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M-mappings and the cellularity of spaces

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Abstract

It is shown that an image X of a product of Lindelöf Σ-spaces or of a countably compact (even countably protocompact) space Π under a continuous M-mapping is $\aleph_0$-cellular in the sense of A.V. Arhangel'skiĭ. The same holds if both Π and X have "good" lattices of continuous mappings. Some generalizations of earlier results of V.V. Uspenskiĭ and M.G. Tkačenko on continuous images of (dense subspaces of) topological groups are also given.

Key words: Lindelöf Σ-space; Countably protocompact; Pseudocompact; Separately continuous function; M-mapping; Strong and weak σ-lattices; Factorizative lattice; $\aleph_0$-cellular space

AMS (MOS) Subj. Class: 54C05, 54B10, 54A25, 54C10, 54H11

1. Introduction

The well-investigated class of dyadic compact spaces has the natural extension considered in [4], where some interesting traits of compact continuous images of σ-compact topological groups are mentioned. Obviously, all dyadic compact spaces fall under consideration in [4], because the generalized Cantor cube D^τ carries the natural structure of a compact topological group. This line of investigation was continued by considering compact continuous images of (dense subsets of) Lindelöf Σ-groups [30] and products of Lindelöf Σ-groups [27]. Later on, Pasynkov [18] suggested to "separate" the structures of a product space and of a topological group. So the basic situations we begin with are the following ones:

(a) Π is a product of Lindelöf Σ-spaces and $f:\Pi\to X$ is a continuous mapping of Π onto a topological group X;

(b) in addition, $g:S\to Y$ is a continuous mapping of a dense subset S of X onto a compact space Y.

In what follows the most of our results will treat essentially more general cases;

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however, one should keep in mind the original nature of spaces Π and X, the product space and a topological group respectively.

The notion of $\aleph_0$-cellularity plays a central rôle in this paper. Following Arhangel'skiĭ [4], we call a space X *$\aleph_0$-cellular* if every family γ of G_δ-sets in X contains a countable subfamily $\mu \subseteq \gamma$ such that $\mathrm{cl}(\bigcup \mu) = \mathrm{cl}(\bigcup \gamma)$. As far as the author knows, it was Efimov [8] who considered this property first (without introducing the term) and who proved that D^τ is $\aleph_0$-cellular for each τ. This easily implies that every dyadic compact space is $\aleph_0$-cellular. Ščepin [19] defined k-metrizable spaces and showed that every k-metrizable compact space is $\aleph_0$-cellular.

By the Ivanovskiĭ–Kuz'minov theorem, a compact topological group is dyadic, and so is $\aleph_0$-cellular. On the other hand, every σ-compact group has the Souslin property [25]. Both results follow from the theorem of Uspenskiĭ [29]: every σ-compact (Lindelöf Σ-) group is $\aleph_0$-cellular. Moreover, a product of arbitrarily many σ-compact (Lindelöf Σ-) groups has the same property [27]. This and some other results are generalized here in "algebraic" and topological directions (see Sections 2 and 3). The majority of our generalizations start from the point (a) above.

In Section 4 we proceed with the consideration of (b), which is inspired by the theory of dyadic compact spaces. By the theorem of Esenin-Vol'pin [11], the weight and character of a dyadic compact space Y coincide. For the same Y, the equality $w(Y) = t(Y)$ was found out by Arhangel'skiĭ and Ponomarev [5]. These results were improved by Efimov, Gerlitz and Hagler [9,12,14]: if the cofinality of weight of a dyadic compact space Y is uncountable, then there exists a continuous mapping of Y onto the Tychonoff cube $I^{w(Y)}$. A similar result was proved in [27] for compact spaces Y which are continuous images of (a dense subspace of) a product of Lindelöf Σ-groups. Moreover, the equality $\chi(y, Y) = t(y, Y)$ holds at each point $y \in Y$ in this case (to be published in [28]). Recently Pasynkov [18] extended some results of [27,30] to the spaces as in (a) and (b).

Our aim is to show that the main cause of the above equalities is the existence of well-structured families of continuous mappings the spaces Π and X as in (a), (b) possess. The family $\mathscr{L}_\Pi$ of all projections of the product space Π onto countable subproducts is a strong σ-lattice for Π (see Definition 3.1) consisting of open mappings onto Lindelöf Σ-spaces. As for X, the family $\mathscr{L}_X$ of all quotient mappings of X onto left coset spaces X/N, where N runs through all closed uniform G_δ-subgroups of X, is a weak σ-lattice for X (Definition 3.1) consisting of open mappings onto submetrizable spaces. Note that under conditions of (a), every submetrizable space X/N has countable network. Indeed, it has a G_δ-diagonal, and by a theorem of Engelking [10], every continuous mapping of Π to a space with G_δ-diagonal depends on at most countably many coordinates. So X/N is a continuous image of a countable subproduct Π_B of Π; therefore X/N is a Lindelöf Σ-space with G_δ-diagonal, which in turn implies $\mathrm{nw}(X/N) \leqslant \aleph_0$.

In what follows we substitute the product and group structures of Π and X by lattices of open (d-open, quotient, etc.) mappings of these spaces onto "good" spaces (Lindelöf Σ-spaces, spaces with G_δ-diagonal, spaces with countable network). The properties of the lattices we require are somewhat weaker than those of the lattices $\mathscr{L}_\Pi$, $\mathscr{L}_X$ above.

All spaces are assumed to be completely regular. The symbols $w(X)$, $\text{nw}(X)$, $c(X)$, $\chi(X)$, $t(X)$ denote the weight, network weight, cellularity, character and tightness of X respectively. The character and π-character of X at a point x are denoted by $\chi(x, X)$ and $\pi(x, X)$. If X admits no continuous mapping onto the Tychonoff cube I^{ω_1}, we write $\text{id}(X) \leqslant \aleph_0$ and say that X has *countable index*.

2. *M*-mappings and $\aleph_0$-cellularity

A continuous mapping $F: X^3 \to X$ is called a Mal'tsev operation on the space X [31] if $F(x, y, y) = F(y, y, x) = x$ for all $x, y \in X$. Every topological group and every retract of a topological group admits a continuous Mal'tsev operation [17,31], i.e., is a Mal'tsev space. The following definition extends the notion of a Mal'tsev operation to a mapping between two distinct spaces.

