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Strong coupling expansion of Baxter equation in $\mathcal{N}=4$ SYM

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Abstract

The anomalous dimensions of single-trace local Wilson operators with covariant derivatives in maximally supersymmetric gauge theory are believed to be generated from a deformed noncompact $sl(2)$ Baxter equation. We perform a systematic expansion of this equation at strong coupling in the single-logarithmic limit of large conformal spin to overcome the limitation of the asymptotic nature of the equation. The analysis is reduced to Riemann–Hilbert problems for corresponding resolvents of Bethe roots in each order of the quasiclassical expansion. We explicitly construct the resolvents in the lowest two orders in strong coupling and find all local conserved charges of the underlying long-range spin chain.

1. Introduction

Until recently, the bulk of our knowledge about QCD or any quantum field theory has been gained from methods relying on perturbative expansions with respect to a small parameter. The lack of nonperturbative techniques overcoming limitations of perturbative considerations was a major obstacle in understanding strong coupling dynamics of gauge theories. The numerical framework of lattice gauge theory partially resolves these problems, but one still lacks reliable analytical methods. A concept stepping forward in recent years as a panacea for these complications is integrability—an idea going back to the Liouville theorem, which states that a mechanical system is integrable provided it has as many conserved quantities as degrees of freedom such that all physical quantities can be calculated exactly.

In four-dimensional gauge theories integrable structures were first observed in studies of multi-reggeon compound states [1,2]. Later analyses of one-loop anomalous dimensions of the maximal-helicity Wilson operators in QCD [3–5] revealed that their dilatation operator is mapped in the multi-color limit into the Hamiltonian of the noncompact XXX Heisenberg spin chain, and its eigenspectrum can be computed exactly by means of the algebraic Bethe ansatz [6]. Integrability observed in QCD anomalous dimensions at one-loop order is a generic phenomenon of four-dimensional Yang–Mills theories. Theories with supersymmetries obviously inherit integrability, although the number of integrable sectors strongly depends on the particle content of the models and is enhanced for theories with more supercharges [7], eventually encompassing all operators in maximally supersymmetric Yang–Mills theory [8–11].

The gauge/string duality for the $\mathcal{N}=4$ super-Yang–Mills theory [12] is of the weak/strong coupling nature and allows one to establish exact correspondence between anomalous dimensions of composite Wilson operators in gauge theory and energies of string excitations on the $\mathrm{AdS}_5\times\mathrm{S}^5$ background [13,14]. It was demonstrated that the classical string sigma models with anti-de Sitter space as a factor of the target space possess an infinite set of integrals of motion and therefore are integrable [15]. On the gauge theory side, this result implies that the all-order dilatation operator for Wilson operators should be integrable. A large amount of evidence had been gathered to date about survival of integrability at higher orders of perturbation theory culminating in a conjecture of asymptotic all-order Bethe equations [16] depending on the Yang–Mills coupling constant $g^2=g^2_{\mathrm{YM}}N_c/(4\pi^2)$ which incorporate a phase factor encoding smooth interpolation from weak to strong coupling.

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Recently we have formulated an alternative framework based on long-range Baxter equations [17–19] to eigenvalue problem of all-order anomalous dimensions in maximally supersymmetric gauge theory. In our present note we will address, generalizing earlier considerations of Ref. [20], the question of their systematic strong-coupling expansion restricting to the noncompact $sl(2)$ sector and determine the spectrum of conserved charges of the magnet in the single-logarithmic asymptotic in the conformal spin in the lowest two orders of inverse-coupling series. One of these charges coincides with the eigenvalue of the dilatation operator and in the large-spin limit defines the cusp anomalous dimension—an observable encoding the physics of soft-gluon emission [21,22]. Recently a closed integral equation was formulated for this quantity [18,23,24] and its solution at strong coupling to the lowest few orders was found numerically in Ref. [29] and analytically to the first [30–33] and eventually at an arbitrary order in Ref. [34]. These findings match nicely with classical [14,20,35], one [36,37] and two-loop calculations [38] in string sigma model, and with the quantum string Bethe Ansatz analysis [39].

2. Asymptotic Baxter equation

The calculation of anomalous dimensions of arbitrary twist operators is a nontrivial task even to one-loop order due to a large size of the mixing matrix. However, as we pointed out above the problem can be overcome thanks to the hidden integrability symmetry of the dilatation operator in $\mathcal{N}=4$ super-Yang–Mills theory in the large-N_c limit. In the closed holomorphic $sl(2)$ subsector of the theory, spanned by twist-L local Wilson operators built from scalar fields $X(0)$ and N light-cone covariant derivatives (schematically)

$$\mathcal{O}_{N,L}(0) = \operatorname{tr}\left\{ D_+^N X^L(0) \right\}, \tag{2.1}$$

the one-loop mixing matrix is mapped into the Hamiltonian of the Heisenberg magnet of length L and spin $s=1/2$ determined by the conformal spin of the field $X(0)$. At higher orders of perturbation theory, the one-loop Bethe Ansatz equations [16] or Baxter equations [18] allow for a consistent deformation with 't Hooft coupling g and yield spectra of anomalous dimensions in agreement with diagrammatic calculations. They are plagued however by asymptotic nature, being inapplicable when the order of perturbation theory exceeds the length L of the local Wilson operator in question.

