On global attractor for \( m \)-Laplacian parabolic equation with local and nonlocal nonlinearity

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Abstract

In this paper, we study the long-time behavior of solutions for \( m \)-Laplacian parabolic equation

\[
\begin{aligned}
  u_t - \Delta_m u + a(x)|u|^{\alpha}u &= f_0(u) \int_{\Omega} K(y)|u(y, t)|^\beta \, dy + g(x), \\
  u(x, 0) &= u_0(x),
\end{aligned}
\]

in \( \Omega \times (0, \infty) \) with the initial data \( u(x, 0) = u_0(x) \in L^q \), \( q \geq 1 \), and zero boundary condition in \( \partial \Omega \). Two cases for \( a(x) \geq a_0 > 0 \) and \( a(x) \geq 0 \) are considered. We obtain the existence and \( L^p \) estimate of global attractor \( \mathcal{A} \) in \( L^p \), for any \( p \geq \max\{1, q\} \). The attractor \( \mathcal{A} \) is in fact a bounded set in \( W^{1, m}_0 \cap L^\infty \) if \( a(x) \geq a_0 > 0 \) in \( \Omega \), and \( \mathcal{A} \) is bounded in \( W^{1, m}_0 \cap L^p \) if \( a(x) \geq 0 \) in \( \Omega \).

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1. Introduction

In this paper we consider the global attractor for the initial boundary value problem of \( m \)-Laplacian parabolic equation

\[
\begin{aligned}
  u_t - \Delta_m u + a(x)|u|^{\alpha}u &= f_0(u) \int_{\Omega} K(y)|u(y, t)|^\beta \, dy + g(x), \\
  u(x, 0) &= u_0(x),
\end{aligned}
\]

for \( x \in \Omega \), \( t > 0 \).

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where $\Delta_m u = \text{div}(|\nabla u|^{m-2} \nabla u)$, $2 < m < N$, $\beta \geq 1$, $\alpha > 0$ and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial \Omega$. $\int_\Omega K(y)|u|^{\beta} \, dy$ represents a nonlocal function dependence in space domain $\Omega$. For $m = 2$, the equation in (1.1) appears in an ignition model for a compressible reactive gas which is a nonlocal reaction–diffusion equation, see [3,5].

The existence and uniqueness, blow-up of nonnegative solutions to the problem of form (1.1) with $m = 2$ have been extensively studied, among others, by Pao [13], Rouchon [14], Souplet [15], Wang and Wang [17]. Li and Xie in [10] considered the global and blow-up solutions to the problem (1.1) with $m > 2$, $\beta \geq 1$, $a(x) \equiv g(x) \equiv 0$, $K(y) \equiv f_0(u) \equiv 1$. Recently, Aassila [1] studied the problem (1.1) with $m = 2$ and proved the existence of solutions by Schauder fixed point theorem and the convergence of the solution towards a steady state by using the dynamical systems point of view.

In this paper, we are interested in the existence of global attractor in $L^p$ for the problem (1.1). It seems like that there is few results in this direction.

Cholewa and Dlotko [7], Temam [16] considered the following problem:

\[
\begin{align*}
&u_t - \Delta_m u + |u|^\alpha u = f_0(u) + g(x), \quad x \in \Omega, \quad t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega; \quad u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0,
\end{align*}
\]

and proved the existence of global attractor in $L^2$ which is in fact a bounded set in $W^{1,m}_0 \cap L^{\alpha+2}$. They apply a general abstract theory of monotone operator and for this the function $f_0(u)$ is assumed to be global Lipschitz continuous. We cannot apply directly their result since the equation in (1.1) contains the local and nonlocal nonlinear term. The difficulty in such a nonlinearity lies in the fact that the estimates derived in [7,16] are not sufficient to assume the uniqueness and the continuous dependence on the initial data of solutions.

In this paper, we first establish the existence and a priori estimate for the solution of (1.1). For the function $f_0(u)$ in (1.1), we suppose

\[
|f_0(u)| \leq k_0 |u|^r, \quad r = 0 \text{ or } r \geq 1, \quad u \in \mathbb{R}^1 = (-\infty, \infty).
\]

Then, if $\beta + r < \alpha + 1$ and $a(x) \geq a_0 > 0$ in $\Omega$, we can in fact derive an estimate like

\[
\|u(t)\|_p \leq C(1 + t^{-1/\alpha}), \quad t > 0,
\]

for any $p \geq 1$, including $p = \infty$. If $a(x) \geq 0$ and $\beta + r < m - 1$, we can derive the estimate like

\[
\|u(t)\|_p \leq C(1 + t^{-1/(m-2)}), \quad t > 0,
\]

for any $p \geq q_0 > 1$.

Our second aim is to prove the existence of global attractor $A$ in $L^p$ for any $p \geq 1$ in the case $a(x) \geq a_0 > 0$ and $p \geq q_0$ in the case $a(x) \geq 0$. We will prove that the attractor $A$ is in fact a bounded set in $W^{1,m}_0 \cap L^{\infty}$ if $a(x) \geq a_0 > 0$ in $\Omega$. Note that if $a(x) \geq 0$ in $\Omega$, the global attractor $A$ is constructed in the largest basic space $L^1$, and if $a(x) \geq 0$ in $\Omega$, $A$ is constructed in the space $L^{q_0}$ for any $q_0 \in (1, 2)$.

2. Preliminaries

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. We denote the space $L^p$ and $W^{1,m}_0$ for $L^p(\Omega)$ and $W^{1,m}_0(\Omega)$ and the relevant norms by $\| \cdot \|_p$ and $\| \cdot \|_{1,m}$, respectively.