Definition 2.1. We call $f: X \to Y$ an *M-mapping* if there exists a continuous mapping $F: X^3 \to Y$ such that $F(x, y, y) = F(y, y, x) = f(x)$ for all $x, y \in X$.

Note that if X or Y admits a Mal'tsev operation, then every continuous mapping of X to Y is an M-mapping. Moreover, $f: X \to Y$ is an M-mapping if there exist a Mal'tsev space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ with $f = hg$. Obviously, X is a Mal'tsev space iff the identity mapping id_X is an M-mapping.

It is known [29, Theorem 6] that a Lindelöf Σ-space X with a Mal'tsev operation is $\aleph_0$-cellular. Countably compact spaces also have this property [30]. We generalize these results in two directions simultaneously: to products of Lindelöf Σ-spaces and to M-mappings.

Theorem 2.2. *Let X be an image of a product Π of Lindelöf Σ-spaces under a continuous M-mapping. Then X is $\aleph_0$-cellular.*

The conclusion of Theorem 2.2 remains valid if one requires a strong σ-lattice of open retractions of Π onto its Lindelöf Σ-subspaces (see Theorem 5.12) instead of a product structure on the space Π.

A space Π is said to be *countably protocompact* [4] if Π contains a dense subset S such that every infinite subset A of S has a cluster point in Π. Every countably compact space is countably protocompact, but not vice versa (the Moore–Mrówka space is a counterexample).

Theorem 2.3. *If X is an image of a countably protocompact space under an M-mapping, then X is $\aleph_0$-cellular.*

One easily sees that a countably protocompact space is pseudocompact, so we have the following.

Problem 2.4. Must an image of a pseudocompact space under an M-mapping be $\aleph_0$-cellular?

A similar problem for pseudocompact Mal'tsev spaces (see [31]) is still unsolved.

Every compact Mal'tsev space is a Dugundji space, and hence dyadic [31]. There are, however, dyadic spaces which are not Dugundji, and since every dyadic space is an image of a compact topological group $D^\tau = (\mathbb{Z}_2)^\tau$ under a continuous M-mapping, we see that the image of a compact dyadic space under an M-mapping need not be Dugundji. This leads to the following problem.

Problem 2.5. If X is an image of a compact space under a continuous M-mapping, must X be dyadic? Must the weight and tightness (character) of X coincide?

3. Lattices of continuous mappings and $\aleph_0$-cellularity

In Section 2 we have collected the results on $\aleph_0$-cellularity of algebraic nature. In many instances, however, $\aleph_0$-cellularity of $X = f(\Pi)$ arises as a result of the closed interaction between the lattices $\mathscr{M}(X)$ and $\mathscr{M}(\Pi)$ of continuous mappings of the spaces X and Π. An extremely simple example of this kind is the case when $\Pi = X$ and X has a σ-lattice of open mappings onto spaces with countable network. For reader's convenience we give the necessary definitions here.

We write $g \prec f$ for $f, g \in \mathscr{M}(X)$ if there exists a continuous mapping $h: g(X) \to f(X)$ such that $f = hg$.

Definition 3.1. A subfamily $\mathscr{L}$ of $\mathscr{M}(X)$ is said to be a *σ-lattice* (a *strong σ-lattice*) for X if the following conditions hold:

(1) $\mathscr{L}$ generates the topology of X;

(2) every finite subfamily of $\mathscr{L}$ has a lower bound in $\mathscr{L}$;

(3) for any decreasing sequence $p_0 \succ p_1 \succ p_2 \succ \cdots$ in $\mathscr{L}$, the diagonal product $p = \Delta_{i=0}^{\infty} p_i$ belongs to $\mathscr{L}$ (and if a sequence $\{x_j: j \in N\} \subseteq X$ has the property $p_i(x_j) = p_i(x_i)$ whenever $i < j$, then $\emptyset \neq \bigcap_{i=0}^{\infty} p_i^{-1} p_i(x_i)$).

The definition of a *weak σ-lattice* for X comes if one replaces (3) by

(3′) for any decreasing sequence $p_0 \succ p_1 \succ p_2 \succ \cdots$ in $\mathscr{L}$, there exist $\bar{p} \in \mathscr{L}$ and a one-to-one continuous mapping ϕ of $\bar{p}(X)$ to $p(X)$ such that $p = \phi \bar{p}$ where $p = \Delta_{i=0}^{\infty} p_i$. We denote $\bar{p} = w \lim_{i=0}^{\infty} p_i$ in this case.

Obviously, strong σ-lattice $\Rightarrow$ σ-lattice $\Rightarrow$ weak σ-lattice. Note that (2) and (3′) together imply that every countable subfamily of a weak σ-lattice $\mathscr{L}$ has a lower bound in $\mathscr{L}$, i.e., is $\aleph_0$-directed by the partial ordering $\prec$. The part of (3) in brackets means that $p(X)$ is homeomorphic to the limit space of the spectrum $\{p_i(X), p_{j,i}: i < j\}$ where $p_{j,i} = p_i p_j^{-1}$.

Now let X be a space with a σ-lattice $\mathscr{L}$ of continuous open mappings onto spaces with countable network. To show that X is $\aleph_0$-cellular, consider an arbitrary family $\mathscr{F}$ of G_δ-sets in X. Since $\mathscr{L}$ is $\aleph_0$-directed by $\prec$, we can assume that each element $F \in \mathscr{F}$ is of the form $F = p_F^{-1} p_F(F)$ for some $p_F \in \mathscr{L}$. One easily defines mappings $p_0 \succ p_1 \succ \cdots$ of $\mathscr{L}$ and countable subfamilies $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots$ of $\mathscr{L}$ that satisfy for each $n \in N$ the conditions:

(i) $p_n(\bigcup \mathscr{F}_n)$ is dense in $p_n(\bigcup \mathscr{F})$;

(ii) $p_{n+1} \prec p_F$, and so $F = p_{n+1}^{-1} p_{n+1}(F)$ for each $F \in \mathscr{F}_n$.