In our analyses we use the method of the Baxter equation [40] which proved to be convenient in analyzing various semiclassical limits including the limit of large L and N at one-loop order [4,5,20,41,42]. The method relies on the existence of an operator $\mathbb{Q}(u)$ which acts on the Hilbert space of the chain and is diagonalized by all eigenstates of the magnet for an arbitrary complex spectral parameter u. For discussion of the spectrum of integrals of motion of the magnet it suffices to study just its eigenvalues $Q(u)$. Thus, the long-range $sl(2)$ Baxter equation is written for the polynomial

$$Q(u) = \prod_{j=1}^{N} (u - u_j), \tag{2.2}$$

with zeros determined by the Bethe roots $u_j = u_j(g)$ admitting an infinite series expansion in 't Hooft coupling constant g,

$$(x^+)^L e^{\frac{1}{2}(\Delta_+(x^+) - \Delta_-(x^-))} Q(u+i) + (x^-)^L e^{\frac{1}{2}(\Delta_-(x^-) - \Delta_+(x^+))} Q(u-i) = \tau(x) Q(u). \tag{2.3}$$

The deformation of the one-loop chain is partially encoded in the renormalized spectral parameter $x \equiv x[u] = \frac{1}{2}(u + \sqrt{u^2 - g^2})$ [16], with adopted convention for $x^\pm = x[u \pm \frac{i}{2}]$. The right-hand side depends on the transfer matrix with the two-dimensional auxiliary space which is a polynomial in x with expansion coefficients determined by the conserved charges of the chain.[1]

The most nontrivial deviations from the nearest-neighbor magnet are accommodated in the dressing factors $\sigma_\pm$ and Θ in the combination

$$\Delta_\pm(x) = \sigma_\pm(x) - \Theta(x), \tag{2.4}$$

with $\sigma_\pm$ responsible for the renormalization of the conformal spin of the Wilson operators (2.1)

$$\sigma_\pm(x) = \int_{-1}^{1} \frac{dt}{\pi} \frac{\ln Q(\pm\frac{i}{2} - gt)}{\sqrt{1-t^2}} \left(1 - \frac{\sqrt{u^2 - g^2}}{u + gt} \right), \tag{2.5}$$

and Θ providing smooth matching of the weak- and strong-coupling expansion

[1] Notice that compared to the equation in Ref. [18] we introduced a redefined transfer matrix τ, $t(x) = e^{\frac{1}{2}\Delta_+(x^+) + \frac{1}{2}\Delta_-(x^-)} \tau(x)$, whose advantage compared to $t(x)$ is the absence of x^{L-1} terms and thus corresponding charge in the transfer matrix [17].

$$\Theta(x) = g \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \ln \frac{Q(-\frac{i}{2} - gt)}{Q(+\frac{i}{2} - gt)} \fint_{-1}^{1} ds \frac{\sqrt{1-s^2}}{s-t}$$
$$\times \int_{C_{[i,i\infty]}} \frac{d\kappa}{2\pi i} \frac{1}{\sinh^2(\pi\kappa)} \ln\left(1 + \frac{g^2}{4xx[\kappa + gs]}\right)\left(1 - \frac{g^2}{4xx[\kappa - gs]}\right). \tag{2.6}$$

This phase factor can be expressed as an infinite series expansion [24]

$$i\theta(x) \equiv \Theta(x^+) - \Theta(x^-) = i \sum_{j=1}^{N} \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} g^{r+s-1} c_{r,s}(g) \big[q_r(x) q_s(x_j) - q_r(x_j) q_s(x)\big] \tag{2.7}$$

in terms of single-excitation charges on the spin chain

$$q_r(x_j) = \frac{i}{r-1}\left(\frac{1}{(x_j^+)^{r-1}} - \frac{1}{(x_j^-)^{r-1}}\right). \tag{2.8}$$

The latter define the local integrals of motion of the chain,

$$\mathcal{Q}_r(g) = \sum_{j=1}^{N} q_r(x_j) = \frac{2i}{r-1}\left(-\frac{2}{g}\right)^{r-2} \int_{-1}^{1} \frac{dt}{\pi} \sqrt{1-t^2}\, U_{r-2}(t) \left(\ln \frac{Q(+\frac{i}{2} - gt)}{Q(-\frac{i}{2} - gt)}\right)', \tag{2.9}$$

where $U_r(t)$ are the Chebyshev polynomials of the second kind. The anomalous dimensions of the operators (2.1) are related to the Hamiltonian $\mathcal{Q}_2$ via

$$\gamma(g) = \frac{g^2}{2} \mathcal{Q}_2(g). \tag{2.10}$$

At strong coupling, the expansion coefficients $c_{r,s}(g)$ were suggested in Refs. [25,27] and [24] at leading, first subleading and all orders, respectively. They represent analytic continuation of the phase (2.6) suitable for perturbative analyses to strong coupling. The idea of using the strong-coupling expansion of the scattering phase (2.7) has been recently explored in Ref. [39] within the framework of the asymptotic Bethe ansatz.

