Automorphisms of cubic Cayley graphs of order $2pq$

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1. Introduction

Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is semiregular on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and regular if $G$ is transitive and semiregular. Throughout this paper a graph means a finite, simple, connected and undirected one. For a graph $X$, we use $V(X)$, $E(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set and full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to $u$ and $v$ in $X$. An $s$-arc in a graph is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of the graph such that any two consecutive vertices are adjacent and any three consecutive vertices are distinct. A 1-arc is called an arc for short and a 0-arc is a vertex. A graph $X$ is said to be $s$-arc-transitive if $\text{Aut}(X)$ is transitive on the set of $s$-arcs in $X$. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A symmetric graph $X$ is said to be $s$-regular if the automorphism group $\text{Aut}(X)$ acts regularly on the set of $s$-arcs in $X$.

Let $G$ be a finite group and $S$ a subset of $G$ such that $1 \not\in S$. The Cayley digraph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is defined to have vertex set $V(\text{Cay}(G, S)) = G$ and arc set $E(\text{Cay}(G, S)) = \{(g, sg) \mid g \in G, s \in S\}$. If $S = S^{-1}$ then $\text{Cay}(G, S)$, called a Cayley graph, is viewed as a graph by identifying two opposite arcs with one edge. It is known that a Cayley digraph $\text{Cay}(G, S)$ is connected if and only if $S$ generates $G$. Furthermore, $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\text{Aut}(\text{Cay}(G, S))$ of $\text{Cay}(G, S)$. Given a $g \in G$, define the permutation $R(g)$ on $G$ by $x \mapsto xg, x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the right regular representation of $G$, is a permutation group isomorphic to $G$. The Cayley digraph $\text{Cay}(G, S)$ is vertex-transitive because it admits $R(G)$ as a regular subgroup of the automorphism group $\text{Aut}(\text{Cay}(G, S))$. A Cayley digraph $\text{Cay}(G, S)$ is said to be normal if $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$. Xu [30, Proposition 1.5] proved that $\text{Cay}(G, S)$ is normal if and only if $\text{Aut}(\text{Cay}(G, S))_1 = \text{Aut}(G, S)$, where $\text{Aut}(\text{Cay}(G, S))_1$ is the stabilizer of $1$ in $\text{Aut}(\text{Cay}(G, S))$. A graph $X$ is isomorphic to a Cayley graph on $G$ if and only if $\text{Aut}(X)$ has a subgroup isomorphic to $G$, acting regularly on vertices (see [3, Lemma 16.3] or [26, Lemma 4]).

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the graph is neither the empty graph nor the complete graph. Du et al. [11] and Dobson et al. [9] determined the normality of Cayley graphs on groups of order twice a prime and prime square, respectively. Wang et al. [27] obtained all disconnected normal Cayley graphs. Let Cay(G, S) be a connected cubic Cayley graph on a non-abelian simple group G. Praeger [23] proved that if \( N_{\text{Aut}(\text{Cay}(G, S))}(R(G)) \) is transitive on edges then the Cayley graph Cay(G, S) is normal, and Fang et al. [12] proved that the vast majority of connected cubic Cayley graphs on non-abelian simple groups are normal. Recently, Wang and Xu [28] determined the normality of 1-regular tetravalent Cayley graphs on dihedral groups and Feng and Xu [15] proved that every connected tetravalent Cayley graph on a regular p-group is normal when \( p \neq 2, 5 \). For more results on the normality of Cayley graphs, we refer the reader to [13, 16, 19, 29, 30]. The normality of cubic Cayley graphs of order 2\( p^3 \) and 4\( p \) was determined in [31, 32] and in this paper we determine the normality of cubic Cayley graphs of order 2\( pq \) for distinct odd primes \( p \) and \( q \). Furthermore, all cubic non-symmetric Cayley graphs of order 2\( pq \) are classified, while the classifications of cubic symmetric graphs and vertex-transitive non-Cayley graphs of order 2\( pq \) were given in [33].

Let \( \mathbb{Z}_n \) be the cyclic group of order \( n \), as well as the ring of integers modulo \( n \). Denote by \( \mathbb{Z}_n^* \) the multiplicative group of \( \mathbb{Z}_n \) consisting of numbers coprime to \( n \) and by \( D_{2n} \) the dihedral group of order \( 2n \). For two groups \( M \) and \( N \), \( N \leq M \) means that \( N \) is a subgroup of \( M \) and \( N < M \) means that \( N \) is a proper subgroup of \( M \). By elementary group theory, we know that, up to isomorphism, there are six groups of order 2\( pq \) (\( p > q > 2 \)) defined as

\[
\begin{align*}
G_1(2pq) &= \langle a \rangle, \\
G_2(2pq) &= \langle a, b \mid a^p = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\
G_3(2pq) &= \langle a, b, c \mid a^p = b^2 = c^2 = 1, ab = ba, cac = a^{-1}, bc = cb \rangle, \\
G_4(2pq) &= \langle a, b, c \mid a^p = b^3 = c^2 = 1, ab = ba, ac = ca, cbc = b^{-1} \rangle, \\
G_5(2pq) &= \langle a, b, c \mid a^p = b^3 = c^2 = 1, ac = ca, bc = cb, b^{-1}ab = a' \rangle, \\
G_6(2pq) &= \langle a, b, c \mid a^p = b^3 = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a' \rangle,
\end{align*}
\]

where \( r \) is an element of order \( q \) in \( \mathbb{Z}_p^* \).

2. Preliminaries

For a subgroup \( H \) of a group \( G \), denote by \( C_G(H) \) the centralizer of \( H \) in \( G \) and by \( N_G(H) \) the normalizer of \( H \) in \( G \). Then \( C_G(H) \) is normal in \( N_G(H) \).

**Proposition 2.1** ([18, I, Theorem 4.5]). The quotient group \( N_G(H)/C_G(H) \) is isomorphic to a subgroup of the automorphism group \( \text{Aut}(H) \) of \( H \).

The following proposition is a basic fact in permutation group theory.

**Proposition 2.2** ([29, Proposition 4.4]). Every transitive abelian group \( G \) on a set \( \Omega \) is regular and the centralizer \( C_G(\Omega) \) of \( G \) in the symmetric group \( S_\Omega \) is \( G \).

In view of [7, pp.285, summary], one may extract the following proposition.

**Proposition 2.3.** Every maximal subgroup of \( \text{PSL}(2, 7) \) is isomorphic to \( \mathbb{Z}_7 \times \mathbb{Z}_3 \) or \( S_5 \). Let \( p = 7, 11 \) or 23. All subgroups of \( \text{PGL}(2, p) \) of order \( p(p - 1) \) are conjugate and isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_{p-1} \), a Frobenius group of degree \( p \).

The following proposition is known as Burnside’s \( p \)-\( q \) Theorem.

**Proposition 2.4** ([25, Theorem 8.5.3]). Let \( p \) and \( q \) be primes and let \( m \) and \( n \) be non-negative integers. Then, any group of order \( p^m q^n \) is solvable.

Let \( p \) and \( q \) be distinct odd primes. The following result gives the number of solutions of the equation \( x^2 + x + 1 = 0 \) in \( \mathbb{Z}_{pq} \).

