscr{I}(c)$ for $c: I \to \mathscr{RA}(\varphi)$, a Path. Let $c: I \to \mathscr{RA}(\varphi)$ be a path. By 2.14(3),(4) there are natural equivalences

$$\overline{(\pi_2 \circ c)}_*: \mathscr{I}(\mathrm{gr}(c)(0)) \simeq \mathscr{I}(c(0)),$$
$$\overline{(\pi_2 \circ c)}_*: \mathscr{I}(\mathrm{gr}(c)(1)) \simeq \mathscr{I}(c(1)),$$

and $\mathrm{gr}(c): I \to \mathscr{RA}(\varphi)$ is a section of $c^*(\pi_2): \mathscr{RA}(\varphi) \to I$. Define $\mathscr{I}(c)$: $\mathscr{I}(c(0)) \to \mathscr{I}(c(1))$ to be the map between connected simple systems which makes the diagram below commutative:

$$\begin{array}{ccc}
\mathscr{I}(\mathrm{gr}(c)(0)) & \xrightarrow{\bar{F}(\mathrm{gr}(c),[0,1])} & \mathscr{I}(\mathrm{gr}(c)(1)) \\
\wr\| & & \wr\| \\
\mathscr{I}(c(0)) & \xrightarrow{\mathscr{I}(c)} & \mathscr{I}(c(1))
\end{array}$$

where $\bar{F}(\mathrm{gr}(c), [0, 1])$ is defined as in 2.12.

2.16. Proposition. *If $c: I \to \mathscr{RA}(\varphi)$ is a path so that $\pi_2 \circ c: I \to \Lambda$ is a homeomorphism onto its image α, then*:

(1) *$\sigma \equiv c \circ (\pi_2 \circ c)^{-1}: \alpha \to \mathscr{RA}(\varphi)$ is a section of π_2 over the arc α.*

(2) *Setting $\lambda_i \equiv \pi_2 \circ c(i)$ for $i = 0$ and 1, $\bar{F}(\sigma, [\lambda_0, \lambda_1]) = \mathscr{I}(c)$.*

Proof. The proof follows from the naturality of the equivalences $\overline{(\pi_2 \circ c)}_*$ relative to the maps of 2.5; i.e., by choosing partitions $P = (0, t_1, \ldots, t_{n-1}, 1)$ of I subordinate to an open cover Q with the equivalence map property and P' of α subordinate to an open cover Q' with the equivalence map property so that P' is the image of P by $\pi_2 \circ c$ with origin λ_0 and end λ_1; whence the diagram below is commutative:

$$\begin{tikzcd}
\mathscr{I}(\mathrm{gr}(c)(0)) \arrow[r, "\overline{F}(\mathrm{gr}(c),[0,1])"] \arrow[d, equal, "(\overline{\pi_2 \circ c})_*"'] & \mathscr{I}(\mathrm{gr}(c)(1)) \arrow[d, equal, "(\overline{\pi_2 \circ c})_*"] \\
\mathscr{I}(c(0)) \arrow[r, "\overline{F}(\sigma,[\lambda_0,\lambda_1])"] & \mathscr{I}(c(1)). \quad \blacksquare
\end{tikzcd}$$

2.17. Proposition. *Let* c_1 *and* c_2 *be paths into* $\mathscr{R}\mathscr{A}(\varphi)$ *with* $c_1(1) = c_2(0)$. *Then* $\mathscr{I}(c_1 * c_2) = \mathscr{I}(c_2) \circ \mathscr{I}(c_1)$.

Proof. First note that $(I, 0) \vee (I, 1)$ is homeomorphic to I with homeomorphism given by

$$g: I \to I \times \{1\} \cup \{0\} \times I = (I, 0) \vee (I, 1),$$
$$t \to (\max\{2t - 1, 0\}, \min\{2t, 1\})$$

which maps $[0, \frac{1}{2}]$ onto $\{0\} \times I$ and $[\frac{1}{2}, 1]$ onto $I \times \{1\}$ preserving orientation.

Note too $g^{-1}(s, t) = 2^{-1}(s + t)$. We write $I \vee I$ for $(I, 0) \vee (I, 1)$, and make the asignments

$$H \equiv c_2 \vee c_1: I \vee I \to \mathscr{R}\mathscr{A}(\varphi),$$
$$h \equiv \pi_2 \circ c_2 \vee \pi_2 \circ c_1: I \vee I \to A,$$
$$G \equiv c_1 * c_2: I \to \mathscr{R}\mathscr{A}(\varphi)$$

and note that $H \circ g = G$. Thus by 2.14 the diagram (1) below commutes:

$$\begin{tikzcd}
\mathscr{R}\mathscr{A}((h \circ g)^* \varphi) \arrow[r, "\bar{g}"] & \mathscr{R}\mathscr{A}(h^*\varphi) \arrow[r, "\bar{h}"] & \mathscr{R}\mathscr{A}(\varphi) \\
I \arrow[u, "\mathrm{gr}(G)"] \arrow[r, "g"] \arrow[urr, "G"] & I \vee I \arrow[u, "\mathrm{gr}(H)"] \arrow[r, "h"] \arrow[ur, "H"] & A. \arrow[u, "\pi_2"']
\end{tikzcd} \tag{1}$$

Let $p_i: I \times I \to I$ be a projection on the ith factor, $i = 1, 2$. Then note that $\mathrm{gr}(H) = \mathrm{gr}(c_1 \circ p_2) \cup \mathrm{gr}(c_2 \circ p_1)$, where

$$I \vee I \supset \{0\} \times I \xrightarrow{\mathrm{gr}(c_1 \circ p_2)} \mathscr{R}\mathscr{A}(h^*\varphi),$$

$$I \vee I \supset I \times \{1\} \xrightarrow{\mathrm{gr}(c_2 \circ p_1)} \mathscr{R}\mathscr{A}(h^*\varphi)$$

$$
\begin{tikzcd}
\mathscr{I}(c_1 * c_2(0)) \arrow[rr, "\mathscr{I}(c_1 * c_2)"] \arrow[d, "\wr", "(\bar{h} \circ \bar{g})_*^{-1}"'] & & \mathscr{I}(c_1 * c_2(1)) \arrow[d, "\wr", "(\bar{h} \circ \bar{g})_*^{-1}"'] \\
\mathscr{I}(\mathrm{gr}(G)(0)) \arrow[rr, "\bar{F}(\mathrm{gr}(G), I)"] \arrow[d, "\wr", "\bar{g}_*"'] & & \mathscr{I}(\mathrm{gr}(G)(1)) \arrow[d, "\wr", "\bar{g}_*"'] \\
\mathscr{I}(\mathrm{gr}(H)(0, 0)) \arrow[rr, "\bar{F}(\mathrm{gr}(H), \alpha)"] \arrow[d, equal] & & \mathscr{I}(\mathrm{gr}(H)(1, 1)) \arrow[d, equal] \\
\mathscr{I}(\sigma_1(0, 0)) \arrow[r, "\bar{F}(\sigma_1, \alpha_1)"] & \mathscr{I}(\sigma_1(0, 1)) = \mathscr{I}(\sigma_2(0, 1)) \arrow[r, "\bar{F}(\sigma_2, \alpha_2)"] & \mathscr{I}(\sigma_2(1, 1))
\end{tikzcd}
$$

FIGURE 3

are sections of $h^*(\pi_2)$ so by 2.11(4) setting

$$\sigma_1 \equiv \mathrm{gr}(c_1 \circ p_2), \qquad \sigma_2 \equiv \mathrm{gr}(c_2 \circ p_1),$$

$$\alpha_1 \equiv [(0, 0), (0, 1)], \qquad \alpha_2 \equiv [(0, 1), (1, 1)], \qquad \alpha \equiv [(0, 0), (1, 1)],$$

we have

$$\bar{F}(\mathrm{gr}(H), \alpha) = \bar{F}(\sigma_2, \alpha_2) \circ \bar{F}(\sigma_1, \alpha_1). \tag{2}$$

Then by the commutativity of the diagram at (1), the naturality of $\bar{g}_*$ and $\bar{h}_*$, Eq. (2), and the definition of $\mathscr{I}(c_1 * c_2)$, the diagram of Fig. 3 is commutative.

On the other hand, defining $a_1, a_2 \colon I \vee I \to I$ by

$$a_1(t) = (t, 0), \qquad a_2(t) = (1, t),$$

as $c_j = H \circ a_j$ $(j = 1, 2)$, by 2.14 the diagram

$$
\begin{tikzcd}
\mathscr{R}\mathscr{A}((h \circ a_j)^* \varphi) \arrow[r, "\bar{a}_j"] & \mathscr{R}\mathscr{A}(h^* \varphi) \arrow[r, "\bar{h}"] & \mathscr{R}\mathscr{A}(\varphi) \\
I \arrow[u, "\mathrm{gr}(c_j)"] \arrow[r, "a_j"] \arrow[urr, "c_j"] & I \vee I \arrow[u] \arrow[r, "h"] \arrow[ur, "H"'] & \Lambda \arrow[u, "\pi_2"']
\end{tikzcd}
$$

is commutative $(j = 1, 2)$; and from this, Eq. (2), the naturality of $(\bar{a}_j)^*$ and $\bar{h}_*$, and the definition of $\mathscr{I}(c_j)$ $(j = 1, 2)$, the diagram of Fig. 4 is commutative; whence juxtaposing the diagrams of Figs. 3 and 4 completes the proof. ∎

$$
\begin{tikzcd}
\mathscr{I}(\sigma_1(0,0)) \arrow[r, "\bar{F}(\sigma_1,\alpha_1)"] \arrow[d, "\wr", "(\bar{a}_1)_*^{-1}"'] & \mathscr{I}(\sigma_1(0,1)) = \mathscr{I}(\sigma_2(0,1)) \arrow[r, "\bar{F}(\sigma_2,\alpha_2)"] \arrow[d, "\wr", "(\bar{a}_1)_*^{-1}"'] & \mathscr{I}(\sigma_2(1,1)) \arrow[d, "\wr"', "(\bar{a}_2)_*^{-1}"] \\
\mathscr{I}(\mathrm{gr}(c_1)(0)) \arrow[r, "\bar{F}(\mathrm{gr}(c_1),I)"] \arrow[d, "\wr", "\bar{h}_*\circ\bar{a}_{1*}"'] & \mathscr{I}(\mathrm{gr}(c_1)(1)) \simeq \mathscr{I}(\mathrm{gr}(c_2)(0)) \arrow[r, "\bar{F}(\mathrm{gr}(c_2),I)"] \arrow[d, "\wr", "\bar{h}_*\circ\bar{a}_{1*}"'] & \mathscr{I}(\mathrm{gr}(c_2)(1)) \arrow[d, "\wr"', "\bar{h}_*\circ\bar{a}_{2*}"] \\
\mathscr{I}(c_1(0)) \arrow[r, "\mathscr{I}(c_1)"] & \mathscr{I}(c_1(1)) = \mathscr{I}(c_2(0)) \arrow[r, "\mathscr{I}(c_2)"] & \mathscr{I}(c_2(1))
\end{tikzcd}
$$