3. Quasiclassical expansion

As we pointed our earlier, due to asymptotic character of the Baxter equation it is well defined only when the interaction range does not exceed the length of the chain. Therefore, to study its strong-coupling expansion we have to consider a limit which does not violate this requirement. It was established in Ref. [20], that the anomalous dimensions of operators (2.1) for large conformal spin N and length L at strong coupling g depend on the "hidden" parameter $\xi_{\rm str} = g^2 \ln^2(N/L)/L^2$ and for $\xi_{\rm str} \gg 1$ the anomalous dimension becomes insensitive to the twist L of the operators thus circumventing limitations of the asymptotic Baxter equation. In this asymptotic domain, the anomalous dimension scales as $\sim \ln N$. Therefore, our goal is to develop the strong-coupling quasiclassical expansion in the limit $g \gg 1$, $N \gg L \gg 1$ such that $\xi_{\rm str} \gg 1$.

To start with, let us construct a systematic expansion at strong coupling g. To this end, we rescale the spectral parameters with a power of the coupling constant

$$u = g\hat{u}, \qquad x = g\hat{x}, \tag{3.1}$$

such that the Baxter equation reads

$$\hat{x}^L \hat{\tau}(\hat{x}) = (\hat{x}^-)^L e^{\frac{1}{2}(\hat{\sigma}_-(\hat{x}^-) - \hat{\sigma}_+(\hat{x}^+)) + i\theta(\hat{x})} \frac{\hat{Q}(\hat{u} - \frac{i}{g})}{\hat{Q}(\hat{u})} + (\hat{x}^+)^L e^{\frac{1}{2}(\hat{\sigma}_+(\hat{x}^+) - \hat{\sigma}_-(\hat{x}^-)) - i\theta(\hat{x})} \frac{\hat{Q}(\hat{u} + \frac{i}{g})}{\hat{Q}(\hat{u})}, \tag{3.2}$$

where $\hat{x}^\pm \equiv \hat{x}[\hat{u} \pm \frac{i}{2g}]$ and the left-hand side depends on the rescaled transfer matrix $\tau(x) = x^L \hat{\tau}(\hat{x})$. The magnon phase $\theta(\hat{x})$ is re-expressed in terms of the local charges

$$\theta(\hat{x}) = \sum_{j=1}^{N} \vartheta(\hat{x}, \hat{x}_j) \equiv g \sum_{j=1}^{N} \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) \big[\hat{q}_r(\hat{x}) \hat{q}_s(\hat{x}_j) - \hat{q}_r(\hat{x}_j) \hat{q}_s(\hat{x})\big], \tag{3.3}$$

with the expansion coefficients $c_{r,s}(g)$ admitting the inverse-coupling expansion

$$c_{r,s}(g) = c_{r,s}^{(0)} + \frac{1}{g} c_{r,s}^{(1)} + \frac{1}{g^2} c_{r,s}^{(2)} + \cdots. \tag{3.4}$$

The Baxter polynomial in the $\hat{x}$-variable,

$$\hat{Q}(\hat{u}) = \prod_{j=1}^{N}(\hat{x} - \hat{x}_j) \prod_{j=1}^{N}\left(1 - \frac{1}{4\hat{x}\hat{x}_j}\right), \tag{3.5}$$

exhibits the double covering nature of the map $\hat{u} \to \hat{x}$, since for each value of $\hat{u}$ there are two corresponding values, $\hat{x}$ and $1/(4\hat{x})$. While the second factor is irrelevant in perturbative considerations, for finite coupling g these zeros do contribute on equal footing. However, as we will see below, their contribution vanishes from the Riemann–Hilbert problems.

For our subsequent discussion, we introduce two resolvents

$$S'(\hat{x}) = \eta \sum_{j=1}^{N} \frac{1}{\hat{x} - \hat{x}_j}, \qquad G(\hat{x}) = \eta \sum_{j=1}^{N} \frac{1}{(\hat{x} - \hat{x}_j)(1 - \frac{1}{4\hat{x}_j^2})}, \tag{3.6}$$

where $\eta = (N + \frac{1}{2}L)^{-1}$, which naturally emerge in the large-g expansion of the Baxter polynomials,