**Lemma 2.5.** Let \( p > q \) be odd primes and \( \mathcal{O}_{pq}^3 \) the set of solutions of the equation \( x^2 + x + 1 = 0 \) in \( \mathbb{Z}_{pq} \). Then,

\[
|\mathcal{O}_{pq}^3| = \begin{cases}
2 & \text{if } q = 3, \\
4 & \text{if } (p - 1) \text{ and } 3 \mid (q - 1), \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Since \( x^2 - 1 = (x - 1)(x + 1) \), a solution of the equation \( x^2 + x + 1 = 0 \) must be an element of order 3 in \( \mathbb{Z}_{pq}^* \), implying that either \( 3 \mid (p - 1) \) and \( q = 3 \) or \( 3 \mid (p - 1) \) and \( 3 \mid (q - 1) \). For \( 3 \mid (p - 1) \) and \( q = 3 \), there are two elements...
of order 3 in $\mathbb{Z}_p^3$, say $x_1$ and $x_2 = x_1^2$. Then, $x_i = 1$ in $\mathbb{Z}_3$ for each $i = 1, 2$. Since $(x_i - 1)(x_i^2 + x_i + 1) = x_i^3 - 1 = 0$ in $\mathbb{Z}_3$, it follows that $x_1$ and $x_2$ are solutions of $x^2 + x + 1 = 0$ in $\mathbb{Z}_3$. That is $|\mathcal{O}_3^2| = 2$. For $3 \mid (p - 1)$ and $3 \mid (q - 1)$, a solution $k$ of $x^2 + x + 1 = 0$ in $\mathbb{Z}_p$ implies that $k$ is an element of order 3 in both $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$. Conversely, for every element, say $k_1$, of order 3 in $\mathbb{Z}_p^*$ and every element, say $k_2$, of order 3 in $\mathbb{Z}_q^*$, there is a unique element $k$ in $\mathbb{Z}_p \times \mathbb{Z}_q$ satisfying the equation $x^2 + x + 1 = 0$ such that $k_1 = k_2$ (mod $p$) and $k = k_3$ (mod $q$) and this can be easily proved by Eq. (2) in the proof of Lemma 3.1 in [21] which claims that for any $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_q$, $(i + p) \cap (j + q) = 1$, where $P = \{sp \mid s \in \mathbb{Z}_q\}$ and $Q = \{sq \mid s \in \mathbb{Z}_p\}$. It follows that $|\mathcal{O}_p^2| = 4$ because there are exactly two elements of order 3 in $\mathbb{Z}_p^*$ and in $\mathbb{Z}_q^*$, respectively.

Let $p > q$ be primes such that $3 \mid (p - 1)$ and $3 \mid (q - 1)$. By Lemma 2.5, there are exactly two elements of order 3, say $\lambda$ and $\lambda^2$, in the ring $\mathbb{Z}_3p$, and exactly four elements, say $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$, of order 3 satisfying the equation $x^3 + x + 1 = 0$ in $\mathbb{Z}_p$. Define

$$\mathcal{S}_{6p} = \text{Cay}(D_{6p}, \{b, ba, ba^{-1}\}),$$
$$\mathcal{S}_{2pq} = \text{Cay}(D_{2pq}, \{b, ba, ba^{-1}\}),$$
$$\mathcal{S}_{2pq} = \text{Cay}(D_{2pq}, \{b, ba, ba^{-2}\}),$$

where $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ with $n = 3p$ or $pq$. It is easy to show that $\mathcal{S}_{6p}$, $\mathcal{S}_{2pq}$ and $\mathcal{S}_{2pq}$ are independent of the choices $\lambda, \lambda_1$ and $\lambda_2$.

Take $H_1 = G_6(2 \cdot 5 \cdot 11) = G_6(110)$ and let $S_1 = \{c, abc, (abc)^{-1}\}$ be a subset of $H_1$. Take $H_2 = G_6(2 \cdot 11 \cdot 13) = G_6(506)$ and let $S_2 = \{c, ab^3c, (ab^3c)^{-1}\}$ be a subset of $H_2$. In the groups $H_1$ and $H_2$ given in Eq. (1), set $r = 3$ because 3 is an element of order 5 in $\mathbb{Z}_3^*$ and an element of order 11 in $\mathbb{Z}_{11}^*$. Define

$$CF_{110} = \text{Cay}(H_1, S_1),$$
$$\mathcal{S}_{506} = \text{Cay}(H_2, S_2).$$

With the help of software package MAGMA [4], one may easily check $Aut(CF_{110}) \cong \text{PGL}(2, 11)$ and $Aut(\mathcal{S}_{506}) \cong \text{PGL}(2, 23)$. By [5], there is a unique cubic 3-regular graph of order 110 and a unique cubic 4-regular graph of order 506. It follows that these two graphs must be $CF_{110}$ and $\mathcal{S}_{506}$ because $|\text{PGL}(2, 11)| = 1320$ and $|\text{PGL}(2, 23)| = 12144$, of which the first is called Coxeter–Frucht graph (see [6]). Note that $\text{PGL}(2, 11)$ and $\text{PGL}(2, 23)$ have subgroups of order 110 and 506 by Proposition 2.3 and since these subgroups are Frobenius, they are isomorphic to $G_6(110)$ and $G_6(506)$, respectively. A classification of cubic symmetric graphs of order 2pq was given in [33] and one may easily extract those which are Cayley.

**Proposition 2.6.** Let $X = \text{Cay}(G, S)$ be a connected cubic symmetric Cayley graph on a group $G$ of order 2pq, where $p > q$ are odd primes. Then, $X$ is s-regular for $s = 1, 3$ or 4. Furthermore,

1. $X$ is 1-regular if and only if either $q = 3$ and $3 \mid (p - 1)$ or $3 \mid (p - 1)$ and $3 \mid (q - 1)$. If $X$ is 1-regular then it is isomorphic to $\mathcal{S}_{6p}$ for $q = 3$ and $3 \mid (p - 1)$, or to $\mathcal{S}_{2pq}$ or $\mathcal{S}_{2pq}$ for $3 \mid (p - 1)$ and $3 \mid (q - 1)$;
2. $X$ is 3-regular if and only if it is isomorphic to $CF_{110}$. In this case, $G = G_6(110), S \equiv \{c, abc, (abc)^{-1}\}$ (take $r = 3$) and $Aut(X) \equiv \text{PGL}(2, 11)$;
3. $X$ is 4-regular if and only if it is isomorphic to $\mathcal{S}_{506}$. In this case, $G = G_6(506), S \equiv \{c, ab^3c, (ab^3c)^{-1}\}$ (take $r = 3$) and $Aut(X) \equiv \text{PGL}(2, 23)$.

Let $X = \text{Cay}(G, S)$ be a Cayley graph on $G$ and $A = Aut(X)$. It is known that $Aut(G, S) = \{a \in Aut(G) \mid S^a = S\}$ is a subgroup of $A$. Normal Cayley graphs are those which have the smallest possible automorphism groups.

**Proposition 2.7 ([30, Propositions 1.3 and 1.5]).** The Cayley graph $X = \text{Cay}(G, S)$ is normal if and only if $A_1 = Aut(G, S)$ if and only if $A = R(G) \rtimes Aut(G, S)$, where $A_1$ is the stabilizer of 1 in $A$ and $R(G)$ is the right regular representation of $G$.

By [10, Theorem 1 and Lemma 3.4], we have the following proposition, which can also be deduced from [14, 22].