FIGURE 4

2.18. PROPOSITION. *For each* $z \in \mathscr{R}\mathscr{A}(\varphi)$, *let* ε_z *denote the constant path at* z. *Then for each* $z \in \mathscr{R}\mathscr{A}(\varphi)$,

$$\mathscr{I}(\varepsilon_z) = 1_{\mathscr{I}(z)}.$$

Proof. Let $\lambda_0 = \pi_2(z)$ and let ε_{λ_0} be the constant path at λ_0. Then by definition of $\mathscr{I}(\varepsilon_z)$ we have the commutative diagram

$$
\begin{tikzcd}
\mathscr{I}(\mathrm{gr}(\varepsilon_z)(0)) \arrow[r, "\bar{F}(\mathrm{gr}(\varepsilon_z),I)"] \arrow[d, "\wr", "(\bar{\varepsilon}_{\lambda_0})_*"'] & \mathscr{I}(\mathrm{gr}(\varepsilon_z)(1)) \arrow[d, "\wr"', "(\bar{\varepsilon}_{\lambda_0})_*"] \\
\mathscr{I}(z) \arrow[r, "\mathscr{I}(\varepsilon_z)"] & \mathscr{I}(z)
\end{tikzcd}
$$

so we must show that $(\bar{\varepsilon}_{\lambda_0})_* \bar{F}(\mathrm{gr}(\varepsilon_z), I)(\bar{\varepsilon}_{\lambda_0})_*^{-1} = 1_{\mathscr{I}(z)}$. Let $N_1 \supset N_2 \supset N_3$ be an index triple for z, and let s be the associated long coexact sequence. By Remark 1 after 2.1, it suffices to show that

$$(\bar{\varepsilon}_{\lambda_0})_* \bar{F}(\mathrm{gr}(\varepsilon_z), I)(\bar{\varepsilon}_{\lambda_0})_*^{-1}(s,s) = 1_s. \tag{1}$$

Writing $z = (S_{\lambda_0}, A^*_{\lambda_0}, A_{\lambda_0})$, because ε_{λ_0} is the constant path, in $\mathscr{R}\mathscr{A}(\varepsilon^*_{\lambda_0}\varphi)$, for each $t \in I$,

$$\mathrm{gr}(\varepsilon_z)(t) = (S_{\lambda_0} \times \{t\}, A^*_{\lambda_0} \times \{t\}, A_{\lambda_0} \times \{t\}),$$

and for each $t \in I$, $N_1 \times \{t\} \supset N_2 \times \{t\} \supset N_3 \times \{t\}$ is an index triple for $\mathrm{gr}(\varepsilon_z)(t)$ with associated long coexact sequence s_t. It follows that $\bar{F}(\mathrm{gr}(\varepsilon_z), I)(s_0, s_1)$ is the map induced by the diagram

$$
\begin{tikzcd}
N_2 \times \{0\}/N_3 \times \{0\} \arrow[r] \arrow[d] & N_1 \times \{0\}/N_3 \times \{0\} \arrow[r] \arrow[d] & N_1 \times \{0\}/N_2 \times \{0\} \arrow[d] \\
N_2 \times \{1\}/N_3 \times \{1\} \arrow[r] & N_1 \times \{1\}/N_3 \times \{1\} \arrow[r] & N_1 \times \{1\}/N_2 \times \{1\},
\end{tikzcd}
$$

where the vertical arrows are all induced from $(\gamma, 0) \to (\gamma, 1)$. The equality of (1) then follows by definition of $(\bar{\varepsilon}_{\lambda_0})_*$. ∎

The following Theorem is the last fact needed to make $\mathcal{J}$ a functor on $\Pi(\mathcal{R}\mathcal{A}(\varphi))$ to $\mathcal{C}\mathcal{S}\mathcal{S}(\mathcal{H}\mathcal{L}\mathcal{C})$.

2.19. THEOREM. *Suppose c_0 and c_1 are two paths into $\mathcal{R}\mathcal{A}(\varphi)$ which are endpoint homotopic with homotopy H; i.e., $c_i(t) = H(t, i)$ and $H(i, t) = H(i, t')$ for every $t, t' \in I$, for $i = 0, 1$. Then*

$$\mathcal{J}(c_0) = \mathcal{J}(c_1).$$

Proof. For $j = 0, 1$ define $\iota_j: I \to I \times I$ by $\iota_j(s) = (s, j)$; whence $c_j = H \circ \iota_j$. Then where $h \equiv \pi_2 \circ H$ by 2.14 we have for $j = 0, 1$, the commutative diagram:

$$\begin{tikzcd}
\mathcal{R}\mathcal{A}((h \circ \iota_j)^* \varphi) \arrow[r, "\bar{\iota}_j"] & \mathcal{R}\mathcal{A}(h^* \varphi) \arrow[r, "\bar{h}"] & \mathcal{R}\mathcal{A}(\varphi) \\
I \arrow[u, "\mathrm{gr}(c_j)"] \arrow[r, "\iota_j"'] \arrow[urr, "c_j"] & I \times I \arrow[u, "\mathrm{gr}(H)"] \arrow[r, "h"'] \arrow[ur, "H"] & \Lambda \arrow[u, "\pi_2"']
\end{tikzcd}$$

whence we have the commutative diagram of connection indices:

$$\begin{tikzcd}
\mathcal{J}(c_j(0)) \arrow[r, "\mathcal{J}(c_j)"] \arrow[d, "(\bar{h} \circ \bar{\iota}_j)_*^{(-1)^{j+1}}"'] & \mathcal{J}(c_j(1)) \arrow[d, "(\bar{h} \circ \bar{\iota}_j)_*^{(-1)^{j+1}}"] \\
\mathcal{J}(\mathrm{gr}(c_j)(0)) \arrow[r, "F(\mathrm{gr}(c_j), I)"] \arrow[d, "(\bar{\iota}_j)_*^{(-1)^j}"'] & \mathcal{J}(\mathrm{gr}(c_j)(1)) \arrow[d, "(\bar{\iota}_j)_*^{(-1)^j}"] \\
\mathcal{J}(\mathrm{gr}(H)(0, j)) \arrow[r, "F(\bar{\sigma}_j, \alpha_j)"] & \mathcal{J}(\mathrm{gr}(H)(1, j)),
\end{tikzcd} \tag{1}$$

where $\alpha_j = I \times \{j\}$ and $\bar{\sigma}_j = \mathrm{gr}(H) \mid \alpha_j$, $j = 0, 1$.

Let $\beta_i = \{i\} \times I$ and $\bar{\tau}_i = \mathrm{gr}(H) \mid \beta_i$, $i = 0, 1$. Suppose the diagram below is commutative:

$$\begin{tikzcd}
\mathcal{J}(\mathrm{gr}(H)(0, 1)) \arrow[r, "F(\bar{\sigma}_1, \alpha_1)"] \arrow[d, "F(\bar{\tau}_0, \beta_0)"'] & \mathcal{J}(\mathrm{gr}(H)(1, 1)) \arrow[d, "F(\bar{\tau}_1, \beta_1)"] \\
\mathcal{J}(\mathrm{gr}(H)(0, 0)) \arrow[r, "F(\bar{\sigma}_0, \alpha_0)"] & \mathcal{J}(\mathrm{gr}(H)(1, 0)).
\end{tikzcd} \tag{2}$$

Then juxtaposing this diagram with the two obtained by taking $j=0, 1$ in (1) yields the commutative diagram

$$
\begin{tikzcd}
\mathscr{I}(c_1(0)) \arrow[r, "\mathscr{I}(c_1)"] & \mathscr{I}(c_1(1)) \\
\mathscr{I}(\mathrm{gr}(H)(0,1)) \arrow[r, "\bar{F}(\bar{\sigma}_1,\alpha_1)"] \arrow[u, "(\bar{h})_*"] & \mathscr{I}(\mathrm{gr}(H)(1,1)) \arrow[u, "(\bar{h})_*"'] \\
\mathscr{I}(\mathrm{gr}(H)(0,0)) \arrow[r, "\bar{F}(\bar{\sigma}_0,\alpha_0)"] \arrow[u, "\bar{F}(\bar{\tau}_0,\beta_0)"] & \mathscr{I}(\mathrm{gr}(H)(1,0)) \arrow[u, "\bar{F}(\bar{\tau}_1,\beta_1)"'] \\
\mathscr{I}(c_0(0)) \arrow[r, "\mathscr{I}(c_0)"] \arrow[u, "(\bar{h})_*^{-1}"] & \mathscr{I}(c_0(1)), \arrow[u, "(\bar{h})_*^{-1}"']
\end{tikzcd}
\tag{3}
$$

and the proof will be complete once we have shown that

$$(\bar{h})_* F(\bar{\tau}_i,\beta_i)\, \bar{h}_*^{-1} = 1_{\mathscr{I}(z_i)}, \tag{4}$$

where $z_i = c_0(i) = c_1(i)$ $(i=0, 1)$.

