Physics Letters B 697 (2011) 271–274

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

On noncommutative minisuperspace and the Friedmann equations

W. Guzmán [a], M. Sabido [b,*], J. Socorro [b]

[a] Centro Brasileiro de Pesquisas Fisicas, Rua Dr. Xavier Sigaud, 150 Urca, CEP 22290-180, Rio de Janeiro, RJ, Brazil
[b] Departamento de Física, DCI-Campus León, Universidad de Guanajuato, A.P. E-143, C.P. 37150, Guanajuato, Mexico

ARTICLE INFO

Article history:
Received 20 May 2009
Received in revised form 2 February 2011
Accepted 2 February 2011
Editor: S. Dodelson

Keywords:
Cosmology
Noncommutativity
Minisuperspace models

ABSTRACT

In this Letter we present a noncommutative version of scalar field cosmology. We find the noncommutative Friedmann equations as well as the noncommutative Klein–Gordon equation, interestingly the noncommutative contributions are only present up to second order in the noncommutative parameter. Finally we conclude that if we want a noncommutative minisuperspace with a constant noncommutative parameter as viable phenomenological model, the noncommutative parameter has to be very small.

The initial interest in noncommutative field theory [1] slowly but steadily permeated in the realm of gravity, from which several approaches to noncommutative gravity [2] were proposed. All of these formulations showed that the end result of a noncommutative theory of gravity is a highly nonlinear theory, finding solutions to the corresponding noncommutative field equations has been complicated. Even if working with a full noncommutative theory of gravity looks like a fruitless ordeal, several attempts were made to understand the effects of noncommutativity on different aspects of the universe. In some cases the effects of noncommutativity on the gravitational degrees of freedom were ignored but imposed in the matter sector. Interesting results were obtained in connection to scalar field cosmologies [3].

An interesting proposal concerning noncommutative cosmology was presented in [4]. The authors noticed that noncommutative deformations modify the noncommutative fields, and conjectured that the effects of the full noncommutative theory of gravity should be reflected in the minisuperspace variables. This was achieved by introducing the Moyal product of functions in the Wheeler–DeWitt equation, in the same manner as is done in noncommutative quantum mechanics. They analyzed the Kantowski–Sachs cosmology and the study was carried out at the quantum level, the authors showed that new states of the universe can be created from the deformations of the quantum phase space. Several works followed with this idea [5,6].

Although the noncommutative deformations of the minisuperspace were originally analyzed at the quantum level by an effective noncommutativity on the minisuperspace, classical noncommutative formulations have been proposed. In [5], the authors considered classical noncommutative relations in the phase space for the Kantowski–Sachs cosmological model and established the classical noncommutative equations of motion. For scalar field cosmology, in [6] the classical minisuperspace is deformed and a scalar field is used as the matter component of the universe. In [7], the study is focused on the consequences the noncommutative deformation has on the slow-roll parameter when an exponential potential for the scalar field is considered, the noncommutative deformation gives a mechanism that ends inflation. The main idea of this classical noncommutativity is based on the assumption that modifying the Poisson brackets of the classical theory gives the noncommutative equations of motion [4,6,5,7]. The main purpose of this Letter is to construct the noncommutative Friedmann equations for noncommutative scalar field cosmology with an arbitrary scalar field potential and analyze the effects of noncommutativity.

We will work with the FRW universe and a scalar field with arbitrary potential as the matter content. The model has been used to explain several aspects of our universe, like inflation, dark energy and dark matter. The main reason for using scalar fields is their flexibility and the simplicity of their dynamics. Noncommutativity in the minisuperspace will be introduced by modifying the symplectic structure (Poisson's algebra of the minisuperspace) in the same manner as in [6,5,7]. Once this is achieved noncommutative equivalents of the Friedmann equations are derived. Interestingly the noncommutative deformations only appear up to second order in the noncommutative parameter. Furthermore, if we want

* Corresponding author.
E-mail addresses: wguzman@cbpf.br (W. Guzmán), msabido@fisica.ugto.mx (M. Sabido), socorro@fisica.ugto.mx (J. Socorro).

0370-2693 © 2011 Elsevier B.V. Open access under CC BY license.
doi:10.1016/j.physletb.2011.02.012

to consider noncommutative minisuperspace based cosmology as a viable phenomenological model the resulting equations seem to favor two very restrictive possibilities, an extremely small value for the noncommutative minisuperspace parameter, or a very high degree of fine tuning in the parameters of the scalar field potential.