**Proposition 2.8.** Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ be a dihedral group of order $2n$. A cubic Cayley graph $\text{Cay}(D_{2n}, S)$ on $D_{2n}$ is 1-regular if and only if $S$ is equivalent to $\{b, ba, ba^{-k}\}$ for $n \geq 13$ and $k^2 + k + 1 \equiv 0 (\text{mod } n)$. Further, these 1-regular Cayley graphs are normal.

Let $X$ and $Y$ be two graphs. The lexicographic product $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X[Y])$, $u$ is adjacent to $v$ in $X[Y]$ whenever $(x_1, x_2) \in E(X)$ or $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$. Denote by $K_n$ the complete graph of order $n$, $C_n$ the cycle of length $n$, and $K_{n, n} - nK_2$ the graph by deleting a one factor from the complete bipartite graph $K_{n, n}$ of order $2n$. The following proposition gives all non-normal connected Cayley graphs of valency at most 4 on cyclic groups.

**Proposition 2.9 ([2, Corollary 1.3]).** All connected Cayley graphs with valency at most 4 on a finite cyclic group are normal, except for $G = \mathbb{Z}_4$ and $X = K_4, G = \mathbb{Z}_6$ and $X = K_3, G = \mathbb{Z}_5$ and $X = K_5, G = \mathbb{Z}_{2n}$ and $X = C_m[2K_1](m \geq 3)$, or $G = \mathbb{Z}_{10}$ and $X = K_{5, 5} - 5K_2$. 


Given a subset $S$ of a group $G$ with $1 \not\in S$, we call $S$ a CI-subset of $G$ and $\text{Cay}(G, S)$ a CI-graph, if $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ implies that $S$ and $T$ are equivalent, that is, there exists a $\gamma \in \text{Aut}(G)$ such that $S' = T$. The following result is a well-known criterion for CI-subset due to Babai [1].

**Proposition 2.10.** Let $G$ be a finite group and $S$ a subset of $G$ not containing the identity element 1. Let $X = \text{Cay}(G, S)$ and $A = \text{Aut}(X)$. Then $S$ is a CI-subset of $G$ if and only if for any $\sigma \in S_3$ with $\sigma^{-1}R(G)\sigma \preceq A$, there exists an $\alpha \in A$ such that $\sigma^{-1}R(G)\sigma = \alpha^{-1}R(G)\alpha$, where $S_3$ denotes the symmetric group on $G$.

Qu and Yu [24] investigated the CI-property of Cayley graphs on dihedral groups.

**Proposition 2.11 ([24, Theorem 3.5]).** Let $G$ be a dihedral group of order $2n$ with $n$ odd and $S$ a subset of $G$ not containing the identity 1. If $|S| \leq 3$ then $S$ is a CI-subset.

3. Automorphism groups of cubic Cayley graphs of order $2pq$

In this section, we shall determine the automorphism groups of cubic Cayley graphs of order $2pq$ for two distinct odd primes $p$ and $q$. First we prove a lemma which will be used later.

**Lemma 3.1.** Let $G$ be a regular subgroup of $\text{Aut}(\mathcal{C}_{6p})$. Then, $G \cong G_2(6p)$ or $G_6(6p)$. Furthermore, as a Cayley graph on $G_2(6p)$, $\mathcal{C}_{6p}$ is normal and as a Cayley graph on $G_6(6p)$, $\mathcal{C}_{6p}$ is non-normal and $\mathcal{C}_{6p} \cong \text{Cay}(G_6(6p), S)$ with $S \equiv \{c, abc, (abc)^{-1}\}$.

**Proof.** Let $X = \mathcal{C}_{6p}$ and $A = \text{Aut}(X)$. We first claim that $A$ contains regular subgroups isomorphic to $G_6(6p)$. By definition of the graph $\mathcal{C}_{6p}$, one may assume that $X = \text{Cay}(G_2(6p), S)$, where $G_2(6p) = \langle a, b \mid a^{6p} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $S = \{b, ba, ba^{-1}\}$ with $k^2 + k + 1 = 0$ in $\mathbb{Z}_{3p}$. Clearly, $k$ has order 3 in $\mathbb{Z}_{3p}$. By Proposition 2.8, $X$ is regular and $\text{Cay}(G_2(6p), S)$ is normal. Thus, $A = \text{Aut}(G_2(6p)) \times \langle \alpha \rangle$, where $\alpha$ is an automorphism of order 3 of $G_2(6p)$ induced by $a^\alpha = a$ and $b^\alpha = ba$.

Note that $R(a^\alpha) \not\leq A$. Since each subgroup of $\langle R(a^\alpha) \rangle$ is characteristic in $R(a^\alpha)$, one has $R(a^\alpha) \not\leq A$ and $R(a^\alpha) \not\leq A$. Thus, $R(a^\alpha)$ has order 3. Note that $k^2 + k + 1 = 0$ (in $\mathbb{Z}_{3p}$) implies that $k = 1$. It follows that $k^2 \equiv 1 \pmod{3}$. Clearly, $k^2 \equiv 1 \pmod{3}$ because $k^2 \equiv 1 \pmod{3}$. Thus, $a^{t-k}b$ has order 3 and since $3 | (p - 1)$, $a^{t-k}b$ also has order 3. Now it is easy to show that $R(a^\alpha)) R(a^\alpha) = R(b^\alpha)$. Furthermore, $R(a^\alpha) R(a^\alpha) = R(a^\alpha)^3 = (R(a^\alpha))^3$ and $R(a^\alpha) R(a^\alpha) = R(a^\alpha)^3 = (R(a^\alpha))^3$. Thus, $H = \langle R(a^\alpha), R(a^\alpha), R(a^\alpha) \rangle \cong G_6(6p)$. If the stabilizer $H$ of the identity 1 in $H$ is not trivial, then $H_1 = \text{Aut}(G_2(6p), S) = \langle \alpha \rangle$, forcing $A = H$, a contradiction. Thus, $H$ is regular on $\text{Aut}(X)$, that is, $A$ contains regular subgroups isomorphic to $G_6(6p)$, as claimed.

Let $M$ be an arbitrary regular subgroup of $A$. If $M \cong G_2(6p)$ then Proposition 2.8 implies that $X$, as a Cayley graph on $G_2(6p)$, is normal. Now assume $M \cong G_2(6p)$. Since $|A| = 18p$, one has $A = \text{Aut}(G_2(6p))$, implying that $|M \cap \text{Aut}(G_2(6p))| = 2p$. Since $G_2(6p)$ has no normal subgroups of order 2p, $M$ is not normal in $A$, namely, $X$, as a Cayley graph on $M \cong G_2(6p)$, is non-normal. Further, since $|M \cap G_2(6p)| / 2p$ and $R(a^\alpha)$ is a normal Sylow $p$-subgroup of $A$, one has $R(a^\alpha) \not\leq M$. As the centralizer $C_A(R(a^\alpha))$ of $R(a^\alpha)$ in $A$ is $\langle R(a^\alpha) \rangle \cong \mathbb{Z}_{3p}$, one has $C_A(R(a^\alpha)) = \mathbb{Z}_{3p} \cap C_A(R(a^\alpha)) = (R(a^\alpha))$. For any given group in Eq. (1), if the centralizer of a Sylow $p$-subgroup of the group is the Sylow $p$-subgroup itself then the group must be $G_6(6p)$. It follows that $M \cong G_6(6p)$. Without loss of generality, let $M = G_6(6p) = \langle a, b, c \mid a^{6p} = b^2 = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = da \rangle$ with $r$ an element of order 3 in $\mathbb{Z}_{3p}$, and let $X = \text{Cay}(G_6(6p), S)$. Since all involutions of $G_6(6p)$ are conjugate and are contained in $(a, c)$, by the connectivity of $X$, one may assume $S = \{c, y, y^{-1}\}$, where $y$ has order 3 or 6. If $y$ has order 3 then there is a 3-cycle $(1, y, y^{-1})$, passing through 1, $y$ and $y^{-1}$, but there is no 3-cycle passing through the vertices 1, $c$, $y$, contrary to the symmetry of $X$. Thus, $y$ has order 6 and one of $y$ and $y^{-1}$ has form $a^ib^j$, $1 \leq i \leq 6$. Since the map $a \mapsto a^i, b \mapsto b^j, c \mapsto c$ induces an automorphism of $G_6(6p)$, one has $S \equiv \{c, abc, (abc)^{-1}\}$. □

The following is the main result of this section.