In general, $\| \cdot \|_X$ denotes the norm of Banach space $X$. In the proof of our result, we will use the following lemmas.
Lemma 1. (See [16].) Let \( y(t) \) be a nonnegative differential function on \((0, \infty)\) satisfying
\[
y'(t) + Ay^{1+\mu}(t) \leq B, \quad t > 0,
\]
with \( A, \mu > 0, B \geq 0 \). Then
\[
y(t) \leq \left( BA^{-1} \right)^{1/(1+\mu)} + (A\mu t)^{-1/\mu}, \quad t > 0.
\]
Further, if \( y(t) \) is continuous on \([0, \infty)\), then
\[
y(t) \leq \left( BA^{-1} \right)^{1/(1+\mu)} + \left( y(0)^{-\mu} + A\mu t \right)^{-1/\mu}, \quad t > 0,
\]
and
\[
y(t) \leq \max\{y(0), \left( BA^{-1} \right)^{1/(1+\mu)}\}.
\]
In this paper, we seek for the weak solution of (1.1) in the class
\[
X_q \equiv C\left( \mathbb{R}_+, L^q \right) \cap C\left( (0, \infty), L^p \right) \cap L_\text{loc}^\infty((0, \infty), W_0^{1,m})
\]
with \( R_+ = [0, \infty) \), \( q \geq 1 \) and \( p > 1 \).

We make the following definition of weak solution.

Definition 1. A function \( u = u(x,t) \in X_q \) is called a weak solution of (1.1) if the function
\[
h(t) = \int_{\Omega} K(y) |u(y,t)|^\beta \, dy \in L_\text{loc}^1(\mathbb{R}_+)
\]
and
\[
\int_0^\infty \int_{\Omega} \left\{ |\nabla u|^{m-2} \nabla u \cdot \nabla \phi + (a(x)|u|^\alpha u - h(t)f_0(u) - g(x))\phi(x,t) \right\} \, dx \, dt
\]
\[
= \int_0^\infty \int_{\Omega} u(x,t)\phi_t(x,t) \, dx \, dt + \int_{\Omega} u_0(x)\phi(x,0) \, dx
\]
is valid for any \( \phi \in C^{1,2}_0(R_+ \times \Omega) \) with compact support and \( u_0 \in L^q \).

We begin with an existence theorem of global solution for the initial data \( u_0 \in L^q \), \( q \geq 1 \), including some a priori estimates.

It is well known that the solution of (1.1) is in fact given as limits of smooth solutions of appropriate approximate problems for (1.1). Hence we first consider the following problem:
\[
\begin{cases}
  u_t - \Delta_{m,i}u + a(x)|u|^\alpha u = f_0(u) \int_{\Omega} K(y)|u(y,t)|^\beta \, dy + g(x), & x \in \Omega, \ t > 0, \\
  u(x,0) = u_{0,i}(x), & x \in \Omega; \ u(x,t) = 0, & x \in \partial \Omega, \ t \geq 0,
\end{cases}
\]
where \( \Delta_{m,i}u = \text{div}\left((|\nabla u|^2 + i^{-1})^{m-2} \nabla u\right), i = 1, 2, \ldots, \) and \( u_{0,i} \in C^2_0(\Omega) \), \( u_{0,i} \rightarrow u_0 \) in \( L^q \) as \( i \rightarrow \infty \).

The problem (2.4) is an initial boundary value problem for a standard nondegenerate quasilinear parabolic equation with a nonlocal term. A similar argument as in [8,9] can be applied to show
that the problem (2.4) admits a unique solution \( u_i(t) = u_i(x, t) \in C(R_+, L^2) \cap L^{m}(R_+, W_0^{1,m}) \).

For each \( i = 1, 2, \ldots \), we will derive various estimates for \( u_i(t) \). For the simplicity of notations, we write \( u \) instead of \( u_i \) and \( u^{p-1} \) for \( |u|^{p-2}u \) with \( p > 1 \).

We first establish

**Lemma 2.** Assume that

\begin{align*}
(H_1) \quad & a(x) \in C(\overline{\Omega}) \text{ and } \exists a_0 > 0, \text{ such that } a(x) \geq a_0 \text{ in } \Omega; \\
(H_2) \quad & f_0(u) \in C^1(R^1) \text{ and either } f_0(u) \equiv 1 \text{ or } |f_0(u)| \leq k_0|u|^r, \ |f_0'(u)| \leq k_0|u|^{r-1} \text{ for some } k_0 > 0, \ r \geq 1; \\
(H_3) \quad & g(x), K(x) \in L^\infty.
\end{align*}

In addition, \( \beta + r < 1 + \alpha \). Then, if \( u(t) \) is the solution of (2.4) with \( u_0 \in L^1 \), we have \( u(t) \in L^\infty(R_+, L^1) \); i.e., \( \exists M > 0 \text{ such that } u(t) \in B_{L^1}(M) = \{v \in L^1 \mid \|v\|_{L^1} \leq M\} \text{ for any } t \geq 0 \).

**Proof.** Without loss of generality, we assume \( r \geq 1 \) in (H2). For \( n = 1, 2, \ldots \), we let

\[ f_n^+(s) = \begin{cases} 1, & s \geq 1/n, \\ n^2 s^2 + 2ns, & 0 \leq s < 1/n. \end{cases} \quad (2.5) \]

Furthermore, let \( f_n(s) \) be an odd extension of \( f_n^+(s) \) in \( R^1 \). It is easy to verify that \( f_n(s) \in C^1(R^1), |f_n(s)| \leq 1 \) in \( R^1 \) and \( F_n(u) = \int_0^u f_n(s)ds \rightarrow |u| \) uniformly in \( R^1 \) as \( n \rightarrow \infty \).

Multiplying the equation in (2.4) by \( f_n(u) \), we obtain

\begin{align*}
& \int_\Omega f_n(u)u_t \, dx + \int_\Omega |\nabla u|^m f_n'(u) \, dx + \int_\Omega a(x)|u|^\alpha u f_n(u) \, dx \\
& \leq \int_\Omega |g(x)||f_n(u)| \, dx + \int_\Omega |f_0(u)||f_n(u)| \, dx \int_\Omega K(y)|u|^\beta \, dy. \quad (2.6)
\end{align*}

Notice that

\begin{align*}
& \int_\Omega |g(x)||f_n(u)| \, dx \leq \|g\|_1 \leq C\|g\|, \\
& a_0\|u\|_1^{1+\alpha} - C|\Omega|n^{-1-\alpha} \leq \int_\Omega a(x)|u|^\alpha u f_n(u) \, dx, \\
& \int_\Omega |f_0(u)||f_n(u)| \, dx \int_\Omega K(y)|u|^\beta \, dy \leq C_1\|u\|_1^{r+\beta} \leq \frac{a_0}{2}\|u\|_1^{1+\alpha} + C_1.
\end{align*}

