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Unified phantom cosmology: Inflation, dark energy and dark matter under the same standard

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Abstract

Phantom cosmology allows to account for dynamics and matter content of the universe tracing back the evolution to the inflationary epoch, considering the transition to the non-phantom standard cosmology (radiation/matter dominated eras) and recovering the today observed dark energy epoch. We develop the unified phantom cosmology where the same scalar plays the role of early time (phantom) inflaton and late-time dark energy. The recent transition from decelerating to accelerating phase is described too by the same scalar field. The (dark) matter may be embedded in this scheme, giving the natural solution of the coincidence problem. It is explained how the proposed unified phantom cosmology can be fitted against the observations which opens the way to define all the important parameters of the model.

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1. According to recent astrophysical data the (constant) effective equation of state (EOS) parameter w_{eff} of dark energy lies in the interval: $-1.48 < w_{\text{eff}} < -0.72$ [1] (see very recent comparison of observational data from different sources in [2], and see also [3]). It is clear that standard Λ -CDM cosmology is in full agreement with observations. Nevertheless, it remains the possibility that universe is currently in its phantom DE phase (for recent study of phantom cosmology, see [4–6] and refs therein). Despite the fact that it remains unclear how decelerating FRW world transformed to the accelerating DE universe, one can try to unify the early time (phantom?) inflation with late time acceleration [4]. In fact, the phantom inflation has been proposed in [7]. The unified inflation/acceleration universe occurs in some versions of modified gravity [8] as well

as for complicated EOS of the universe [9] (for recent discussion of similar (phantom) EOS, see [10] and time-dependent viscous EOS [11]).

In the present Letter we consider unified phantom cosmology with the account of dark matter. Due to the presence of scalar dependent function in front of kinetic term, the same scalar field may correspond to the (phantom) inflaton at very early universe, quintessence at the intermediate epoch and DE phantom at the late universe. The recent transition from decelerating phase to the accelerating phase is naturally described there too. On the same time it is shown that both phantom phases are stable against small perturbations, and that coincidence problem may be naturally solved in our unified model. The equivalent description of the same unified phenomena via the (multi-valued) EOS is given too. In the final section we explain how the proposed unified phantom cosmology can be fitted against the observations which gives the way to define all the important parameters of the model.

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2. Let us start from the following action:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\} + S_m. \tag{1}$$

Here $\omega(\phi)$ and $V(\phi)$ are functions of the scalar field ϕ and S_m is the action for matter field. Without matter, such an action has been proposed in [12] for the unification of early-time inflation and late-time acceleration in frames of phantom cosmology. We now assume the spatially-flat FRW metric $ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2$. Let the scalar field ϕ only depends on the time coordinate t . Then the FRW equations are given by

$$\frac{3}{\kappa^2} H^2 = \rho + \rho_m, \quad -\frac{2}{\kappa^2} \dot{H} = p + \rho + p_m + \rho_m. \tag{2}$$

Here ρ_m and p_m are the energy density and the pressure of the matter respectively. The energy density ρ and the pressure p for the scalar field ϕ are given by

$$\rho = \frac{1}{2} \omega(\phi) \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \omega(\phi) \dot{\phi}^2 - V(\phi). \tag{3}$$

Combining (3) and (2), one finds

$$\omega(\phi) \dot{\phi}^2 = -\frac{2}{\kappa^2} \dot{H} - (\rho_m + p_m),$$

$$V(\phi) = \frac{1}{\kappa^2} (3H^2 + \dot{H}) - \frac{\rho_m - p_m}{2}. \tag{4}$$

As usually ρ_m and p_m satisfy the conservation of the energy:

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \tag{5}$$

As clear from the first equation (2), in case without matter ($\rho_m = p_m = 0$), when \dot{H} is positive, which corresponds to the phantom phase, ω should be negative, that is, the kinetic term of the scalar field has non-canonical sign. On the other hand, when \dot{H} is negative, corresponding to the non-phantom phase, ω should be positive and the sign of the kinetic term of the scalar field is canonical. If we restrict in one of phantom or non-phantom phase, the function $\omega(\phi)$ can be absorbed into the field redefinition given by $\varphi = \int^\phi d\phi \sqrt{\omega(\phi)}$ in non-phantom phase or $\varphi = \int^\phi d\phi \sqrt{-\omega(\phi)}$ in phantom phase. Usually, at least locally, one can solve ϕ as a function of φ , $\phi = \phi(\varphi)$. Then the action (1) can be rewritten as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R \mp \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \tilde{V}(\varphi) \right\} + S_m. \tag{6}$$

Here $\tilde{V}(\varphi) \equiv V(\phi(\varphi))$. In the sign \mp of (6), the minus sign corresponds to the non-phantom phase and the plus one to the phantom phase. Then both of $\omega(\phi)$ and $V(\phi)$ in the action (1) do not correspond to physical degrees of freedom but only one combination given by $\tilde{V}(\varphi)$ has real freedom in each of the phantom or non-phantom phase and defines the real dynamics of the system. The redefinition, however, has a discontinuity between two phases. When explicitly keeping $\omega(\phi)$, the two phases are smoothly connected with each other (kind of phase transitions). Hence, the function $\omega(\phi)$ gives only redundant degree of freedom and does not correspond to the extra degree of freedom of the system (in the phantom or non-phantom phase).