$$\begin{aligned}\ln \frac{\hat{Q}(\hat{u} \pm \frac{i}{g})}{\hat{Q}(\hat{u})} &= \pm \frac{i}{g\eta} \frac{4\hat{x}^2 S'(\hat{x}) - S'(\frac{1}{4\hat{x}})}{4\hat{x}^2 - 1} + \frac{\eta}{2!}\left(\frac{\pm i}{g\eta}\right)^2 \frac{4\hat{x}^2 G'(\hat{x}) - G'(\frac{1}{4\hat{x}})}{4\hat{x}^2 - 1} \\ &\quad + \frac{\eta^2}{3!}\left(\frac{\pm i}{g\eta}\right)^3 \left(\frac{16\hat{x}^4 G''(\hat{x}) + G''(\frac{1}{4\hat{x}})}{(4\hat{x}^2 - 1)^2} - \frac{32\hat{x}^3[G'(\hat{x}) - G'(\frac{1}{4\hat{x}})]}{(4\hat{x}^2 - 1)^3} + \right) + \mathcal{O}(g^{-4}).\end{aligned} \tag{3.7}$$

They are not independent though and are related to each other as follows

$$G(\hat{x}) = \frac{4\hat{x}^2 S'(\hat{x})}{4\hat{x}^2 - 1} + \frac{S'(-\frac{1}{2})}{2(2\hat{x} + 1)} - \frac{S'(\frac{1}{2})}{2(2\hat{x} - 1)}. \tag{3.8}$$

The phases also admit the inverse-coupling expansion

$$\begin{aligned}\frac{1}{2}\sigma_-(x^-) - \frac{1}{2}\sigma_+(x^+) &= \frac{i}{g\eta}\sigma_0(\hat{x}) + \frac{i}{g^3\eta}\sigma_2(\hat{x}) + \mathcal{O}(g^{-5}), \\ \vartheta(\hat{x}, \hat{y}) &= \frac{1}{g\eta}\vartheta_0(\hat{x}, \hat{y}) + \frac{1}{g^2\eta}\vartheta_1(\hat{x}, \hat{y}) + \frac{1}{g^3\eta}\vartheta_2(\hat{x}, \hat{y}) + \mathcal{O}(g^{-4}),\end{aligned} \tag{3.9}$$

with

$$\vartheta_0(\hat{x}, \hat{y}) = \sum_{r=2} \sum_{s=r+1} c_{r,s}^{(0)} \big[\hat{q}_r^{(0)}(\hat{x})\hat{q}_s^{(0)}(\hat{y}) - \hat{q}_r^{(0)}(\hat{y})\hat{q}_s^{(0)}(\hat{x})\big], \tag{3.10}$$

$$\vartheta_1(\hat{x}, \hat{y}) = \sum_{r=2} \sum_{s=r+1} c_{r,s}^{(1)} \big[\hat{q}_r^{(0)}(\hat{x})\hat{q}_s^{(0)}(\hat{y}) - \hat{q}_r^{(0)}(\hat{y})\hat{q}_s^{(0)}(\hat{x})\big], \tag{3.11}$$

$$\begin{aligned}\vartheta_2(\hat{x}, \hat{y}) &= \sum_{r=2} \sum_{s=r+1} c_{r,s}^{(2)} \big[\hat{q}_r^{(0)}(\hat{x})\hat{q}_s^{(0)}(\hat{y}) - \hat{q}_r^{(0)}(\hat{y})\hat{q}_s^{(0)}(\hat{x})\big] \\ &\quad + \sum_{r=2} \sum_{s=r+1} c_{r,s}^{(0)} \big[\hat{q}_r^{(0)}(\hat{x})\hat{q}_s^{(2)}(\hat{y}) + \hat{q}_r^{(2)}(\hat{x})\hat{q}_s^{(0)}(\hat{y}) - \hat{q}_r^{(0)}(\hat{y})\hat{q}_s^{(2)}(\hat{x}) - \hat{q}_r^{(2)}(\hat{y})\hat{q}_s^{(0)}(\hat{x})\big].\end{aligned} \tag{3.12}$$

These are expressed in terms of the strong-coupling coefficients [25–28]

$$c_{r,s}^{(0)} = \frac{1}{2^{2r}}\delta_{r,s-1}, \tag{3.13}$$

$$c_{r,s}^{(1)} = -\frac{4}{\pi} \frac{1 - (-1)^{r+s}}{2^{r+s}} \frac{(r-1)(s-1)}{(s+r-2)(s-r)}, \tag{3.14}$$

$$c_{r,s}^{(2)} = \frac{1}{6} \frac{1 - (-1)^{r+s}}{2^{r+s}} (r-1)(s-1), \tag{3.15}$$

and local integrals of motion

$$\hat{q}_r(\hat{x}) = \frac{1}{g}\hat{q}_r^{(0)}(\hat{x}) + \frac{1}{g^3}\hat{q}_r^{(2)}(\hat{x}) + \mathcal{O}(g^{-5}), \tag{3.16}$$

where

$$\hat{q}_r^{(0)}(\hat{x}) = \frac{4\hat{x}^{2-r}}{4\hat{x}^2 - 1}, \tag{3.17}$$

$$\hat{q}_r^{(2)}(\hat{x}) = -8\hat{x}^{4-r}\left(\frac{r(r+1)}{3(4\hat{x}^2-1)^3} + \frac{2(r+1)}{(4\hat{x}^2-1)^4} + \frac{4}{(4\hat{x}^2-1)^5}\right). \tag{3.18}$$