Then (2.6) becomes

\[ \int_\Omega f_n(u)u_t \, dx + \frac{a_0}{2}\|u\|_1^{1+\alpha} \leq C|\Omega|n^{-1-\alpha} + C_1. \quad (2.7) \]

Here and in the sequel, we let \( C_1 \) be a constant depending only on \( g, K, u_0 \) and \( C \) be a generic constant, which is independent of \( g, K, u_0 \) and changeable from line to line.
Letting $n \to \infty$ in (2.7), we get
\[ \frac{d}{dt} \|u(t)\|_1 + \frac{a_0}{2} \|u(t)\|_{1+\alpha}^{1+\alpha} \leq C_1. \]  
(2.8)

By Lemma 1,
\[ \|u(t)\|_1 \leq C_1 + C_2(\|u_0\|_1)(1 + t)^{-1/\alpha}, \quad t \geq 0, \]
where $C_2(y)$ is a monotone increasing function for $y \geq 0$. This completes the proof of Lemma 2. \(\square\)

**Remark 1.** It is possible that, if $1 + \alpha \leq \beta + r$, there is no global solution for (1.1), see [2].

**Lemma 3.** Assume $(H_2)$ and $a(x) \in C(\overline{\Omega})$, $a(x) \geq 0$ in $\Omega$. In addition, $\beta + r < m - 1$. Then if $u(t)$ is the solution of (2.4) with $u_0 \in L^{q_0}$ and $q_0 \in (1, 2)$, $u(t) \in L^\infty(R_+, L^{q_0})$.

The proof of Lemma 3 is similar to that of Proposition 1 in [6] and is omitted here.

**Remark 2.** If $a(x) \geq 0$, $\beta + r \geq m - 1$ and either $|\Omega|$ or $u_0(x)$ is sufficiently large, the problem (1.1) maybe has no global solution, see [10].

**Lemma 4.** Under the assumptions of Lemma 2, for any $T > 0$, the solution $u(t)$ of (2.4) also satisfies
\[ \|u(t)\|_\infty \leq C_1 + C_3 t^{-1/\alpha}, \quad t > 0, \]  
(2.9)
\[ \|\nabla u(t)\|^m_m \leq C_4(T) t^{-1-1/\alpha}, \quad 0 < t \leq T, \]  
(2.10)
\[ \int_0^T s^{1+\mu} \|u_t(s)\|^2_2 \, ds \leq C_4(T), \]  
(2.11)
where $\mu > 1/\alpha$, $C_1 = C_1(\|g\|_\infty, \|K\|_\infty)$, $C_3 = C_3(a_0, \alpha, |\Omega|)$, and $C_4(T) = C_4(T, \|g\|_\infty, \|K\|_\infty)$.

**Proof.** Multiplying the equation in (2.4) by $u^{p-1}$ ($p > 1$), we have
\[ \frac{1}{p} \frac{d}{dt} \|u(t)\|^p_p + Cp^{1-m} \|\nabla u\|^m_m \|u\|_{p+\alpha}^{p+\alpha} \leq \frac{a_0}{2} \|u\|_{p+\alpha}^{p+\alpha} + A_p \]  
(2.12)
which $C$ is independent of $p$, $i$, and
\[ A_p = \left( \frac{a_0}{4} \right)^{-\frac{p+\alpha}{1+\alpha}} |\Omega| \|g\|_{\frac{p+\alpha}{1+\alpha}} + \left( \frac{a_0}{4} \right)^{-\frac{p+\alpha}{1+\alpha-r}} (h_0 \|K\|_\infty)^{\frac{p+\alpha}{1+\alpha-r}}. \]

Since
\[ \|u\|_{p+\alpha}^{p+\alpha} \geq \|u\|_{p+\alpha}^{p+\alpha} |\Omega|^{-\alpha/p}, \]
then (2.12) becomes
\[ \frac{d}{dt} \|u(t)\|^p_p + \frac{p a_0}{2} |\Omega|^{-\alpha/p} \|u(t)\|_{p+\alpha}^{p+\alpha} \leq p A_p. \]  
(2.13)
Applying Lemma 1 to (2.13), we obtain
\[ \|u(t)\|_p \leq \left(2a_0^{-1}|\Omega|^p\right)^{\frac{1}{p}} A_p^{\frac{1}{p}} + (\alpha a_0)^{\frac{1}{\alpha}} |\Omega|^{\frac{1}{\alpha}} |t|^{-\frac{1}{\alpha}}, \quad t > 0. \] (2.14)

Letting \( p \to \infty \), we have (2.9).

In order to derive (2.10)–(2.11), we choose \( \mu > 1/\alpha \), \( \eta(t) \in C[0, \infty) \cap C^1(0, \infty) \) such that \( \eta(t) = t^\mu \) for \( t \in [0, 1] \), \( \eta(t) = 2 \) for \( t \geq 2 \) and \( \eta(t), \eta'(t) \geq 0 \) in \([0, \infty)\).

Multiplying the equation in (2.4) by \( \eta(t)u(t) \), we have
\[ \frac{1}{2} \eta(t)\|u(t)\|^2_2 + \int_0^t \int_\Omega \eta(s)|\nabla u|^m dx \, ds + \int_0^t \int_\Omega \eta(s)a(x)|u|^{2+\alpha} dx \, ds \leq I + II \]
where
\[ I = \frac{1}{2} \int_0^t \eta'(s)\|u(s)\|^2_2 ds \leq \int_0^t \eta'(s)\|u(s)\|_\infty \|u(s)\|_1 ds \leq C_1 \int_0^t s^{\mu-1-\frac{1}{2}} ds \leq C_1 t^{\mu-\frac{1}{2}} \] (2.15)
and
\[ II = \int_0^t \eta(s) \left( \int_\Omega f_0(u(x,s))u(x,s) \, dx \int_\Omega |K(y)||u(y,s)|^{\beta} \, dy \right) \, ds \]
\[ \leq k_0 \|K\|_\infty \int_0^t \eta(s)\|u(s)\|_\infty^{\beta+\beta-2} \|u(s)\|^2_1 \, ds \leq C_1 t^{\mu+1-(r+\beta-2)/\alpha}. \] (2.16)

Hence we have from (2.15)–(2.16) that
\[ \int_0^t \eta(s)\|\nabla u(s)\|^m_m \, ds + \int_0^t \eta(s)|u(s)|^{2+\alpha} dx \, ds \leq C_1 \left(t^{\mu-1/\alpha} + t^{1+\mu-(r+\beta-2)/\alpha}\right) \] (2.17)
with \( t \in [0, T] \).