Put $\mathscr{F}^* = \bigcup_{n=0}^{\infty} \mathscr{F}_n$ and $p = \Delta_{n=0}^{\infty} p_n$, $p \in \mathscr{L}$. Then $\mathscr{U} = p(\bigcup \mathscr{F}^*)$ is dense in $p(\bigcup \mathscr{F})$, and $F = p^{-1}p(F)$ for all $F \in \mathscr{F}^*$. Since p is an open mapping, we have $\text{cl} \bigcup \mathscr{F}^* = p^{-1}(\text{cl } \mathscr{U})$, which in turn implies $\bigcup \mathscr{F} \subseteq p^{-1}p(\bigcup \mathscr{F}) \subseteq \text{cl} \bigcup \mathscr{F}^*$.

One of the most simple examples of spaces with a strong σ-lattice of open mappings is a product space $\Pi = \prod_{\alpha \in A} X_\alpha$ whose σ-lattice $\mathscr{L}_\Pi$ consists of all projections p_B of Π onto countable subproducts $\Pi_B = \prod_{\alpha \in B} X_\alpha$. The restrictions $p_B | \tilde{\Pi}$ of projections p_B to a subspace $\tilde{\Pi}$ of Π constitute a σ-lattice of continuous mappings for $\tilde{\Pi}$. If $\tilde{\Pi}$ is dense in Π, then this σ-lattice consists of d-open mappings [23].

Another basic example is a k-metrizable compact space, which has, by [19], a strong σ-lattice of open mappings onto second-countable spaces. Therefore, every k-metrizable compact space is $\aleph_0$-cellular (see [19]).

The $\aleph_0$-cellularity of Lindelöf Σ-groups arises from a different reason. It was mentioned in the introduction that a Lindelöf Σ-group G has a weak σ-lattice of open mappings onto spaces with countable network. However, the existence of this weak σ-lattice itself does not imply $\aleph_0$-cellularity: the presence of a continuous algebraic operation on G was used in [29] to conclude that G is $\aleph_0$-cellular. Nevertheless, the lattice approach works in this case: every Lindelöf Σ-space is a continuous image of a Lindelöf p-space Π and the latter has a strong σ-lattice of perfect mappings onto second-countable spaces. The existence of such lattices for Π and G implies $\aleph_0$-cellularity of G as we will see below (Theorems 3.3 and 3.4).

For generality, we use the following definition.

Definition 3.2. We call X an *OD-space* (*D-space*) if X has a weak σ-lattice of open (d-open) mappings onto spaces with G_δ-diagonal.

Every Hausdorff topological group (more generally, a Hausdorff paratopological group, i.e., an algebraic group with continuous multiplication) is an OD-space; a dense subspace of an OD-space is a D-space. In the following theorems, the union of an arbitrary family of G_δ-sets in X is called a $G_{\delta,\Sigma}$-set.

Theorem 3.3. *Let an OD-space X be a continuous image of a space Π with a strong σ-lattice of open retractions onto Lindelöf Σ-subspaces. Then X is $\aleph_0$-cellular and the closure of any $G_{\delta,\Sigma}$-set in X is a G_δ-set.*

A product of Lindelöf Σ-spaces has a strong σ-lattice of open retractions onto Lindelöf Σ-subspaces; more generally, every subspace of this product that contains a Σ-product $\Sigma(p)$ with a base point p has the requisite σ-lattice.

Theorem 3.4. *Let an OD-space X with the corresponding weak σ-lattice $\mathscr{L}_X$ be a continuous image of a space Π with a factorizative strong σ-lattice $\mathscr{L}_\Pi$ of quotient mappings onto Lindelöf Σ-spaces. If* $\mathrm{nw}\,\phi(X) \leqslant \aleph_0$ *for each* $\phi \in \mathscr{L}_X$, *then X is $\aleph_0$-cellular and the closure of a $G_{\delta,\Sigma}$-set in X is a G_δ-set.*

The term *factorizative lattice* applied to a given lattice $\mathscr{L}_\Pi$ means that for every continuous real-valued function h on Π there exists $p \in \mathscr{L}_\Pi$ with $p \prec h$. Clearly, for any continuous mapping ϕ of Π to a second-countable space one can find $p \in \mathscr{L}_\Pi$ such that $p \prec \phi$ (provided that the factorizative lattice $\mathscr{L}_\Pi$ is $\aleph_0$-directed).

Note that Corollary 5.7 reduces Theorem 3.3 to Theorem 3.4. Since every paratopological group is an OD-space, Theorem 3.3 implies the following.

Corollary 3.5. *If a paratopological group H is a continuous image of a product of Lindelöf Σ-spaces, then H is $\aleph_0$-cellular* (*even τ-cellular for each cardinal $\tau \geqslant \aleph_0$*).

This result implies Theorem 2 of [29] and the first part of Corollary 1.8 of [27].

4. Regular mappings onto compact spaces

The first result concerning compact continuous images of dense subsets of D^τ was proved in [8]: if there exists a continuous mapping of a Σ-product $\Sigma(p) \subseteq D^\tau$ onto a compact space Y, then Y is metrizable. The same holds if Y is a continuous image of an arbitrary dense subset $S \subseteq \Sigma(p)$ (see [24]). Answering a question of Arhangel'skiĭ, Shirokov [21] proved the equality $w(Y) = t(Y)$ for every compact continuous image Y of a dense subset of D^τ. Later on, Uspenskiĭ [30] and the author [27] generalized Shirokov's theorem by showing that the same equality remains valid if Y is a compact continuous image of a dense subset of a σ-compact (Lindelöf Σ-) group. We extend Theorem 10 of [30] and the main results of [27, Section 2] to compact *regular* images of subsets of D_g*-spaces*.

Definition 4.1. A space X is said to be a D_g-space if X has a weak σ-lattice of d-open mappings onto spaces with strict G_δ-diagonal.

Obviously, every D_g-space is a D-space; a dense subset of a (para)topological group is a D_g-space. Furthermore, every d-space in the sense of Uspenskiĭ [30] is a D_g-space.

