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On the Discrepancy of Some Special Sequences

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We obtain estimates for the discrepancy of the sequence $(xs^{(d)}(q;n))_{n=0}^{\infty}$, where $s^{(d)}(q;n)$ denotes the sum of the dth powers of the q-ary digits of the nonnegative integer n and x is an irrational number of finite approximation type. Furthermore metric results for a similar type of sequences are given.

1. INTRODUCTION

A sequence $(x_n)_{n=0}^{\infty}$ of real numbers is said to be uniformly distributed mod 1 (u.d. mod 1) if and only if the number $A(I,N)=\text{card}\{0\leqslant n<N: \{x_n\}\in I\}$ is asymptotically N times the length $|I|$ of I (where I denotes an arbitrary subinterval of $[0,1)$; the fractional part $\{t\}$ is defined by $\{t\}=t-[t]$ and $[t]$ is the greatest integer $\leqslant t$). As a quantitative measure of the distribution behaviour of (x_n) the discrepancy

$$D_N(x_n)=\sup_I\left|\frac{A(I,N)}{N}-|I|\right|$$

can be introduced and it is well known (cf. the monographs [5] and [6]) that (x_n) is u.d. mod 1 if and only if

$$\lim_{N\to\infty} D_N(x_n)=0.$$

In several papers [1, 2, 3] Coquet (*et al.*) investigated the distribution behaviour mod 1 of some sequences of the form $(x\varphi(n))_{n=0}^{\infty}$, where x is an irrational number and $\varphi(n)$ is a number-theoretic function that is additive with respect to a given digit representation (and $\varphi(0)=0$). In the case of the usual q-ary representation

$$n=\sum_{k=0}^{\infty}\varepsilon_k q^k \qquad (q\geqslant 2,\ n\geqslant 0),$$

$$0\leqslant\varepsilon_k=\varepsilon_k(q;n)\leqslant q-1$$

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the most interesting type of an additive function is the sum of dth powers of the digits

$$\varphi(n)=s^{(d)}(q;n)=\sum_{k=0}^{\infty}(\varepsilon_k(q;n))^d \qquad (d, q\geqslant 2 \text{ integral with } d\geqslant 1, q\geqslant 2).$$

(It is obvious that $\varphi(n)=\sum_{k=0}^{\infty}\varphi(\varepsilon_k(q;n)\,q^k)$, i.e., φ is additive.) It follows from [1] that $(xs^{(1)}(q;n))_{n=0}^{\infty}$ is uniformly distributed mod 1 if (and only if) x is an arbitrary real irrational.

For a special type of irrationals we prove a quantitative version of Coquet's result. We consider real numbers x of finite approximation type η, i.e., irrationals x such that for every $\varepsilon>0$,

$$\|hx\|\geqslant\frac{c(x,\varepsilon)}{h^{\eta+\varepsilon}}$$

for all positive integers h; $c(x,\varepsilon)$ is a positive constant only depending on x and ε and $\|t\|$ is defined by $\|t\|=\min(\{t\}, 1-\{t\})$. In Section 2 we prove

THEOREM 1. *Let x be of finite approximation type η. Then for every* $\varepsilon>0$,

$$D_N(xs^{(d)}(q;n))\leqslant\frac{c(q,x,\varepsilon)}{(\log N)^{1/2\eta-\varepsilon}}$$

for all integers $N>1$. If x is not of approximation type η' for any $\eta'<\eta$ then for every $\varepsilon>0$ and infinitely many N

$$D_N(xs^{(d)}(q;n))\geqslant\frac{1}{(\log N)^{1/2\eta+\varepsilon}}.$$

Furthermore for every irrational x and infinitely many N

$$D_N(xs^{(d)}(q;n))\geqslant\frac{c'(q,d,x)}{(\log N)^{1/2}}.$$

In Section 3 we consider sequences $(x_n(t))_{n=0}^{\infty}$ $(0\leqslant t<1)$ of the form $x_n(t)=xs_n^{(d)}(q;t)+a_n$, where x is an irrational number, $(a_n)_{n=0}^{\infty}$ an arbitrary sequence of reals, and (for real $d>0$ and integral $q\geqslant 2$)

$$s_n^{(d)}(q;t)=\sum_{k=1}^{n}(\varepsilon_k(q;t))^d \text{ for } t=\sum_{k=1}^{\infty}\varepsilon_k(q;t)\,q^{-k}$$
$$(0\leqslant\varepsilon_k(q,t)<q).$$

Note that the q-ary representation of t is assumed to be infinite. We prove that $(x_n(t))_{n=0}^{\infty}$ is u.d. mod 1 for almost all real numbers $t \in [0, 1)$ (in the sense of the Lebesgue measure). If x is of finite approximation type we obtain the following more precise result.