Finally, we assume that the resolvents and the transfer matrix admit systematic expansion in inverse powers of the coupling

$$S(\hat{x}) = S_0(\hat{x}) + \frac{1}{g}S_1(\hat{x}) + \frac{1}{g^2}S_2(\hat{x}) + \cdots, \tag{3.19}$$

$$G(\hat{x}) = G_0(\hat{x}) + \frac{1}{g}G_1(\hat{x}) + \frac{1}{g^2}G_2(\hat{x}) + \cdots, \tag{3.20}$$

$$\tau(\hat{x}) = \tau_0(\hat{x}) + \frac{1}{g}\tau_1(\hat{x}) + \frac{1}{g^2}\tau_2(\hat{x}) + \cdots, \tag{3.21}$$

with each terms uniformly bounded. The coefficients $\tau_k(\hat{x})$ are entire functions in the complex plane $\hat{x}$ with an essential singularity at $\hat{x} = 0$. Substituting these expansions into the Baxter equation and equating the coefficients in front of powers of g^{-1}, find a set of equations for the resolvents.

3.1. Leading order

In explicit form, the leading contributions to the phase factors are

$$\sigma_0(\hat{x}) = \frac{S_0'(\frac{1}{2})}{2(2\hat{x}-1)} - \frac{S_0'(-\frac{1}{2})}{2(2\hat{x}+1)} - \frac{2S_0'(\frac{1}{4\hat{x}})}{(4\hat{x}^2-1)}, \tag{3.22}$$

$$\theta_0(\hat{x}) = -\frac{S_0'(\frac{1}{2})}{2(4\hat{x}^2-1)} - \frac{S_0'(-\frac{1}{2})}{2(4\hat{x}^2-1)} + \frac{S_0'(\frac{1}{4\hat{x}})}{(4\hat{x}^2-1)}, \tag{3.23}$$

so that the zero-order quasiclassical form of the Baxter equation reads

$$\hat{\tau}_0(\hat{x}) = 2\cos p_0(\hat{x}), \tag{3.24}$$

in terms of the quasimomentum

$$p_0(\hat{x}) = \frac{1}{g\eta}\left(\frac{2\beta\hat{x}}{4\hat{x}^2-1} + \frac{4\hat{x}^2 S_0'(\hat{x}) - \hat{x}[S_0'(\frac{1}{2}) - S_0'(-\frac{1}{2})]}{4\hat{x}^2-1}\right), \tag{3.25}$$

where $\beta = L\eta$. From Eq. (3.24) we find that $\sin p_0(\hat{x})$ is a double-valued function on the complex $\hat{x}$-plane with the square-root branching points $\hat{x}_j$ obeying the condition $\tau_0(\hat{\xi}_j) = \pm 2$. It becomes single-valued on a hyperelliptic Riemann surface defined by gluing together two copies of the complex $\hat{x}$-planes along the cuts running between the branching points $\hat{\xi}_j$, $S = [\hat{\xi}_1, \hat{\xi}_2] \cup [\hat{\xi}_3, \hat{\xi}_4] \cup \cdots$. In the quasiclassical limit, these cuts accumulate the Bethe roots and correspond to regions of the allowed classical motion of the system in separated variables.

From the single-valuedness of the transfer matrix, we can write down the Riemann–Hilbert problem

$$\not{p}_0(\hat{x}) = \pi m, \tag{3.26}$$

where here and below the principal value is defined as

$$\not{p}_0(\hat{x}) \equiv \frac{1}{2}\big(p_0(\hat{x} + i0) + p_0(\hat{x} - i0)\big). \tag{3.27}$$

In this study we are interested in eigenstates possessing the minimal possible energy for a given total spin N. For a given total spin N, this trajectory is realized when all cuts but two in S shrink into points and the Bethe roots are located on two symmetric cuts on the real axis $[-a, -b] \cup [b, a]$ most distant from the origin. From the point of view of separated variables, this means that classically all but two collective degrees of freedom are frozen and the classical motion is confined to the two intervals. As a consequence, the complex spectral curve gets reduced to the genus one curve [20]. In Eq. (3.26), the integer m defines the position of the interval $[b, a]$ inside S and $m - 1$ counts how many collapsed cuts are situated to the right from the interval $[b, a]$ on the real axis. The minimal value of the energy is achieved for $m = 1$.

