Available online at www.sciencedirect.com
SCIENCE DIRECT
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 47 (2004) 1915–1920
www.elsevier.com/locate/camwa

Series Solution to the Pochhammer-Chree Equation and Comparison with Exact Solutions

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(Received June 2002; revised and accepted February 2003)

Abstract—In this study, a decomposition method for approximating the solution of the Pochhammer-Chree equation is implemented. By using this scheme, explicit exact solution is calculated in the form of a convergent power series with easily computable components. To illustrate the application of this method numerical results are derived by using the calculated components of the decomposition series. The obtained results are found to be in good agreement with the exact solutions known for some special cases.

Keywords—Decomposition method, Pochhammer-Chree equation, Traveling wave solution, Solitary wave solution

1. INTRODUCTION

The generalized Pochhammer-Chree (PC) equation

$$u_{tt} - u_{ttxx} - u_{xx} - (f(u))_{xx} = 0,$$

represent a nonlinear model of longitudinal wave propagation of elastic rods [1,2]. In the work of Bogolubsky [1], the author obtained exact solitary wave solutions to equation (1) for $f(u) = u^p$ for the values $p = 2, 3, 5$, respectively, [3–5]. Li *et al.* [4] and Zhang *et al.* [5] derived some explicit solitary wave solutions of (1) using the method of solving algebraic equations for the cases $f(u) = a_1u + a_3u^3 + a_5u^5$ and $f(u) = a_1u + a_2u^2 + a_3u^3$.

Finding explicit exact and numerical solutions of nonlinear equations efficiently is of major importance and has widespread applications in numerical analysis and applied mathematics. In this study, we will implement the Adomian decomposition method (in short ADM) [6–8] to find exact solution and approximate solutions to the PC equation for a given nonlinear $f(u)$.

Unlike classical techniques, the decomposition method leads to an analytical approximate and exact solutions of the nonlinear equations easily and elegantly without transforming the equation

0898-1221/04/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2003.02.012

or linearizing the problem and with high accuracy, minimal calculation and avoidance of physically unrealistic assumptions. As a numerical tool, the method provide us with numerical solution without discretization of the given equation, and therefore, it is not effected by computation round-off errors and one is not faced with necessity of large computer memory and time. The method has features in common with many other methods, but it is distinctly different on close examination and one should not be mislead by apparent simplicity into superficial conclusions [7].

In this paper, various PC equations [4] can be handled more easily, quickly, and elegantly by implementing the ADM rather than the traditional methods for finding analytical as well as numerical solutions.

2. ANALYSIS OF THE METHOD

In this section, we outline the steps to obtain analytic solution of PC equation (1) using the ADM. First, we write the PC equation in the standard operator form

$$L_t u - L_x(L_t u) - L_x u - L_x f(u) = 0, \tag{1}$$

where the notations $L_t = \frac{\partial^2}{\partial t^2}$ and $L_x = \frac{\partial^2}{\partial x^2}$ symbolize the linear differential operators. The inverse operator L_t^{-1} exists and it can conveniently be taken as the twofold integration operator L_t^{-1}. Thus, applying the inverse operator L_t^{-1} to (1) yields

$$L_t^{-1} L_t u = L_t^{-1}(L_x(L_t u) + L_x u + L_x f(u)). \tag{2}$$

Therefore, it follows that

$$u(x,t) = u(x,0) + t u_t(x,0) + L_t^{-1}(L_x(L_t u) + L_x u + L_x f(u)). \tag{3}$$

Now, we decompose the unknown function $u(x,t)$ a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{4}$$

The zeroth component is usually taken to be all terms arise from the initial conditions, i.e.,

$$u_0 = u(x,0) + t u_t(x,0). \tag{5}$$

The remaining components $u_n(x,t)$, $n \geq 1$, can be completely determined such that each term is computed by using the previous term. Since u_0 is known,

$$u_n = L_t^{-1}(L_x(L_t u_{n-1}) + L_x u_{n-1} + L_x A_{n-1}), \qquad n \geq 1, \tag{6}$$

where $f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$. The components A_n are called the Adomian polynomials, these polynomials can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [6,9]. In this specific nonlinearity, we use the general formula for A_n polynomials as

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right]_{\lambda=0}, \qquad n \geq 0. \tag{7}$$

This formula make it easy to set computer code to get as many polynomial as we need in the calculation of the numerical as well as analytical solutions. The first few Adomian polynomials for the nonlinearity $f(u)$

$$A_0 = f(u_0), \qquad A_1 = u_1 f^{(1)}(u_0), \qquad A_2 = u_2 f^{(1)}(u_0) + \frac{1}{2!} u_1^2 f^{(2)}(u_0),$$
$$A_3 = u_3 f^{(1)}(u_0) + u_2 u_1 f^{(2)}(u_0) + \frac{1}{3!} u_1^3 f^{(3)}(u_0),$$

and so on, the rest of the polynomials can be constructed in a similar manner.