THEOREM 2. *Let x be of finite approximation type η and let $(a_n)_{n=0}^{\infty}$ be an arbitrary sequence of real numbers. Then for almost all $t \in [0, 1)$ and every $\varepsilon > 0$,*

$$D_N(xs_n^{(d)}(q; t) + a_n) \leqslant c(t, q, x, \varepsilon)\, N^{-(1/2\eta)+\varepsilon}$$

for all positive integers N.

This result has the following curious consequence: There exists a nondecreasing sequence $(s_n)_{n=0}^{\infty}$ of integers with $0 \leqslant s_n \leqslant n$ such that the sequence $(\sqrt{2}\, s_n + \pi^n)$ is u.d. mod 1.

2. Proof of Theorem 1

We will use the inequality of Erdös and Turan for proving our upper bound for $D_N(xs^{(d)}(q; n))$. Hence we begin to investigate the exponential sums

$$\frac{1}{N} \sum_{n=0}^{N-1} e(hxs^{(d)}(q; n)) \qquad (h = 1, 2, ...)$$

where $e(t) = e^{2\pi i t}$ for real t. By the next Lemma we may restrict ourselves to the case $N = q^k$.

LEMMA 1. *Let $g\colon \mathbb{N}_0 \to \mathbb{C}$ be a function such that $g(0) = 1$, $|g(n)| \leqslant 1$, and*

$$g(n) = \prod_{k=0}^{\infty} g(\varepsilon_k(q; n)\, q^k) \qquad (\textit{for } n \in \mathbb{N}).$$

Assume that

$$\left| \frac{1}{q^k} \sum_{n=0}^{q^k - 1} g(n) \right| \leqslant \frac{1}{f(q^k)} \qquad \textit{for } k = 1, 2, ...,$$

where $f\colon [1, \infty) \to (0, \infty)$ is continuous nondecreasing, and $f(u) \leqslant u$. Then we have

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} g(n) \right| \leqslant \frac{q+1}{f(\sqrt{N})} \qquad \textit{for } N = 1, 2, ...$$

Proof of Lemma 1. Let m be the largest index such that $\varepsilon_m = \varepsilon_m(q; N) \neq 0$, and define

$$N(j) = \sum_{k=j}^{m} \varepsilon_k(q; N)\, q^k.$$

Then

$$\sum_{n=0}^{N-1} g(n) = \sum_{n=0}^{N(m)-1} g(n) + \sum_{j=0}^{m-1} \sum_{n=N(j+1)}^{N(j)-1} g(n),$$

where

$$\sum_{n=0}^{N(m)-1} g(n) = \sum_{l=0}^{\varepsilon_m - 1} \sum_{n=lq^m}^{(l+1)q^m - 1} g(n) = \sum_{l=0}^{\varepsilon_m - 1} g(lq^m) \sum_{n=0}^{q^m - 1} g(n)$$

and

$$\sum_{n=N(j+1)}^{N(j)-1} g(n) = g(N(j+1)) \sum_{n=0}^{\varepsilon_j q^j - 1} g(n)$$

$$= g(N(j+1)) \sum_{l=0}^{\varepsilon_j - 1} g(lq^j) \sum_{n=0}^{q^j - 1} g(n).$$

Hence

$$\left|\sum_{n=0}^{N-1} g(n)\right| \leqslant \sum_{j=0}^{m} \left|\sum_{l=0}^{\varepsilon_j - 1} g(lq^j)\right| \left|\sum_{n=0}^{q^j - 1} g(n)\right|$$

$$\leqslant \sum_{j=0}^{m} \varepsilon_j q^j \frac{1}{q^j} \left|\sum_{n=0}^{q^j - 1} g(n)\right|$$

$$\leqslant \sum_{j=0}^{r-1} \varepsilon_j q^j + \sum_{j=r}^{m} \varepsilon_j q^j \frac{1}{f(q^r)} \leqslant q^r + \frac{N}{f(q^r)}$$

for arbitrary $r \in \mathbb{N}$. Let t be the unique real number such that $(t/q)\, f(t/q) = N$; then $t/q \geqslant \sqrt{N}$ because of $f(t/q) \leqslant t/q$.