It plays the important role just in the point of the transition between the phantom phase and non-phantom phase. By using the redundancy of $\omega(\phi)$, in any physically equivalent model, one may choose, just for example, $\omega(\phi)$ as $\omega(\phi) = \omega_0(\phi - \phi_0)$ with constants ω_0 and ϕ_0 . If we further choose ω_0 to be positive, the region given by $\phi > \phi_0$ corresponds to the non-phantom phase, the region $\phi < \phi_0$ to the phantom phase, and the point $\phi = \phi_0$ to the point of the transition between two phases.

First we consider the case that the parameter w_m in the matter EOS is a constant: $w_m = p_m/\rho_m$. (In principle, such dark matter may be presented via the introduction of one more scalar field.) Then by using (5), one gets $\rho_m = \rho_{m0} a^{-3(1+w_m)}$. Here ρ_{m0} is a constant. If $\omega(\phi)$ and $V(\phi)$ are given by a single function $g(\phi)$ as

$$\omega(\phi) = -\frac{2}{\kappa^2} g''(\phi) - \frac{w_m + 1}{2} g_0 e^{-3(1+w_m)g(\phi)},$$

$$V(\phi) = \frac{1}{\kappa^2} (3g'(\phi)^2 + g''(\phi)) + \frac{w_m - 1}{2} g_0 e^{-3(1+w_m)g(\phi)}, \tag{7}$$

with a positive constant g_0 , we find a solution of (2) or (4) given by

$$\phi = t, \quad H = g'(t) \quad \left(a = a_0 e^{g(t)}, \quad a_0 \equiv \left(\frac{\rho_{m0}}{g_0} \right)^{\frac{1}{3(1+w_m)}} \right). \tag{8}$$

Hence, even in the presence of matter, any required cosmology defined by $H = g'(t)$ can be realized by (7).

More generally, one may consider the generalized EOS like [10]: $p_m = -\rho_m + F(\rho_m)$. Here $F(\rho)$ is a proper function of ρ_m . Using the conservation of the energy (5) gives $a = a_0 e^{-\frac{1}{3} \int \frac{d\rho_m}{F(\rho_m)}}$. Let us assume the above equation can be solved with respect to ρ_m as $\rho_m = \rho_m(a)$. Then if we may choose $\omega(\phi)$ and $V(\phi)$ by a single function $g(\phi)$ as

$$\omega(\phi) = -\frac{2}{\kappa^2} g''(\phi) - F(\rho_m(a_0 e^{g(\phi)})),$$

$$V(\phi) = \frac{1}{\kappa^2} (3g'(\phi)^2 + g''(\phi)) - \rho_m(a_0 e^{g(\phi)}) + \frac{1}{2} F(\rho_m(a_0 e^{g(\phi)})), \tag{9}$$

with a positive constant a_0 , we find a solution of (2) or (4) again:

$$\phi = t, \quad H = g'(t) \quad (a = a_0 e^{g(t)}). \tag{10}$$

Hence, any cosmology defined by $H = g'(t)$ can be realized by (9).

Since the second FRW equation is given by

$$p = -\frac{1}{\kappa^2} (2\dot{H} + 3H^2), \tag{11}$$

by combining the first FRW equation, the EOS parameter w_{eff} looks as

$$w_{\text{eff}} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2}. \tag{12}$$

Now it is common to believe that about 5 billion years ago, the deceleration of the universe has turned to the acceleration. We now show that the model describing such a transition could

be easily constructed in the present formulation. As an example, we consider the model with constant w_m (7). It is also assumed $w_m > -1$. Choosing $g(\phi)$ as

$$g(\phi) = \frac{2}{3(w_m + 1)} \ln\left(\frac{\phi}{t_s - \phi}\right), \quad (13)$$

we obtain

$$\begin{aligned} \omega(\phi) &= -\frac{4}{3(w_m + 1)\kappa^2} \left(-\frac{1}{\phi^2} + \frac{1}{(t_s - \phi)^2} \right) \\ &\quad - \frac{w_m + 1}{2} g_0 \frac{(t_s - \phi)^2}{\phi^2}, \\ V(\phi) &= \frac{1}{\kappa^2} \left\{ \frac{4}{3(w_m + 1)^2} \left(\frac{1}{\phi} + \frac{1}{t_s - \phi} \right)^2 \right. \\ &\quad \left. + \frac{2}{3(w_m + 1)} \left(-\frac{1}{\phi^2} + \frac{1}{(t_s - \phi)^2} \right) \right\} \\ &\quad + \frac{w_m - 1}{2} g_0 \frac{(t_s - \phi)^2}{\phi^2}. \end{aligned} \quad (14)$$

Then the Hubble rate H is given by

$$H = \frac{2}{3(w_m + 1)} \left(\frac{1}{t} + \frac{1}{t_s - t} \right). \quad (15)$$