1. Noncommutative cosmological equations

As already suggested, cosmology presents an attractive arena for noncommutative models, both in the quantum as well as classical level. One of the features of noncommutative field theories is UV/IR mixing, this effectively mixes short scales with long scales, from this fact one may expect that even if noncommutativity is present at a small scale, by this UV/IR mixing, the effects might be present at an older time of the universe. Furthermore the presence of the noncommutativity could be related to a minimal size, this idea is from the analogy with quantum mechanics where uncertainty relation between the momentum and coordinates is present.

Let us start by introducing the phase space for a homogeneous and isotropic universe with Friedmann–Robertson–Walker metric

$$ds^2 = -N^2(t)\,dt^2 + e^{2\alpha}\left[dr^2 + r^2\,d\Omega\right], \tag{1}$$

here we consider a flat universe, $a(t) = e^{\alpha}$ is the scale factor of the universe and $N(t)$ is the lapse function, finally we will use a scalar field ϕ as the matter content for the model. The Hamiltonian function is obtained from the action

$$S = \int dx^4 \sqrt{-g}\left[R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \tag{2}$$

where we have used the units $8\pi G = 1$.

The Hamiltonian is calculated as usual and is given by

$$\mathcal{H} = e^{-3\alpha}\left[\frac{1}{12}P_\alpha^2 - \frac{1}{2}P_\phi^2 - e^{6\alpha}V(\phi)\right], \tag{3}$$

where $V(\phi)$ is the potential for the scalar field, we also set $N(t) = 1$, this means that we will be using the cosmic time.

The phase space coordinates for this model are given by $\{\alpha, \phi; P_\alpha, P_\phi\}$, using Eq. (3), we find the equations of motion

$$\dot{\alpha} = \frac{1}{6}e^{-3\alpha}P_\alpha, \qquad \dot{P}_\alpha = 6e^{3\alpha}V(\phi),$$
$$\dot{\phi} = -e^{-3\alpha}P_\phi, \qquad \dot{P}_\phi = e^{3\alpha}\frac{dV(\phi)}{d\phi}. \tag{4}$$

From the equations for α and ϕ and the Hamiltonian we construct the Friedmann equation

$$3H^2 = \frac{1}{2}\dot{\phi}(t) + V(\phi), \tag{5}$$

the Klein–Gordon equation follows from the Hamilton's equations for ϕ and P_ϕ

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV(\phi)}{d\phi}. \tag{6}$$

As mentioned in the introduction of this Letter, we want to apply *noncommutativity* to this cosmological model. To analyze noncommutative cosmology one should start with a noncommutative theory of gravity, and from this theory derive the equivalent of the Friedmann equations. Unfortunately this is a very difficult ordeal due to the highly nonlinear character of noncommutative theories of gravity [2]. In [4], the authors circumvent this problem by using an effective noncommutativity in the minisuperspace. They start by calculating the Hamiltonian from Hilbert–Einstein Lagrangian, together with the metric for the Kantowski–Sachs cosmology. This gives a Hamiltonian as a function of the minisuperspace variables (β, Ω) and their canonical momenta. Furthermore, they propose that the minisuperspace variables do not commute $[\beta, \Omega] = i\theta$ and derive a noncommutative Wheeler–DeWitt equation. Based on this idea several papers have introduced this effective noncommutativity in different cosmological models [5–7]. Following [4], the introduction of noncommutativity is achieved by a noncommutative deformation of the minisuperspace spawned by the minisuperspace variables (α, ϕ), which is encoded in the commutation relation

$$[\alpha, \phi] = i\theta, \tag{7}$$

this is an effective noncommutativity that arises from a fundamental noncommutative theory of gravity. If we start with the Lagrangian derived in [2] (where noncommutativity is derived by using the Seiberg–Witten map), the noncommutative fields are a consequence of noncommutativity among the coordinates [1] and then the minisuperspace variables would inherit some effective noncommutativity, this we assume to be encoded in Eq. (7).