Choosing r such that $q^{r+1} > t \geqslant q^r$ we obtain

$$\left|\sum_{n=0}^{N-1} g(n)\right| \leqslant q^r + \frac{N}{f(q^r)} \leqslant t + \frac{N}{f(t/q)} = \frac{qN}{f(t/q)} + \frac{N}{f(t/q)}$$

$$\leqslant (q+1) \frac{N}{f(\sqrt{N})},$$

thus proving the lemma.

We want to apply the lemma to $g(n)=e(hxs^{(d)}(q;n))$. In order to verify the assumptions on $g(n)$ in Lemma 1 it remains to prove an inequality of the form

$$\left|\sum_{n=0}^{q^k-1} g(n)\right| \leqslant \frac{q^k}{f(q^k)} \qquad (k=1,2,\ldots).$$

We have

$$\sum_{n=0}^{q^{k+1}-1} g(n)=\sum_{j=0}^{q-1}\sum_{n=0}^{q^k-1} g(jq^k+n)=\sum_{j=0}^{q-1} g(j)\cdot\sum_{n=0}^{q^k-1} g(n)$$

and so

$$\sum_{n=0}^{q^k-1} g(n)=\left(\sum_{j=0}^{q-1} g(j)\right)^k=\left(\sum_{j=0}^{q-1} e(hxj^d)\right)^k. \qquad (*)$$

For estimating $|\sum_{n=0}^{q^k-1} g(n)|$ we will apply the following simple

LEMMA 2. *For reals α and $d>0$ and integral $q\geqslant 2$ we have*

$$\left|\sum_{j=0}^{q-1} e(\alpha j^d)\right| \leqslant q-2\pi\,\|\alpha\|^2.$$

Proof of Lemma 2. A simple argument shows

$$\left|\sum_{j=0}^{q-1} e(\alpha j^d)\right| \leqslant q-2+2\,|\cos\pi\alpha| \qquad \text{for } q\geqslant 2.$$

Next we observe that $|\cos\pi\alpha|=\cos\pi\|\alpha\|$. Hence the result of the lemma immediately follows from the inequality $\cos x\leqslant 1-(x^2/\pi)$ for $|x|\leqslant\pi/2$ which is valid since

$$\cos x=1-\int_0^x \sin t\,dt\leqslant 1-\int_0^x \frac{2}{\pi}\,t\,dt=1-\frac{x^2}{\pi} \qquad \left(|x|\leqslant\frac{\pi}{2}\right).$$

Thus we obtain

$$\left|\sum_{n=0}^{q^k-1} g(n)\right| \leqslant (q-2\pi\,\|hx\|^2)^k\leqslant (q-4\,\|hx\|^2)^k.$$

Now we choose the function f in Lemma 1 such that

$$f(q^k)=\left(\frac{q}{q-4\,\|hx\|^2}\right)^k,$$

i.e.,

$$f(u)=\left(\frac{q}{q-4\,\|hx\|^2}\right)^{\log u/\log q}$$

Since $q-4\,\|hx\|^2\geqslant 1$ (for $q\geqslant 2$; for $q\geqslant 3$ we could work with the bound $q-2\pi\,\|hx\|^2$), we have $f(u)\leqslant q^{\log u/\log q}=u$ for $u\geqslant 1$; obviously f is non-decreasing. Hence Lemma 1 and the well-known inequality $(1-(1/u))^u\leqslant 1/e$ give

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=0}^{N-1} e(hxs^{(d)}(q;n))\right|\\
&\quad\leqslant\frac{q+1}{f(\sqrt{N})}\\
&\quad=(q+1)\left(1-\frac{4\,\|hx\|^2}{q}\right)^{(\log N)/(2\log q)}\\
&\quad=(q+1)\left(1-\frac{1}{q/4\,\|hx\|^2}\right)^{(q/4\,\|hx\|^2)\cdot(\log N)/(2q\log q/4\,\|hx\|^2)}\\
&\quad\leqslant(q+1)\exp\left(-\frac{2\,\|hx\|^2\log N}{q\log q}\right)\qquad(\exp t=e^t).
\end{aligned}$$

Now we apply the inequality of Erdös and Turan (cf. [6, p. 112]) and make use of $\|hx\|\geqslant c_0(x,\varepsilon)\,h^{-\eta-\varepsilon}$ $(\varepsilon>0)$,

$$\begin{aligned}
&D_N(xs^{(d)}(q;n))\\
&\quad\leqslant 6\left(\frac{1}{H}+\sum_{h=1}^{H}\frac{q+1}{h}\exp(-2c_0(x,\varepsilon)^2\,h^{-2\eta-2\varepsilon}\log N/(q\log q))\right)\\
&\quad\leqslant 6\left(\frac{1}{H}+(1+\log H)(q+1)\exp(-c_1(q,x,\varepsilon)\,H^{-2\eta-2\varepsilon}\log N)\right).
\end{aligned}$$