We begin by showing that the diagram of (2) is commutative. By 2.8, let Q be an open cover of $I \times I$ with the equivalence map property ($h^*\varphi\colon X \times I \times I \to h^*\Phi$ is a product parametrization and $\mathrm{gr}(H)$ a section of $\mathscr{R}\mathscr{A}(h^*\varphi)$), and let ε be a Lebesgue number for Q relative to the standard Euclidean metric. Then choose a positive integer n with $\sqrt{2}/n < \varepsilon$, and define a partition P of I with $n+1$ partition points by $P = (t_0, t_1, \dots, t_n)$ with $t_i = i/n$, $i = 0, 1, \dots, n$. Then $P \times P$ is a grid for $I \times I$ which is subordinate to the cover Q. Label the (i,j)th point of the grid $\lambda_{i,j}$, label the jth horizontal arc of the grid a_j, and label the ith vertical arc of the grid b_i $(i = 0, \dots, n;$ $j = 0, \dots, n)$. Also, label the (i,j)th cell of the grid $e_{i,j}$, label the bottom and top arcs of $e_{i,j}$ by $a_{i,j-1}$ and $a_{i,j}$, respectively, and label the left and right

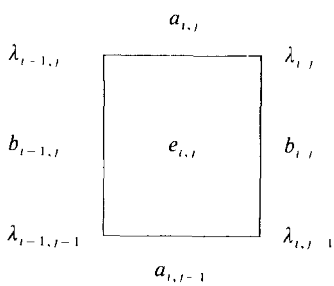

FIGURE 5

vertical arcs of $e_{i,j}$ by $b_{i-1,j}$ and $b_{i,j}$, respectively ($i=1,\dots,n$; $j=1,\dots,n$). (See Fig. 5.) Referring to 2.8, let $(S;A^*,A)=\bigcup\{\mathrm{gr}(H)(s,t)\colon (s,t)\in I\times I\}$ and to help simplify the notation we purposely confuse $e_{i,j}$ with $(S_{e_{i,j}};A^*_{e_{i,j}},A_{e_{i,j}})$ and similarly for the left, right, bottom, and top arcs of the cell.

Then as Q has the equivalence map property and $P\times P$ is subordinate to Q by 2.5(iii), we obtain the identities

(i) $F(\lambda_{p,q},a_{i,q})=F(a_{i,q},e_{i,j})^{-1}\circ F(\lambda_{p,q},e_{i,j})$,

(ii) $F(\lambda_{p,q},b_{p,j})=F(b_{p,j},e_{i,j})^{-1}\circ F(\lambda_{p,q},e_{i,j})$,

where $p=i-1,i$; $q=j-1,j$; $i=1,\dots,n$; $j=1,\dots,n$. Then using these we get the identities

(iii) $F(\lambda_{i,q},a_{i,q})^{-1}\circ F(\lambda_{i-1},a_{i,q})=F(\lambda_{i,q},e_{i,j})^{-1}\circ F(\lambda_{i-1,q},e_{i,j})$,

(iv) $F(\lambda_{p,j},b_{p,j})^{-1}\circ F(\lambda_{p,j-1},b_{p,j})=F(\lambda_{p,j},e_{i,j})^{-1}\circ F(\lambda_{p,j-1},e_{i,j})$,

where $p=i-1,i$; $q=j-1,j$; $i=1,\dots,n$; $j=1,\dots,n$. It follows that

(v) $F(\lambda_{i,j-1},a_{i,j-1})^{-1}\circ F(\lambda_{i-1,j-1},a_{i,j-1})=(F(\lambda_{i,j-1},b_{i,j})^{-1}$
$\circ F(\lambda_{i,j},b_{i,j}))\circ(F(\lambda_{i,j},a_{i,j})^{-1}\circ F(\lambda_{i-1,j},a_{i,j}))$
$\circ(F(\lambda_{i-1,j},b_{i-1,j})^{-1}\circ F(\lambda_{i-1,j-1},b_{i-1,j}))$ for $i=1,\dots,n$,
$j=1,\dots,n$.

Defining

$$\sigma_{i,j}\equiv\mathrm{gr}(H)\mid a_{i,j},\qquad \tau_{i,j}\equiv\mathrm{gr}(H)\mid b_{i,j},$$

the identity of (v) becomes for $i=1,\dots,n$, $j=1,\dots,n$,

(vi) $\bar F(\sigma_{i,j-1},a_{i,j-1})=\bar F(\tau_{i,j},b_{i,j})^{-1}\circ\bar F(\sigma_{i,j},a_{i,j})\circ\bar F(\tau_{i-1,j},b_{i-1,j})$,

and by composing the left-hand sides of (vi) it follows that for $j=1,\dots,n$,

(vii) $\bar F(\sigma_{n,j-1},a_{n,j-1})\circ\cdots\circ\bar F(\sigma_{1,j-1},a_{1,j-1})=\bar F(\tau_{n,j},b_{n,j})^{-1}$
$\circ\bar F(\sigma_{n,j},a_{n,j})\circ\cdots\circ\bar F(\sigma_{1,j},a_{1,j})\circ\bar F(\tau_{0,j},b_{0,j})$.

Hence defining for $i=0,\dots,n$, $j=0,\dots,n$,

$$\sigma_j\equiv\mathrm{gr}(H)\mid a_j,\qquad \tau_i\equiv\mathrm{gr}(H)\mid b_i$$

by 2.11(4), (vii) becomes

(viii) $\bar F(\sigma_{j-1},a_{j-1})=\bar F(\tau_{n,j},b_{n,j})^{-1}\circ\bar F(\sigma_j,a_j)\circ\bar F(\tau_{0,j},b_{0,j})$;

hence after n recursive substitutions we get that

(ix) $\bar F(\sigma_0,a_0)=\bar F(\tau_n,b_n)^{-1}\circ\bar F(\sigma_n,a_n)\circ\bar F(\tau_0,b_0)$.

However, by definition $a_0=\alpha_0$, $a_n=\alpha_1$, $b_0=\beta_0$, $b_n=\beta_1$, $\sigma_0=\bar\sigma_0$, $\sigma_n=\bar\sigma_1$, $\tau_0=\bar\tau_0$, and $\tau_n=\bar\tau_1$; whence (ix) says the diagram of (2) is commutative.

It remains to show that (4) holds. Accordingly, for $i = 0, 1$. Define $v_i : I \to I \times I$ by $v_i(t) = (i, t)$, and note that $H \circ v_i$ is the constant path at z_i, ε_{z_i}. Thus the diagram

$$\begin{tikzcd}
 & & \mathscr{R}\mathscr{A}(\varphi) \\
I \arrow[r, "v_i"] \arrow[urr, "\varepsilon_{z_i}"] & I \times I \arrow[r, "h"] \arrow[ur, "H"'] & \Lambda \arrow[u, "\pi_2"']
\end{tikzcd}$$

is commutative for $i = 0, 1$; hence by 2.14 and the definition of ε_{z_i}, there is the commutative diagram $(i = 0, 1)$

$$\begin{tikzcd}
\mathscr{I}(\mathrm{gr}(\varepsilon_{z_i})(0)) \arrow[r, "\bar{F}(\mathrm{gr}(\varepsilon_{z_i}), I)"] \arrow[d, "\bar{v}_{i*}"'] & \mathscr{I}(\mathrm{gr}(\varepsilon_{z_i})(1)) \arrow[d, "\bar{v}_{i*}"] \\
\mathscr{I}(\mathrm{gr}(H)(i, 0)) \arrow[r, "\bar{F}(\bar{\tau}_i, \beta_i)"] \arrow[d, "\bar{h}_*"'] & \mathscr{I}(\mathrm{gr}(H)(i, 1)) \arrow[d, "\bar{h}_*"] \\
\mathscr{I}(z_i) \arrow[r, "\mathscr{I}(\varepsilon_{z_i})"] & \mathscr{I}(z_i)
\end{tikzcd}$$

whence (4) is immediate. ∎

Remark. If in the statement of Theorem 2.19, $\mathscr{R}\mathscr{A}(\varphi)$ and $\mathscr{I}$ are replaced by $\mathscr{S}(\varphi)$ and $\mathscr{I}$, respectively, then the resulting statement is [2, Chap. IV, Theorem 2.5] which is given without proof. However, John Montgomery does prove a version of the theorem for $\mathscr{S}(\varphi)$ [22, Theorem 4] in the context of flows on compact metric spaces wherein only index spaces arising from isolating blocks are admitted. Montgomery's terminology and proof seem to be motivated by techniques of analytic continuation in complex function theory, in particular the proof of the monodromy theorem. Our proof given above is motivated by standard proofs of Van Kampen's theorem; cf. [11].

2.20. THEOREM. *$\mathscr{I}$ is a contravariant functor on the fundamental groupoid of $\mathscr{R}\mathscr{A}(\varphi)$ to the category of connected simple systems in $\mathscr{H}\mathscr{L}\mathscr{C}$ the homotopy category of long coexact sequences of pointed spaces and infinite commutative ladders between them.*

Proof. That $\mathscr{I}$ is well-defined on homotopy classes of paths is 2.19, and 2.18 shows $\mathscr{I}(\varepsilon_z) = 1_{\mathscr{I}(z)}$ for each $z \in \mathscr{R}\mathscr{A}(\varphi)$, and 2.17 shows that $\mathscr{I}(c_1 * c_2) = \mathscr{I}(c_2) \circ \mathscr{I}(c_1)$. ∎

2.21. COROLLARY. *Let* $z \in \mathscr{R}\mathscr{A}(\varphi)$. *Then* $\mathscr{I}$ *defines an anti-homomorphism on* $\pi_1(\mathscr{R}\mathscr{A}(\varphi), z)$ *into* $\operatorname{Aut}(\mathscr{I}(z))$, *where* $\operatorname{Aut}(\mathscr{I}(z))$ *is the group of equivalences from* $\mathscr{I}(z)$ *to itself, each equivalence a morphism in* $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H}\mathscr{L}\mathscr{C})$.