We write $\mathrm{id}_c(X) \leqslant \aleph_0$ if $\mathrm{id}(K) \leqslant \aleph_0$ for each compact subset K of X. It is clear that $\mathrm{id}_c(X) \leqslant \mathrm{id}(X)$ and $\mathrm{id}_c(X) \leqslant t(X)$. A gap between $\mathrm{id}_c(X)$ and $\mathrm{id}(X)$ can be arbitrarily large; however, $\mathrm{id}(X) = \mathrm{id}_c(X)$ for any Lindelöf Σ-space X [27, Assertion 2.12]. The following theorem is the main result of this paper.

Theorem 4.2. *Let a D_g-space X be a continuous image of a space Π that has a strong σ-lattice of open retractions onto Lindelöf Σ-subspaces, and suppose that a space Y of pointwise-countable type is an image of a dense subset S of X. Then*:

(a) *the set $M = \{y \in Y\colon t(y, Y) \leqslant \aleph_0\}$ has countable network, and $\chi(y, Y) \leqslant \aleph_0$ for each $y \in M$*;

(b) *if* $\mathrm{id}_c(Y) \leqslant \aleph_0$ *then* $\mathrm{nw}(Y) \cdot \chi(Y) \leqslant \aleph_0$.

Theorem 4.2 remains true if the space Y is assumed to be an image of an *arbitrary* subset S of X under a *regular with respect to* X mapping g. The latter means (see [27]) that there exists an operator e assigning to each open subset O of Y an open subset $e(O)$ of X so that $e(O) \cap S = g^{-1}(O)$ and $e(O) \cap e(U) = \emptyset$ whenever $O \cap U = \emptyset$. One easily sees that any continuous mapping $g: S \to Y$ of a dense subset S of X is regular with respect to X. Making use of Corollary 5.7 below, we will reduce Theorem 4.2 to the following.

Theorem 4.3. *Let a space X with a weak σ-lattice of d-open mappings onto spaces with countable network be a continuous image of a space Π that has a factorizative strong σ-lattice of quotient mappings onto Lindelöf Σ-spaces. If a space Y of pointwise-countable type is an image of a subset S of X under a regular with respect to X mapping, then*

(a) *the set* $M = \{y \in Y\colon t(y, Y) \leqslant \aleph_0\}$ *has countable network, and* $\chi(y, Y) \leqslant \aleph_0$ *for each* $y \in M$;

(b) *if* $\mathrm{id}_c(Y) \leqslant \aleph_0$ *then* $\mathrm{nw}(Y) \cdot \chi(Y) \leqslant \aleph_0$.

It seems surprising that the space X in Theorems 4.2 and 4.3 need not be $\aleph_0$-cellular: all earlier results of [4,30] and [27] generalized here depend heavily on the $\aleph_0$-cellularity of X. In addition, the assertion (b) in Theorem 4.2, the pointwise coincidence of the character and tightness of Y, is new; it generalizes a similar coincidence theorem for dyadic compact spaces and may be compared with Theorem 9 of [13] on continuous images of product spaces.

For simplicity, all our results are formulated in the countable case. However, one easily extends them to the general case. Keeping this in mind and making use of the fact that product spaces and paratopological groups have "good" τ-directed lattices of open mappings for each $\tau \geqslant \aleph_0$, we infer the following corollary of Theorem 4.2.

Corollary 4.4. *Let a paratopological group X be a continuous image of a product of Lindelöf Σ-spaces,* $S \subseteq X$ *and suppose that a compact space Y is an image of S under a regular with respect to X mapping. Then*

(a) *for each* $\tau \geqslant \aleph_0$, *the set* $M_\tau = \{y \in Y\colon t(y, Y) \leqslant \tau\}$ *satisfies* $\mathrm{nw}(M_\tau) \cdot \chi(M_\tau) \leqslant \tau$;

(b) $w(Y) = t(Y) = \mathrm{id}(Y)$.

The assertion (a) of Corollary 4.4 has been proved also by Pasynkov [18] in the case X is a topological group.

Our last result is a very special case of Theorem 4.3.

Corollary 4.5. *Suppose that a space X has a σ-lattice of d-open mappings onto spaces with countable network,* $S \subseteq X$ *and* $g: S \to Y$ *is a regular with respect to X mapping onto a compact space Y with* $\mathrm{id}(Y) \leqslant \aleph_0$. *Then Y is metrizable.*

Indeed, set $\Pi = X$ in Theorem 4.3. Note that the σ-lattice $\mathscr{L}_X$ for X is factorizative by Theorem 1 of [26]. The assumption of Theorem 4.3 that the mappings of the lattice $\mathscr{L}_\Pi$ for Π are quotient is unnecessary in this case, because the family $\mathscr{L}_X$ $(=\mathscr{L}_\Pi)$ is assumed to be a σ-lattice.

5. Proofs

Here we prove the results formulated in Sections 2 and 3. Throughout this section the symbol $\mathscr{L}_\Pi$ denotes a strong σ-lattice of quotient mappings of Π onto Lindelöf Σ-spaces. Let $f : \Pi \to X$ be a continuous mapping of Π onto a space X with a weak σ-lattice $\mathscr{L}_X$ of continuous mappings (onto spaces with countable network usually), and $\mathscr{F}(X)$ be the family of all sets in X of the form $\phi^{-1}(R)$ where $\phi \in \mathscr{L}_X$ and $R \subseteq \phi(X)$. Choose a subfamily $\mathscr{F} \subseteq \mathscr{F}(X)$ and denote by $\mathscr{M}$ the set of all triples (p, ϕ, γ) where $p \in \mathscr{L}_\Pi$, $\phi \in \mathscr{L}_X$ and $\gamma \subseteq \mathscr{F}$, $|\gamma| \leqslant \aleph_0$. Define the partial ordering $<$ on $\mathscr{M}$ by

$$(p_1, \phi_1, \gamma_1) < (p_0, \phi_0, \gamma_0) \quad \text{if } p_1 \prec p_0,\ \phi_1 \prec \phi_0 \text{ and } \gamma_0 \subseteq \gamma_1.$$

Clearly, every countable subset of $(\mathscr{M}, <)$ has a lower bound in $\mathscr{M}$.