Thus, to leading order of the semiclassical expansion the function $S_0'(\hat{x})$ is determined by the above analytical properties and prescribed asymptotic behavior at the origin and infinity

$$S_0'(\hat{x} \to 0) = o(\hat{x}), \qquad S_0'(\hat{x} \to \infty) = \frac{\eta N}{\hat{x}} + \mathcal{O}\big(\hat{x}^{-2}\big), \tag{3.28}$$

for symmetric, $S_0'(-\hat{x}) = -S_0'(\hat{x})$, two-cut solution. It reads

$$S_0'(\hat{x}) = \frac{1}{4\hat{x}}\oint_{C_1}\frac{d\hat{z}}{2\pi i}\frac{W_0(\hat{z})}{\sqrt{(\hat{z}^2-a^2)(\hat{z}^2-b^2)}}\left\{\frac{1-4\hat{x}^2}{\hat{z}^2-\hat{x}^2}\sqrt{\big(\hat{x}^2-a^2\big)\big(\hat{x}^2-b^2\big)} + \frac{ab}{4\hat{z}^2}\right\}, \tag{3.29}$$

where the contour C_1 goes around the cut $[b, a]$ in counterclockwise direction and

$$W_0(\hat{z}) = 2(g\eta)\pi m - \frac{4\beta\hat{z}}{4\hat{z}^2 - 1}. \tag{3.30}$$

From the asymptotic behavior of $S_0'(\hat{x} \to \infty)$ one deduces conditions on the end-points of the cuts

$$\oint_{C_1} \frac{d\hat{z}}{2\pi i} \frac{W_0(z)}{\sqrt{(\hat{z}^2 - a^2)(\hat{z}^2 - b^2)}} = 0, \qquad \oint_{C_1} \frac{d\hat{z}}{2\pi i} \frac{W_0(z)}{\sqrt{(\hat{z}^2 - a^2)(\hat{z}^2 - b^2)}} \left(\hat{z}^2 + \frac{ab}{4\hat{z}^2} \right) = \eta N. \tag{3.31}$$

As follows from these conditions, the branching points of the curve, $\pm b$ and $\pm a$, depend on the ratio L/N and the coupling constant g. As was demonstrated in Ref. [20], the single-logarithmic asymptotics emerges for the configuration when b approaches its minimal value $\frac{1}{2}$ so that the inner boundaries of two cuts $[-a, -b]$ and $[b, a]$ coincide with the position of poles at $|\hat{x}| = \frac{1}{2}$ and the outer points $\pm a$ run away to infinity:

$$a \to \infty, \qquad b \to \frac{1}{2} + \varepsilon. \tag{3.32}$$

From Eqs. (3.31) we find the following parametric dependence of a and b on N, L and g

$$\frac{1}{\sqrt{\varepsilon}} = \frac{4mg}{L} \ln(4a), \quad a = \frac{N}{2mg}, \tag{3.33}$$

with the parameter ε related to the string parameter introduced in Ref. [20] as follows $\varepsilon \sim \xi_{\text{str}}^{-2}$. The analysis of the solution (3.29) yields in the single-logarithmic regime

$$S_0'(\hat{x}) = -\frac{\beta}{2\sqrt{\varepsilon}} \frac{i\sqrt{\hat{x}^2 - b^2} + b}{\hat{x}}, \tag{3.34}$$

or in terms of the resolvent G_0,

$$G_0(\hat{x}) = -\frac{2i\beta}{\sqrt{\varepsilon}} \frac{\hat{x}\sqrt{\hat{x}^2 - b^2}}{4\hat{x}^2 - 1}, \tag{3.35}$$

which agrees with Ref. [39].

3.2. Next-to-leading order

Collecting the terms at the order $1/g$ in the expansion of the Baxter equation, we get the equation for the next-to-leading correction to the resolvent

$$G_1(\hat{x}) = \frac{g\eta}{2} \frac{p_0'(\frac{1}{4\hat{x}}) - 4\hat{x}^2 p_0'(\hat{x})}{4\hat{x}^2 - 1} \cot p_0(\hat{x}) + \theta_0^{(1)}(\hat{x}) + \frac{2\beta\hat{x}^2(4\hat{x}^2 + 1)}{(4\hat{x}^2 - 1)^3} \cot p_0(\hat{x}) - \frac{g\eta}{2} \frac{\tau_1(\hat{x})}{\sin p_0(\hat{x})}. \tag{3.36}$$

The Riemann–Hilbert problem for the subleading resolvent follows from this assuming that $G_1(\hat{x})$ possesses the same analytic properties as the leading $G_0(\hat{x})$. Therefore, since $p_0'(\hat{x})$ and $\sin p_0(\hat{x})$ change sign across the cuts as a consequence of the leading order Riemann–Hilbert problems (3.26), we immediately find

$$\not{G}_1(\hat{x}) = -\frac{g\eta}{2} \frac{4\hat{x}^2}{4\hat{x}^2 - 1} p_0'(\hat{x} + i0) \cot p_0(\hat{x} + i0) + \theta_0^{(1)}(\hat{x}), \tag{3.37}$$