In order to analyze the classical evolution of the noncommutative cosmological model, we modify the Poisson algebra [5,7], the new commutation relations are

$$\{\alpha, \phi\} = \theta, \qquad \{p_\alpha, p_\phi\} = 0,$$
$$\{\alpha, p_\alpha\} = 1, \qquad \{\phi, p_\phi\} = 1. \tag{8}$$

This particular choice of the noncommutative relations is inspired and consistent with the effective noncommutativity used in noncommutative quantum cosmology. Now we use Hamilton's equations to find the classical dynamics. We might ask ourselves, if for the modified Poisson algebra, the equations of motion are given by $\dot{x_i} = \{x, \mathcal{H}\}$, $\dot{P}_{x_i} = \{P_{x_i}, \mathcal{H}\}$, where $x_i = (\alpha, \phi)$ and $P_{x_i} = (P_\alpha, P_\phi)$. To answer this question, we turn to the symplectic formalism where the equations of motion can be easily determined. Among the advantages of this formalism is the fact that the equations of motion for a particular algebra can be easily calculated, also nonconstant deformation parameters can be analyzed, furthermore it is not necessary to do an expansion on the noncommutative parameters (as when using the Moyal product of functions) to get a closed equations (details are presented in Appendix A).

Using the new algebra we calculate the deformed equations that govern the dynamics

$$\dot{\alpha} = \{\alpha, \mathcal{H}\} = \frac{1}{6}e^{-3\alpha}P_\alpha - \theta e^{3\alpha}\frac{dV(\phi)}{d\phi},$$
$$\dot{\phi} = \{\phi, \mathcal{H}\} = -e^{-3\alpha}P_\phi + 6\theta e^{3\alpha}V(\phi), \tag{9}$$

we omitted writing the equations for the momenta and the Hamiltonian, as they remain unchanged under the noncommutative deformation. In order to arrive at Eq. (9) we used the following formulas

$$\{\alpha, f(\alpha,\phi)\} = \theta\frac{\partial f}{\partial\phi}, \qquad \{\phi, f(\alpha,\phi)\} = -\theta\frac{\partial f}{\partial\alpha}, \tag{10}$$

which are calculated from the noncommutative relations (8). Using Eq. (9) and the Hamiltonian we arrived at the *deformed Friedmann's equation*

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 6\theta a^3\left[H\frac{dV}{d\phi} + \dot{\phi}V\right] - 3\left(\theta a^3\right)^2\left[\left(\frac{dV}{d\phi}\right)^2 - 6V^2\right]. \tag{11}$$

The Klein–Gordon equation for this noncanonical 2-form can be calculated from Eqs. (3) and (9) giving

$$\ddot{\phi}+3H\dot{\phi}=-\frac{dV}{d\phi}+6\theta a^3\left[\dot{\phi}\frac{dV}{d\phi}+6HV\right], \tag{12}$$

we clearly see that in the limit $\theta\to 0$ we recover the usual equations of scalar field cosmology. These are the noncommutative Friedmann equations for scalar field cosmology, these equations are derived for an arbitrary potential of the scalar field.

Exact solutions to the noncommutative Friedmann equations are not easy to find for a general potential. In the simplest case, that of a free scalar field ($V=0$), the noncommutative effect disappears and the cosmological equations are the usual ones.

The other example which has an exact solution, is for a constant scalar potential ($V=\Lambda$)

$$3H^2=\frac{1}{2}\dot{\phi}^2+\Lambda-6\theta a^3\dot{\phi}\Lambda+18\theta^2a^6\Lambda^2,$$
$$\ddot{\phi}+3H\dot{\phi}=36\theta\Lambda\dot{a}a^2. \tag{13}$$

It is not difficult to find solutions to the above system of equations, the behavior of the scale factor a and the scalar field ϕ is given by

$$a(t)\sim\exp(\sqrt{\Lambda/3}t), \tag{14}$$
$$\phi(t)\sim 2\theta\sqrt{3\Lambda}\exp(\sqrt{3\Lambda}t). \tag{15}$$

Unfortunately for this case, the Hubble parameter is a constant ($H=\dot{a}/a=\sqrt{\Lambda/3}$) and the solutions do not give the appropriate dynamics of the universe.

2. Discussion and outlook

In this Letter we have constructed a model of noncommutative scalar field cosmology. We used a modified Poisson algebra among the minisuperspace variables, that is consistent with the assumptions taken in noncommutative quantum cosmology [4–7], and applied them to scalar field cosmology.