**Theorem 2.3.** Let $p > q$ be odd primes and let $X = \text{Cay}(G, S)$ be a connected cubic Cayley graph of order $2pq$. Then either $\text{Aut}(X) \cong \text{Aut}(G, S)$ or one of the following holds:

1. $G = G_6(6p)$ with $3 | (p - 1)$, $S \equiv \{c, abc, (abc)^{-1}\}$ and $\text{Aut}(X) \cong G_6(6p) / 3$;
2. $G = G_6(110), S \equiv \{c, abc, (abc)^{-1}\}$ (take $r = 3$) and $\text{Aut}(X) \cong \text{PGL}(2, 11)$;
3. $G = G_5(506), S \equiv \{a, b, c \mid (abc)^{-1}\}$ (take $r = 3$) and $\text{Aut}(X) \cong \text{PGL}(2, 23)$;
4. $G = G_4(42), S \equiv \{c, ab, (ab)^{-1}\}$ and $\text{Aut}(X) \cong \text{PGL}(2, 7)$.

**Proof.** Let $A = \text{Aut}(X)$. Assume that $\text{Aut}(X) > R(G) \times \text{Aut}(G, S)$, that is, $R(G)$ is not normal in $A$. We deal with two cases depending on the symmetry of $X$.

Case I: $X$ is symmetric.

By Proposition 2.6, $X$ is isomorphic to $\text{CF}_{110}, \mathcal{C}_{506}, \mathcal{C}_{6p}, \mathcal{C}_{2pq}$ or $\mathcal{C}_{2pq}$. If $X \cong \mathcal{C}_{6p}$, then by Lemma 3.1, $G \cong G_6(6p)$ and $S \equiv \{c, abc, (abc)^{-1}\}$, that is the case (1) in the theorem. Assume $X \cong \text{CF}_{110}$. Then $\text{Aut}(X) \cong \text{PGL}(2, 11)$, and by Proposition 2.3, one may assume that $X = \text{Cay}(G_6(110), S)$, where $G_6(110) = \langle a, b, c \mid a^{11} = b^5 = c^2 = 1, cac = \rangle$. The remaining cases are similar.
Proposition 2.3

and so there are seven involutions in $G$. Then $|X| \leq 30$, implying that $X$ cannot be a Cayley graph on $G_i(2pq)$ for $i = 1, 3, 4, 5$ or 6.

Case II: $X$ is non-symmetric.

In this case, the stabilizer $A_v$ of $v \in X$ in $A$ is a 2-group and hence $|A| = 2^f \cdot p \cdot q$ with $f \geq 2$. We claim that $A$ has no normal 2-subgroups. Suppose to the contrary that $H$ is a normal 2-subgroup of $A$. Let $X_H$ be the quotient graph of $X$ relative to $H$, that is, the graph with vertices the orbits of $H$ in $X$ and with two orbits adjacent if there is an edge in $X$ between those two orbits. Let $K$ be the kernel of $A$ acting on $V(X_H)$. Then, $H \leq K$ and $A/K$ is transitive on $V(X_H)$. Since $|V(X_H)| = 2pq$, every orbit of $H$ in $V(X)$ has length 2, implying $|V(X_H)| = pq$. As $X$ has valency 3 and $H \leq K$, $X_H$ has valency 2 or 3, and since $pq$ is odd, $X_H$ has valency 2. By the connectivity of $X$, $X_H$ is a cycle of length $pq$, say $V(X_H) = \{B_0, B_1, \ldots, B_{pq-1}\}$, where $B_i$ is adjacent to $B_{i+1}$ for each $i \in \mathbb{Z}_{pq}$. If there is no edge in each $B_i$, then one may assume that each vertex in $B_i$ is adjacent to one vertex in $B_0$ and two vertices in $B_2$. By the transitivity of $A/K$ on $V(X_H)$, the length of the cycle $X_H$ must be even, contrary to the fact that $pq$ is odd. If there is an edge in some $B_i$, then there is an edge in each $B_i$, $0 \leq i \leq pq - 1$, because of the transitivity of $A/K$ on $V(X_H)$. Since $K$ fixes each orbit of $H$, $K$ is isomorphic to $H$ and hence $|A : A/K| = 2$, implying $|A/K| \leq 2$, a contradiction.

 fermented. By the transitivity of $A/K$ on $V(X_H)$, since $K$ fixes each orbit of $H$, $K$ is isomorphic to $H$ and hence $|A : A/K| = 2$, implying $|A/K| \leq 2$, a contradiction.

Assume that $N$ is solvable. Then $N \cong Z_p$ or $Z_q$, by Proposition 2.4, $A/N$ is solvable. Let $C = C_4(N)$. By Proposition 2.1, $A/C \leq \text{Aut}(N) \cong Z_p$ or $Z_q$. Clearly, $N \cong C$. There are two subcases: $N = C$ and $N < C$, that is, $N$ is a proper subgroup of $C$.

Suppose $N = C$. Then $A/N \leq \text{Aut}(N) \cong Z_p$ or $Z_q$. Since $|N/C| = 2$, one has $N \cong Z_p$ and $A/N \leq Z_p$. Let $X_H$ be the quotient graph of $X$ relative to the orbits of $N$, and $K$ the kernel of $A$ acting on $V(X_H)$. Then, $N \leq K$ and $A/K$ is transitive on $V(X_H)$. Since $N$ is normal in $A$, $X_H$ has valency 3 at most, and since $N \cong Z_p$, one has $|V(X_H)| = 2q > 1$, implying that $X_H$ has valency 2 or 3. If $X_H$ has valency 3 then $K$ has trivial stabilizers and hence $K = N$. By Proposition 2.2, $A/N$ is regular on $V(X_H)$ because $A/N \leq Z_p$. It follows that $|A| = 2pq$, forcing $R(G) \not\leq A$, a contradiction. If $X_H$ has valency 2 then $X_H$ is a cycle of length $2pq$ because of the connectivity of $X$. Let $V(X_H) = \{B_0, B_1, B_2, \ldots, B_{pq-1}\}$ with $B_i$ adjacent to $B_{i+1}$ for each $i \in \mathbb{Z}_{pq}$. If there is an edge of $X$ in each $B_i$, then the induced subgraph $(B_i)$ of $B_i$ in $X$ must be a cycle of length $p$ because $|B_i| = p$ is odd. In this case, $X_H$ has valency 1, a contradiction. Thus, there is no edge in each $B_i$, and one may assume that each vertex in $B_i$ connects one vertex in $B_0$ and two vertices in $B_2$. It follows that the induced subgraph $(B_0 \cup B_1)$ of $B_0 \cup B_1$ in $X$ is a perfect matching and the induced subgraph $(B_1 \cup B_2)$ of $B_1 \cup B_2$ in $X$ is a cycle of length $2p$ because $|B_i| = p$ is odd. Thus, $A/K$ is not arc-transitive on $X_H$, and hence $A/K < \text{Aut}(X)/N \cong D_{4pq}$, implying $|A/K| = 2q$ by the vertex-transitivity of $A/K$ on $V(X_H)$. Further, $K$ acts faithfully on $B_i$ and $K < \text{Aut}(B_1 \cup B_2) \cong D_{4pq}$. It follows that $|K| \leq 2p$ and hence $|A| \leq 4pq$. Thus, $R(G) \leq A$ because $|A : R(G)| \leq 2$, a contradiction.