Next, let \( \rho(t) = \int_0^t \eta(s) \, ds \), \( t \geq 0 \). Similarly, multiplying the equation in (2.4) by \( \rho(t)u_t \), we get
\[ \rho(t)\|u_t(t)\|^2_2 + \frac{d}{dt}(\rho(t)G(t)) \]
\[ \leq \eta(t)G(t) + \frac{1}{2} \rho(t)\|u_t(t)\|^2_2 + C_1 \rho(t)\left(\|g\|^2_2 + \|u(t)\|^2_2 + \|u(t)\|^2_\beta\right), \] (2.18)
where
\[ G(t) = \frac{1}{m} \int_\Omega \left(|\nabla u(t)|^2 + i^{-1}\right)^\frac{m}{2} \, dx + \frac{1}{2+\alpha} \int_\Omega a(x)|u(t)|^{2+\alpha} \, dx. \]

By (2.9) and \( u(t) \in L^\infty(R_+, L^1) \), we obtain
\[ \frac{1}{2} \rho(t)\|u_t(t)\|^2_2 + \frac{d}{dt}(\rho(t)G(t)) \leq \eta(t)G(t) + C_1 \rho(t)\left(\|g\|^2_2 + t^{-2(r+\beta-1)/\alpha}\right). \] (2.19)
Now, the application of (2.17) and the integration of (2.19) on \([0, t]\) yield
\[
\int_0^t \rho(s)\|u_t(s)\|_2^2\,ds + \rho(t)\left(\frac{1}{m}\|\nabla u(t)\|_m^m + \frac{\alpha_0}{2 + \alpha}\|u(t)\|_2^{2+\alpha}\right) \\
\leq \int_0^t \eta(s)G(s)\,ds + C_1\int_0^t \rho(s)(1 + s^{-2(r+\beta-1)/\alpha})\,ds \\
\leq C_1\left(t^{\mu-1/\alpha} + t^{\mu+1-(r+\beta)/\alpha} + t^{\mu+2} + t^{\mu+2(1-(r+\beta)-1)/\alpha}\right).
\]
(2.20)

This implies
\[
\|\nabla u(t)\|_m^m \leq C_4(T)t^{-1-1/\alpha}, \quad 0 < t \leq T,
\]
(2.21)
and
\[
\int_0^T \rho(s)\|u_t(s)\|_2^2\,ds \leq C_4(T),
\]
(2.22)
where \(C_4(T)\) depends on \(T\), \(\|g\|_\infty\) and \(\|K\|_\infty\). Hence we complete the proof of Lemma 4. \(\square\)

**Remark 3.** The estimate (2.9) implies that for any \(p \geq 1\),
\[
\|u(t)\|_p \leq \|\Omega\|_{1/p}^{1/p}(C_1 + C_3t^{-1/\alpha}), \quad t > 0.
\]
**Lemma 5.** Let the assumptions in Lemma 3 hold and \(p \geq q_0 > 1\). Then the solution \(u(t)\) of (2.4) with \(u_0 \in L^{q_0}\) also satisfies \(u(t) \in L^\infty(R_+, L^{q_0})\) and
\[
\|u(t)\|_p \leq A_0\left(1 + t^{-1/(m-2)}\right), \quad t > 0, \quad \text{if } p > q_0,
\]
(2.23)
and for any \(T > 0\),
\[
\|u(t)\|_\infty \leq A_1(T)t^{-\lambda}, \quad \|\nabla u(t)\|_m \leq A_1(T)t^{-(1+\lambda(2-q_0))/m}, \quad 0 < t \leq T,
\]
(2.24)
\[
\int_0^T \frac{d}{dt}\|u_t(s)\|_2^2\,ds \leq A_1(T),
\]
(2.25)
where \(\tau > \lambda, \lambda = \frac{N}{m(q_0+(m-2)N)}\), \(A_0 = A_0(p, \|g\|_\infty, \|K\|_\infty)\), \(A_1(T) = A_1(T, \|u_0\|_{q_0})\).

**Proof.** Multiplying the equation in (2.4) by \(u^{p-1}\) \((p \geq q_0 > 1)\), we have
\[
\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + Cp^{1-m}\|\nabla u\|_m^{p+m-2} + \int_\Omega a(x)|u|^{p+\alpha}\,dx \\
\leq \int_\Omega |g(x)||u|^{p-1}\,dx + \int_\Omega |f_0(u)||u|^{p-1}\,dx \int_\Omega K(y)||u|^\beta\,dy.
\]
(2.26)
By Sobolev’s inequality,
\[
\|\nabla u\|_m^{p+m-2} \geq \mu_0\|u\|_p^{p+m-2} \geq \mu_1\|u\|_p^{p+m-2}
\]
(2.27)
with some \(\mu_0, \mu_1 > 0\).
Furthermore, it follows from Young’s inequality that
\[ \int_{\Omega} |g(x)| |u|^{p-1} \, dx \leq \varepsilon \|u\|^{\frac{p+m-2}{p+m-2}} + C \varepsilon \|g\|_\infty^{\mu_3} \] (2.28)
and
\[ \int_{\Omega} |f_0(u)| |u|^{p-1} \, dx \int_{\Omega} |K(y)| |u|^{\beta} \, dy \leq k_0 \|K\|_\infty \|u\|^{\frac{p+r-1}{p+r-1}} \|u\|^{\beta} \]
\[ \leq \frac{\mu_0}{4} \|u\|^{\frac{p+m-2}{p+m-2}} + C \|K\|_\infty^{\mu_4} \] (2.29)
with a small \( \varepsilon > 0 \), \( \mu_3 = \frac{p+m-2}{m-1} \), \( \mu_4 = \frac{p+m-2}{m-1-\beta-r} > 1 \). Then (2.26) becomes
\[ \frac{d}{dt} \|u(t)\|_p + \mu_5 \|u(t)\|_p^{\frac{p+m-2}{p}} \leq C (\|K\|_\infty^{\mu_4} + \|g\|_\infty^{\mu_3}), \quad t > 0, \] (2.30)
for some \( \mu_5 > 0 \). Applying Lemma 1 to (2.30), we obtain \( u(t) \in L^\infty(R^+, L^{q_0}) \) if \( p = q_0 \) and
\[ \|u(t)\|_p \leq A_0 (1 + t^{-1/(m-2)}), \quad t > 0, \] (2.31)
if \( p > q_0 \). This is (2.23). The proof of (2.24)–(2.25) is similar to that of Propositions 1, 2 in [6] and is omitted here. \( \square \)