Definition 5.1. A triple $(p, \phi, \gamma) \in \mathscr{M}$ is called *dense* if

(1) $F = \phi^{-1}\phi(F)$ for each $F = \gamma$;

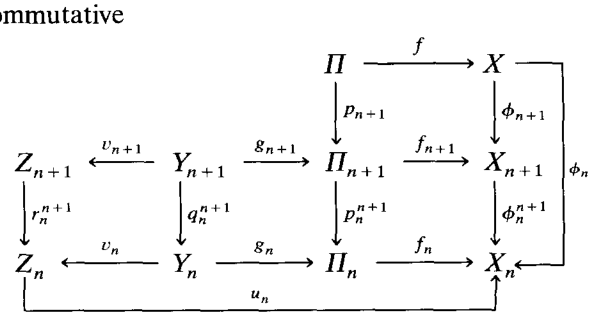
(2) $p \prec \phi f$;

(3) the unique continuous mapping $f_0 : p(\Pi) \to \phi(X)$ with $f_0 p = \phi f$ (see (2)) satisfies $\phi(\bigcup \mathscr{F}) \subseteq f_0(\mathrm{cl}\ pf^{-1}(\bigcup \gamma))$.

It is easy to verify that $\phi(\bigcup \mathscr{F}) \subseteq \mathrm{cl}\ \phi(\bigcup \gamma)$ whenever (p, ϕ, γ) is a dense triple. The following key lemma is used in the proof of most of our results.

Lemma 5.2. *Suppose that $\mathscr{L}_\Pi$ is factorizative and that $\phi(X)$ has countable network for each $\phi \in \mathscr{L}_X$. Then the set $\mathscr{M}^*$ of all dense triples of $\mathscr{M}$ is cofinal in $(\mathscr{M}, <)$, and the greatest lower bound in $\mathscr{M}$ of any decreasing sequence in $\mathscr{M}^*$ belongs to $\mathscr{M}^*$.*

Proof. The second statement of the lemma immediately follows from Definition 5.1 and the assumption that all mappings of $\mathscr{L}_\Pi$ are quotient. Only the cofinality of $\mathscr{M}^*$ in $\mathscr{M}$ requires proof. Pick an element $t_0 = (p_0, \phi_0, \gamma_0)$ of $\mathscr{M}$ with $p_0 \prec \phi_0 f$. For every integer n define spaces X_n, Y_n, Z_n, Π_n, continuous mappings ϕ_n, f_n, g_n, p_n, q_n, r_n, u_n, v_n and a countable subfamily γ_n of $\mathscr{F}$ so that the following diagram is commutative



(1)

(the mapping $u_{n+1}: Z_{n+1} \to X_{n+1}$ is not depicted here) and the conditions (1)–(5) below are fulfilled:

(1) $w(Z_n) \leqslant \aleph_0$, $\mathrm{nw}(X_n) \leqslant \aleph_0$;

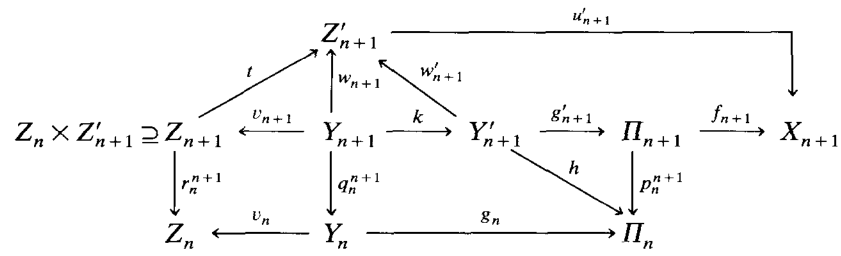
(2) $\phi_n \in \mathscr{L}_X$, $p_n = p_n^{n+1} p_{n+1} \in \mathscr{L}_\Pi$;

(3) v_n is a perfect mapping;

(4) the set $v_n g^{-1} p_n f^{-1}(\Gamma_n)$ is dense in $v_n g^{-1} p_n f^{-1}(\Gamma)$, where $\Gamma_n = \bigcup \gamma_n$ and $\Gamma = \bigcup \mathscr{F}$;

(5) $\Gamma_n = \phi^{-1}\phi_{n+1}(\Gamma_n)$.

The mapping $\phi_0 \in \mathscr{L}_X$ and the countable subfamily $\gamma_0 \subseteq \mathscr{F}$ are determined by the triple t_0. Put $X_0 = \phi_0(X)$. It is easy to define spaces $\Pi_0, \ldots, Z_0$ and mapping $p_0, \ldots, v_0$ that satisfy (1)–(4) and the commutativity of the appropriate part of the above diagram for $n = 0$. So assume that all requisite spaces, mappings and subfamilies of $\mathscr{F}$ are already defined for some $n \in N$. Since the lattice $\mathscr{L}_X$ is $\aleph_0$-directed by $\prec$ and γ_n is a countable subfamily of $\mathscr{F}(X)$, there exists $\phi_{n+1} \in \mathscr{L}_X$ such that $\phi_{n+1} \prec \phi_n$ and $F = \phi_{n+1}^{-1}\phi_{n+1}(F)$ for each $F \in \gamma_n$. This implies (5). Put $X_{n+1} = \phi_{n+1}(X)$. Then X_{n+1} has a countable network, so one can find a continuous bijection j of X_{n+1} onto a second-countable space T_{n+1}. Since $\mathscr{L}_\Pi$ is factorizative, there exist $p_{n+1} \in \mathscr{L}_\Pi$, $p_{n+1} \prec p_n$, and continuous mapping f'_{n+1} of $\Pi_{n+1} = p_{n+1}(\Pi)$ to T_{n+1} such that $f'_{n+1} p_{n+1} = j\phi_{n+1}$. Then the mapping $f_{n+1} = j^{-1} f'_{n+1}$ is continuous, because p_{n+1} is quotient. The equality $f_{n+1} p_{n+1} = \phi_{n+1} f$ is immediate. Making use of $\phi_{n+1} \prec \phi_n$ and $p_{n+1} \prec p_n$, define continuous mappings ϕ_n^{n+1} and p_n^{n+1} so that $\phi_n = \phi_n^{n+1}\phi_n$ and $p_n = p_n^{n+1} p_{n+1}$. The following diagram illustrates the remaining construction.