Suppose $N < C$. Take a minimal normal subgroup of $A/N$, say $M/N$, in $C/N$. Since $A/N$ is solvable, $M/N$ is elementary abelian. It follows that either $M/N$ is a 2-group, or $M/N \cong Z_2$ or $Z_3$. For the former, one has $|M| = 2^f \cdot p$ or $2^f \cdot q$ for some integer $f \geq 1$. Since $M \leq C$, a Sylow 2-subgroup of $M$ is characteristic in $M$, and hence normal in $A$ because $M \leq A$. This is impossible because $A$ has no normal 2-subgroups. Thus, $M/N \cong Z_2$ or $Z_3$, and hence $M \cong Z_2$ or $Z_3$ because $M \leq C$. Clearly, $M \leq C_4(M)$. If $M = C_4(M)$ then, by Proposition 2.1, $A/M \leq \text{Aut}(M) \cong Z_2 \times Z_2$. Since $M \leq A$, one has $M \leq R(G)$, implying $R(G)/M \leq A/M$, that is, $R(G) \leq A$, a contradiction. If $M < C_4(M)$ then $C_4(M)/M$ must be a 2-group. It follows that $C_4(M) = M \times Q$, where $Q$ is a Sylow 2-subgroup of $C_4(M)$. Thus, $Q$ is characteristic in $C_4(M)$ and normal in $A$ because $C_4(M) \leq A$, contrary to the fact that $A$ has no normal 2-subgroups.

Assume that $N$ is solvable. Since $|N| = 2^f \cdot p \cdot q$ and $p > q > 2$, $N$ must be a non-abelian simple and by [17, pp. 12–14], $N$ is one of the following groups:

$$A_5, A_6, PSL(2, 7), PSL(2, 8), PSL(2, 17), PSL(3, 3), PSU(3, 3) \text{ and } PSU(4, 2).$$

Since $p^2 \nmid |N|$ and $q^2 \nmid |N|$, by checking the orders of the above groups, one has $N = A_5$ or $PSL(2, 7)$. Let $C = C_4(N)$. Then $N \cap C = 1$ because $N$ is simple. It follows that either $C$ is a 2-subgroup or $C = 1$. Thus, $C = 1$ because $A$ has no normal 2-subgroups, and by Proposition 2.1, one has $A \leq \text{Aut}(N)$. If $N = A_5$ then $A = A_5$ or $S_5$. However, both $S_5$ and $A_5$ have no subgroups of order 30, implying that $X$ is a non-Cayley graph, a contradiction. It follows that $N = PSL(2, 7)$ and $A \leq \text{Aut}(N) \cong PGL(2, 7)$. Since $X$ is a Cayley graph, $A$ contains a regular subgroup of order 42 and by Proposition 2.3, $PGL(2, 7)$ has no subgroup of order 42, implying $A = PGL(2, 7)$. By Proposition 2.3, every subgroup of order 42 in $PGL(2, 7)$ is conjugate to $G_2(42)$. Without loss of generality, let $G = G_2(42) = \{a, b, c | a^2 = b^2 = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a^b \}$. Clearly, all involutions in $G$ are conjugate and hence one may assume $c \in S$. Note that the centralizer of $c$ in $G$ has order 6 and so there are seven involutions in $G$, of which all are contained in $(a, c)$. Since $S$ generates $G$, $S = \{c, y, y^{-1}\}$, where $y$ has
order 3 or 6. If y has order 6 then one of y and $y^{-1}$ has the form $a'bc$, $1 \leq i \leq 6$, and since the map $a \mapsto a'$, $b \mapsto b$, $c \mapsto c$ induces an automorphism of $G_0(6p)$, one may further assume $S = \{c, abc, (abc)^{-1}\}$. If y has order 3, one of y and $y^{-1}$ has the form $a'b$, $1 \leq i \leq 6$, and similarly one may assume $S = \{c, ab, (ab)^{-1}\}$. With the help of computer software package MAGMA [4], $|\text{Aut}(X)| = 3 \cdot 42$ for $S = \{c, abc, (abc)^{-1}\}$ and $\text{Aut}(X) \cong \text{PGL}(2, 7)$ for $S = \{c, ab, (ab)^{-1}\}$. For the former, X is arc-transitive, a contradiction, and for the latter, X is not normal because PGL(2, 7) has no normal subgroup of order 42, which is the Case (4) in the theorem. □

4. Cubic non-symmetric Cayley graphs of order $2pq$

Let $p > q$ be odd primes. In this section we shall classify connected cubic non-symmetric Cayley graphs of order $2pq$. For $x \in \mathbb{Z}_{2pq}$ denote $x^2$ the inverse of x in the multiplicative group $\mathbb{Z}_{2pq}$. Let $\Theta_{pq}^3$ be the set of solutions of the equation $x^2 + x + 1 = 0$ in $\mathbb{Z}_{2pq}$. By Lemma 2.5, $|\Theta_{pq}^3| = 2$ for $3 | (p - 1)$ and $q = 3$, $|\Theta_{pq}^3| = 4$ for $3 | (p - 1)$ and $3 | (q - 1)$, and $|\Theta_{pq}^3| = 0$ otherwise. There are exactly three involutions in $\mathbb{Z}_{2pq}^*$, denoted by $\lambda_1, \lambda_2$ and $\lambda_3$. Set

$$A = \{\lambda_1, \lambda_2, \lambda_3\},$$
$$\Theta = \mathbb{Z}_{2pq} - \{(0, 1), 2^{-1}, \lambda_1, \lambda_2, \lambda_3, 1 - \lambda_1, 1 - \lambda_2, 1 - \lambda_3\} \cup \{-\Theta_{pq}^3\}.$$ (2)

Now we introduce some cubic non-symmetric Cayley graphs of order $2pq$.

**Example 4.1.** Let $G = \langle a, b | a^p = b^q = 1, b^{-1}ab = a^{-1} \rangle$. Define

$$C_{2pq}^1 = \text{Cay}(G, \{a, a^{-1}\}),$$
$$C_{2pq}^2, C_{2pq}^3 = \text{Cay}(G, \{ba, ba^{-1}\}), \lambda \in \Lambda$$
$$C_{2pq}^3 = \text{Cay}(G, \{ba, ba^\lambda\}), \mu \in \Theta.$$

Then we have the following:

(1) For each $\lambda \in \{2^{-1}, 1 - \lambda, 1 - \lambda | \lambda \in \Lambda\}$, the Cayley graph $\text{Cay}(G, \{ba, ba^\lambda\})$ is isomorphic to one of $C_{2pq}^1, \lambda \in \Lambda$.