By Lemma 4, we can establish

**Theorem 1.** Let the assumptions in Lemma 2 hold. Then (1.1) admits a unique global weak solution \( u(t) \) which satisfies \( u_i \in L^2_{\text{loc}}((0, \infty), L^2) \) and
\[ u(t) \in C(R^+, L^1) \cap L^\infty(R^+, L^1) \cap C((0, \infty), L^p) \cap L^\infty_{\text{loc}}((0, \infty), W^{1,m}_0) \] (2.32)
and the estimates (2.9)–(2.11), where \( p \geq 1 \). Further, we have for some \( \lambda_1 > 0 \) that
\[ \|\nabla u(t)\|_m \leq C_5 (1 + \exp(\lambda_1 (t - 1))), \quad t \geq 1, \] (2.33)
where \( C_5 \) depends on \( \|g\|_2, \|K\|_\infty \).

**Proof.** Noticing that the estimate constants \( C_1, C_3, C_4 \) in (2.9)–(2.11) are independent of \( i \), we see that there exists a subsequence of \( \{u_i(t)\} \) (again denoted by \( \{u_i(t)\} \)), such that as \( i \to \infty \),
\[ u_i(t) \to u(t) \quad \text{strongly in } C(R^+, L^1), \]
\[ u_i(t) \to u(t) \quad \text{weakly* in } L^\infty_{\text{loc}}((0, \infty), L^\infty) \cap L^\infty(R^+, L^1) \cap L^\infty_{\text{loc}}((0, \infty), W^{1,m}_0), \]
\[ \frac{\partial}{\partial t} u_i(t) \to \frac{\partial}{\partial t} u(t) \quad \text{weakly in } L^2_{\text{loc}}((0, \infty); L^2). \]

Furthermore, by a standard monotonicity argument we have (see [5,11,12])
\[- \text{div}((|\nabla u_i|^2 + i^{-1}) \frac{m^2}{\mu^2} \nabla u_i) \to - \text{div}(|\nabla u|^\frac{m^2}{\mu^2} \nabla u) \]
weakly* in \( L^\infty_{\text{loc}}((0, \infty), W^{-1, \frac{m^2}{\mu^2}}) \).

Then \( u(t) \) is a weak solution of (1.1) with \( u_0 \in L^1 \) and also satisfies (2.9)–(2.11).
It remains to prove the uniqueness. Let \( u_1(t), u_2(t) \) be two solutions of (1.1) which satisfies (2.9)–(2.11). Denote \( u(t) = u_1(t) - u_2(t), \ F_n(u) = \int_0^u f_n(s) \, ds, \ u \in \mathbb{R}^1, \) which the function \( f_n(s) \) is in the proof of Lemma 2. Then \( u(t), u_1(t), u_2(t) \) satisfy

\[
\begin{align*}
    u_t - \Delta_m u_1 + \Delta_m u_2 + a(x)(|u_1|^\alpha u_1 - |u_2|^\alpha u_2) \\
  = f_0(u_1) \int K(y)|u_1|^\beta \, dy - f_0(u_2) \int K(y)|u_2|^\beta \, dy.
\end{align*}
\]

(2.34)

Multiplying (2.34) by \( f_n(u) \), we have from Lemma 4.4 in [8, Chapter 1] that

\[
\begin{align*}
    \frac{d}{dt} \int \frac{d}{dx} \left( f_n(u(t)) \right) \, dx + \gamma_0 \int |\nabla|^m f_n(u) \, dx + \int a(x)(|u_1|^\alpha u_1 - |u_2|^\alpha u_2) f_n(u) \, dx \\
  \leq \int \left( f_0(u_1) \int K(y)|u_1|^\beta \, dy - f_0(u_2) \int K(y)|u_2|^\beta \, dy \right) f_n(u) \, dx
\end{align*}
\]

(2.35)

with some \( \gamma_0 > 0 \). Since \( f_n'(s) \geq 0 \) and \( f_n(s) \) is odd in \((-\infty, \infty)\), we have \((|u_1|^\alpha u_1 - |u_2|^\alpha u_2) f_n(u) \geq 0\) in \( \Omega \) and

the right-hand side of (2.35) \leq k_0 \|K\|_\infty \left( \|u_1\|_{r-1}^{r-1} + \|u_2\|_{r-1}^{r-1} \right) \|u_2\|_\beta \|u(t)\|_1

+ k_0 \|K\|_\infty \left( \|u_1\|_{r}^{r-1} + \|u_2\|_{r}^{r-1} \right) \|u_1\|_\beta \|u(t)\|_1. \quad (2.36)

By the estimate (2.9), we notice that

\[
\begin{align*}
    \|u_1(t)\|_r &\leq \|u_1(t)\|_{\infty}^{r-1} \|u_1(t)\|_1 \leq C_1 t^{(1-r)/\alpha}, \quad \|u_2(t)\|_\beta \leq C_1 t^{(1-\beta)/\alpha}, \quad 0 < t \leq T.
\end{align*}
\]

(2.37)

Therefore,

the right-hand side of (2.35) \leq C_1 \|K\|_\infty \|u(t)\|_1 t^{-(\beta + r - 2)/\alpha}, \quad 0 < t \leq T. \quad (2.38)

Then it follows from (2.35) that

\[
\begin{align*}
    \frac{d}{dt} \int \frac{d}{dx} \left( f_n(u(t)) \right) \, dx &\leq C_1 \|K\|_\infty \|u(t)\|_1 t^{-(\beta + r - 2)/\alpha}.
\end{align*}
\]

(2.39)

where \( C_1 > 0 \), independent of \( n \) and \( i \). Integrating (2.39) on \([0, t] \) and letting \( n \to \infty \), we get from \( \|u(0)\|_1 = 0 \) that

\[
\begin{align*}
    \|u(t)\|_1 &\leq C_1 \|K\|_\infty \int_0^t s^{-(\beta + r - 2)/\alpha} \|u(s)\|_1 \, ds, \quad 0 \leq t \leq T.
\end{align*}
\]

(2.40)

Note that \((r + \beta - 2)/\alpha < 1\), then the application of Gronwall’s Lemma gives

\[
\|u(t)\|_1 \equiv 0, \quad 0 \leq t \leq T. \quad (2.41)
\]

Thus, \( u_1(t) \equiv u_2(t) \) in \([0, T] \) and the proof of uniqueness is completed.