A slight modification to the ADM was proposed by Wazwaz [8] that gives some flexibility in the choice of the zeroth component u_0 to be any simple term and modify the term u_1 accordingly. Since the computations in (6) depends heavily on u_0 the whole computations to find the solution will be simplified considerably. For example an alternative scheme to (6) might be

$$\begin{aligned} u_0 = u(x,0), \qquad u_1 = tu_t(x,0) + L_t^{-1}(L_x(L_t u_0) + L_x u_0 + L_x A_0), \\ u_n = -L_t^{-1}(L_x(L_t u_{n-1}) + L_x u_{n-1} + L_x A_{n-1}), \qquad n \geq 2, \end{aligned} \tag{8}$$

Finally an N-term approximate solution is given by

$$\Phi_N = \sum_{n=0}^{N-1} u_n, \qquad N \geq 1, \tag{9}$$

and the exact solution is $u(x,t) = \lim_{N\to\infty} \Phi_N$.

Numerical computations of the PC equation have often been repeated in the literature. However, to show the effectiveness of the proposed decomposition method and to give a clear overview of the methodology some examples of the generalized PC equation (1) will be discussed in the following section.

3. APPLICATIONS OF THE PC EQUATION

In this section, we will be concerned with the solitary wave solutions of the generalized PC equation

$$u_{tt} - u_{ttxx} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0. \tag{10}$$

Existence and derivations of such solutions have been discussed for particular values of the constants [1–5].

In the first example, we will consider equation (10) for the special case $a_3 = 0$ associated the initial conditions

$$u(x,0) = \sqrt{B\,\mathrm{sech}(2kx)}, \qquad u_t(x,0) = -\sqrt{B}kc\sqrt{\mathrm{sech}(2kx)}\tanh(2kx), \tag{11}$$

where $B = \sqrt{3(c^2 - a_1)/a_5}$, $k = \sqrt{(c^2 - a_1)/c^2}$. For simplicity, we take $a_1 = 1$, $a_5 = 1/5$. To find the solution of the initial value problem (10) and (11), we apply the scheme (8). The Adomian polynomials A_n are computed according to (7) with $f(u) = u^5$ and this gives

$$A_0 = u_0^5, \qquad A_1 = 5u_1 u_0^4, \qquad A_2 = 5u_2 u_0^4 + 10u_1^2 u_0^3,$$

Performing the integration, we obtain the following

$$u_0 = \sqrt{B\,\mathrm{sech}(2kx)}, \tag{12}$$

$$\begin{aligned} u_1 = {} & -\sqrt{B}kct\,\mathrm{sech}(2kx)^{3/2}\sinh(2kx) + \frac{Bk^2t^2}{16}[-9 - 36B^2 - 8\cosh(4kx) \\ & + 20B^2\cosh(4kx) + \cosh(8kx)]\,\mathrm{sech}^{9/2}(2kx), \end{aligned} \tag{13}$$

$$\begin{aligned} u_2 = {} & \frac{Bk^4t^2}{64\sqrt{B}}[754 + 11804B^2 + (447 - 9872B^2)\cosh(4kx) - (306 - 500B^2)\cosh(8kx) \\ & + \cosh(12kx)]\,\mathrm{sech}^{13/2}(2kx) + \frac{Bk^4t^4}{3072\sqrt{B}}[1955 + 20400B^2 + 86768B^4 \\ & + (2096 + 7080B^2 - 81472B^4)\cosh(4kx) - (164 + 12720B^2 - 6480B^4)\cosh(8kx) \\ & - (304 - 600B^2)\cosh(12kx) + \cosh(16kx)]\,\mathrm{sech}^{17/2}(2kx) + \frac{\sqrt{B}k^3ct^3}{48}[57 \\ & + 404B^2 + (56 - 100B^2)\cosh(4kx) - \cosh(8kx)]\,\mathrm{sech}^{11/2}(2kx)\sinh(2kx), \end{aligned} \tag{14}$$