Since

$$\dot{H} = \frac{2}{3(w_m + 1)} \left(-\frac{1}{t^2} + \frac{1}{(t_s - t)^2} \right), \quad (16)$$

the EOS parameter w_{eff} (12) goes to $w_m > -1$ when $t \rightarrow 0$ and goes to $-2 - w_m < -1$ at large times. The crossing $w_{\text{eff}} = -1$ occurs when $\dot{H} = 0$, that is, $t = t_s/2$. Note that

$$\frac{\ddot{a}}{a} = \frac{16t_s}{27(w_m + 1)^3(t_s - t)^2t^2} \left\{ t - \frac{(3w_m + 1)t_s}{4} \right\}. \quad (17)$$

Hence, if $w_m > -1/3$, the deceleration of the universe turns to the acceleration at $t = t_a \equiv (3w_m + 1)t_s/4$. The energy density of the scalar field ϕ and that of the matter are given by

$$\begin{aligned} \rho &= \frac{4t_s^2}{3\kappa^2(w_m + 1)^2(t_s - t)^2t^2} - g_0 \frac{(t_s - t)^2}{t^2}, \\ \rho_m &= g_0 \frac{(t_s - t)^2}{t^2}. \end{aligned} \quad (18)$$

Then if the coincidence time t_c is defined by $\rho|_{t=t_c} = \rho_m|_{t=t_c}$, we find $t_c = t_s - \{2t_s^2/3g_0\kappa^2(w_m + 1)^2\}^{1/4}$. We may assume t_s could be of the order of the age of the universe, $t_s \sim 10^{10} \text{ yr} \sim (10^{-33} \text{ eV})^{-1}$. On the other hand $\kappa \sim (10^{19} \text{ GeV})^{-1} \sim (10^{28} \text{ eV})^{-1}$. Then there is a mixing of very large parameter κ and small one $1/t_s$ in (14), which can be unnatural.

We now show that the above problem of the unnaturalness could be also avoided in the present formulation. We now propose the second example in (7) given by

$$g(\phi) = -\alpha \ln\left(1 - \beta \ln \frac{\phi}{\kappa}\right). \quad (19)$$

Here α and β are dimensionless positive constants. As we explain soon, we choose $\beta \sim \mathcal{O}(10^{-2})$. Note that the parameter order of 10^{-2} is not unnatural since, say $\pi^4 \sim \mathcal{O}(10^2)$. Eq. (19)

gives the following expressions for $\omega(\phi)$ and $V(\phi)$:

$$\begin{aligned} \omega(\phi) &= -\frac{2\alpha\beta(\beta - 1 + \beta \ln \frac{\phi}{\kappa})}{\kappa^2(1 - \beta \ln \frac{\phi}{\kappa})^2\phi^2} \\ &\quad - \frac{w_m + 1}{2} g_0 \left(1 - \beta \ln \frac{\phi}{\kappa}\right)^{3(w_m + 1)\alpha}, \\ V(\phi) &= \frac{\alpha\beta(3\alpha\beta + \beta - 1 + \beta \ln \frac{\phi}{\kappa})}{\kappa^2(1 - \beta \ln \frac{\phi}{\kappa})^2\phi^2} \\ &\quad + \frac{w_m - 1}{2} g_0 \left(1 - \beta \ln \frac{\phi}{\kappa}\right)^{3(w_m + 1)\alpha}. \end{aligned} \quad (20)$$

Supposing $g_0 \sim \mathcal{O}(\kappa^{-2})$, there does not appear small parameter like $1/t_s$ in (20). Now the Hubble rate is given by $H = \alpha\beta/(1 - \beta \ln(t/\kappa))t$, which is positive if

$$0 < t < t_s \equiv \kappa e^{1/\beta}, \quad (21)$$

and has a Big Rip type singularity at $t = t_s$ (which apparently may not occur due to account of quantum effects [6]). Since $10^{61} \sim e^{140}$, with the choice $\beta \sim 1/140$, we obtain $t_s \sim \kappa \times 10^{61} \sim (10^{-33} \text{ eV})^{-1}$, whose order is that of the age of the present universe. Then now, due to the property of the exponential function (or logarithmic function), the small scale like t_s appears rather naturally. We should also note that if $\alpha\beta \sim \mathcal{O}(10^{0-2})$ and t is a present age of the universe $t \sim (10^{-33} \text{ eV})^{-1}$, the observed value of the Hubble rate $H \sim 10^{-33} \text{ eV}$ could be also reproduced. Since

$$\frac{\ddot{a}}{a} = \frac{\alpha\beta^2(\ln \frac{t}{\kappa} + \alpha + 1 - \frac{1}{\beta})}{(1 - \beta \ln \frac{t}{\kappa})^2t^2}, \quad (22)$$

the universe turns to the acceleration from the deceleration when

$$t = t_a \equiv \kappa e^{\frac{1}{\beta} - \alpha - 1} < t_s. \quad (23)$$

Since the energy density of the scalar field ϕ and that of the matter are given by

$$\begin{aligned} \rho &= \frac{3\alpha^2\beta^2}{\kappa^2(1 - \beta \ln \frac{t}{\kappa})^2t^2} - g_0 \left(1 - \beta \ln \frac{t}{\kappa}\right)^{3(w_m + 1)\alpha}, \\ \rho_m &= g_0 \left(1 - \beta \ln \frac{t}{\kappa}\right)^{3(w_m + 1)\alpha}, \end{aligned} \quad (24)$$

the coincidence time t_c could be given by solving the following equation:

$$\left(1 - \beta \ln \frac{t_c}{\kappa}\right)^{3(w_m + 1)\alpha + 2} t_c^2 = \frac{3\alpha^2\beta^2}{\kappa^2 g_0}. \quad (25)$$