In the following, we show that the continuity of solution \( u(t) \) in \( L^p, \ p \geq 1 \). Let \( t > s > 0 \). First, we note that
\[
\|u(t) - u(s)\|_2^2 = \int_\Omega \left( \int_s^t u_\tau(\tau) \, d\tau \right)^2 \, dx \leq \int_s^t \|u_\tau(\tau)\|_2^2 \, d\tau (t-s).
\] (2.42)

Then it follows from (2.11) that \(\|u(t) - u(s)\|_2^2 \to 0\) as \(t \to s\). So, \(u(t) \in C((0, \infty), L^2)\).

When \(1 \leq p < 2\), it follows from Hölder’s inequality that
\[
\|u(t) - u(s)\|_p^p = \int_\Omega \left| u(t) - u(s) \right|^p \, dx \leq \left( \int_\Omega \|u_\tau(\tau)\|_2^2 \, d\tau \right)^{p/2} (t-s)^{p-2}\|\Omega\|^{2-p}. \] (2.43)

This shows that \(\|u(t) - u(s)\|_p^p \to 0\) as \(t \to s\).

When \(p \geq 2\), it follows from (2.9) that
\[
\|u(t) - u(s)\|_p^p = \int_\Omega \left| u(t) - u(s) \right|^p \, dx \leq \|u(t) - u(s)\|_\infty^{p-2} \|u(t) - u(s)\|_2^2.
\]
\[
\leq Cs^{-(p-2)/\alpha} \|u(t) - u(s)\|_2^2 \to 0 \quad \text{as} \quad t \to s. \] (2.44)

This shows that \(u(t) \in C((0, \infty), L^p)\).

Since \(u_i(t) \in C(R_+, L^2)\) and \(\|u_i(t) - u(t)\|_1 \to 0\) as \(i \to \infty\), and
\[
\|u(t) - u_0\|_1 \leq \|u(t) - u_i(t)\|_1 + \|u_i(t) - u_{0,i}\|_1 + \|u_{0,i} - u_0\|_1
\] (2.45)

we see that \(\|u(t) - u_0\|_1 \to 0\) as \(t \to 0\). Hence \(u(t) \in C(R_+, L^1)\).

Finally, we derive the estimate (2.33). Denote
\[
F(t) = \frac{1}{m} \int_\Omega \left| \nabla u(x,t) \right|^m \, dx + \frac{1}{\alpha + 2} \int_\Omega a(x)|u(x,t)|^{\alpha+2} \, dx.
\]

Multiplying (1.1) by \(u_t\), we obtain
\[
\|u_t(t)\|_2^2 + F'(t) = \int_\Omega f_0(u)u_t \, dx \int_\Omega K(y)|u|^\beta \, dy + \int_\Omega g(x)u_t(x,t) \, dx
\]
\[
\leq k_0 \|K\|_\infty \|u_t(t)\|_2 \|u(t)\|_\beta + \|g\|_2 \|u_t(t)\|_2
\]
\[
\leq \frac{1}{2} \|u_t(t)\|_2^2 + C_1 \|u(t)\|_{2^\prime}^2 \|u(t)\|_\beta^2 + C \|g\|_2^2. \] (2.46)

This implies that
\[
\frac{1}{2} \|u_t\|_2^2 + F'(t) \leq C_1 \|u(t)\|_{2^\prime}^2 \|u(t)\|_\beta^2 + C \|g\|_2^2. \] (2.47)

Further, multiplying (1.1) by \(u_t\), we obtain
\[
\left| \nabla u(t) \right|_m^m + a_0 \|u(t)\|_{\alpha+2} \leq \int_\Omega g(x)u \, dx - \int_\Omega u_t u \, dx + \int_\Omega f_0(u)u \, dx \int_\Omega K(y)|u|^\beta \, dy
\]
\[
\leq \|u_t(t)\|_2 \|u(t)\|_2 + C_1 \|u(t)\|_{r+1} \|u(t)\|_\beta^\prime + \|g\|_2 \|u(t)\|_2
\]
\[
\leq \frac{1}{2} \|u_t(t)\|_2^2 + \frac{a_0}{2} \|u(t)\|_{\alpha+2}^\alpha + C_1. \] (2.48)
where \( C_1 \) depends on \( \| K \|_{\infty} \) and \( \| g \|_2 \). Hence it follows from (2.47) and (2.48) that

\[
\lambda_1 F(t) \leq C_1 \| u(t) \|_{2r}^{2r} u(t) \|_{2 \beta}^{2 \beta} - F'(t) + C_1
\]

for some \( \lambda_1 > 0 \). By (2.9) we have

\[
F'(t) + \lambda_1 F(t) \leq C_1 (1 + \| u(t) \|_{2r}^{2r} u(t) \|_{2 \beta}^{2 \beta}) \\
\leq C_1 (1 + t^{-2(\beta+r-1)/\alpha}) \leq 2C_1, \quad t \geq 1.
\]

This implies that

\[
F(t) \leq F(1) \exp(-\lambda_1 (t - 1)) + 2C_1/\lambda_1, \quad t \geq 1.
\]

Noticing that (2.9)–(2.10) and

\[
F(t) \geq \alpha_0 \| \nabla u(t) \|_m^m
\]

with some \( \alpha_0 > 0 \), we obtain (2.33) and finish the proof of Theorem 1. \( \Box \)

Similarly, we can prove by Lemmas 3 and 5 that

**Theorem 2.** Let the assumptions in Lemma 3 hold. Then (1.1) admits a unique global weak solution \( u(t) \) which satisfies the estimates (2.23)–(2.25), \( u_t \in L^{2}_{\text{loc}}((0, \infty), L^2) \) and

\[
u(t) \in C\left(R_+, L^{q_0}\right) \cap L^{\infty}\left(R_+, L^{q_0}\right) \cap C\left((0, \infty), L^p\right) \cap L^{\infty}_{\text{loc}}\left((0, \infty), W^{1,m}_0\right),
\]

where \( p \geq q_0 \). Further, if \( p \geq 2 \max\{\beta, r\} \), we have

\[
\| \nabla u(t) \|_m^{m} \leq A_2 \left(1 + \exp(-\lambda_2 (t - 1))\right), \quad t \geq 1,
\]

which \( A_2 \) depends on \( A_1(1) \) in (2.24) and with some \( \lambda_2 > 0 \).