$$\begin{aligned}
u_3 = {} & \frac{Bk^6t^2}{256\sqrt{B}}[-204085 - 8070376B^2 - (63364 - 8257772B^2)\cosh(4kx) \\
& + (132916 - 956632B^2)\cosh(8kx) - (7804 - 12500B^2)\cosh(12kx) \\
& + \cosh(16kx)]\operatorname{sech}^{17/2}(2kx) + \frac{Bk^4t^4}{6144\sqrt{B}}[-471534k^2 - 10345608B^2k^2 \\
& - 74503968B^4k^2 + 7896B^2c^2 - (401982k^2\cosh(4kx) + 905184B^2k^2\cosh(4kx) \\
& - 82618736B^4k^2\cosh(4kx) - 6208B^2c^2)\cosh(4kx) + (194664k^2 \\
& + 8267424B^2k^2 - 12705760B^4k^2 - 4000B^2c^2)\cosh(8kx) \\
& + (117309k^2 - 1159200B^2k^2 + 343440B^4k^2 - 2112B^2c^2)\cosh(12kx) \\
& - (7802k^2 - 200B^2c^2)\cosh(16kx) + k^2\cosh(20kx)]\operatorname{sech}^{21/2}(2kx) \\
& + \frac{Bk^6t^6}{1474560\sqrt{B}}[-1345050 - 30125800B^2 - 189007600B^4 - 805302400B^6 \\
& - (1552368 + 20446120B^2 - 6079296B^4 - 943215040B^6)\cosh(4kx) \\
& + 104655\cosh(8kx) + (19438048B^2 + 162041280B^4 - 175733632B^6)\cosh(8kx) \\
& + (421480 + 8297468B^2 - 32170816B^4 + 7625280B^6)\cosh(12kx) \\
& + (101706 - 1444600B^2 + 874800B^4)\cosh(16kx) - (7800 - 16300B^2)\cosh(20kx) \\
& + \cosh(24kx)]\operatorname{sech}^{25/2}(2kx) + \frac{\sqrt{B}k^5ct^3}{192}[-10450 - 188940B^2 - (8895 \\
& - 96848B^2)\cosh(4kx) + (1554 - 2500B^2)\cosh(8kx) \\
& - \cosh(12kx)]\operatorname{sech}^{15/2}(2kx)\sinh(2kx) + \frac{\sqrt{B}k^5ct^5}{15360}[-29795 - 523184B^2 \\
& - 2760688B^4 - (37136 + 356136B^2 - 1927232B^4)\cosh(4kx) - (5788 - 163248B^2 \\
& + 110160B^4)\cosh(8kx) + (1552 - 3800B^2)\cosh(12kx) \\
& - \cosh(16kx)]\operatorname{sech}^{19/2}(2kx)\sinh(2kx),
\end{aligned} \tag{15}$$

in this manner the components of the decomposition series (4) are obtained as far as we like. This series is exact to the last term, as one can verify, of the Taylor series of the exact closed form solution $u(x,t) = \sqrt{B\operatorname{sech}[2k(x-ct)]}$, [9].

In the second example, we will consider the PC equation (10) with the initial conditions

$$u(x,0) = \sqrt{D(1-\tanh kx)}, \qquad u_t(x,0) = \frac{Dkc\operatorname{sech}^2 kx}{\sqrt{D(1-\tanh kx)}}, \tag{16}$$

where

$$c = \sqrt{a_1 - \frac{3a_3^2}{16a_5}}, \qquad D = \frac{2(c^2-a_1)}{a_3}, \qquad k = \sqrt{\frac{c^2-a_1}{c^2}}.$$

Again, to find the solution of this equation, we substitute in the scheme (8)

$$u_0 = \sqrt{D(1-\tanh kx)}, \tag{17}$$

$$u_1 = \frac{tDkc\operatorname{sech}^2 kx}{2\sqrt{D(1-\tanh kx)}} + \int_0^t\int_0^t \left[\left(a_1u_0 + a_3\hat{A}_0 + a_5A_0\right)_{xx} + (u_0)_{xxtt}\right] dt\,dt, \tag{18}$$

$$\vdots$$

$$u_n = \int_0^t\int_0^t \left[\left(a_1u_{n-1} + a_3\hat{A}_{n-1} + a_5A_{n-1}\right)_{xx} + (u_{n-1})_{xxtt}\right] dt\,dt, \qquad n \ge 2, \tag{19}$$