The modified equations have the correct commutative limit when the noncommutative parameter vanishes. An intriguing feature is that the corrections only appear up to second order in θ without any approximations, from this observation we can see that even if the noncommutative parameter is large the effective noncommutative equations have rather simple modifications. A simplification arises for the exponential potential, the quadratic term of θ in Eq. (11) can vanish if we take an exponential potential $V(\phi)=V_0e^{-\lambda\phi}$ and choose $\lambda=\sqrt{6}$, the equations are further simplified. Furthermore there would be epochs when the terms in the brackets multiplied by the noncommutative parameter may vanish, giving dynamics similar to the commutative universe, but again this will only be achieved under very particular conditions on the potential.

To study the effects of noncommutative in dark energy, dark matter or inflation, we only need to solve Eqs. (11) and (12) for the particular potential that explains each of the aspects mentioned before. Even if the noncommutative terms look simple, analytical solutions to the equations are difficult to find, but a complete analysis can be done numerically. Unfortunately things are not as simple as that, taking a closer look on the noncommutative corrections, we see that these are weighted by the product θa^3. Being these terms proportional to the volume of the universe the noncommutative corrections would dominate the dynamics at late times. It seems that in order to have some plausible evolution, the minisuperspace noncommutative parameter should be very small, of order of the inverse of the current volume of the universe. Taking this into account the effects of noncommutativity will almost disappear at the early epochs of the universe and would be relevant to the current epoch. This might seem awkward, but scale mixing is a feature that appears in noncommutative field theory, so this might be an effect of the UV/IR mixing.

In conclusion, following [4–7], noncommutative versions of the Friedmann equations were constructed in accordance with symplectic mechanics (see Appendix A) and argued that the only way that these equations could be phenomenological sensible is by using very specific and fined tuned potentials or an extremely small value of θ, rendering noncommutativity irrelevant at very early stages of the universe with its effects appearing at older stages of cosmological evolution. Then in order to believe that minisuperspace noncommutativity with a constant noncommutative parameter is viable phenomenologically we only have one option, that the noncommutative parameter is almost zero. This might be an unattractive result, as one would expect that the effects of noncommutativity be present at early times or scales and disappear as we go to a larger universe, this picture can be realized if the noncommutativity parameter changes in time. Research in this direction is being constructed and will be reported elsewhere.

Acknowledgements

This work was partially supported by CONACYT grants 62253, 135023; CONCYTEG grant 07-16-K662-062 A01 and DAIP 18/10. W. Guzmán is supported by CONACyT postdoctoral grant 147822 and CNPq/CLAF grant 160021/2009-0.

Appendix A

In order to work with the noncommutative deformations of the minisuperspace, we start by analyzing deformations of classical mechanics. In order to find the correct equations of motion once we modify the Poisson algebra, for this we need an appropriate formulation of classical dynamics. For our purposes we will use the symplectic formalism of classical mechanics.

It is well known that a Hamiltonian classical mechanic can be formulated in a $2n$-dimensional differential manifold $\mathcal{M}$ with a *symplectic structure*. This means that a differential 2-form ω which is closed and nondegenerate exists, the pair formed by $(\mathcal{M},\omega)$ is called a *symplectic manifold*. In the Hamiltonian manifold, Hamilton's function $\mathcal{H}$ satisfies

$$i_{X_\mathcal{H}}\omega=-d\mathcal{H} \tag{A.1}$$

where X_H is called Hamiltonian vector field. Specifying local coordinates on $\mathcal{M}$, $x^\mu=\{q^i,p^i\}$, the above condition takes an explicit dependence on the 2-form ω

$$\frac{dx^\mu(t)}{dt}=\omega^{\mu\nu}\frac{\partial\mathcal{H}}{\partial x^\nu}, \tag{A.2}$$

where $\omega^{\mu\nu}$ are the components of ω^{-1} in the local coordinates x^μ.

In the symplectic manifold there is a general expression for the Poisson brackets between two functions in $\mathcal{M}$ based on Hamiltonian fluxes $\{f,g\}=\omega(X_f,X_g)$, which in local coordinates has the familiar form

$$\{f,g\}=\frac{\partial f}{\partial x^\mu}\omega^{\mu\nu}\frac{\partial g}{\partial x^\nu}. \tag{A.3}$$

It is easy to check that the last equation generates the following commutation relations $\{x^\mu,x^\nu\}=\omega^{\mu\nu}$.

If we consider the *canonical symplectic structure* ω_c defined by $\omega_c=dp^i\wedge dq^i$, where $i=1,\dots,n$, we recover the usual Poisson brackets and Eqs. (A.2) are just Hamilton's equations of classical mechanics.