2.22. COROLLARY. *By restriction,* $\mathscr{I}$ *defines a contravariant functor* $\mathscr{I}$ *on the fundamental groupoid of* $\mathscr{S}(\varphi)$ *to the category of connected simple systems in* $\mathscr{H}\mathscr{T}^*$.

Proof. For any isolated invariant set S, $(S, \varnothing)$ is an R–A pair of S, and if $\langle N_1, N_2\rangle$ is a nested index pair for S, $\langle N_1, N_2, N_2\rangle$ is a nested index triple for $(S, \varnothing)$. $\mathscr{I}(S)$ is of course the Morse index of S, and if b is a path in $\mathscr{S}(\varphi)$, claim $b \oplus b \oplus e$ is a path in $\mathscr{R}\mathscr{A}(\varphi)$, where e is the path at the empty invariant set over $\pi_2 \circ b$; i.e., $e(t) = (\pi_2 \circ b(t), \varnothing)$. Assuming the claim, $\mathscr{I}(b)$ is defined to be that "slice" of the diagram for $\mathscr{I}(b \oplus b \oplus e)$ corresponding to the arrow between the indices of the repellers.

To show the claim only requires showing e continuous. Accordingly, suppose $e(t) \in \sigma_N(U)$, $\Lambda(N) \supset U$ open. Then also

$$e(t) = (\pi_2 \circ b(t), \varnothing) \in \sigma_\varnothing(\Lambda(\varnothing)) \cap \sigma_N(U);$$

in particular, $\pi_2 \circ b(t) \in U \cap \Lambda(\varnothing) \subset \Lambda(N) \cap \Lambda(\varnothing)$. Whence by 1.5, there is an open W about $\pi_2 \circ b(t)$ so that for $\lambda \in W$, $\varphi_\lambda(N)$ and $\varphi_\lambda(\varnothing)$ have the same maximal invariant set, namely, $\varnothing$. The continuity of $\pi_2 \circ b$ then guarantees an open $\mathscr{L}$ about t so that for $t' \in \mathscr{L}$

$$\pi_2 \circ b(t') \in W \cap U;$$

hence

$$e(t') = (\pi_2 \circ b(t'), \varnothing) \in \sigma_N(W \cap U) \subset \sigma_N(U). \quad \blacksquare$$

As an aid to the reader's understanding, there is the following partial reformulation of 2.20.

2.23. COROLLARY (Naturality of the Connection Map). *Let* β *be a path (class) in* $\mathscr{R}\mathscr{A}(\varphi)$. *Then for every pair of nested index triples* $\langle N_1(0), N_2(0), N_3(0)\rangle$, $\langle N_1(1), N_2(1), N_3(1)\rangle$ *for* $\beta(0)$ *and* $\beta(1)$, *respectively, the diagram below commutes*:

$$\begin{tikzcd}
N_2(0)/N_3(0) \arrow[r] \arrow[d, "\mathscr{I}(\beta_2)"'] & N_1(0)/N_3(0) \arrow[r] \arrow[d, "\mathscr{I}(\beta_0)"'] & N_1(0)/N_2(0) \arrow[r, "c(0)"] \arrow[d, "\mathscr{I}(\beta_1)"'] & S(N_2(0)/N_3(0)) \arrow[r] \arrow[d] & \cdots \\
N_2(1)/N_3(1) \arrow[r] & N_1(1)/N_3(1) \arrow[r] & N_1(1)/N_2(1) \arrow[r, "c(1)"] & S(N_2(1)/N_3(1)) \arrow[r] & \cdots
\end{tikzcd} \tag{1}$$

where $c(0)$ and $c(1)$ are the connection maps, where β_0, β_1, and β_2 are the paths into $\mathscr{S}(\varphi)$ which are the components of β, i.e., $\beta = \beta_0 \oplus \beta_1 \oplus \beta_2$, and where every third vertical arrow after the first three is the suspension of the one three places to the left.

In particular for $i = 0, 1, 2,\ldots$, deleting the arguments,

$$S^{i+1}\,\mathscr{I}(\beta_2) \circ S^i c(0) = S^i c(1) \circ S^i\,\mathscr{I}(\beta_1);$$

i.e., $\mathscr{I}(\beta_1)$ and $\mathscr{I}(\beta_2)$ define a natural transformation between the connection maps at either end of the path.

Proof. If $s(0)$ and $s(1)$ are the long coexact sequences which are the top and bottom rows, respectively, of the diagram (1), then the diagram (1) is just $\mathscr{I}(\beta)(s(0), s(1))$ by definition of $\mathscr{I}(\beta)$ as a morphism in $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H}\mathscr{L}\mathscr{C})$. ∎

Remark. Conley states 2.23 without proof [2, Chap. IV, Proposition 3.1].

2.24. Theorem (Naturality of the Splitting Map). *Let β be a path class in $\mathscr{R}\mathscr{A}(\varphi)$ so that writing $\beta(t) = (\lambda(t), S(t), A^*(t), A(t))$ for $t \in I$,*

$$S(t) = A^*(t) \cup A(t).$$

Then for each $t \in I$, $\mu(t)$, the splitting class of $\mathscr{I}(S(t), A^*(t), A(t))$ is defined, and for $i = 0, 1, 2,\ldots$,

$$S^i\,\mathscr{I}(\beta_0) \circ S^i \mu(0) = S^i \mu(1) \circ S^i\,\mathscr{I}(\beta_1);$$

i.e., if $\langle N_1(j), N_2(j), N_3(j)\rangle$ is a nested index triple for $\beta(j)$ $(j = 0, 1)$, the diagram below is commutative;

$$\begin{tikzcd}
N_1(0)/N_3(0) \arrow[d, "\mathscr{I}(\beta_0)"'] & N_1(0)/N_2(0) \arrow[l, "\mu(0)"'] \arrow[d, "\mathscr{I}(\beta_1)"] \\
N_1(1)/N_3(1) & N_1(1)/N_2(1) \arrow[l, "\mu(1)"']
\end{tikzcd}$$

i.e., $\mathscr{I}(\beta_0)$ and $\mathscr{I}(\beta_1)$ define a natural transformation between $\mu(0)$ and $\mu(1)$, where β_0 and β_1 are as in 2.23 above.

Proof. This follows immediately from 2.11(5), 2.12, and the definition of $\mathscr{I}(\beta)$. ∎

3. Homology of Connected Simple Systems in $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H}\mathscr{T}^*)$

It is clear that any reduced homology theory $\tilde{H}_*$ (with coeficients in a ring R) on $\mathscr{H}\mathscr{T}^*$ defines a functor on $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H}\mathscr{T}^*)$ to $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{G}_R\mathscr{M})$, also denoted $\tilde{H}_*$, where $\mathscr{G}_R\mathscr{M}$ is the category of graded left R-modules, by applying $\tilde{H}_*$ to the objects and morphisms of a $\mathscr{C} \in \mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H}\mathscr{T}^*)$.

It is therefore convenient to make the following definition.

3.1. Definition (Homology of a Connected Simple System). If $\mathscr{C}$ is a connected simple system in $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H}\mathscr{T}^*)$, by an homology class $\bar{\alpha}$ of $\mathscr{C}$ we mean an equivalence class of homology classes of objects of $\mathscr{C}$ defined by

$$\alpha \sim \alpha'$$

if, and only if,

$$\tilde{H}_*(f)\alpha = \alpha',$$

where $f\colon A \to A'$ is a morphism of $\mathscr{C}$ and $\alpha \in \tilde{H}_*(A)$ and $\alpha' \in \tilde{H}_*(A')$. It is immediate that this is a well-defined equivalence relation and that the collection of equivalence classes can be given the structure of a graded R-module which is isomorphic to $\tilde{H}_*(A)$ for each object A of $\mathscr{C}$. This collection of equivalence classes with the imposed R-module structure will be denoted $\tilde{H}_*(\mathscr{C})$. Note that this creates an ambiguity because $\tilde{H}_*(\mathscr{C})$ is already being used to denote a connected simple system in $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{G}_R\mathscr{M})$. Any ambiguity is resolved in context, although any confluence of the two definitions arising in one's mind is probably helpful.

Note that the equivalence relation defined above is preserved by morphisms of $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{G}_R\mathscr{M})$. Thus if $F\colon \mathscr{C} \to \mathscr{D}$ is a map between connected simple systems in $\mathscr{H}\mathscr{T}^*$, there is a map on homology $F_*\colon \tilde{H}_*(\mathscr{C}) \to \tilde{H}_*(\mathscr{D})$. Again there is an ambiguity: F_* denotes both a map between connected simple systems and a graded R-module homomorphism.

Of course our particular interest in the above situation arises when $\mathscr{C}$ and $\mathscr{D}$ are Morse indices of isolated invariant sets. Specifically, each of the arrows in a long coexact sequence for an index triple of an R–A pair (A^*, A) of S, by Remark 1 after Definition 2.1, defines a map between connected simple systems in $\mathscr{C}\mathscr{S}\mathscr{S}(\mathscr{H}\mathscr{T}^*)$.

For a connection map $c\colon \mathscr{I}(A^*) \to S\mathscr{I}(A)$ (here $S\mathscr{I}(A)$ denotes the reduced suspension) passing to homology and composing with the well-known isomorphism of degree -1 from the homology of the suspension of a space to the homology of the space (apply the axiomatic Meyer–Veitoris sequence to the upper and lower reduced cones of the reduced suspension—cf. [18, pp. 190 and 209]) yields the algebraic connection homomorphism of degree -1

$$\tilde{H}_{*+1}(\mathscr{I}(A^*)) \xrightarrow{c_*} \tilde{H}_*(\mathscr{I}(A))$$

which coincides with the usual algebraic homomorphism ∂_* of the triple $(N_1/N_3, N_2/N_3, *)$—this is immediate from the definitions in the case of spectral reduced theory, cf. [11, 19] and the proof of [14, Theorem 3.2], and hence follows also for the singular construction as given in [11, 19].