where the Adomian polynomials A_{n-1} are given same as in the first example and $\hat{A}_{n-1}$ are given as

$$\hat{A}_0 = u_0^3, \qquad \hat{A}_1 = 3u_1u_0^2, \qquad \hat{A}_2 = 3u_2u_0^2 + 3u_1^2u_0,$$

the Adomian polynomials are constructed by (7) for the function $f(u) = a_3u^3 + a_5u^5$ with $a_1 = 1$, $a_3 = 1$, $a_5 = -1$. Performing the calculations in (8) using MATHEMATICA and substituting into (4) gives the exact solution

$$u(x,t) = \sqrt{D(1 - \tanh k(x - ct)}, \qquad (20)$$

in a series form [9].

4. EXPERIMENTAL RESULTS FOR THE PC EQUATION

The convergence of the decomposition series have investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the literature [10–14]. They obtained some results about the speed of convergence of this method. In recent work of Abbaoui *et al.* [15] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [15].

In order to verify numerically whether the proposed methodology lead to accurate solutions, we will evaluate the ADM solutions using the N-term approximation for some examples of the PC equations solved in the previous section. First, we consider equation (10) with $a_1 = 1$, $a_3 = 0$, $a_5 = 1/5$ with initial conditions (11). The differences between the 5-terms solution and the exact solution for some values of the constant c are shown in Tables 1 and 2.

The solution (20) of equation (10) and initial conditions (16) is evaluated for the values $a_1 = 1$, $a_3 = 1$, $a_5 = -1$ and the differences between the approximate 4-terms solution and the exact

Table 1. The absolute difference between the present solution $\Phi_5(x,t)$ and the analytical solution of the equation (10) with initial values (11) when $c = 1.01$.

t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.62343E − 04	6.61116E − 04	5.73209E − 04	1.62443E − 04	6.21100E − 04
0.2	1.05693E − 03	1.86111E − 03	2.38637E − 03	2.59403E − 03	2.43226E − 03
0.3	1.65274E − 03	3.07289E − 03	4.23211E − 03	5.09069E − 03	5.59681E − 03
0.4	2.24761E − 03	4.29091E − 03	6.10035E − 03	7.63655E − 03	8.84953E − 03
0.5	2.83918E-03	5.50892E-03	7.97933E-03	1.02125E − 02	1.21621E − 02

Table 2. The absolute difference between the present solution $\Phi_5(x,t)$ and the analytical solution of the equation (10) with initial values (11) when $c = 1.001$.

t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	3.32780E − 05	6.65452E − 05	9.97919E − 05	1.33009E − 04	1.66187E − 04
0.2	6.65474E − 05	1.33075E − 04	1.99563E − 04	2.65994E − 04	3.32348E − 04
0.3	9.97983E − 05	1.99567E − 04	2.99278E − 04	3.98904E − 04	4.98416E − 04
0.4	1.33021E − 04	2.66004E − 04	3.39891E − 04	5.31703E − 04	6.64346E − 04
0.5	1.66207E − 04	3.32366E − 04	4.98431E − 04	6.64354E − 04	8.30090E − 04

Table 3. The absolute difference between the present solution $\Phi_5(x,t)$ and the analytical solution of the equation (10) with initial values (16).

t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	3.69187E − 04	1.44302E − 03	3.19550E − 03	5.63246E − 03	8.79062E − 03
0.2	2.14845E − 04	8.15019E − 04	1.74908E − 04	2.98429E − 02	4.50513E − 03
0.3	5.27426E − 05	1.57764E − 04	2.40127E − 04	2.29654E − 04	5.88702E − 05
0.4	1.06828E − 04	4.87030E − 04	1.23547E − 03	2.45603E − 03	4.26410E − 03
0.5	2.54173E-04	1.08021E-03	2.58802E-03	4.90917E − 03	8.19960E − 03

solution are summarized in Tables 3. Tables 1–3 show that we achieved a very good approximation to the actual solution of the equations by using only 4- and 5-terms of the decomposition series solution derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

The solutions are very rapidly convergent by utilizing the ADM. The numerical results we obtained justify the advantage of this methodology. Furthermore, as the decomposition method does not require discretization of the variables, i.e., time and space, it is not effected by computation round off errors and necessity of large computer memory and time. Clearly, the series solution methodology can be applied to various type of linear or nonlinear ordinary differential equations [17,18] and partial differential equations [19–26], as well.

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