**Proof.** The proof of existence and continuity of weak solution for (1.1) can be proceeded as the proof of Theorem 1. Hence we consider only the uniqueness and the estimate (2.52). Let \( u_1(t), u_2(t) \) be two solutions of (1.1) which satisfies (2.23)–(2.25). Denote \( u(t) = u_1(t) - u_2(t) \). By Lemma 5,

\[
\| u_1(t) \|_r^{r} \leq \| u_1(t) \|_{\infty}^{r_0} \leq A_1 t^{-\lambda r'}, \quad \| u_2(t) \|_p^{\beta} \leq A_2 t^{-\lambda \beta'}, \quad 0 < t \leq T,
\]

with \( r' = \max\{r - q_0, 0\}, \beta' = \max\{\beta - q_0, 0\} \). Then

\[
\text{the right-hand side of (2.35) \leq A_1 \phi(t) \| u(t) \|_1, \quad 0 < t \leq T,}
\]

where

\[
\phi(t) = t^{-\lambda(\beta'+r-1)} + t^{-\lambda(r'+\beta-1)}.
\]

Then for the present, it follows from (2.35) that

\[
\frac{d}{dt} \int_{\Omega} F_n(u(t)) \, dx \leq A_1 \phi(t) \| u(t) \|_1
\]
\[
\|u(t)\|_1 \leq \|u(0)\|_1 + A_1 \int_0^t \phi(s) \|u(s)\|_1 \, ds, \quad 0 \leq t \leq T. \tag{2.56}
\]

Since \(\phi(s) \in L^1[0, T]\) for any \(T > 0\), we have from Gronwall’s Lemma that
\[
\|u(t)\|_1 \equiv 0, \quad 0 \leq t \leq T. \tag{2.57}
\]

Thus, \(u_1(t) \equiv u_2(t)\) in \([0, T]\).

In order to obtain the estimate (2.52), we note that if \(q \geq 2 \max\{\beta, r\}\), then
\[
\|u(t)\|_2^r \|u(t)\|_\beta^\beta \leq C \|u(t)\|_q^{2r+2\beta} \leq A_0(1 + t^{\frac{2(q-\beta)}{\alpha}}) \leq 2A_0, \quad t \geq 1, \tag{2.58}
\]
where the estimate (2.23) has been used. Hence, (2.50) gives the estimate (2.52). We completes
the proof of Theorem 2. \(\square\)

So, from Theorem 1, we obtain that the solution operator \(S(t)u_0 = u(t), t \geq 0,\) of the problem (1.1) generates a semigroup on \(L^1\), which satisfies the following properties:

1. \(S(t): L^1 \to L^1\) for every \(t \geq 0;\)
2. \(S(t + s) = S(t)S(s)\) for \(t, s \geq 0;\)
3. \(S(t)\phi \to S(s)\phi\) in \(L^1\) as \(t \to s\) for any \(\phi \in L^1.\)

By Theorem 2, the solution operator \(S(t)u_0 = u(t), t \geq 0,\) of the problem (1.1) satisfies

4. \(S(t): L^{q_0} \to L^{q_0}\) for every \(t \geq 0;\)
5. \(S(t + s) = S(t)S(s)\) for \(t, s \geq 0;\)
6. \(S(t)\phi \to S(s)\phi\) in \(L^{q_0}\) as \(t \to s\) for any \(\phi \in L^{q_0}.\)

We are now in a position to establish some continuity of \(S(t)\) with respect to the initial data \(u_0\) which will be needed for the proof of the existence of global attractor.

**Lemma 6.** Assume that the conditions in Theorem 1 (Theorem 2) are satisfied. Let \(S(t)\phi_n\) and \(S(t)\phi\) be the solutions of the problem (1.1) with the initial data \(\phi_n\) and \(\phi\), respectively. If \(\phi_n \to \phi\) in \(L^1 (L^{q_0})\) as \(n \to \infty,\) then \(S(t)\phi_n \to S(t)\phi\) in \(L^1 (L^{q_0})\) as \(n \to \infty.\)

**Proof.** Let \(u_n(t) = S(t)\phi_n, u(t) = S(t)\phi, n = 1, 2, \ldots.\) Then \(w_n(t) = u_n(t) - u(t)\) satisfies
\[
w_{nt} - \Delta_n u_n + \Delta_m u + a(x)(|u_n|^\alpha u_n - |u|^\alpha u) = f_0(u_n) \int_\Omega K(y)|u_n|^\beta \, dy - f_0(u) \int_\Omega K(y)|u|^\beta \, dy \tag{2.59}
\]
and \(w_n(0) = \phi_n - \phi.\) If \(\phi_n \to \phi\) in \(L^1,\) we use the same argument as in the proof of uniqueness in Theorem 1 and obtain
\[
\|w_n(t)\|_1 \leq \|w_n(0)\|_1 + C\|K\|_\infty \int_0^t s^{-(r+\beta-2)/\alpha} \|w_n(s)\|_1 \, ds, \quad 0 \leq t \leq T. \tag{2.60}
\]
Since \(r + \beta - 2 < \alpha,\) we have
\[ \| S(t)\phi_n - S(t)\phi \|_1 = \| w_n(t) \|_1 \leq C_0(T, \| K \|_\infty) \| w_n(0) \|_1, \quad 0 \leq t \leq T. \] (2.61)

Letting \( n \to \infty \), we have the desired result.