(2) The graphs $C_{2pq}^1, C_{2pq}^2, C_{2pq}^3$ are connected cubic non-symmetric Cayley graphs of order $2pq$. Moreover, $\text{Aut}(C_{2pq}^1) \cong \text{Aut}(C_{2pq}^2) \cong G \times \mathbb{Z}_2$ and $\text{Aut}(C_{2pq}^3) \cong G$.

(3) The graphs $C_{2pq}^1, C_{2pq}^2, \lambda \in \Lambda$, are pairwise non-isomorphic.

(4) For $\mu_1, \mu_2 \in \Theta$, $C_{2pq}^{3, \mu_1} \cong C_{2pq}^{3, \mu_2}$ if and only if one of the following holds in the ring $\mathbb{Z}_{2pq}$: $\mu_1\mu_2 = 1$, $\mu_1 + \mu_2 = 1$.

**Proof.** The automorphism of G induced by $b \mapsto ba$ and $a \mapsto a^{-1}$ maps $\{ba, ba^\lambda\}$ to $\{ba, ba^{1-\lambda}\}$, and the automorphism of G induced by $b \mapsto ba^{-1}$ and $a \mapsto a^{-2}$ maps $\{ba, ba^{-1}\}$ to $\{ba, ba^{-1}\}$. Since one of $\lambda_1, \lambda_2$ and $\lambda_3$ must be $-1, 1$ follows.

Set $S_1 = \{a, a^{-1}\}, S_2 = \{ba, ba^\lambda\}$ and $S_3 = \{ba, ba^{a}\}$, where $\lambda \in \Lambda$ and $\mu \in \Theta$. Since $S_i = G(1 \leq i \leq 3)$, the graphs $C_{2pq}^1, C_{2pq}^2$ are connected cubic Cayley graphs, which are normal by Theorem 3.2. Thus, $\text{Aut}(\text{Cay}(G, S_i)) = R(G) \rtimes \text{Aut}(G, S_i)$. To prove (2), it suffices to show that $\text{Aut}(G, S_1) \cong \text{Aut}(G, S_2) \cong \mathbb{Z}_2$ and $\text{Aut}(G, S_3) = 1$. Since $S_1$ contains only one involution, $\text{Aut}(G, S_1) \cong \mathbb{Z}_2$. Since $S_1$ contains only one involution, $\text{Aut}(G, S_1) \cong \mathbb{Z}_2$ and since the automorphism of G induced by $b \mapsto b$ and $a \mapsto a^{-1}$ fixes $S_1$, one has $\text{Aut}(G, S_1) \cong \mathbb{Z}_2$. Let $S = \{ba, ba^\lambda\}$ with $k \neq 0, 1$. It is easy to check that $S \cong \text{Aut}(S)$ if and only if there is $a \in \text{Aut}(S)$ such that $a$ permutes $\{ba, ba^\lambda\}$ cyclically if and only if $-k \in \Theta_{pq}^3$. It follows that $\text{Aut}(G, S_3) \cong \mathbb{Z}_2$ because the map $a \mapsto a^{-k}$ and $b \mapsto b$ induces an automorphism of $G$ of order 2 that fixes $S_3$. Furthermore, $\text{Aut}(G, S) \cong \mathbb{Z}_2$ if and only if there is an element of order 2 in $\text{Aut}(G)$ that fixes one element in $S$ and interchanges the other two in $S$. Thus, if and only if one of the following holds in $\mathbb{Z}_{2pq}$: $k^2 = 1$, $k(k - 2) = 0$ and $2k - 1 = 0$. Thus, $\text{Aut}(G, S) = 1$ if and only if $k \in \Theta$, which implies that $\text{Aut}(G, S_3) = 1$.

By Proposition 2.11, any 3-subset of $G$ not containing the identity is a CC-subset. Thus, for each $\lambda \in \Lambda$ we have $C_{2pq}^1 \neq C_{2pq}^2$, because $S_1$ contains only one involution and $S_2$ consists of involutions. Also, it is easy to check that $\{ba, ba^\mu\}$, $\{ba, ba^{a}\}$ and $\{ba, ba^{a\mu}\}$ are pairwise non-equivalent. Thus, $C_{2pq}^1, C_{2pq}^2, \lambda \in \Lambda$, are pairwise non-isomorphic.

Note that $\text{Cay}(G, \{ba, ba^{a\mu}\}) \cong \text{Cay}(G, \{ba, ba^{a\mu}\})$ if and only if there exists $\beta \in \text{Aut}(G)$ such that $\{ba, ba^{a\mu}\} = \{ba, ba^{a\beta}\}$. This is true if and only if one of the following holds in the ring $\mathbb{Z}_{2pq}$: $\mu_1\mu_2 = 1$, $\mu_1 + \mu_2 = 1$, $\mu_1(1 - \mu_1 - 1, \mu_1 + \mu_2 - \mu_1\mu_2 = 0$. The proof is straightforward. For example, there exists an automorphism of $G$ that maps $ba^{a\mu}$ to $ba^{-a\mu}$ and interchanges $b$ and $ba$ if and only if $\mu_1\mu_2 = 1$. □

**Example 4.2.** Let $G = \langle a, b, c | a^p = b^q = c^2 = 1, ab = ba, cac = a^{-1}, bc = cb \rangle$. Define

$$C_{2pq}^4 = \text{Cay}(G, \{c, ab, (ab)^{-1}\}).$$

Then $C_{2pq}^4$ is a cubic non-symmetric Cayley graph and $\text{Aut}(C_{2pq}^4) \cong G \times \mathbb{Z}_2$. 
Proof. Set $S = \{c, ab, (ab)^{-1}\}$. One may easily show that $\text{Aut}(G, S) = \{\alpha\} \cong \mathbb{Z}_2$, where $\alpha$ is the automorphism of $G$ induced by $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ and $c \mapsto c$. By Theorem 3.2, Cay($G, S$) is normal, and hence $\text{Aut}(C_{2p}^4) \cong R(G) \times \mathbb{Z}_2$, implying that $C_{2p}^4$ is a cubic non-symmetric Cayley graph. □

Example 4.3. Let $G = (a, b, c \mid a^p = b^q = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a')$ where $r$ is an element of order $q$ in $\mathbb{Z}_p^*$, and set $r = 3$ for $(p, q) = (11, 5)$ or $(23, 11)$. Define

$$\begin{align*}
C_{2pq}^{5,\xi} &= \text{Cay}(G, \{c, ab\xi, (ab\xi)^{-1}\}), \\
C_{2pq}^{6,\xi} &= \text{Cay}(G, \{c, ab\xi c, (ab\xi c)^{-1}\}),
\end{align*}$$

where $1 \leq \xi, \zeta \leq \frac{q-1}{2}$. Then we have the following:

1. The graph $C_{2pq}^{5,\xi}$ is non-symmetric. Furthermore,

$$\text{Aut}(C_{2pq}^{5,\xi}) = \begin{cases}
R(G) & \text{if } (p, q) \neq (7, 3), \\
PGL(2, 7) & \text{if } (p, q) = (7, 3);
\end{cases}$$

2. The graph $C_{2pq}^{6,\xi}$ is non-symmetric if and only if $p > q > 3$ and $(p, q, \zeta) \neq (11, 5, 1), (23, 11, 3)$. If $C_{2pq}^{6,\xi}$ is non-symmetric then $\text{Aut}(C_{2pq}^{6,\xi}) = R(G)$;

3. The graphs $C_{2pq}^{5,\xi}$ and $C_{2pq}^{6,\xi}$, $1 \leq \xi, \zeta \leq \frac{q-1}{2}$, are pairwise non-isomorphic.