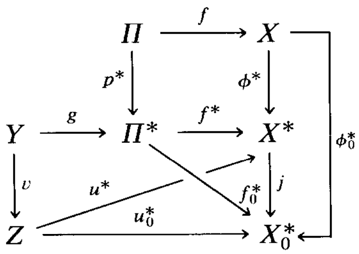
(2)

Since Π_{n+1} is a Lindelöf Σ-space, we can find spaces Y'_{n+1} and Z'_{n+1} and continuous onto mappings g'_{n+1} and w'_{n+1} such that Z'_{n+1} is second-countable and w'_{n+1} is perfect. One can assume that $w'_{n+1} \prec f_{n+1} g_{n+1}$; otherwise replace w'_{n+1} by the diagonal product $w'_{n+1} \Delta (f_{n+1} g'_{n+1})$, which is perfect because w'_{n+1} is. Denote by u'_{n+1} a continuous mapping that satisfies $u'_{n+1} w'_{n+1} = f_{n+1} g'_{n+1}$, and put $h = p_n^{n+1} g'_{n+1}$.

Now let Y_{n+1} be the fan product (see [1, Addendum to Ch. 1]) of the spaces Y_n and Y'_{n+1} with respect to the mappings h and g_n, i.e., $Y_{n+1} = \{(x, y) \in Y_n \times Y'_{n+1}: g_n(x) = h(y)\}$. Denote by q_n^{n+1} and k the restrictions to Y_{n+1} of projections of the product $Y_n \times Y'_{n+1}$ onto the first and the second factors respectively. Also put $g_{n+1} = g'_{n+1} k$. Obviously, Y_{n+1} is closed in $Y_n \times Y'_{n+1}$; hence the restriction of the perfect mapping $v_n \times w'_{n+1}$ to Y_{n+1}, call it v_{n+1}, is perfect. Note that the space

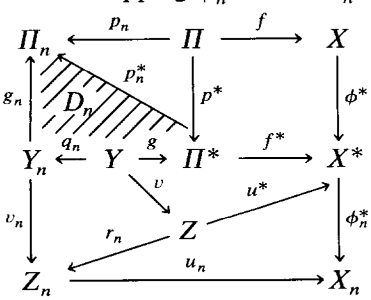
$Z_{n+1} = v_{n+1}(Y_{n+1})$ is second-countable. Denote by r_n^{n+1} and t the restrictions to Z_{n+1} of projections of the product $Z_n \times Z'_{n+1}$ onto the first and the second factors. It remains to put $u_{n+1} = u'_{n+1}t$. The existence of a countable subfamily $\gamma_{n+1} \subseteq \mathscr{F}$ such that $v_{n+1}g_{n+1}^{-1}p_{n+1}f^{-1}(\bigcup \gamma_{n+1})$ is dense in $v_{n+1}g_{n+1}^{-1}p_{n+1}f^{-1}(\bigcup \mathscr{F})$ follows from $w(Z_{n+1}) \leqslant \aleph_0$. This completes our construction at the $(n+1)$th step. It is important to note that for every $n \in N$, the square diagram determined by the equality $p_n^{n+1}g_{n+1} = g_n q_n^{n+1}$ is bicommutative in the sense of [20], i.e., $q_n^{n+1}g_{n+1}^{-1}(x) = g_n^{-1}p_n^{n+1}(x)$ for each $x \in \Pi_{n+1}$. This follows immediately from the definition of Y_{n+1} as a fan product.

Put $p^* = \Delta_{n=0}^{\infty} p_n$, $\phi_0^* = \Delta_{n=0}^{\infty}\phi_n$, $\Pi^* \in p(\Pi)$ and $X_0^* = \phi_0^*(X)$. Then $p^* = \mathscr{L}_\Pi$ and there exist $\phi^* \in \mathscr{L}_X$ and a continuous bijection j of $X^* = \phi^*(X)$ onto X_0^* such that $\phi_0^* = j\phi^*$. Since $\mathscr{L}_\Pi$ is a strong σ-lattice, Π^* is homeomorphic to the limit space $\Pi_0^* = \underleftarrow{\lim}\{\Pi_n, p_n^{n+1}: n \in N\}$, so one can identify Π^* and Π_0^*. Denote $Y = \underleftarrow{\lim}\{Y_n, q_n^{n+1}: n \in N\}$ and $Z = \underleftarrow{\lim}\{Z_n, r_n^{n+1}: n \in N\}$. Let q_n, r_n, and $p_n^* : \Pi^* \to \Pi_n$ be the limit projections, $n \in N$. Define continuous mappings $g: Y \to \Pi^*$ and $v: Y \to Z$ to be the limit mappings of the morphisms $\{g_n: n \in N\}$ and $\{v_n: n \in N\}$. Since all mappings v_n are perfect, so is v [1, Addendum to Ch. 1]. Define similarly the mappings $u_0^* = \underleftarrow{\lim}\{u_n: n \in N\}$ and $f_0^* = \underleftarrow{\lim}\{f_n: n \in N\}$. This gives the following commutative diagram.



(3)

Here $f^* = j^{-1}f_0^*$ and $u^* = j^{-1}u_0^*$. All mappings are onto. Since $\phi^* f$ is continuous and p^* is quotient, the mapping f^* is continuous. Similarly, since $f^* g$ is continuous and v is perfect, u^* is continuous. It is easy to see that there exists a continuous mapping $\phi_n^* : X^* \to X_n$ that makes the following diagram commutative.



(4)

(The mapping $f_n : \Pi_n \to X_n$ is not depicted here.)

Note that for each $n \in N$, the mappings g_n, q_n, g and p_n^* constitute the bicommutative diagram D_n, because so are all "nonlimit" diagrams determined by the mappings g_{k+1}, q_k, q_k^{k+1} and p_k^{k+1}, $k \geqslant n$.