One may regard ρ_m as the sum of the energy density of usual matter, like baryons, and that of (cold) dark matter. If ρ corresponds to the energy density of the dark energy, the current data indicate that $\rho : \rho_m \sim 7 : 3$. Then in the present universe, it follows

$$\left(1 - \beta \ln \frac{t}{\kappa}\right)^{3(w_m + 1)\alpha + 2} t^2 \sim \frac{9\alpha^2\beta^2}{10\kappa^2 g_0}. \quad (26)$$

Thus, in the model (19), the acceleration of the present universe and the coincidence problem seem to be explained rather naturally.

Besides the present acceleration of the universe, there was a period of the accelerated expansion of universe, which is the inflation of the early universe. One can further extend the model (19), as the third example, to explain on the same time also the inflation of the early universe. By introducing a new dimensionless positive constant γ , the following $g(\phi)$ can be proposed:

$$g(\phi) = -\alpha \ln\left(1 - \frac{\beta}{2} \ln\left(\gamma + \frac{\phi^2}{\kappa^2}\right)\right). \tag{27}$$

As in (19), α and β are dimensionless positive constants and it is assumed $\beta \sim \mathcal{O}(10^{-2})$. In the limit of large t ($t^2/\kappa^2 \gg \gamma$) being still less than t_s in (21), $g(t)$ coincides with that in (19). Since the scale factor a is given by $a = a_0 e^{g(t)}$ as in (8), the universe is invariant under the time reversal $t \rightarrow -t$ in (27). Then the universe is shrinking when $t < 0$ and expanding when $t > 0$. The scale factor has minimum at $t = 0$ as $a = (1 - (\beta/2) \ln \gamma)^{-\alpha}$. In the model (27), the Hubble rate is given by

$$H(t) = \frac{\alpha\beta t}{\kappa^2(1 - \frac{\beta}{2} \ln(\gamma + \frac{t^2}{\kappa^2}))(\gamma + \frac{t^2}{\kappa^2})}. \tag{28}$$

Since

$$\frac{\ddot{a}}{a} = \frac{\alpha\beta}{\kappa^2(1 - \frac{\beta}{2} \ln(\gamma + \frac{t^2}{\kappa^2}))^2(\gamma + \frac{t^2}{\kappa^2})^2} \times \left\{ \left(1 - \frac{\beta}{2} \ln\left(\gamma + \frac{t^2}{\kappa^2}\right)\right) \left(\gamma - \frac{t^2}{\kappa^2}\right) + \frac{\beta(1 + \alpha)t^2}{\kappa^2} \right\}, \tag{29}$$

if $t > 0$, there are two solutions of $\ddot{a} = 0$, one corresponds to late time and another corresponds to early time. The late time solution of $\ddot{a} = 0$ is obtained by neglecting γ and coincides with (23): $t = t_l \sim \kappa e^{1/\beta - \alpha - 1}$. On the other hand, the early time solution could be found by neglecting β , which is $\mathcal{O}(10^{-2})$, to be $t = t_e \sim \kappa \sqrt{\gamma}$. Then the universe undergoes accelerated expansion when $0 < t < t_e$ and $t_l < t < t_s$. Here t_s is Rip time: $t_s = \kappa \sqrt{-\gamma + e^{2/\beta}} \sim \kappa e^{1/\beta}$. t_e may be identified with the time when the inflation ended. One is able to define the number of the e-foldings N_e as $N_e = \ln(a(t_e)/a(0))$. Then we obtain

$$N_e = -\alpha \ln\left(\frac{1 - \frac{\beta}{2} \ln(2\gamma)}{1 - \frac{\beta}{2} \ln(\gamma)}\right). \tag{30}$$

It is known that N_e should be equal or larger than 60. We should note that ln-function in (30) cannot be so large naturally since this requires $(1 - (\beta/2) \ln(2\gamma))/(1 - (\beta/2) \ln(\gamma)) \sim e^{60} \sim 10^{25}$. Then the ln-function in (30) should be of order of unity, which requires that the parameter α should be $\alpha \sim 10^2$. For example, since we can rewrite (30) as $\gamma = \exp(2/\beta - \ln 2/(1 - e^{-N_e/\alpha}))$, we find that, when $\alpha = 1/\beta = 240$, we have $N_e = 60$ if we choose γ as $\gamma = 0.043925\dots$. In the same way one can define number of e-foldings in other similar models. For instance, in [12], without matter, the following unified

model has been considered:

$$f(\phi) \equiv g'(\phi) = h_0^2 \left(\frac{1}{t_0^2 - \phi^2} + \frac{1}{\phi^2 + t_1^2} \right). \tag{31}$$