We put $H=[(\log N)^{(1/2\eta)-\varepsilon}]$ for $0<\varepsilon<1/4\eta$ and sufficiently large N and obtain (with positive constants c_1, c_2, c_3, c_4 depending on q, x, and ε)

$$\begin{aligned}
&(1+\log H)\exp(-c_1H^{-2\eta-2\varepsilon}\log N)\\
&\quad\leqslant\left(\frac{1}{2\eta}-\varepsilon\right)\log\log N\exp(-c_2(\log N)^{-(\varepsilon/\eta)+2\varepsilon\eta+2\varepsilon^2})\\
&\quad\leqslant\frac{1}{2\eta}\log\log N\exp(-c_2(\log N)^{-\varepsilon+2\varepsilon+2\varepsilon^2})
\end{aligned}$$

$$\leqslant c_3(\log N)^{\varepsilon}\exp(-c_2(\log N)^{\varepsilon})$$

$$\leqslant \frac{c_3}{H}\exp\left(\frac{1}{2\eta}\log\log N - c_2(\log N)^{\varepsilon}\right)$$

$$\leqslant \frac{c_4}{H};$$

thus $D_N(xs^{(d)}(q;n)) \leqslant c_5/H \leqslant c(q,x,\varepsilon)/(\log N)^{1/2\eta-\varepsilon}$.

In order to prove the lower bounds we note that

$$\|hx\| \leqslant c/h^{\kappa} \text{ for infinitely many } h \in \mathbb{N} \qquad (**)$$

with $\kappa = \eta - \varepsilon$ ($\eta > \varepsilon > 0$ arbitrary) if x is not of approximation type η' (for any $\eta' < \eta$) and with $\kappa = 1$ for arbitrary x (by Dirichlet's approximation theorem). Applying Koksma's inequality (cf. [6, Theorem 5.1, p. 143 and Example 5.1, p. 144]) to the function $t \mapsto \exp(2\pi iht)$ (of variation $2\pi h$), we obtain for all positive integers h, N,

$$\left|\frac{1}{N}\sum_{0\leqslant n<N}\exp(2\pi ihxs^{(d)}(q;n))\right| \leqslant 2\pi h D_N(xs^{(d)}(q;n)).$$

By $(*)$ we have for $N = q^k$

$$\left|\frac{1}{N}\sum_{0\leqslant n<N}\exp(2\pi ihxs^{(d)}(q;n))\right|$$

$$= \left|\frac{1}{q}\sum_{0\leqslant n<q}\exp(2\pi ihxn^d)\right|^k$$

$$\geqslant \left|\frac{1}{q}\sum_{0\leqslant n<q}\cos(2\pi hxn^d)\right|^k = \left|\frac{1}{q}\sum_{0\leqslant n<q}\cos(2\pi n^d\|hx\|)\right|^k.$$

Since, by $(**)$, $\|hx\| \leqslant ch^{-\kappa} \leqslant 1/(\pi\sqrt{2}(q-1)^d)$ for infinitely many h, the inequality $\cos t \geqslant 1 - t^2/2$ yields

$$\left|\frac{1}{q}\sum_{0\leqslant n<q}\cos(2\pi n^d\|hx\|)\right|^k$$

$$= \left(\frac{1}{q}\sum_{0\leqslant n<q}\cos(2\pi n^d\|hx\|)\right)^k$$

$$\geqslant \cos(2\pi(q-1)^d\|hx\|)^k$$

$$\geqslant (1 - 2\pi^2(q-1)^{2d}c^2h^{-2\kappa})^k,$$

hence

$$D_N(xs^{(d)}(q;n)) \geqslant \frac{1}{2\pi h}\,(1-2\pi^2(q-1)^{2d}\,c^2h^{-2\kappa})^k, \qquad N=2^k$$

for infinitely many h. Choosing $k=[h^{2\kappa}]$ yields

$$D_N(xs^{(d)}(q;n)) \geqslant \frac{1}{4\pi}\left(\frac{\log q}{\log N}\right)^{1/2\kappa} \exp(-2\pi^2(q-1)^{2d}\,c^2), \qquad N=q^{[h^{2\kappa}]}$$

for infinitely many h, thus establishing both lower bounds in Theorem 1.