Put $\gamma^* = \bigcup_{n=0}^{\infty}\gamma_n$ and $\Gamma^* = \bigcup \gamma^* \subseteq X$. Clearly, $|\gamma^*| \leqslant \aleph_0$, and by virtue of (5), $\Gamma^* = (\phi^*)^{-1}\phi^*(\Gamma^*)$. We claim that the triple $t^* = (p^*, \phi^*, \gamma^*)$ belongs to $\mathscr{M}^*$, and $t^* < t_0$. Only the first assertion requires a proof. It suffices to show that

$$\phi^*(\Gamma) \subseteq f^*(\mathrm{cl}_{\Pi^*} p^* f^{-1}(\Gamma^*)) \quad \text{where } \Gamma = \bigcup \mathscr{F}. \tag{$*$}$$

Indeed, by (4), the set $v_n g_n^{-1} p_n f^{-1}(\Gamma^*)$ is dense in $v_n g_n^{-1} p_n f^{-1}(\Gamma)$ for each $n \in N$. We have $v_n g^{-1} p_n f^{-1}(\Gamma^*) = v_n (g_n^{-1} p_n) p^* f^{-1}(\Gamma^*)$, and since the diagram D_n is bicommutative, the latter set coincides with $v_n(q_n g^{-1}) p^* f^{-1}(\Gamma^*) = r_n v g^{-1} p^* f^{-1}(\Gamma^*)$. The same is true for Γ instead of Γ^*. Thus, for each $n \in N$, the set $r_n v(T^*)$ is dense in $r_n v(T)$ where $T^* = g^{-1} p^* f^{-1}(\Gamma^*)$ and $T = g^{-1} p^* f^{-1}(\Gamma)$. By the definition of Z, $v(T^*)$ is dense in $v(T)$. Hence the inclusion $v(T) \subseteq v(\mathrm{cl}\ T^*)$ follows from the fact that v is perfect. Consequently, $u^* v(T) \subseteq u^* v(\mathrm{cl}\ T^*) = f^* g(\mathrm{cl}\ T^*) \subseteq f^*(\mathrm{cl}\ i\, g(T^*)) = f^*(\mathrm{cl}\ p^* f^{-1}(\Gamma^*))$. Making use of the commutativity of diagram (4), we come to the equality $u^* v(T) = \phi^*(\Gamma)$. This completes the proof of $(*)$. Thus, t^* is a dense triple, $t^* \in \mathscr{M}^*$ and $t^* < t_0$. □

The simplest application of Lemma 5.2 is in the proof of Theorem 3.4.

Proof of Theorem 3.4. Let $\mathscr{L}_X$ be a weak σ-lattice of open mappings of X onto spaces with countable network. Consider an arbitrary family $\mathscr{F}$ of G_δ-sets in X. Since $\mathscr{L}_X$ is $\aleph_0$-directed by $\prec$, every G_δ-set in X is a union of sets of the form $\phi^{-1}(y)$, where $\phi \in \mathscr{L}_X$ and $y \in \phi(X)$. Thus, we can assume without loss of generality that $\mathscr{F} \subseteq \mathscr{F}(X)$ (see the beginning of this section). Pick an element $t_0 \in \mathscr{M}$. By Lemma 5.2, there exists a dense triple $t \in \mathscr{M}^*$, say $t = (p, \phi, \gamma)$, with $t < t_0$. Put $\Gamma^* = \bigcup \gamma$ and $\Gamma = \bigcup \mathscr{F}$. Since t is dense, one can find a continuous mapping $f_0 : p(\Pi) \to \phi(X)$ so that $f_0 p = \phi f$ and $\phi(\Gamma) \subseteq f_0(\mathrm{cl}\ p f^{-1}(\Gamma^*))$. The continuity of f_0 implies that $\phi(\Gamma) \subseteq \mathrm{cl}\ f_0 p f^{-1}(\Gamma^*) = \mathrm{cl}\ \phi(\Gamma^*)$. By the choice of t, we have $\Gamma^* = \phi^{-1}\phi(\Gamma^*)$, and since ϕ is an open mapping, $\mathrm{cl}\ \Gamma^* = \mathrm{cl}\ \phi^{-1}\phi(\Gamma^*) = \phi^{-1}(\mathrm{cl}\ \phi(\Gamma^*)) \supseteq \phi^{-1}\phi(\Gamma) \supseteq \Gamma$. Thus, $\Gamma^* = \bigcup \gamma$ is dense in $\Gamma = \bigcup \mathscr{F}$ and $|\gamma| \leqslant \aleph_0$. This means that X is $\aleph_0$-cellular.

The second claim of the theorem readily follows from the first one. Indeed, using the same notation, we have $\mathrm{cl}\ \Gamma = \mathrm{cl}\ \Gamma^* = \phi^{-1}(\mathrm{cl}\ \phi(\Gamma^*))$. The set $F = \mathrm{cl}\ \phi(\Gamma^*)$ is closed in $\phi(X)$ and $\mathrm{nw}\ \phi(X) \leqslant \aleph_0$, so F is a G_δ-set in $\phi(X)$. Being the preimage of F under the mapping ϕ, $\mathrm{cl}\ \Gamma$ is a G_δ-set in X. □

We defer the proof of Theorem 3.3, similar to the proof of Theorem 3.4, till later, for it requires an additional result on factorization of continuous functions (see Lemma 5.5). The following result is known in folklore.

Lemma 5.3. *Let X be a Lindelöf Σ-space with G_δ-diagonal. Then X has countable network.*

Proof. Obviously, X is Lindelöf. Every Lindelöf space with G_δ-diagonal admits a continuous one-to-one mapping onto a second-countable space. Furthermore, a

Lindelöf Σ-space is $\aleph_0$-stable in the sense of Arhangel'skiĭ [3], i.e., every continuous image Y of this space that admits a continuous bijection onto a second-countable space has countable network. These facts imply the lemma. □

For generality, we use the following notion (see [6]).

Definition 5.4. A subset T of Π satisfies the inequality $wl(T, \Pi) \leqslant \aleph_0$ ($l(T, \Pi) \leqslant \aleph_0$) if every open cover γ of Π contains a countable subfamily $\mu \subseteq \gamma$ such that $T \subseteq \mathrm{cl} \bigcup \mu$ ($T \subseteq \bigcup \mu$).

Obviously, $c(T) \leqslant \aleph_0$ or $c(\Pi) \leqslant \aleph_0$ implies $wl(T, \Pi) \leqslant \aleph_0$, and $l(T) \leqslant \aleph_0$ implies $l(T, \Pi) \leqslant \aleph_0$.