Here h_0 , t_0 , and t_1 are positive constants. It is assumed $t_0 > t_1$. It has been found [12] that H has two minima at $t = t_{\pm} \equiv \pm \sqrt{(t_0^2 - t_1^2)}/2$ and at $t = 0$, H has a local maximum. Hence, (early and late) accelerating phantom phase occurs when $t_- < t < 0$ and $t > t_+$. The number of e-foldings may be defined as

$$N_e = \ln \frac{a(0)}{a(t_-)} = -\frac{h_0^2}{2t_0} \ln\left(\frac{t_0 - \sqrt{\frac{t_0^2 - t_1^2}{2}}}{t_0 + \sqrt{\frac{t_0^2 - t_1^2}{2}}}\right) + \frac{h_0^2}{t_1} \left(\text{Arctan}\left(t_1 \sqrt{\frac{2}{t_0^2 - t_1^2}}\right) + \pi \right). \tag{32}$$

Since ln and Arctan-functions should be the order of unity, we find h_0^2/t_1 (Note that we have assumed $t_0 > t_1 > 0$.) should be $\mathcal{O}(10^2)$ so that N_e could be equal or larger than 60. For example, when $t_0 \gg t_1$, we find $N_e \sim \frac{h_0^2 \pi}{t_1}$. Then if we choose $\frac{h_0^2 \pi}{t_1} \sim 60$, we find $N_e \sim 60$. Hence, it is demonstrated that scalar field may play the role of phantom inflaton in the early universe and phantom DE in the late universe even in the presence of matter. In the intermediate phase of the universe evolution the scalar has the standard canonical sign for kinetic energy.

3. It is interesting to investigate the stability of the solution (8). It is easier to work without matter, that is, we put g_0 in (7) or F in (9) to be equal zero. One defines $d/dN \equiv (H^{-1}) d/dt$, $X \equiv \dot{\phi}$, $Y \equiv f(\phi)/H$. Note that $X = Y = 1$ in the solution (8). By using the first FRW equation (2), we find, for the solution (8),

$$\mu \equiv \frac{1 - Y^2}{1 - X^2} = \frac{H^2}{\dot{H}}. \tag{33}$$

Then by using the FRW equations (2), (3) with (7) or (9), and the scalar field equation

$$0 = \omega(\phi)\ddot{\phi} + \frac{1}{2}\omega'(\phi)\dot{\phi}^2 + 3H\omega(\phi)\dot{\phi} + V'(\phi), \tag{34}$$

one finds

$$\frac{dX}{dN} = Y - X, \quad \frac{dY}{dN} = \mu X(1 - XY). \tag{35}$$

Consider the perturbations from the solution $X = Y = 1$: $X = 1 + \delta X$, $Y = 1 + \delta Y$. From (35), it follows

$$\frac{d}{dN} \begin{pmatrix} \delta X \\ \delta Y \end{pmatrix} = M \begin{pmatrix} \delta X \\ \delta Y \end{pmatrix}, \quad M \equiv \begin{pmatrix} -1 & 1 \\ -\mu & -\mu \end{pmatrix}. \tag{36}$$

If the real parts of all the eigenvalues of the matrix M are negative, the solution $X = Y = 1$ is stable. The eigenvalues λ_{\pm} are given by $\lambda_{\pm} = \{-(1 + \mu) \pm \sqrt{(1 + \mu)^2 - 8\mu}\}/2$. Then the solution (8) is stable if and only if $\mu > 0$. From (33), it is seen

the positive μ means positive \dot{H} . Hence, in the phantom phase ($\dot{H} > 0$), the solution is stable.¹

One can check what kind of the EOS for the scalar field appears. For simplicity, the universe without matter is considered. For the solution (8), by using the first FRW equation (2), we find

$$\frac{3}{\kappa^2} H^2 = \frac{3}{\kappa^2} f(\phi)^2 = \rho, \quad f(\phi) \equiv g'(\phi), \quad (37)$$

which may be solved as $\phi = f^{-1}(\kappa\sqrt{\frac{\rho}{3}})$. Here f^{-1} is the inverse of f , that is, if $y = f(x)$, then $x = f^{-1}(y)$. By using (4) and (7) with $\rho_m = p_m = g_0 = 0$, it follows $\omega(\phi) = -\frac{2}{\kappa^2} f'(\phi) = \rho + p$. Combining the above equations, the following EOS may be obtained

$$p = -\rho - \frac{2}{\kappa^2} f' \left(f^{-1} \left(\kappa \sqrt{\frac{\rho}{3}} \right) \right). \quad (38)$$

Note that f^{-1} could be multi-valued function in general. For example, in case of (13), EOS is

$$p = -\rho \mp \frac{(w_m + 1)\rho}{t_s} \sqrt{t_s^2 - \frac{8}{3(w_m + 1)\kappa} \sqrt{\frac{3}{\rho}}}. \quad (39)$$

Similarly, EOS for other types of scalar couplings may be constructed which shows that scalar field dynamics may be always mapped into the (complicated) EOS.