When $S = A^* \cup A$, any connection map for an index triple of $(S; A^*, A)$ is inessential and it follows that $c_* \equiv 0$; hence there is the short exact sequence

$$0 \longrightarrow \tilde{H}_*(\mathscr{I}(A)) \longrightarrow \tilde{H}_*(\mathscr{I}(S)) \xrightarrow{p_*} \tilde{H}_*(\mathscr{I}(A^*)) \longrightarrow 0.$$

Moreover this sequence is split as the splitting class μ of $\mathscr{I}(S; A^*, A)$ induces a map

$$\mu_*: \tilde{H}_*(\mathscr{I}(A^*)) \to \tilde{H}_*(\mathscr{I}(S))$$

which is a right inverse of p_*.

For convenience we make the following definition after which we give a theorem which provides for the convenient exploitation of the splitting class for the purpose of proving existence theorems for parametrized families of differential equations.

3.2. DEFINITION. Let $\varphi: X \times \Lambda \to \Gamma_0$ be a product parametrization of a local flow Φ. Suppose $\beta = \beta_0 \oplus \beta_1 \oplus \beta_2$ is a path from $[0, 1]$ into $\mathscr{R}\mathscr{A}(\varphi)$, and note that we may write $\beta(t) = (\lambda(t), S(t), A^*(t), A(t))$ as in Theorem 2.24 above. Suppose for $i = 0, 1$, $S(i) = A^*(i) \cup A(i)$, so that the splitting class $\mu(i)$ of $\mathscr{I}(\beta(i))$ is defined. Then as a map on homology define

$$[\mathscr{I}(\beta), \mu]_*: \tilde{H}_*(\mathscr{I}(A^*(0))) \to \tilde{H}_*(\mathscr{I}(S(1)))$$

by

$$[\mathscr{I}(\beta), \mu]_* = \mathscr{I}(\beta_0)_* \, \mu(0)_* - \mu(1)_* \, \mathscr{I}(\beta_1)_*;$$

i.e., $[\mathscr{I}(\beta), \mu]_*$ is the difference on homology between the two possible compositions afforded by the diagram of Theorem 2.24 from the upper right-hand to the lower left-hand lattice point.

3.3. THEOREM. *With φ and β as in Definition 3.2 above, if $[\mathscr{I}(\beta), \mu]_* \alpha \neq 0$ for some $\alpha \in \tilde{H}_*(\mathscr{I}(A^*(0)))$, then for some t, $0 < t < 1$, $C(A^*(t), A(t)) \neq \varnothing$; i.e., there is a heteroclinic orbit from $A^*(t)$ to $A(t)$.*

Proof. Because $[\mathscr{I}(\beta), \mu]_* \alpha \neq 0$, the diagram in the statement of Theorem 2.24 does not commute. Hence, the splitting map for $\mathscr{I}(\beta(t))$ is not defined for some t, $0 < t < 1$. By [14, Proposition 4.1],

$$\varnothing \neq C(A^*(t), A(t)) \equiv S(t) \backslash (A^*(t) \cup A(t)).$$

For $\gamma \in C(A^*(t), A(t))$, we have $\omega^*(\gamma) \subset A^*(t)$ and $\omega(\gamma) \subset A(t)$, for instance, by [2, Chap. II, Sect. 5.1.A]. The desired heteroclinic orbit is $\gamma \cdot \mathbf{R}$. ∎

4. The Space of Morse Decompositions

The sole purpose of this section is to define the space $\mathscr{M}(\varphi)$ of Morse decompositions for a product parametrization $\varphi: X \times \Lambda \to \Gamma_0$ and show that there is a local homeomorphism $\pi_\infty: \mathscr{M}(\varphi) \to \Lambda$. The results needed to carry this out are generalizations of those appearing in Section 1 for R–A pairs and essentially follow from the latter by induction.

Interesting and important applications of Morse decompositions are given in [24, 25]. Selgrade studies Morse decompositions for linearized flows on the projectivized tangent bundle of a manifold and obtains important results on hyperbolic chain-recurrent sets. Young applies Morse decompositions to the study of the semi-flow induced by the evolution equation of burning gas theory.

4.1. Definition of the Space of Morse Decompositions. Let $\varphi: X \times \Lambda \to \Gamma_0$ be a product parametrization with X locally compact.

(1) For each positive integer n, let

$$\bar{\pi}_{n+1}: \bigoplus^{n+1} \mathscr{S}(\varphi) \to \Lambda$$

denote the Whitney sum with $\mathscr{S}(\varphi)$ taken as a summand $n+1$ times; i.e., $\bigoplus^{n+1} \mathscr{S}(\varphi)$ is the pullback by the generalized diagonal of the cartesian product with $n+1$ factors of $\pi: \mathscr{S}(\varphi) \to \Lambda$; and let $\mathscr{M}(\varphi; n)$ denote the space of Morse decompositions of length n defined by $(S, M_1, ..., M_n)$ is in $\mathscr{M}(\varphi; n)$ if, and only if, for some $\lambda \in \Lambda$, S is a Φ_λ-isolated invariant set and $\{M_1, ..., M_n\}$ is a Morse decomposition of S corresponding to some sequence of attractors $S \supset A_0 \supset A_1 \supset \cdots \supset A_n = \varnothing$; and let $\pi_n: \mathscr{M}(\varphi; n) \to \Lambda$ be $\Delta^*(\bar{\pi}_{n+1}) \mid \mathscr{M}(\varphi; n)$.

(2) Given a sequence of attractors $S = A_0 \supset A_1 \supset \cdots \supset A_n = \varnothing$, the sequence can be extended to a sequence of length $n+1$ by making $A_{n+1} = \varnothing$. If $\{M_1, ..., M_n\}$ is the Morse decomposition of the original sequence, the extended sequence has Morse decomposition $\{M_1, ..., M_{n+1}\}$,

where $M_{n+1} = \varnothing$. It is straight-forward to check that this results in an embedding of $\mathscr{M}(\varphi; n)$ into $\mathscr{M}(\varphi; n+1)$.

In this manner each $\mathscr{M}(\varphi; n)$ can be regarded as a subset of $\oplus^{\infty} \mathscr{S}(\varphi)$, the pullback by the diagonal of the countable cartesian product of $\mathscr{S}(\varphi)$, with $\mathscr{M}(\varphi; n) \subset \mathscr{M}(\varphi; n+1)$. Define

$$\mathscr{M}(\varphi) \equiv \bigcup_{n=1}^{\infty} \mathscr{M}(\varphi; n)$$

and

$$\pi_{\infty}: \mathscr{M}(\varphi) \to \Lambda$$

by

$$\pi_{\infty} \equiv \bigcup_{n=1}^{\infty} \pi_n.$$

Then $\mathscr{M}(\varphi)$ has the relative topology of $\oplus^{\infty} \mathscr{S}(\varphi)$ and π_{∞} is well-defined and continuous.

The following lemma due to Conley was mentioned in [14], and is implicitly used in 4.3 and 4.4 below. For a proof see [2].

4.2. Lemma [2, Chap. II, Proposition 5.3.D]. *Let S be a compact Hausdorff invariant set, and suppose A_1 is an attractor of S and A_2 is an attractor of A_1. Then A_2 is an attractor of S.*

Remark. The interested reader can provide his own proof by choosing U to be a compact S-neighborhood of A_2 disjoint from $M_2 \cup A_1^*$ and applying [14, Proposition 1.4] to show U^+ is the required attractor neighborhood, where M_2 is the dual repeller of A_2 relative to A_1 and A_1^* the dual repeller of A_1 in S.

4.3. Proposition. *Let $S \subset \Gamma_0$ be a compact Hausdorff invariant set with Morse decomposition $\{M_1, \ldots, M_n\}$ associated to the sequence of attractors $S = A_0 \supset A_1 \supset \cdots \supset A_n = \varnothing$ $(n \geqslant 1)$. Suppose $W_1, \ldots, W_n$ are pairwise disjoint open sets with $M_i \subset W_i$, $i = 1, \ldots, n$. Then there exists V_0 open so that $S \subset V_0 \subset \Gamma_0$ and if $S' \subset V_0$ and S' is a compact invariant set, then, setting M_i^* to be the maximal invariant set of $W_i \cap S'$, $\{M_1', \ldots, M_n'\}$ is a Morse decomposition of S' associated to a sequence of attractors $S' = A_0' \supset \cdots \supset A_n' = \varnothing$. In particular, along with V_0, there exist open sets $V_1, \ldots, V_n$; $U_1, \ldots, U_n$; $U_1^*, \ldots, U_n^*$ satisfying for $i = 1, \ldots, n$,*

(i) *$M_i \subset U_i^* \subset W_i$ and $A_i \subset U_i \subset V_i$ and $U_i^* \cap U_i = \varnothing$;*

(ii) *$A_i' = \omega(U_i \cap A_{i-1}'; A_{i-1}')$ and $M_i' = \omega^*(U_i^* \cap A_{i-1}'; A_{i-1}')$.*

Proof. If $n = 1$, then $M_1 = S$ and defining $V_0 \equiv W_1$, $V_1 \equiv \varnothing$, $U_1^* \equiv W_1$, and $U_1 \equiv \varnothing$, note that (i) is satisfied, and if $S' \subset V_0$ is a compact invariant set, then $M_1' = S'$ and (ii) is trivially satisfied.