Darboux's theorem states that every symplectic structure can be driven to canonical one by a suitable choice of local coordinates in the neighborhood of any point $x \in \mathcal{M}$. However it is possible to find new effects if we consider a noncanonical symplectic structure, for example a magnetic field can appear considering the appropriate ω (see [8]).

This formalism of classical mechanics gives the mathematical framework to construct the noncommutative deformation of the minisuperspace. Using this formalism we can calculate the deformed Poisson brackets, from which we will determine the corresponding equations of motion and the resulting algebra is consistent with NCQM. Being the deformation constructed in the tangent bundle $T\mathcal{M}$ instead of the symplectic manifold $\mathcal{M}$ all the original symmetries are left intact. This feature is attractive because the classical symmetries used to construct a commutative theory would be present in the deformed theory.

In order to find the effects of noncommutativity on the cosmological equations of motion, we follow the symplectic formalism on the phase space to the FRW cosmology with the scalar field. Let us first consider the following 2-form $\omega_{nc} = \omega_c + \theta\, dp_\alpha \wedge dp_\phi$, evidently if θ is constant, ω_{nc} is closed and invertible, thus ω_{nc} and the cosmological phase space define a symplectic manifold. The components of $\omega_{nc}^{\mu\nu}$ are

$$\omega_{nc}^{\mu\nu} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{A.4}$$

From Eq. (A.3), the Poisson commutation relations are

$$\{\alpha, \phi\} = \theta, \qquad \{p_\alpha, p_\phi\} = 0,$$
$$\{\alpha, p_\alpha\} = 1, \qquad \{\phi, p_\phi\} = 1, \tag{A.5}$$

and because the 2-form ω_{nc} is exact the equations of motions are given by Eq. (A.2) where $x^\mu = \{\alpha, \phi, P_\alpha, P_\phi\}$, from which we easily get $\dot{\alpha} = \{\alpha, \mathcal{H}\}$, $\dot{\phi} = \{\phi, \mathcal{H}\}$, $\dot{P}_\alpha = \{P_\alpha, \mathcal{H}\}$, $\dot{P}_\phi = \{P_\phi, \mathcal{H}\}$.

References

[1] N. Seiberg, E. Witten, JHEP 9909 (1999) 032;
A. Connes, M.R. Douglas, A. Schwarz, JHEP 9809 (1998) 003;
M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977.

[2] H. Garcia-Compean, O. Obregon, C. Ramirez, M. Sabido, Phys. Rev. D 68 (2003) 044015;
H. Garcia-Compean, O. Obregon, C. Ramirez, M. Sabido, Phys. Rev. D 68 (2003) 045010;
P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Class. Quant. Grav. 23 (2006) 1883;
X. Calmet, A. Kobakhidze, Phys. Rev. D 72 (2005) 045010;
L. Alvarez-Gaume, F. Meyer, M.A. Vazquez-Mozo, Nucl. Phys. B 753 (2006) 92;
S. Estrada-Jimenez, H. Garcia-Compean, O. Obregon, C. Ramirez, Phys. Rev. D 78 (2008) 124008.

[3] S. Alexander, R. Brandenberger, J. Magueijo, Phys. Rev. D 67 (2003) 081301(R);
S. Koh, R.H. Brandenberger, JCAP 0706 (2007) 021.

[4] H. Garcia-Compean, O. Obregon, C. Ramirez, Phys. Rev. Lett. 88 (2002) 161301.

[5] G.D. Barbosa, N. Pinto-Neto, Phys. Rev. D 70 (2004) 103512.

[6] L.O. Pimentel, C. Mora, Gen. Rel. Grav. 37 (2005) 817;
L.O. Pimentel, O. Obregon, Gen. Rel. Grav. 38 (2006) 553;
M. Aguero, J.A. Aguilar, S.C. Ortiz, M. Sabido, J. Socorro, Int. J. Theor. Phys. 46 (2007) 2928;
W. Guzman, C. Ortiz, M. Sabido, J. Socorro, M.A. Aguero, Int. J. Mod. Phys. D 16 (2007) 1625;
B. Vakili, N. Khosravi, H.R. Sepangi, Class. Quant. Grav. 24 (2007) 931.

[7] W. Guzman, M. Sabido, J. Socorro, Phys. Rev. D 76 (2007) 087302.

[8] V. Guillemin, S. Sternberg, Symplectic Techniques in Physics, Cambridge Univ. Press, UK, 1990, 468 pp.