4. The description of unified phantom dynamics in terms of EOS can be fitted against observations if one selects suitable sets of data at low redshift ($z \sim 0-1$), medium redshift ($1 \ll z \ll 100$) and extremely high redshift ($100 \ll z \ll 1000$). Specifically, observational evidences point out that the evolutionary history of the universe comprises two periods of accelerated expansion, namely the inflationary epoch and the present day dark energy dominated phase with an intermediate decelerated phase where a component of cosmic fluid (dark/baryonic matter) has given rise to clustered large scale structure. As we have seen, a single (effective) fluid may indeed be responsible of both periods of accelerated expansion. At the same time, this fluid should be subdominant during the radiation/matter dominated epochs to give rise to baryogenesis and structure formation. In any case, whatever the fluid is, in order to achieve a unified model which could be matched with observations, the cosmological energy density has to scale as

$$\rho(a) = \mathcal{N} a^{-3} \left(1 + \frac{a_1}{a} \right)^\eta \left(1 + \frac{a}{a_{DE}} \right)^\chi \quad (40)$$

¹ The scalar model of deceleration/acceleration transition has been considered in [13]. As also discussed here, the solution (8) with $g_0 = 0$ is stable in the phantom phase but unstable in the non-phantom phase. As one of the eigenvalues λ_\pm becomes very large when crossing $w = -1$, the instability is high there. In order to avoid this problem for above toy model, we may consider two scalar model as in Appendix A. In case of one scalar model, the instability becomes infinite at the crossing $w = -1$ point, which occurs since the coefficient of the kinetic term $\omega(\phi)$ in (1) vanishes at the point. In the two scalar model, one can choose the corresponding coefficients do not vanish anywhere. Then we may expect that such a divergence of the instability would not occur, which can be checked in Appendix A.

with \mathcal{N} a normalization constant, (η, χ) slope parameters and $a_1 \ll a_{DE}$ two scaling values of the scale factor which lead to early inflationary epoch (I) and late dark energy epoch (DE). It is convenient to rewrite Eq. (40) in terms of the redshift z :

$$\rho(z) = \mathcal{N} (1+z)^3 \left(1 + \frac{1+z}{1+z_I} \right)^\eta \left(1 + \frac{1+z_{DE}}{1+z} \right)^\chi. \quad (41)$$

To match this formalism with the above results, we have to consider $z = a_0/a - 1$, setting, as standard, $a_0 = 1$ with the subscript 0 denoting quantities evaluated at the present day, i.e., $z = 0$. Immediately, the Hubble rate is $H = -\frac{\dot{z}}{z+1}$. Then, taking into account Eq. (8), all results in terms of time can be translated in terms of redshift. However, the phenomenological parameters η and χ , assigning the slope of ρ , are derived approximating the function $g(t)$, while the rip time t_s and the coincidence time t_c are, for each given $g(t)$ -model, related respectively to z_I and z_{DE} . In the same way, using (8), the other characteristic times as t_a or t_e can be translated in terms of redshift. Keeping in mind these considerations, the following discussion holds, in principle, for any unified phantom cosmology, comprising dark matter, like those which we have discussed. Essentially, $g(t)$ leads the slope of ρ and the transition between the various epochs. From Eq. (40), it is easy to see that

$$\begin{cases} \rho \sim a^{-(\eta+3)} & \text{for } a \ll a_1 \ll a_{DE}, \\ \rho \sim a^{-3} & \text{for } a_1 \ll a \ll a_{DE}, \\ \rho \sim a^{\chi-3} & \text{for } a_1 \ll a_{DE} \ll a. \end{cases}$$

Such an energy density scales as dust matter in the range $a_1 \ll a \ll a_{DE}$. This means that the fluid follows matter along a large part of the universe history, while it scales differently only during the very beginning ($a \ll a_1$) and the present period ($a \gg a_{DE}$). Moreover, choosing $\eta = -3$, the fluid energy density remains constant for $a \ll a_1$ thus behaving as the usual cosmological constant Λ during the early epoch of the universe evolution. Finally, the slope parameter χ determines how the fluid energy density scales with a in the present epoch. Considering an effective EOS $w_{\text{eff}} \equiv p/\rho$, we get, in terms of redshift,

$$w_{\text{eff}} = \frac{\eta}{3} \left(\frac{1+z}{2+z+z_I} \right) - \frac{\chi}{3} \left(\frac{1+z_{DE}}{2+z+z_{DE}} \right). \quad (42)$$

It is worth noting that w_{eff} does not depend neither on χ nor on z_{DE} for high values of z , that is for early epochs. On the contrary, these two parameters play a key role in determining the behavior of the EOS over the redshift range (0, 100) which represents most of the history of the universe (in terms of time) and the interesting period for structure formation. The role of the different quantities $(\eta, \chi, z_I, z_{DE})$ is better understood considering the asymptotic limits of the EOS: $\lim_{z \rightarrow \pm\infty} w_{\text{eff}}(z) = \frac{\eta}{3}$, where $z \rightarrow -\infty$ refers to the asymptotic future. Setting $\eta = -3$, the fluid equation of state asymptotically approaches that of the cosmological constant, i.e., $w_\Lambda = -1$. In general, if we impose $\eta < -1$, we get a fluid having a negative pressure in the far past so that it is able to drive the accelerated expansion occurring during the inflationary epoch. It is now clear that z_I controls the transition towards the past asymptotic value in the sense that the larger is z with respect to z_I , the smaller is the difference

between $w_{\text{eff}}(z)$ and its asymptotic limit $\eta/3$. This consideration suggests that z_1 has to take quite high values (indeed, far greater than 10^3) since, for $z \gg z_1$, the universe is in its inflationary phase. The present day value of w_{eff} is