Proof. Suppose there is an $\alpha \in \text{Aut}(G)$ such that $(ab\xi)^\alpha = (a^rb^s)^{-1}$, where $1 \leq k_1, k_2 \leq \frac{q-1}{2}$ and $\delta = \pm 1$. Since $(a)$ is characteristic in $G$, one has $a^\alpha = a$ for some $t \in \mathbb{Z}_p^*$. Clearly, $(ab\xi)^{-1} a(ab\xi) = a^{-1}$. It follows that $(a^rb^s)a^{-1}(a^rb^s)^{-1} = a^\alpha$, namely, $a^\alpha = a^r = a^\xi$. Then $k_1 + k_2 = 1(\text{mod } p)$ and hence $q \mid (k_1 + k_2)$ because $r$ is an element of order $q$ in $\mathbb{Z}_p^*$. This is impossible because $2 \leq k_1 + k_2 < q$. Thus, there is no $\alpha \in \text{Aut}(G)$ such that $(ab\xi)^\alpha = (a^rb^s)^{-1}$ for any $1 \leq k_1, k_2 \leq \frac{q-1}{2}$ and $\delta = \pm 1$.

For each $1 \leq \xi, \zeta \leq \frac{q-1}{2}$, $ab\xi$ has order $q$ and $ab\xi c$ has order $2q$. This implies that $\{a, ab\xi, (ab\xi)^{-1}\} \neq \{a, ab\xi c, (ab\xi c)^{-1}\}$. Let $S = \{c, ab\xi, (ab\xi)^{-1}\}$ and $A = \text{Aut}(C_{2pq}^{5,\xi})$. Note that if $q = 3$ then $\xi = 1$. By Theorem 3.2, if $(p, q) \neq (7, 3)$ then $C_{2pq}^{5,\xi} = \text{Cay}(G, S)$ is normal and if $(p, q) = (7, 3)$ then $\text{Aut}(C_{2pq}^{5,\xi}) = \text{Aut}(C_{42}^{5,\xi}) \cong PGL(2, 7)$. Since $|PGL(2, 7)| = 42 \times 8$, $C_{2pq}^{5,\xi}$ is non-symmetric. Assume $(p, q) \neq (7, 3)$. By Proposition 2.7, $A = R(G)\text{Aut}(G, S)$. Let $\alpha \in \text{Aut}(G, S)$. As $c$ is the unique involution in $S$, $\alpha$ fixes $c$. By the first proof there is no $\alpha \in \text{Aut}(G)$ interchanging $ab\xi$ and $(ab\xi)^{-1}$. Thus, $(ab\xi)^\alpha = (ab\xi)$ and since $G = \langle c, ab\xi \rangle$, one has $\alpha = 1$. This implies that $A = R(G)$ and hence $C_{2pq}^{5,\xi}$ is non-symmetric.

By Theorem 3.2, $C_{2pq}^{5,\xi} \cong C_{2q+2,2q}^{5,\xi}$ and $C_{2pq}^{6,\xi}$ are symmetric. Note that if $q = 3$ then $\xi = 1$. It follows that if $C_{2pq}^{5,\xi}$ is non-symmetric then $p > q > 3$ and $(p, q, \xi) \neq (11, 5, 1), (23, 11, 3)$. Conversely, assume $p > q > 3$ and $(p, q, \xi) \neq (11, 5, 1), (23, 11, 3)$. To finish the proof of (2), it suffices to show that $\text{Aut}(C_{2pq}^{6,\xi}) = R(G)$. If $(p, q) = (11, 5)$, with the help of computer software package MAGMA [4], one can compute that $|\text{Aut}(C_{2pq}^{6,\xi})| = 110$ for $\xi = 2$ and hence $\text{Aut}(C_{2pq}^{6,\xi}) = R(G)$. Similarly, if $(p, q) = (23, 11)$ then $\text{Aut}(C_{2pq}^{6,\xi}) = R(G)$ for $\xi = 1, 2, 4, 5$. Thus, one may assume that $(p, q) \neq (11, 5), (23, 11)$. Since $p > q > 3$, by Theorem 3.2, $C_{2pq}^{6,\xi}$ is normal. Let $S = \{c, ab\xi c, (ab\xi c)^{-1}\}$ and $A = \text{Aut}(C_{2pq}^{6,\xi})$. By Proposition 2.7, $A = R(G)\text{Aut}(G, S)$. Let $\beta \in \text{Aut}(G, S)$. Clearly, $\beta^2 = c$. Suppose that $(ab\xi c)^\beta = (ab\xi c)^{-1}$.

Then, $(ab\xi)^\beta = (a^{-1}b^{-1})^{-1} = (a^{-1}b^{-1})^{-1}$, which is impossible because of the argument in the first paragraph. Thus, $(ab\xi c)^\beta = ab\xi c$ and hence $\beta = 1$ because $G = \langle c, ab\xi c \rangle$, which implies that $\text{Aut}(C_{2pq}^{6,\xi}) = R(G)$, as required.

If $q = 3$ then $\xi = \zeta = 1$. To prove (3), one may assume that $p > q > 3$ and $(p, q, \xi) \neq (11, 5, 1), (23, 11, 3)$. Thus, $\text{Aut}(C_{2pq}^{6,\xi}) = \text{Aut}(C_{2pq}^{6,\xi}) = R(G)$. By Proposition 2.10, $C_{2pq}^{6,\xi}$ and $C_{2pq}^{6,\xi}$ are CI-graphs. Since $\{c, ab\xi, (ab\xi)^{-1}\} \neq \{c, ab\xi c, (ab\xi c)^{-1}\}$, it suffices to show that $C_{2pq}^{6,\xi}$ and $C_{2pq}^{6,\xi}$ are pairwise non-isomorphic, respectively. Assume $C_{2pq}^{6,\xi} \cong C_{2pq}^{6,\xi}$ for some $1 \leq \xi_1, \xi_2 \leq \frac{q-1}{2}$. The CI-property of $C_{2pq}^{6,\xi}$ implies that there is an $\alpha \in \text{Aut}(G)$ such that $\{c, ab\xi, (ab\xi)^{-1}\} = \{c, ab\xi, (ab\xi)^{-1}\}$. Clearly, $c^a = c$. By the argument in the first paragraph, $(ab\xi)^a = ab\xi b$. Then $(a^{-1}b^{-1})a = (ab\xi b)^a = a^{-1}b^{-1}b\xi$ and $(a^2)^a = (ab\xi b)(a^{-1}b^{-1}b\xi)^{-1}a = ab\xi b(a^{-1}b^{-1}b\xi)^{-1} = a^{-1}$, implying $a^\alpha = a$. It follows that $(b^{-1}a^2)^a = b^{-1}a^{-1}a$ and since $(b^{-1}a^{-1}a)^{b^{-1}} = b^{-1}a^{-1}$, one may obtain $r^{-1} = 1 = 2\xi_2$. Since $r$ has order $q$ in $\mathbb{Z}_p^*$ and $1 \leq \xi_1, \xi_2 \leq \frac{q-1}{2}$, one has $\xi_1 = \xi_2$.