Having disposed of this special case, suppose that $n > 1$; the proof is a construction by induction. Begin the construction by making the following assignments:

$$U_n \equiv \varnothing, \qquad U_n^* \equiv W_n, \qquad V_n \equiv \varnothing,$$
$$U_{n-1} \equiv W_n, \qquad U_{n-1}^* \equiv W_{n-1}, \qquad V_{n-1} \equiv W_{n-1}.$$

Note that $(M_n, A_n) = (A_{n-1}, \varnothing)$ and is an R–A pair of A_{n-1}, and also, $M_n = \omega^*(U_n^* \cap A_{n-1}; A_{n-1})$ and $A_n = \varnothing = \omega(U_n \cap A_{n-1}; A_{n-1})$. Now $M_n \equiv A_{n-1}$ and (M_{n-1}, A_{n-1}) is an R–A pair of A_{n-2}, and as $U_{n-1}^* \supset M_{n-1}$ and $U_{n-1} \supset A_{n-1}$ and $U_{n-1}^* \cap U_{n-1} = \varnothing$, by 1.10 there is an open set $V_{n-2} \supset A_{n-2}$ so that if $A_{n-2}' \subset V_{n-2}$ is a compact invariant set, then defining

$$M_{n-1}' \equiv \omega^*(U_{n-1}^* \cap A_{n-2}'; A_{n-2}') \qquad \text{and} \qquad A_{n-1}' \equiv \omega(U_{n-1} \cap A_{n-2}'; A_{n-2}'),$$

(M_{n-1}', A_{n-1}') is an R–A pair of A_{n-2}' with $M_{n-1}' \subset U_{n-1}^*$ and $A_{n-1}' \subset U_{n-1}$.

Note that (i) holds for $i = n-1$, n. Suppose by induction that the open sets $V_{k-1}, \ldots, V_n$; $U_k, \ldots, U_n$; $U_k^*, \ldots, U_n^*$ have been defined satisfying (i) for $i = k, \ldots, n$, where $1 < k \leqslant n$, so that if $A_{j-1}' \subset V_{j-1}$ is a compact invariant set, then

$$(\omega^*(U_j^* \cap A_{j-1}'; A_{j-1}'), \omega(U_j \cap A_{j-1}'; A_{j-1}'))$$

is an R–A pair of A_{j-1}' with $\omega^*(U_j^* \cap A_{j-1}'; A_{j-1}') \subset U_j^*$ and $\omega(U_j \cap A_{j-1}'; A_{j-1}') \subset U_j$, $j = k, \ldots, n$. Then as M_{k-1} and A_{k-1} are disjoint compact sets in the open, Hausdorff Γ_0, there are disjoint open sets $U_{k-1}^* \supset M_{k-1}$ and $U_{k-1} \supset A_{k-1}$, and as $M_{k-1} \subset W_{k-1}$ and $A_{k-1} \subset V_{k-1}$, taking intersections if necessary, U_{k-1}^* and U_{k-1} can also be chosen to satisfy $U_{k-1}^* \subset W_{k-1}$ and $U_{k-1} \subset V_{k-1}$. Also as (M_{k-1}, A_{k-1}) is an R–A pair of A_{k-2}, by 1.10, there exists V_{k-2} open so that if $A_{k-2}' \subset V_{k-2}$ is a compact invariant set then

$$(\omega^*(U_{k-1}^* \cap A_{k-2}'; A_{k-2}'), \omega(U_{k-1} \cap A_{k-2}'; A_{k-2}'))$$

is an R–A pair of A_{k-2}'.

This completes the construction of the V's, U^*'s, and U's. If $S' \subset V_0$ is a compact invariant set, setting $A_0' = S'$ and defining A_i', M_i' inductively as in (ii) for $i = 1, \ldots, n$ note that by construction as $A_i' \subset U_i \subset V_i$, (M_{i+1}', A_{i+1}') is an R–A pair of A_i'. This yields that $S' = A_0' \supset A_1' \supset \cdots \supset A_n' = \varnothing$ is a sequence of attractors with Morse decomposition $\{M_1', \ldots, M_n'\}$. Finally M_i' is the maximal invariant set in $W_i \cap S'$. For if $\gamma \cdot \mathbf{R} \subset W_i \cap S'$, for some $p \leqslant q$ $\omega^*(\gamma; S') \subset M_p' \subset W_p$ and $\omega(\gamma; S') \subset M_q' \subset W_q$, and hence for some $\bar{t} > 0$,

$\gamma \cdot]-\infty, -\bar{t}] \subset W_p$ and $\gamma \cdot [\bar{t}, \infty[\subset W_q$. As $W_p \cap W_q = \varnothing$ if $p \neq q$, it follows that $p = q = i$. Hence as

$$\omega(\gamma; S') \subset M_i' = A_{i-1}' \cap (A_i')^*,$$

$\gamma \in (A_i')^*$, and as $\omega^*(\gamma; S') \subset A_{i-1}'$, $\gamma \in A_{i-1}'$; i.e., $\gamma \in M_i'$. ∎

4.4. PROPOSITION. *$\pi_n: \mathscr{M}(\varphi; n) \to \Lambda$ is a surjective local homeomorphism.*

Proof. The empty invariant set has a Morse decomposition of length n associated to the sequence of attractors $\varnothing = A_0 \supset A_1 \supset \cdots \supset A_n = \varnothing$ so that each Morse set of the decomposition is empty. It follows that π_n is surjective.

As in 1.11, to show π_n is a local homeomorphism it suffices to show that $\mathscr{M}(\varphi; n)$ is open in $\bigoplus^{n+1} \mathscr{S}(\varphi)$. Accordingly, suppose $(S, M_1, ..., M_n) \in \mathscr{M}(\varphi; n)$, and let $S = A_0 \supset \supset \cdots \supset A_n = \varnothing$ be the sequence of attractors giving rise to this Morse decomposition so that (M_i, A_i) is an R–A pair of A_{i-1}, $i = 1, ..., n$.

Because $M_1, ..., M_n$ are pairwise disjoint compact subsets of Γ_0, there are $W_1', ..., W_n'$ open and pairwise disjoint subsets of Γ_0 with $M_i \subset W_i'$, $i = 1, ..., n$. Then choose $N_1', ..., N_n'$ compact subsets of X so that $\varphi_\lambda(N_i')$ isolates M_i and $\varphi_\lambda(N_i') \subset W_i'$, $i = 1, ..., n$. By compactness of $N_1', ..., N_n'$ and by the continuity of φ, choose B_0 Λ-open so that

$$\lambda \in B_0 \subset \bigcap_{i=1}^{n} \Lambda(N_i') \qquad \text{and} \qquad \varphi(N_i' \times B_0) \subset W_i',$$

$i = 1, ..., n$.

By definition of the relative topology, there are Γ-open sets, $W_i \subset W_i'$ so that $\varphi(\operatorname{int} N_i' \times B_0) = \Phi \cap W_i$, $i = 1, ..., n$. Thus 4.3 applies yielding Γ_0-open sets

$$V_0, ..., V_n; \qquad U_1, ..., U_n; \qquad U_1^*, ..., U_n^*$$

satisfying the conclusions of 4.3. In particular

(i) $S \subset V_0$ and $A_i \subset U_i \subset V_i$ and $M_i \subset U_i^* \subset W_i$ and $U_i^* \cap U_i = \varnothing$, $i = 1, ..., n_i$.

(ii) If $S' \subset V_0$ is compact and invariant, then, for $i = 1, ..., n$, defining $M_i' \equiv \omega^*(S' \cap U_i; S')$ and $A_i' \equiv \omega(S' \cap U_i; S')$, (M_i', A_i') is an R–A pair of S' and $M_i' \subset U_i^*$, $A_i' \subset U_i$.

By (i) choose compact subsets of X, $K_0 \supset K_1 \supset \cdots \supset K_n = \varnothing$ so that $\varphi_\lambda(K_0) \subset V_0$ and $\varphi_\lambda(K_i) \subset U_i$, $i = 1, ..., n$, and $\varphi_\lambda(K_i)$ isolates A_i, $i = 0, ..., n$, and then choose compact subsets of X, $N_1, ..., N_n$ so that $N_i \subset K_{i-1}$ and $\varphi_\lambda(N_i) \subset U_i^*$ and $\varphi_\lambda(N_i)$ isolates M_i, $i = 1, ..., n$.

Because $\varphi(N_i) \subset U_i^* \subset W_i$, $N_i \subset \operatorname{int} N_i'$, and as $\varphi_\lambda(N_i)$ and $\varphi_\lambda(N_i')$ both isolate M_i, $i = 1, \ldots, n$, by 1.5 choose B_1 Λ-open so that $\lambda \in B_1 \subset \bigcap_{i=1}^n (\Lambda(N_i') \cap \Lambda(N_i))$ and if $\eta \in B_1$ then $\varphi_\eta(N_i')$ and $\varphi_\eta(N_i)$ have the same maximal invariant set, $i = 1, \ldots, n$.

Finally by continuity of φ and compactness of $K_0, \ldots, K_n$, choose B_2 Λ-open so that $\lambda \in B_2 \subset \bigcap_{i=0}^n \Lambda(K_i)$ and if $\eta \in B_2$, then $\varphi_\eta(K_0) \subset V_0$ and $\varphi_\eta(K_i) \subset U_i$, $i = 1, \ldots, n$.

Set $B = B_0 \cap B_1 \cap B_2$ and define Q by

$$Q = B \times \sigma_{K_0}(B) \times \sigma_{N_1}(B) \times \sigma_{N_2}(B) \times \cdots \times \sigma_{N_n}(B).$$

Note that $Q \cap \oplus^{n+1} \mathscr{S}(\varphi)$ is an open neighborhood of $(\lambda, S, M_1, \ldots, M_n)$ in $\oplus^{n+1} \mathscr{S}(\varphi)$. To finish it suffices to show that

$$Q \cap \overset{n+1}{\oplus} \mathscr{S} \subset \mathscr{M}(\varphi; n).$$

Accordingly, suppose $(\eta, S', M_1'', \ldots, M_n'') \in Q \cap \oplus^{n+1} \mathscr{S}$. Then $(\eta, S') \in \sigma_{K_0}(B)$ so that $S' \subset \varphi_\eta(K_0) \subset V_0$. Then with M_1' and A_1' defined as in (ii) above, (M_1', A_1') is an R–A pair of S' and $M_1' \subset S' \cap U_1^* \subset \varphi_\eta(N_1')$. However, $\varphi_\eta(N_1')$ isolates the same invariant set as $\varphi_\eta(N_1)$ which is M_1'' as $(\eta, M_1'') \in \sigma_{N_1}(B)$; thus $M_1' \subset M_1''$. On the other hand, since $N_1 \subset K_0$, it follows that $M_1'' \subset S' \cap U_i^*$; whence

$$M_1'' \subset \omega^*(S' \cap U_i^*; S') = M_1'.$$

Hence $M_1' = M_1''$ and (M_1'', A_1') is an R–A pair of S'. Now by (ii), $A_1' \subset U_1' \subset V_1$ so that with A_1' playing the role of S' in (ii), with M_2' and A_2' defined as in (ii), (M_2', A_2') is an R–A pair of A_2', and arguing as above gives $M_2' = M_2''$. Continuing by induction we get a sequence of attractors

$$S' = A_0' \supset A_1' \supset \cdots \supset A_n' = \varnothing$$

so that (M_i'', A_i') is an R–A pair of A_{i-1}', $i = 1, \ldots, n$. It follows that

$$(\eta, S', M_1'', \ldots, M_n'') \in \mathscr{M}(\varphi; n). \quad \blacksquare$$