$$w_0 = \frac{\eta}{3(2+z_1)} - \frac{\chi}{3} \left(\frac{1+z_{\text{DE}}}{2+z_{\text{DE}}} \right) \simeq -\frac{\chi}{3} \left(\frac{1+z_{\text{DE}}}{2+z_{\text{DE}}} \right), \quad (43)$$

where we have used the fact that z_1 is very large. Being $z_{\text{DE}} > 0$, in order to have the present day accelerated expansion, w_0 should be negative so that we get $\chi > 0$. Moreover, depending on the values of χ and z_{DE} , w_0 could also be smaller than w_Λ so that we may recover phantom-like models. The parameter z_{DE} then regulates the transition to the dark energy-like dominated period. It is worth noting that, since ρ scales with a as dust matter for a long period, the coincidence problem is naturally solved in any theoretical model endowed with these features, as the unified phantom ones presented above. Moreover, although w_0 could be smaller than -1 as for a phantom field, the equation of state asymptotically tends to a value larger than -1 (provided that $-3 \leq \eta \leq -1$) so that the Big Rip is significantly delayed or does not occur at all. In order to take into account the dust-matter-dominated era, we have to ask for an EOS (42) of the form $w_{\text{eff}}(z_M) = 0$, which gives

$$z_M = \frac{y_{\text{DE}}\chi - (2 + y_{\text{DE}})\eta \pm \sqrt{y_{\text{DE}}\mathcal{Z}(\eta, \chi, z_{\text{DE}}, z_1)}}{2\eta} \quad (44)$$

with $y_{\text{DE}} = 1 + z_{\text{DE}}$ and

$$\mathcal{Z}(\eta, \chi, z_{\text{DE}}, z_1) = y_{\text{DE}}\eta^2 + 2(2 - y_{\text{DE}} + 2z_1)\eta\chi + y_{\text{DE}}\chi^2. \quad (45)$$

It is easy to check that, for reasonable values of η and z_1 , z_M is always a complex number and hence the equation of state never vanishes and it is always negative. Thus, even if its energy density scales as that of dark matter over the most of the universe life, dark energy cannot play the same role of matter since its equation of state is always significantly different from null. As a result, we have to include also the dark matter in the total energy budget. This intrinsic difference in the behavior of dark energy and dark matter equations of state is the reason why the former, essentially, contributes to the cosmic acceleration while the latter gives rise to the clustered large scale structure. This kind of analysis is particularly useful in the case of solution (39) where the dominance of dark matter or dark energy is ruled by the parameters t_s and w_m .

Summarizing, we have proposed an unifying approach to the problem of inflation, dark matter and dark energy in the same theoretical framework. As we have seen, it is possible to construct *exact* phantom-like cosmological models where *all* peculiar eras of cosmic evolution are achieved. Such an evolution can be matched with observations using the approach outlined in [9] where it is considered the dimensionless coordinate distance to gold SNeIa sample [14] and a dataset comprising 20 radio galaxies [16,17], the shift parameter [18,19] and the baryonic acoustic peak in the LRG correlation function [20]. By this approach, it is possible not only to show the viability of the present unifying phantom model, but also to constrain its main parameters which are, essentially, the times (i.e., the redshift) of

transition between the various epochs. As an independent cross check, we can also use a recently proposed method to constrain the model parameters with the lookback time to galaxy clusters and the age of the universe thus obtaining consistent estimates for the model parameters [21]. In this case we are not resorting to distance indicators but to cosmic clocks which allow to fix with good accuracy the transition scales (at least the one from dark matter to dark energy dominated eras). Furthermore, the unified phantom model can be compared and contrasted with the CMBR anisotropy and polarization spectrum, with the data of the matter power spectrum and the growth index [15,18,19]. These tests make it possible to check the model over different redshift ranges than the SNeIa and radio galaxies data, offering also the possibility to tighten the ranges for the different parameters (in particular z_1 and z_{DE} , or t_i with $i = \{e, c, s, a\}$ in the above “time” description). As further remark, this unified phantom cosmology can be related to fundamental theories as discussed in [12] where generalized holographic dark energy can be constructed in this scheme. In a forthcoming paper, the detailed comparison with observations will be developed and discussed.

Appendix A. Two scalar model

As seen in (33)–(36), the solution (8) with $g_0 = 0$ is stable in the phantom phase but unstable in the non-phantom phase. As one of the eigenvalues λ_\pm of the matrix M in (36) becomes very large when crossing $w = -1$, the instability is very high there. In order to avoid this problem, we may consider two scalar model like

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \eta(\chi) \partial_\mu \chi \partial^\mu \chi - V(\phi, \chi) \right\}. \quad (\text{A.1})$$

Here $\eta(\chi)$ is a function of the scalar field χ . The FRW equations give

$$\begin{aligned} \omega(\phi) \dot{\phi}^2 + \eta(\chi) \dot{\chi}^2 &= -\frac{2}{\kappa^2} \dot{H}, \\ V(\phi, \chi) &= \frac{1}{\kappa^2} (3H^2 + \dot{H}). \end{aligned} \quad (\text{A.2})$$