Remark 1. Lemma 1 is a quantitative refinement of [2, Lemme 1].

Remark 2. By the theorem of Thue-Siegel-Roth every irrational real algebraic number x is of approximation type $\eta=1$; in this case

$$D_N(xs^{(d)}(q;n)) \leqslant \frac{c(q,x,\varepsilon)}{\log N^{1/2-\varepsilon}}$$

and $\frac{1}{2}$ cannot be replaced by a larger exponent.

3. Proof of Theorem 2

Our main tool will be the method of Gál and Koksma [4]. For this purpose we have to establish an upper bound for

$$\int_0^1 (ND_N(x_{n+M}(t)))^2\,dt \qquad (x_n(t)=xs_n^{(d)}(q;t)+a_n)$$

uniformly in $M=0, 1, 2,\ldots$. We need the following lemma.

Lemma 3. *For $m>n$, real α and $d>0$, we have*

$$\int_0^1 e(\alpha(s_m^{(d)}(q;t)-s_n^{(d)}(q;t)))\,dt=\left(\frac{1}{q}\sum_{j=0}^{q-1} e(\alpha j^d)\right)^{m-n}.$$

Proof of Lemma 3. We observe that for $j\leqslant m$ the function $\varepsilon_j(q;\cdot)$ is constant on every open interval $I_k=(k/q^m, (k+1)/q^m)$ $(0\leqslant k<q^m)$. Hence for $t\in I_k$, $k=\sum_{i=0}^{m-1} c_i(k)\,q^i$, we obtain

$$\varepsilon_j(q;t)=\varepsilon_j(q;(k+(1/q))q^{-m})=\varepsilon_j(q;(qk+1)q^{-m-1})=c_{m-j}(k).$$

Thus

$$\int_0^1 e(\alpha(s_m^{(d)}(q;t)-s_n^{(d)}(q;t)))\,dt$$

$$=\sum_{k=0}^{q^m-1}\int_{I_k} e\left(\alpha\sum_{j=n+1}^{m}\varepsilon_j(q;t)^d\right)dt$$

$$=\sum_{k=0}^{q^m-1}\frac{1}{q^m}\,e\left(\alpha\sum_{j=n+1}^{m}c_{m-j}(k)^d\right)$$

$$=\frac{q^n}{q^m}\sum_{0\leqslant c_0,\ldots,c_{m-n-1}<q} e(\alpha c_0^d)\cdots e(\alpha c_{m-n-1}^d)$$

$$=q^{n-m}\sum_{c_0=0}^{q-1}e(\alpha c_0^d)\cdots\sum_{c_{m-n-1}=0}^{q-1}e(\alpha c_{m-n-1}^d)=\left(\frac{1}{q}\sum_{j=0}^{q-1}e(\alpha j^d)\right)^{m-n}.$$

From the inequality of Erdös and Turan we derive (applying the inequality of Cauchy–Schwarz),

$$\int_0^1 N^2 D_N(x_{n+M}(t))^2\,dt$$

$$\leqslant 36\int_0^1\left(\frac{N}{H}+\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{n=1}^{N}e(hx_{n+M}(t))\right|\right)^2 dt$$

$$\leqslant 72\left(\frac{N^2}{H^2}+\sum_{1\leqslant h,k\leqslant H}\frac{1}{hk}\int_0^1\left|\sum_{n=1}^{N}e(hx_{n+M}(t))\right|\left|\sum_{n=1}^{N}e(kx_{n+M}(t))\right|dt\right)$$

$$\leqslant 72\left(\frac{N^2}{H^2}+\left(\sum_{h=1}^{H}\frac{1}{h}\sqrt{\int_0^1\left|\sum_{n=1}^{N}e(hx_{n+M}(t))\right|^2 dt}\right)^2\right).$$