4.5. COROLLARY. *$\pi_\infty : \mathscr{M}(\varphi) \to \Lambda$ is a surjective local homeomorphism.*

Proof. π_∞ is surjective as each π_n is. Regarding each point of $\mathscr{M}(\varphi)$ as an infinite-tuple $(S_\lambda, M_{1,\lambda}, \ldots)$, at most a finite number of coordinates are non-empty, and each point lies in some $\mathscr{M}(\varphi; n)$; i.e., $M_{n+i} = \varnothing$, $i = 1, 2, \ldots$. By 4.4 there is a neighborhood Q' of $(S_\lambda, M_{1,\lambda}, \ldots, M_{n,\lambda})$ in $\oplus^{n+1} \mathscr{S}(\varphi)$, $Q' \subset \mathscr{M}(\varphi; n)$, so that $\pi_n \mid Q'$ is a homeomorphism onto its image. Because open sets in the cartesian product are defined by what happens on a finite number of coordinates, Q' defines an open set in $\oplus^\infty \mathscr{S}(\varphi)$, again call it Q', and $Q' \cap \mathscr{M}(\varphi) \subset \mathscr{M}(\varphi; n)$ since associated to each point of Q' is a sequence of attractors $S' = A'_0 \supset A'_1 \supset \cdots \supset A'_n = \varnothing$ determining its first $n+1$ coordinates, and any extension must be by empty sets. Thus $\pi_\infty \mid Q' = \pi_n \mid Q'$ so $\pi_\infty \mid Q'$ is a local homeomorphism since $\pi_n \mid Q'$ is. ∎

APPENDIX: CORRECTION TO "HOMOTOPY INVARIANTS OF REPELLER–ATTRACTOR PAIRS, I"

In the paper named above, the predecessor of the one to which this appendix is attached, the author has made the same error twice in two different proofs: first in the proof of functoriality of the Püppe sequence of an R–A pair (A^*, A) of S relative to the Morse indices $\mathscr{I}(A)$, $\mathscr{I}(S)$, $\mathscr{I}(A^*)$ [14, Theorem 3.2], and second, in the proof that the splitting class μ is natural relative to the connection index $\mathscr{J}(S; A^*, A)$ [14, Proposition 4.4]. However, the idea behind these proofs is sound, and both of these results are correct as stated, except for a printing error: the horizontal arrows in the first diagram of [14, Proposition 4.4] should point in the opposite direction.

The idea behind these proofs is to reduce the proof of commutativity of a diagram relating the index spaces of two arbitrary nested index triples for an R–A pair (A^*, A) of S to the case where one of the triples is included in the other, in which case commutativity of the diagram becomes obvious due to the functorial nature of inclusion maps.

The error that is made in these proofs is the assertion that the reduction can, in general, be carried out by using the intersection of corresponding members of the two arbitrary triples to get an index triple which mediates between them. Precisely, if, for $j = 1, 2$, $\langle N_{1,j}, N_{2,j}, N_{3,j} \rangle$ is a nested index triple for the R–A pair (A^*, A) of S, and if we define, for $i = 1, 2, 3$,

$$N_{i,3} = N_{i,1} \cap N_{i,2},$$

then *in general, the nested sets*

$$N_{1,2} \supset N_{2,3} \supset N_{3,3}$$

do not form an index triple for (A^*, A). For in general what happens is that the pairs $(N_{1,3}, N_{3,3})$, $(N_{1,3}, N_{2,3})$, and $(N_{2,3}, N_{3,3})$ do not have the exit property of an index pair. For instance, $N_{3,3}$ will often be the empty set as in the following example of a local flow on $\mathbf{R}$.

Consider the local flow of the differential equation $\dot{x} = x(1-x)$, and set $S = [0, 1]$, $A^* = \{0\}$, and $A = \{1\}$. It is clear that S is an isolated invariant set of this local flow and that (A^*, A) is an R–A pair of S. Also, for $j = 1, 2$, set

$$N_{1,j} = [-j, j+1], \qquad N_{2,j} = [\tfrac{1}{2}, j+1] \cup \{-j\}, \qquad N_{3,j} = \{-j\}.$$

Then it is easy to verify that, for $j = 1, 2$, $\langle N_{1,j}, N_{2,j}, N_{3,j}\rangle$ is a nested index triple for the R–A pair (A^*, A) of S, but taking intersections as described above, we get

$$N_{1,3} = [-1, 2], \qquad N_{2,3} = [\tfrac{1}{2}, 2], \qquad \text{and} \qquad N_{3,3} = \varnothing,$$

and the pairs $(N_{1,3}, N_{3,3})$ and $(N_{1,3}, N_{2,3})$ fail to have the exit property.

However, the following proposition provides index triples which mediate between two arbitrary index triples in the desired manner, but three triples must be used rather than just one. We assume for the proposition that $\Phi \subset \Gamma_0$ is a local (semi) flow and S is Φ-isolated.

1. Proposition. *Suppose (A^*, A) is an* R–A *pair of S, and suppose for $j = 1, 2$, $\langle N_{1,j}, N_{2,j}, N_{3,j}\rangle$ is a nested index triple for (A^*, A). Then there exists a nested index triple for (A^*, A), call it $\langle L_1, L_2, L_3\rangle$, and $s_0 > 0$ so that for $s \geqslant s_0$ there is an inclusion of nested index triples*

$$(L_1, L_2, L_3) \subset (N_{1,j}, N_{2,j} \cup N_{3,j}^{-s}, N_{3,j}^{-s}). \tag{A1}$$

Hence for $j = 1, 2$, for $s \geqslant s_0$, there is a commutative diagram of index spaces

$$\begin{tikzcd}
L_2/L_3 \arrow[r] \arrow[d, "\iota_{A,j}"'] & L_1/L_3 \arrow[r] \arrow[d, "\iota_{S,j}"'] & L_1/L_2 \arrow[d, "\iota_{A^*,j}"] \\
(N_{2,j} \cup N_{3,j}^{-s})/N_{3,j}^{-s} \arrow[r] & N_{1,j}/N_{3,j}^{-s} \arrow[r] & N_{1,j}/(N_{2,j} \cup N_{3,j}^{-s})
\end{tikzcd} \tag{A2}$$

with all arrows being functorial inclusion induced maps. The vertical arrows are homotopy equivalences since each defines a morphism in the appropriate Morse index $\mathscr{I}(A)$, $\mathscr{I}(S)$, or $\mathscr{I}(A^*)$, respectively, proceeding from left to right.

Before proving the proposition let us note how it allows us to correct the proofs of the aforementioned results.

2. Correction to the Proof of [14, Theorem 3.2]. In the proof given in [14, Theorem 3.2] it is correctly observed that the functoriality of the sequence relative to the Morse indices holds if the diagram

$$\begin{tikzcd}
N_{2,1}/N_{3,1} \arrow[r] \arrow[d, "h_A"'] & N_{1,1}/N_{3,1} \arrow[r] \arrow[d, "h_S"'] & N_{1,1}/N_{2,1} \arrow[d, "h_{A^*}"'] \\
N_{2,2}/N_{3,2} \arrow[r] & N_{1,2}/N_{3,2} \arrow[r] & N_{1,2}/N_{2,2}
\end{tikzcd} \tag{A3}$$

is homotopy commutative for any choice of nested index triples $\langle N_{1,j}, N_{2,j}, N_{3,j}\rangle$, $j = 1, 2$, for (A^*, A), where h_A, h_S, and h_{A^*} are the unique morphisms in the appropriate Morse indices. This follows immediately upon applying the above proposition by juxtaposing appropriately the diagrams (A2) for $j = 1, 2$, with the diagrams

$$\begin{tikzcd} N_{2,j}/N_{3,j} \arrow[r] \arrow[d, "\rho_{A,j}"'] & N_{1,j}/N_{3,j} \arrow[r] \arrow[d, "\rho_{S,j}"'] & N_{1,j}/N_{2,j} \arrow[d, "\rho_{A^*,j}"] \\ (N_{2,j} \cup N_{3,j}^{-s})/N_{3,j}^{-s} \arrow[r] & N_{1,j}/N_{3,j}^{-s} \arrow[r] & N_{1,j}/(N_{2,j} \cup N_{3,j}^{-s}) \end{tikzcd} \tag{A4}$$

for $j = 1, 2$, in which all arrows are functorial inclusion induced maps (whence the diagrams are commutative) and in which the vertical arrows are homotopy equivalences because each defines a morphism in the appropriate Morse index. Specifically, by the uniqueness property of morphisms in the Morse index [13, Proposition 1.2] necessarily,

$$h_A = \rho_{A,2}^{-1} \circ \iota_{A,2} \circ \iota_{A,1}^{-1} \circ \rho_{A,1} \tag{A5}$$

(equality here is between homotopy classes of maps), and since the analogous equalities hold for h_S and h_{A^*}, (A3) is homotopy commutative because the diagrams of (A2) and (A4) are. ∎

3. Correction to the Proof of [14, Proposition 4.4]. We must show that if $S = A^* \cup A$ and if, for $j = 1, 2$, $\langle N_{1,j}, N_{2,j}, N_{3,j} \rangle$ is a nested index triple for (A^*, A), then the diagram

$$\begin{tikzcd} N_{1,1}/N_{3,1} \arrow[d, "h_S"'] & N_{1,1}/N_{2,1} \arrow[l, "\mu_1"'] \arrow[d, "h_{A^*}"] \\ N_{1,2}/N_{3,2} & N_{1,2}/N_{2,2} \arrow[l, "\mu_2"'] \end{tikzcd} \tag{A6}$$

is homotopy commutative, where μ_1 and μ_2 are the splitting maps and where h_S and h_A are the unique morphisms in the Morse indices $\mathscr{I}(S)$ and $\mathscr{I}(A^*)$, respectively, between the indicated index spaces.