where we used the fact that while $S'(\hat{x})$ (and $G(\hat{x})$) is discontinuous on the cuts $x \in [-a, -b] \cup [b, a]$, $S'(\frac{1}{4\hat{x}})$ (and $G(\frac{1}{4\hat{x}})$) for the reflected argument is continuous. Here

$$\theta_0^{(1)}(\hat{x}) = \frac{1}{4\hat{x}^2 - 1} \frac{2}{\pi} \oint_C \frac{d\hat{z}}{2\pi i} \frac{S_0'(\hat{z})}{4\hat{z}^2 - 1} \vartheta_1(\hat{x}, \hat{z}), \tag{3.38}$$

with the contour C wrapping around the cuts $[-a, -b] \cup [b, a]$ and where

$$\begin{aligned} \vartheta_1(\hat{x}, \hat{z}) &= -4 \sum_{r=2}^{\infty} \sum_{m=0}^{\infty} \frac{(r-1)(2m+r)}{(2m+2r-1)(2m+1)} \big[(2\hat{x})^{2-r} (2\hat{z})^{2m+r-1} - (2\hat{z})^{2-r} (2\hat{x})^{2m+r-1} \big] \\ &= (4\hat{x}\hat{z})^2 \left\{ \frac{1}{(4\hat{x}\hat{z} - 1)(\hat{x} - \hat{z})} + \left(\frac{1}{(4\hat{x}\hat{z} - 1)^2} + \frac{1}{4(\hat{x} - \hat{z})^2} \right) \ln \frac{(2\hat{x} + 1)(2\hat{z} - 1)}{(2\hat{x} - 1)(2\hat{z} + 1)} \right\}. \end{aligned} \tag{3.39}$$

The solution to the Riemann–Hilbert problem (3.37) is

$$G_1(\hat{x}) = \hat{x} \oint_{C_1} \frac{d\hat{z}}{2\pi i} W_1(\hat{z}) \left\{ \frac{1}{\hat{x}^2 - \hat{z}^2} + \frac{1}{1-4ab} \frac{1}{\hat{z}^2} \right\} \frac{\sqrt{(\hat{z}^2 - a^2)(\hat{z}^2 - b^2)}}{\sqrt{(\hat{x}^2 - a^2)(\hat{x}^2 - b^2)}}, \tag{3.40}$$

where

$$W_1(\hat{x}) = 2\theta_0^{(1)}(\hat{x}) - g\eta \frac{4\hat{x}^2}{4\hat{x}^2 - 1} p_0'(\hat{x} + i0) \cot p_0(\hat{x} + i0), \tag{3.41}$$

in agreement with Ref. [39]. A tedious calculation yields the next-to-leading resolvent in the single-logarithmic asymptotics

$$\begin{aligned} G_1(\hat{x}) = \frac{i\beta}{8\pi\sqrt{\varepsilon}} \Bigg\{ & -\frac{2\hat{x}(2b-1)(12\hat{x}^2+1)}{(4\hat{x}^2-1)^2\sqrt{\hat{x}^2-b^2}} - \frac{64\hat{x}^2(4\hat{x}^2+1)\sqrt{4b^2-1}}{(4\hat{x}^2-1)^3} \cot^{-1}\left(\hat{x}\frac{\sqrt{4b^2-1}}{\sqrt{\hat{x}^2-b^2}}\right) \\ & - \frac{128\hat{x}^3[1-2b^2(4\hat{x}^2+1)]}{(4\hat{x}^2-1)^3\sqrt{16\hat{x}^2b^2-1}} \cot^{-1}\left(\frac{\sqrt{16\hat{x}^2b^2-1}}{2\sqrt{\hat{x}^2-b^2}}\right) - \frac{2\hat{x}(4b^2-1)}{(4\hat{x}^2-1)\sqrt{\hat{x}^2-b^2}} \ln\left(1-\frac{1}{4b^2}\right) \\ & + \frac{\hat{x}[2(16\hat{x}^4-1)-(4b^2-1)(1+24\hat{x}^2+16\hat{x}^4)]}{(4\hat{x}^2-1)^3\sqrt{\hat{x}^2-b^2}} \ln\left[\left(1+\frac{1}{2b}\right)^2\left(1+\frac{1}{4b^2}\right)\right] \\ & + \frac{32\hat{x}^2[2\hat{x}^2-b^2(4\hat{x}^2+1)]}{(4\hat{x}^2-1)^3\sqrt{\hat{x}^2-b^2}} \ln\frac{(\hat{x}-b)(2\hat{x}+1)}{(\hat{x}+b)(2\hat{x}-1)} \Bigg\}. \end{aligned} \tag{3.42}$$

While, the resolvent $S_1'(\hat{x})$ can be found from it making use of the relation (3.8).