Thus, $\{c, ab\xi, (ab\xi)^{-1}\} = \{c, ab\xi, (ab\xi)^{-1}\}$ for some $1 \leq \xi_1, \xi_2 \leq \frac{q-1}{2}$ if and only if $\xi_1 = \xi_2$. Similarly, one may show that $C_{2pq}^{6,\xi} \cong C_{2pq}^{6,\xi}$ for some $1 \leq \xi_1, \xi_2 \leq \frac{q-1}{2}$ if and only if $\xi_1 = \xi_2$.

The following theorem is the main result of this section.
Theorem 4.4. Let \( p > q \) be odd primes. A connected cubic Cayley graph of order \( 2pq \) is non-symmetric if and only if it is isomorphic to one of the following graphs: \( e^{\frac{1}{2}}_{2pq}, e^{\frac{1}{4}}_{2pq} \), \( e^{\frac{1}{2}}_{2pq} \), \( e^{\frac{1}{4}}_{2pq} \), \( (1 \leq \xi \leq \frac{p-1}{2}) \) and \( e^{\frac{1}{2}}_{2pq} \), \( (1 \leq \zeta \leq \frac{q-1}{2}, q > 3, (p, q, \zeta) \neq (11, 5, 2), (23, 11, 2)) \), where \( \Lambda \) and \( \Theta \) are given in Eq. (2).

Proof. Let \( X = \text{Cay}(G, S) \) be a connected cubic non-symmetric Cayley graph on a group \( G \) of order \( 2pq \). Then \( \notin G, S^{-1} = S \) and \( S = G \). Since \( X \) has valency \( 3 \), \( S \) contains an involution, say \( x \). Let \( A = \text{Aut}(X) \) and \( A_1 \) the stabilizer of \( 1 \in G \) in \( A \). To finish the proof, by Examples 4.1–4.3, it suffices to show that \( X \) is isomorphic to one of the graphs listed in the theorem.

Recall that \( G \) is one of the groups \( G_1(2pq), G_2(2pq), G_3(2pq), G_4(2pq), G_5(2pq) \) and \( G_6(2pq) \) given in Eq. (1).

Let \( G = G_1(2pq) = \{ e \} \). Then \( S = (x, a^{pq}, y^{pq}, y^{-pq}) \), where \( y \) is an element of order \( pq \) or \( 2pq \). By Proposition 2.9, \( X \) is normal, and by Proposition 2.7, \( A_1 = \text{Aut}(G, S) \). Since \( X \) is non-symmetric, \( \text{Aut}(G_1(2pq), S) \subseteq \mathbb{Z}_2 \), and since the automorphism \( \alpha \) of \( G_2(2pq) \) induced by \( a \mapsto a^{-1} \) fixes \( S \) setwise, \( A_1 = (\alpha) \subseteq \mathbb{Z}_2 \) and \( R(G_1(2pq)) \times (\alpha) \). It is easy to show that \( (R(a^\alpha), R(a^{pq}\alpha)) \) acts regularly on \( V \), which implies that \( X \) is isomorphic to a Cayley graph on \( G_2(2pq) \).

Let \( G = G_2(2pq) = \{ (a, b) | a^b = b^2 = 1, b^{-1}ab = a^{-1} \} \). Since all involutions of \( G_2(2pq) \) are conjugate, one may let \( x = b \). If \( S = \{ a, a^2 = 1 \} \), then \( (i, pq) = 1 \) because \( S = G \). Let \( \alpha_i \) be the automorphism of \( G_2(2pq) \) induced by \( b \mapsto a \) and \( a^i \mapsto a \).

Then \( S^q = \{ a, a^2 \} \) and hence \( X \cong \mathbb{C}_{2pq} \). Since all \( \alpha_i \) are involutions of \( G_2(2pq) \) conjugate by \( (a, b) \), and by the connectivity of \( X \), one may assume that \( S = \{ x = c, y^\nu \} \), where \( y \) has order \( pq \). Clearly, there exists an automorphism of \( G_3(2pq) \) which fixes \( c \) and maps \( y \) to \( ab \). It follows that \( S \cong \{ c, ab, (ab)^{-1} \} \), and hence \( X \cong \mathbb{C}_{2pq} \).

Let \( G = G_4(2pq) = \{ (a, b, c) | c^a = 1, c^b = b^2 \} \). By a similar argument to the above paragraph, one may let \( S = \{ c, ab, (ab)^{-1} \} \). By Theorem 3.2, \( X = \text{Cay}(G, S) \) is normal, and hence \( A_1 = \text{Aut}(G, S) \). It is easy to check that \( \text{Aut}(G, S) = (\alpha) \subseteq \mathbb{Z}_2 \), where \( \alpha \) is the automorphism of \( G \) induced by \( c \mapsto c, a \mapsto a^{-1} \) and \( b \mapsto b \). Let \( H = (R(a), R(b), R(c)) \). Direct calculation shows that \( R(G_4(2pq)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) (the group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts regularly on \( V \) and hence \( X \) is also a Cayley graph on \( G_5(2pq) \), which is discussed in the previous paragraph.

Let \( G = G_5(2pq) \). Then \( c \) is in the center of \( G \). Since \( G \) has no elements of order \( pq \), there is no connected cubic Cayley graph on \( G_5(2pq) \).

Let \( G = G_6(2pq) \). Since all involutions of \( G \) are conjugate and contained in the subgroup \( (a, c) \), the connectivity of \( X \) implies that \( S = \{ c, y^\nu \} \), where \( y \) has order \( pq \) or \( 2pq \). If \( y \) has order \( 2pq \) then \( y = ab^c \) with \( 1 \leq i \leq p-1 \) and \( 1 \leq k \leq q-1 \). Let \( \alpha_i \) be the automorphism of \( G \) induced by \( a \mapsto a \), \( b \mapsto b \) and \( c \mapsto c \). Then \( S^i = \{ c, ab^c, (ab^c)^{-1} \} \). One may assume \( 1 \leq i \leq \frac{pq-1}{2} \) because the map \( \beta_i \) defined by \( a \mapsto a^{-1} \), \( b \mapsto b \) and \( c \mapsto c \) induces an automorphism of \( G \) and \( (ab^c)^{-1} = a^{-1}b^{-1}c = b^{-1}a^{-1} \). Thus, \( X \cong \mathbb{C}_{2pq} \) \( 1 \leq \zeta \leq 2 \). If \( y \) has order \( 2pq \) then \( y = ab^c \) with \( 1 \leq i \leq p-1 \) and \( 1 \leq k \leq q-1 \). Clearly, \( S_i = \{ c, ab^c, (ab^c)^{-1} \} \), and one may assume \( 1 \leq k \leq \frac{pq-1}{2} \) because the map \( \beta_i \) defined by \( a \mapsto a^{-1} \), \( b \mapsto b \) and \( c \mapsto c \) induces an automorphism of \( G_6(2pq) \) and \( (ab^c)^{-1} = (ab^{-1})^{-1} \). It follows that \( X \cong \mathbb{C}_{2pq} \) \( 1 \leq \zeta \leq \frac{pq-1}{2} \). Since \( X \) is non-symmetric, by Example 4.3, \( p > q > 3 \) and \( (p, q, \zeta) \neq (11, 5, 1), (23, 11, 3) \). □

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References