Although the proof given in [14, Proposition 4.4] incorrectly reduces the proof of commutativity of (A6) to the case where we have two nested index triples $\langle M_1, M_2, M_3 \rangle$ and $\langle N_1, N_2, N_3 \rangle$ for an R–A pair (A^*, A) of S which satisfy $M_i \subset N_i$ for $i = 1, 2, 3$, in the event we do have two such triples and if $S = A^* \cup A$, then the proof just cited does correctly show that the splitting classes μ_M and μ_N defined for the sequence of each triple are appropriately related; i.e., there is the homotopy commutative diagram

$$\begin{tikzcd} N_1/N_3 & N_1/N_2 \arrow[l, "\mu_N"'] \\ M_1/M_3 \arrow[u] & M_1/M_2 \arrow[l, "\mu_M"'] \arrow[u] \end{tikzcd}$$

with the vertical arrows being the morphisms in the Morse indices induced by the inclusions of the corresponding pairs.

Thus, we can use an argument analogous to that of the previous correction, Section 2, to show that diagram (A6) is homotopy commutative. That is, via Proposition 1 again chose $s > 0$ and an index triple $\langle L_1, L_2, L_3\rangle$ so that (A1) holds and construct diagrams analogous to those in (A2) and (A4). More specifically, construct the diagrams which have the same lattice points and vertical arrows as the right-hand squares of those diagrams, but replace the horizontal arrows with the appropriate splitting class. Each diagram so constructed is homotopy commutative as observed above, and the juxtaposition of these diagrams yields that (A6) is homotopy commutative since again necessarily

$$h_S = \rho_{S,2}^{-1} \circ \iota_{S,2} \circ \iota_{S,1}^{-1} \circ \rho_{S,1}$$

and

$$h_{A^*} = \rho_{A^*,2}^{-1} \circ \iota_{A^*,2} \circ \iota_{A^*,1}^{-1} \circ \rho_{A^*,1}. \quad \blacksquare$$

Proof of Proposition 1. Set $N_{i,3} = N_{i,1} \cap N_{i,2}$ for $i = 1, 2$. We will show that for $s > r > 0$ large enough, we can let $\langle L_1, L_2, L_3\rangle$ be the nested triple defined by

$$L_1 \equiv N_{1,3}^r, \qquad L_2 \equiv (L_1 \cap N_{2,3}) \cup L_3, \qquad L_3 \equiv L_1 \cap (N_{3,1} \cup N_{3,2}).$$

This is carried out by first showing that

$$\langle N_{1,3}, N_{1,3} \cap (N_{3,1} \cup N_{3,2})\rangle \tag{A7}$$

is a nested index pair for S. Having shown this, note that for $j = 1, 2$, since $N_{1,3} \subset N_{1,j}$, since both $N_{1,3}$ and $N_{1,j}$ isolate S, and since $\langle N_{1,j}, N_{3,j}\rangle$ is an index pair for S, by [13, Lemma 3.3(4)] there exist $s > r > 0$ so that for $j = 1, 2$,

$$N_{1,3}^r \cap (N_{1,3} \cap (N_{3,1} \cup N_{3,2}))^{-r} \subset N_{3,j}^{-s}.$$

Hence by the obvious containment relation, also for $j = 1, 2$,

$$N_{1,3}^r \cap (N_{3,1} \cup N_{3,2}) \subset N_{3,j}^{-s}. \tag{A8}$$

It is an immediate consequence of this inclusion that the inclusion (A1) holds, and as a consequence of the fact that (A7) defines an index pair for S, in particular that the exit property holds for this pair, it follows easily that $\langle L_1, L_2, L_3\rangle$ is an index triple for (A^*, A). Note that in the case we are only dealing with a local semi-flow, in general, this will not be an isolating index triple for (A^*, A); i.e., L_1 and L_2 may not isolate S and A respectively. See [13, Sect. 4] for an example.

Thus we must show that (A7) gives an index pair for S. The isolating property and the relative positive invariance property follow easily from the fact that these properties hold for the index pair $\langle N_{1,j}, N_{3,j}\rangle$ for $j = 1,2$. We check the exit property.

Accordingly, suppose $\gamma \in N_{1,3}\backslash A^+(N_{1,3})$ so that $\sigma|N_{1,3}(\gamma) < \infty$. To the contrary suppose

$$\gamma \cdot [0, \sigma|N_{1,3}(\gamma)] \subset N_{1,3}\backslash(N_{3,1} \cup N_{3,2}).$$

Because

$$N_{1,3}\backslash(N_{3,1} \cup N_{3,2}) = (N_{1,1}\backslash N_{3,1}) \cap (N_{1,2}\backslash N_{3,2})$$

and because for $j = 1, 2$, the exit property holds for $\langle N_{1,j}, N_{3,j}\rangle$ it follows that

$$\sigma|N_{1,3}(\gamma) < \sigma|N_{1,j}(\gamma).$$

Choose $\varepsilon > 0$ so that for $j = 1, 2$,

$$\sigma|N_{1,3}(\gamma) + \varepsilon < \sigma|N_{1,j}(\gamma).$$

Then by definition of $\sigma|N_{1,j}(\gamma)$,

$$\gamma \cdot [0, \sigma|N_{1,3}(\gamma) + \varepsilon] \subset N_{1,1} \cap N_{1,2} = N_{1,3}$$

which is impossible by definition of $\sigma|N_{1,3}(\gamma)$. Hence, $\gamma \cdot \sigma|N_{1,3}(\gamma) \in N_{1,3} \cap (N_{3,1} \cup N_{3,2})$, and the exit property holds. ∎

References

1. R. Churchill, Isolated invariant sets in compact metric spaces, *J. Differential Equations* **12** (1972), 350–352.
2. C. C. Conley, "Isolated Invariant Sets and the Morse Index," C.B.M.S.–Amer. Math. Soc., Providence, R.I., 1978.
3. C. C. Conley, "The Gradient Structure of a Flow, I," I.B.M. Research Report, Yorktown Heights, July 1972.
4. C. C. Conley, A new statement of Wazewski's theorem and an example, *in* "Proceedings of the Fourth Conference Held at Dundee, Scotland, March 30–April 2, 1976" (W. Everitt and B. O. Steeman, Eds.), Lecture Notes in Mathematics, Vol. 564, Springer-Verlag, New York, 1976.
5. C. C. Conley, Some applications of topology to differential equation, mimeographed notes, NSF-CBMS Regional Conference, Boulder, Colo., May 31–June 4, 1976.
6. C. C. Conley and R. W. Easton, Isolated invariant sets and isolating blocks, *Trans. Amer. Math. Soc.* **158** (1971).
7. A. Dold, "Lectures on Algebraic Topology," Springer-Verlag, New York, 1972.

8. J. DUGUNDJI, "Topology," Allyn & Bacon, Boston, 1967.
9. R. W. EASTON, Isolating blocks and symbolic dynamic, *J. Differential Equations* **17** (1975), 96–118.
10. P. C. FIFE, Singular perturbation and wave front techniques in reaction-diffusion problems, *in* "Proceedings of the A.M.S. Symposium on Asymptotic Methods and Singular Perturbation, New York, 1976."
11. B. GRAY, "Homotopy Theory–An Introduction to Algebraic Topology," Academic Press, New York, 1975.
12. H. L. KURLAND, "Homotopy Invariants of Repeller–Attractor Pairs with Application to Fast-Slow Systems," Ph.D. dissertation, University of Wisconsin, Madison, May 1979.
13. H. L. KURLAND, The Morse index of an isolated invariant set is a connected simple system, *J. Differential Equations* **42** (1981), 234–259.
14. H. L. KURLAND, Homotopy invariants of repeller-attractor pairs. I. The Püppe sequence of an R–A pair, *J. Differential Equations* **46** (1982), 1–31.
15. H. L. KURLAND, Solutions to boundary value problems of fast-slow systems by continuing homology in the Morse index along a path of isolated invariant sets of the fast systems, *J. Differential Equations*, in press.
16. H. L. KURLAND, A bounded traveling wave for a system of non-linear reaction-diffusion equations via the connection index of a repeller-attractor pair, to appear.
17. J. T. MONTGOMERY, Cohomology of isolated invariant sets under perturbation, *J. Differential Equations* **13** (1973), 257–299.
18. E. H. SPANIER, "Algebraic Topology," McGraw–Hill, New York, 1966.
19. R. M. SWITZER, "Algebraic Topology-Homotopy and Homology," Springer-Verlag, New York, 1975.
20. T. WAZEWSKI, Sur un principe topologique de l'examen de l'allure asymptotique des integrals des equations differentielles, *Ann. Soc. Math. Polon.* **20** (1947), 279–313.
21. T. WAZEWSKI, Sur un method topologique de l'examen de l'allure asymptotique des integrals des equations differentielles, *in* "Proceedings, International Congress of Mathematicians, Amsterdam, III (1954)."
22. J. T. MONTGOMERY, On the homotopy index of Conley, *J. Differential Equations* **32** (1979), 32–40.
23. S. LANG, "Differential Manifolds," Addison–Wesley, Reading, Mass. 1972.
24. J. SELGRADE, Isolated invariant sets for flows on vector bundles, *Trans. Amer. Math. Soc.* **203** (1975), 359–390.
25. T. G. YOUNG, "Isolated Invariant Sets of Semi-Flows on Compact Metric Spaces," Ph.D. dissertation, University of Wisconsin, Madison, 1975.

Printed in Belgium