Then if

$$\begin{aligned} \omega(t) + \eta(t) &= -\frac{2}{\kappa^2} f'(t), \\ V(t, t) &= \frac{1}{\kappa^2} (3f(t)^2 + f'(t)), \end{aligned} \quad (\text{A.3})$$

the explicit solution follows

$$\phi = \chi = t, \quad H = f(t). \quad (\text{A.4})$$

One may choose that ω should be always positive and η be always negative, for example

$$\omega(\phi) = -\frac{2}{\kappa^2} \left\{ f'(\phi) - \sqrt{\alpha^2 + f'(\phi)^2} \right\} > 0,$$

$$\eta(\chi) = -\frac{2}{\kappa^2} \sqrt{\alpha^2 + f'(\chi)^2} < 0. \quad (\text{A.5})$$

Here α is a constant. We now define a new function $\tilde{f}(\phi, \chi)$ by

$$\tilde{f}(\phi, \chi) \equiv -\frac{\kappa^2}{2} \left(\int d\phi \omega(\phi) + \int d\chi \eta(\chi) \right), \quad (\text{A.6})$$

which gives $\tilde{f}(t, t) = f(t)$. If $V(\phi, \chi)$ is given by using $\tilde{f}(\phi, \chi)$ as

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left(3\tilde{f}(\phi, \chi)^2 + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \phi} + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \chi} \right), \quad (\text{A.7})$$

not only the FRW equations but also the scalar field equations are also satisfied:

$$0 = \omega(\phi) \ddot{\phi} + \frac{1}{2} \omega'(\phi) \dot{\phi}^2 + 3H\omega(\phi) \dot{\phi} + \frac{\partial \tilde{V}(\phi, \chi)}{\partial \phi},$$

$$0 = \eta(\chi) \ddot{\chi} + \frac{1}{2} \eta'(\chi) \dot{\chi}^2 + 3H\eta(\chi) \dot{\chi} + \frac{\partial \tilde{V}(\phi, \chi)}{\partial \chi}. \quad (\text{A.8})$$

In case of one scalar model, the instability becomes infinite at the crossing $w = -1$ point, which occurs since the coefficient of the kinetic term $\omega(\phi)$ in (1) vanishes at the point. In the two scalar model in (A.1), the coefficients $\omega(\phi)$ and $\eta(\phi)$ do not vanish anywhere, as in (A.5). Then we may expect that such a divergence of the instability would not occur. We now check this explicitly in the following.

By introducing the new quantities, X_ϕ , X_χ , and Y as

$$X_\phi \equiv \dot{\phi}, \quad X_\chi \equiv \dot{\chi}, \quad Y \equiv \frac{\tilde{f}(\phi, \chi)}{H}, \quad (\text{A.9})$$

the FRW and the scalar equations (A.8) are rewritten as follows:

$$\frac{dX_\phi}{dN} = -\frac{\omega'(\phi)}{2H\omega(\phi)} (X_\phi^2 - 1) - 3(X_\phi - Y),$$

$$\frac{dX_\chi}{dN} = -\frac{\eta'(\chi)}{2H\eta(\chi)} (X_\chi^2 - 1) - 3(X_\chi - Y),$$

$$\frac{dZ}{dN} = \frac{\kappa^2}{2H^2} \{ X_\phi(X_\phi Y - 1) + X_\chi(X_\chi Y - 1) \}. \quad (\text{A.10})$$

Here $d/dN \equiv H^{-1} d/dt$. For the solution (A.4), $X_\phi = X_\chi = Y = 1$. The perturbations are considered as

$$X_\phi = 1 + \delta X_\phi, \quad X_\chi = 1 + \delta X_\chi, \quad Y = 1 + \delta Y. \quad (\text{A.11})$$

Then

$$\frac{d}{dN} \begin{pmatrix} \delta X_\phi \\ \delta X_\chi \\ \delta Y \end{pmatrix} = M \begin{pmatrix} \delta X_\phi \\ \delta X_\chi \\ \delta Y \end{pmatrix},$$

$$M \equiv \begin{pmatrix} -\frac{\omega'(\phi)}{H\omega(\phi)} - 3 & 0 & 3 \\ 0 & -\frac{\eta'(\chi)}{H\eta(\chi)} - 3 & 3 \\ \frac{\kappa^2}{2H^2} & \frac{\kappa^2}{2H^2} & \frac{\kappa^2}{H^2} \end{pmatrix}. \quad (\text{A.12})$$

The eigenvalues of the matrix M are given by solving the following eigenvalue equation

$$0 = \left(\lambda + \frac{\omega'(\phi)}{H\omega(\phi)} + 3 \right) \left(\lambda + \frac{\eta'(\chi)}{H\eta(\chi)} + 3 \right) \left(\lambda - \frac{\kappa^2}{H^2} \right) + \frac{3\kappa^2}{2H^2} \left(\lambda + \frac{\omega'(\phi)}{H\omega(\phi)} + 3 \right) + \frac{3\kappa^2}{2H^2} \left(\lambda + \frac{\eta'(\chi)}{H\eta(\chi)} + 3 \right). \quad (\text{A.13})$$

The eigenvalues are clearly finite. Then even if there is an instability, it could be finite. More complicated models along this line may be presented as well.

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