4. Local integrals of motion

For the determination of the local integrals of motion $\mathcal{Q}_r(g)$, it suffices to use reduced resolvents with inner end points of the cuts collided with the poles at $|\hat{x}| = \frac{1}{2}$. They immediately follow from Eqs. (3.35) and (3.42),

$$G_0(\hat{x}) = -\frac{i\beta}{\sqrt{\varepsilon}} \frac{\hat{x}}{\sqrt{4\hat{x}^2-1}}, \tag{4.1}$$

$$G_1(\hat{x}) = \frac{i\beta}{2\pi\sqrt{\varepsilon}} \frac{\hat{x}[4\pi\hat{x}^2 + 3(4\hat{x}^2+1)\ln 2]}{\sqrt{(4\hat{x}^2-1)^5}}. \tag{4.2}$$

Since the discontinuity of the resolvent generates the distribution of Bethe roots, we conclude from the subleading order in the strong-coupling expansion that the enhanced singularity at $|\hat{x}| = \frac{1}{2}$ implies stronger accumulation of Bethe roots around these poles. This is to be contrasted with weak-coupling distributions where the poles are not visible. Then the strong-coupling expansion of the charges

$$\mathcal{Q}_r(g) = \mathcal{Q}_r^{(0)} + \frac{1}{g}\mathcal{Q}_r^{(1)} + \mathcal{O}(g^{-2}) \tag{4.3}$$

is obtained by evaluating residues of the resolvents at $\hat{x} = 0$,

$$\mathcal{Q}_r^{(k)}(g) = -\frac{1}{g^r\eta} \oint_{|\hat{x}|<\delta} \frac{d\hat{x}}{2\pi i} \hat{x}^{-r} G_k(\hat{x}), \tag{4.4}$$

with $k = 1, 2$. Explicitly, they read

$$\mathcal{Q}_r^{(0)} = \frac{L}{g^r\sqrt{\varepsilon}} \frac{\Gamma(r-1)}{\Gamma^2(\frac{r}{2})}, \tag{4.5}$$

$$\mathcal{Q}_r^{(1)} = -\frac{L}{g^r\sqrt{\varepsilon}} \frac{\Gamma(r)}{\Gamma^2(\frac{r}{2})} \left[\frac{r-2}{6} + (2r-1)\frac{\ln 2}{2\pi}\right]. \tag{4.6}$$

The eigenvalue of the Hamiltonian $\mathcal{Q}_2$ agrees with previous calculations of the cusp anomalous dimension at strong coupling [14,20,30–37,39].

5. Conclusions

In the present Letter we have suggested a systematic quasiclassical expansion of all-order Baxter equation in the noncompact $sl(2)$ sector of the maximally supersymmetric Yang–Mill theory at strong coupling. We have focused on the single-logarithmic asymptotics in the conformal spin corresponding to leading order contribution in the parameter ε, see Eq. (3.33). We found the generating function for all local conserved charges $\mathcal{Q}_r$ in the first two orders of the inverse-coupling expansion. For $r = 2$, we reproduce cusp anomalous dimension at strong coupling found by other techniques.

Our approach can immediately be used to compute corrections order-by-order in ε-expansion. It can further be employed to find subsequent terms in the inverse-coupling expansion by computing higher-order terms to the resolvent. For instance, at next-to-next-to-leading order, we find the following Riemann–Hilbert problem

$$\not{G}_2(\hat{x}) = -\frac{\hat{x}^2}{4\hat{x}^2-1}\big[\big(G_1(\hat{x}+i0) - W_1(\hat{x})\big)\cot p_0(\hat{x})\big]' + \theta_1^{(1)}(\hat{x}) + g_2(\hat{x}),$$

where $g_2(\hat{x})$ is

$$g_2(\hat{x}) = -\frac{8\beta\hat{x}^3(1+16\hat{x}^2+16\hat{x}^4)}{(4\hat{x}^2-1)^5} - \frac{1}{1536}\frac{\hat{x}S_0^{(5)}(\frac{1}{2})}{(4\hat{x}^2-1)} - \frac{5}{384}\frac{\hat{x}S_0^{(4)}(\frac{1}{2})}{(4\hat{x}^2-1)} - \frac{1}{384}\frac{\hat{x}(21-296\hat{x}^2+336\hat{x}^4)S_0'''(\frac{1}{2})}{(4\hat{x}^2-1)^3}$$
$$- \frac{1}{64}\frac{\hat{x}((1+4\hat{x}^2)^2-144\hat{x}^2)S_0''(\frac{1}{2})}{(4\hat{x}^2-1)^3} + \frac{1}{96}\frac{\hat{x}(3+104\hat{x}^2+48\hat{x}^4)S_0'(\frac{1}{2})}{(4\hat{x}^2-1)^3}, \tag{5.1}$$

and

$$\theta_1^{(1)}(\hat{x}) = \frac{1}{4\hat{x}^2-1}\frac{2}{\pi}\oint_C \frac{d\hat{z}}{2\pi i}\frac{S_1'(\hat{z})}{4\hat{z}^2-1}\vartheta_1(\hat{x},\hat{z}). \tag{5.2}$$

Its analysis deserves, however, a separate study.

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