If \( \phi_n \to \phi \) in \( L^{q_0} \), we multiply (2.59) by \( |w_n|^q - 2 w_n \) \( (q = q_0) \) and obtain
\[
\frac{1}{q} \frac{d}{dt} \| w_n(t) \|_q^q + \gamma \int_\Omega |\nabla w_n \|_m^m dx + \int_\Omega a(x)(|u_n|^{\alpha} u_n - |u|^{\alpha} u) |w_n|^q - 2 w_n dx \\
\leq \int_\Omega \left( f_0(u_n) \int K(y) |u_n|^\beta dy - f_0(u) \int K(y) |u|^\beta dy \right) |w_n|^q - 2 w_n dx \\
\leq A \equiv k_0 \| K \|_\infty \| w_n(t) \|_q^q (\| u \|_\beta^\beta (\| u_n \|_\infty^{\gamma - 1} + \| u \|_\infty^{\gamma - 1}) + \| u \|_\infty^\gamma (\| u_n \|_\infty^{\beta - 1} + \| u \|_\infty^{\beta - 1})).
\] (2.62)

It follows from the first estimate in (2.24) that
\[
A \leq C \| K \|_\infty \| w_n(t) \|_q^q t^{-\lambda(r + \beta - 1)}, \quad 0 < t \leq T.
\] (2.63)

Then (2.62) and (2.63) imply
\[
\frac{d}{dt} \| w_n(t) \|_q^q \leq C \| K \|_\infty \| w_n(t) \|_q^q t^{-\lambda(r + \beta - 1)}, \quad 0 < t \leq T.
\] (2.64)

Since \( \lambda(r + \beta - 1) < 1 \), (2.64) gives
\[
\| w_n(t) \|_q \leq \| w_n(0) \|_q C_0(T, \| K \|_\infty), \quad 0 \leq t \leq T.
\] (2.65)

Letting \( n \to \infty \), we get \( S(t)\phi_n \to S(t)\phi \) in \( L^{q_0} \). Then the proof of Lemma 6 is completed. \( \square \)

3. Global attractors for the problem (1.1)

In this section, we will prove the existence of the global \((L^q, L^p)\)-attractor for the problem (1.1), which \( q = 1 \), or \( q = q_0 > 1 \) and \( p > 1 \). To this end, we first give some definitions about the bi-spaces global attractor, then prove the asymptotic compactness of \( \{ S(t) \} \) \( t \geq 0 \) in \( L^p \) and the existence of the global \((L^q, L^p)\)-attractor by a priori estimates established in Section 2.

Definition 2. (See [4].) Let \( X, Y \) be Banach spaces and \( \{ S(t) \} \) \( t \geq 0 \) be a semigroup on \( X \). A set \( B_0 \subset Y \), which satisfies that for any bounded subset \( B \) of \( X \), there is \( t_0(B) \) such that \( S(t)B \subset B_0 \) for any \( t \geq t_0 \), is called an \( (X, Y) \)-bounded absorbing set.

Definition 3. (See [4].) A set \( A \subset X \), which is invariant, closed in \( X \), compact in \( Z \) and attracts bounded subsets of \( X \) in the topology of \( Z \) is called a global \((X, Z)\)-attractor.

We now can prove our main result.

Theorem 3. Let that the assumptions in Lemma 2 hold. Then the semigroup \( \{ S(t) \} \) \( t \geq 0 \) generated by the solution of problem (1.1) with \( u_0 \in L^1 \) has a global \((L^1, L^p)\)-attractor \( A \subset L^\infty \cap W_0^{1,m} \) for any \( p > 1 \), more precisely,
\[
A \subset B_{L^\infty}(C_1)
\] (3.1)
and
dist$_{L^p}(S(t)u_0, B_{L^p}(2C_1|\Omega|^{1/p})) \leq |\Omega|^{1/p}C_3t^{-1/\alpha}, \quad t \geq 1, \tag{3.2}

where the constants $C_1, C_3$ are in Lemma 4.

**Proof.** The existence of the global attractor in $L^p$ and the estimates easily follow from Theorem 1. Indeed, for $\phi \in L^1$, we denote the unique solution of (1.1) by $u(t) = S(t)\phi$. By Lemma 6, $S(t)$ is continuous with respect to initial data $u_0$ in $L^1$. Let $B_0 \subset L^p$ be a bounded set, i.e., $\exists R > 0$ such that $\|\phi\|_p \leq R$ for any $\phi \in B_0$. We set

$$S(t)B_0 = \{S(t)\phi \mid \phi \in B_0\}. \tag{3.3}$$

We know by Theorem 1 that $\bigcup_{t \geq 1} S(t)B_0$ is bounded in $W^{1,m}_0 \cap L^\infty$. Since any bounded set in $W^{1,m}_0 \cap L^\infty$ is compact set in $L^p$, $S(t)$ is completely continuous as a map from $L^p$ to $L^p$ ($p \geq 1$). Further we see by Remark 3 that the set $B_0 = B_{L^p}(C_1|\Omega|^{1/p})$ is an absorbing set of $S(t)$ in $L^p$, where the constant $C_1$ is in Lemma 4. By a general theory (see [7,16]), we conclude that $S(t)$ admits a global attractor $\mathcal{A}$ defined by

$$\mathcal{A} = \omega(B_0) = \bigcap_{\tau \geq 0} \left[ \bigcup_{t \geq \tau} S(t)B_0 \right]_{L^p}, \tag{3.4}$$

where $[\mathcal{D}]_{L^p}$ is the closure of $\mathcal{D}$ in $L^p$. Further, by the estimates in Lemma 4 and Theorem 1, we see that $\mathcal{A}$ is in fact a bounded set in $W^{1,m}_0 \cap L^\infty$. Obviously, (3.1) and (3.2) follows from (2.9). Then the proof of Theorem 3 is finished. □

Similarly, we have

**Theorem 4.** Assume that the assumptions in Lemma 3 hold. Then the semigroup $\{S(t)\}_{t \geq 0}$ generated by the solution of problem (1.1) with $u_0 \in L^{q_0}$ has a global $(L^{q_0}, L^p)$-attractor $\mathcal{A} \subset L^\infty((1, \infty), L^p)$ for any $p > q_0$, more precisely,

$$A \subset B_{L^p}(A_0), \quad \text{dist}_{L^p}(S(t)u_0, B_{L^p}(A_0)) \leq A_0t^{-1/(m-2)}, \quad t \geq 1, \tag{3.5}$$

where the constant $A_0$ is in Lemma 5.

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**References**