The inner integral is equal to

$$\int_0^1\sum_{1\leqslant m,n\leqslant N}e(h(x_{m+M}(t)-x_{n+M}(t)))\,dt$$

$$=N+2\,\mathrm{Re}\left(\sum_{1\leqslant n<m\leqslant N}e(h(a_{m+M}-a_{n+M}))\right.$$

$$\left.\times\int_0^1 e(hx(s_{m+M}^{(d)}(q;t)-s_{n+M}^{(d)}(q;t)))\,dt\right)$$

$$\leqslant N+2\sum_{1\leqslant n<m\leqslant N}\left|\frac{1}{q}\sum_{j=0}^{q-1}e(hxj^d)\right|^{m-n},$$

where the last step follows from Lemma 3. Applying Lemma 2 we obtain

$$\sum_{1 \leqslant n < m \leqslant N} \left| \frac{1}{q} \sum_{j=0}^{q-1} e(hxj^d) \right|^{m-n} \leqslant \sum_{m=2}^{N} \sum_{n=1}^{m} \left(1 - \frac{2\pi \ \|hx\|^2}{q}\right)^{m-n}$$

$$\leqslant \sum_{m=2}^{N} \frac{q}{2\pi \ \|hx\|^2} \leqslant \frac{Nq}{2\pi \ \|hx\|^2}.$$

Hence we derive

$$\int_0^1 N^2 D_N(x_{n+M}(t))^2 \, dt \leqslant 72 \left(\frac{N^2}{H^2} + \left(\sum_{h=1}^{H} \frac{1}{h} \sqrt{N + (Nq/\pi \ \|hx\|^2)} \right)^2 \right)$$

$$\leqslant 72 \left(\frac{N^2}{H^2} + N \left(\sum_{h=1}^{H} \frac{1}{h} (1 + \sqrt{q/\pi \ \|hx\|^2}) \right)^2 \right)$$

$$= 72 \left(\frac{N^2}{H^2} + N \left(\sum_{h=1}^{H} \frac{1}{h} + \sqrt{q/\pi} \sum_{h=1}^{H} \frac{1}{h \ \|hx\|} \right)^2 \right).$$

Since x is of finite approximation type η, we have for every $\varepsilon > 0$

$$\sum_{h=1}^{H} \frac{1}{h \ \|hx\|} \leqslant c_0(x, \varepsilon) \, H^{\eta - 1 + \varepsilon/2} \qquad \text{(cf. [6, p. 123])},$$

and so we obtain with $H = [N^{1/2\eta}]$ for every $\varepsilon > 0$,

$$\int_0^1 N^2 D_N(x_{n+M}(t))^2 \, dt \leqslant c_1(q, x, \varepsilon) \left(\frac{N^2}{H^2} + N(\log H + H^{\eta - 1 + \varepsilon/2})^2 \right)$$

$$\leqslant c_2(q, x, \varepsilon) \left(\frac{N^2}{N^{1/\eta}} + N \cdot N^{\frac{\eta - 1}{\eta} + \varepsilon} \right)$$

$$\leqslant c_3(q, x, \varepsilon) \, N^{\frac{2\eta - 1}{\eta} + \varepsilon}.$$

By the following lemma this yields

$$N^2 D_N(x_n(t))^2 = O(N^{\frac{2\eta - 1}{\eta} + \varepsilon})$$

for almost all t and this completes the proof of Theorem 2.

LEMMA 4 (Special case of [4, Théorème 3]). *Put* $F(M, N; t) = N D_N(x_{n+M}(t))$. *If*

$$\int_0^1 F(M, N; t)^2 \, dt = O(\psi(N)) \qquad \textit{uniformly in } M = 0, 1, 2, \ldots$$

and $\psi(N)/N$ is nondecreasing then for almost all t and every $\varepsilon > 0$ there exists a positive constant $c(t, \varepsilon)$ such that

$$F(0, N; t) \leqslant c(t, \varepsilon) \sqrt{\psi(N)} (\log N)^{3/2 + \varepsilon}.$$

Remark 3. In the proof of Theorem 2 we have shown that

$$\int_0^1 \left| \sum_{n=1}^{N} e(hx_n(t)) \right|^2 dt \leqslant N \left(1 + \frac{q}{\pi \|hx\|^2} \right) \qquad (h = 1, 2, \ldots)$$

for every irrational number x. By the theorem of Davenport, Erdös and Le Veque (cf. [6, p. 33, Theorem 4.2]) we conclude that $(x_n(t))_{n=0}^{\infty}$ is u.d. mod 1 for almost all $t \in [0, 1)$.

Remark 4. Let x be an irrational number and let $(a_n)_{n=0}^{\infty}$ be an arbitrary sequence of reals. Then there exists a nondecreasing sequence (s_n) of integers with $0 \leqslant s_n \leqslant n$ such that $(xs_n + a_n)_{n=0}^{\infty}$ is u.d. mod 1. (This follows immediately from the previous remark for $d = 1$ and $q = 2$, since $s_n = s_n^{(1)}(2; t)$ is nondecreasing and $s_n^{(1)}(2; t) \leqslant n$.)

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