Lemma 5.5. *Suppose a space Π has a σ-lattice $\mathscr{L}$ of open mappings onto Lindelöf Σ-spaces and that for every $p \in \mathscr{L}$ there exists a subset $T \subseteq \Pi$ such that $wl(T, \Pi) \leqslant \aleph_0$ ($l(T, \Pi) \leqslant \aleph_0$) and $p(T)$ is dense in $p(\Pi)$. Then*

(a) *the lattice $\mathscr{L}$ is factorizative;*

(b) *every continuous image X of Π with strict G_δ-diagonal (G_δ-diagonal) has countable network.*

Remark 5.6. Items (a) and (b) without brackets of Lemma 5.5 remain true even if one weakens the requirement that the mappings of $\mathscr{L}$ are open to "quotient and d-open".

Proof of Lemma 5.5 and Remark 5.6. We first prove (b) in the case $wl(T, \Pi) \leqslant \aleph_0$ and $\mathscr{L}$ consists of quotient, d-open mappings. Let f be a continuous mapping of Π onto a space X with strict G_δ-diagonal. Then there exists a family $\{U_k\colon k \in N\}$ of open neighborhoods of the diagonal Δ_X in X^2 such that $\Delta_X = \bigcap_{k=0}^{\infty} \mathrm{cl}\, U_k$. For every integer n define a closed subset F_n of Π, a family $\mathscr{R}_n$ of open sets in Π and a mapping $p_n \in \mathscr{L}$ that satisfy the following conditions:

(1) $|\mathscr{R}_n| \leqslant \aleph_0$ and $\mathscr{R}_n = \bigcup_{k=0}^{\infty} \mathscr{R}_n(k)$;

(2) $f(O) \times f(O) \subseteq U_k$ for each $O \in \mathscr{R}_n(k)$;

(3) $p_n(V_n(k))$ is dense in $p_n(\Pi)$ where $V_n(k) = \bigcup \mathscr{R}_n(k)$;

(4) $p_{n+1} \prec p_n$;

(5) $O = p_{n+1}^{-1} p_{n+1}(O)$ for each $O \in \mathscr{R}_n$.

Let $n \in N$ and suppose that we have already defined a mapping $p_n : \Pi \to \Pi_n$, $p_n \in \mathscr{L}$. Then there exists a subset $T_n \subseteq \Pi$ such that $wl(T_n, \Pi) \leqslant \aleph_0$ and $p_n(T_n)$ is dense in Π_n. The family $\mathscr{B}$ of all open sets in Π which are of the form $O = p^{-1}p(O)$ for some $p \in \mathscr{L}$, constitutes a base for Π. By the continuity of f, for every $k \in N$ there exists a subfamily γ_k of $\mathscr{B}$ such that $\Pi = \bigcup \gamma_k$ and $f(O) \times f(O) \subseteq U_k$ for each $O \in \gamma_k$. Since $wl(T_n, \Pi) \leqslant \aleph_0$, one can find a countable subfamily $\mathscr{R}_n(k) \subseteq \gamma_k$ with $T_n \subseteq \mathrm{cl}(\bigcup \mathscr{R}_n(k))$. Put $\mathscr{R}_n = \bigcup_{k=0}^{\infty} \mathscr{R}_n(k)$. Since $\mathscr{R}_n \subseteq \mathscr{B}$ and $\mathscr{L}$ is $\aleph_0$-directed, there exists $p_{n+1} \in \mathscr{L}$ such that $p_{n+1} \prec p_n$ and $O = p_{n+1}^{-1} p_{n+1}(O)$ for each $O \in \mathscr{R}_n$. It is easy to see that the conditions (1)–(5) are satisfied.

For each $k \in N$ denote $\mathscr{R}(k) = \bigcup_{n=0}^{\infty} \mathscr{R}_n(k)$, $V_k = \bigcup \mathscr{R}(k)$ and put $p = \Delta_{n=0}^{\infty} p_n$, $p \in \mathscr{L}$. Then $V_k = p^{-1}p(V_k)$ by (5), and $p(V_k)$ is dense in $\Pi^* = p(\Pi)$ by (3), for all $k \in N$. We claim that for any $x, y \in \Pi$, $p(x) = p(y)$ implies $f(x) = f(y)$. Suppose not; fix x and y so that $p(x) = p(y)$ and $f(x) \neq f(y)$. Then $(f(x), f(y)) \notin \text{cl } U_k$ for some $k \in N$. There exist open neighborhoods V_x, V_y of the points x, y in Π such that $(f(V_x) \times f(V_y)) \cap \text{cl } U_k = \emptyset$. Since p is a d-open mapping, $p(V_x)$ and $p(V_y)$ are dense in some open sets W_x and W_y in Π^* respectively (see [23]). The equality $p(x) = p(y)$ implies $W = W_x \cap W_y \neq \emptyset$. Since $p(V_k)$ is dense in Π^*, there exists $O \in \mathscr{R}(k)$ such that $p(O) \cap W \neq \emptyset$. The latter set is open in Π^* because of the equality $O = p^{-1}p(O)$ and the fact that p is quotient. Therefore, the definition of W implies $p(V_x) \cap p(O) \neq \emptyset$ and $p(V_y) \cap p(O) \neq \emptyset$. Making use of the equality $O = p^{-1}p(O)$, pick two points $x_1 \in V_x \cap O$ and $y_1 \in V_y \cap O$. Then $(f(x_1), f(y_1)) \in f(V_x) \times f(V_y) \subseteq X^2 \setminus \text{cl } U_k$, which contradicts the facts that $O \in \mathscr{R}(k)$ and $f(O) \times f(O) \subseteq U_k$.

Thus, we have proved the existence of a mapping q of Π^* to X such that $f = qp$. Since p is quotient, q is continuous, i.e., $p \prec f$. This along with Lemma 5.3 proves the nonbracket case of (b). Since every second-countable space has strict G_δ-diagonal, (a) is immediate.

It remains to prove (b) in the case $l(T, \Pi) \leqslant \aleph_0$ and X has G_δ-diagonal (all mappings of $\mathscr{L}$ are assumed open). In this case we carry out a similar construction of p_n and $\mathscr{R}_n(k)$ that satisfy the same conditions (1)–(5). However, we define $\mathscr{R}_n(k)$ to be a cover of T_n for each $k \in N$. Then put $R_n = \bigcap_{k=0} V_n(k)$ and $R = \bigcup_{n=0}^{\infty} R_n$ where $V_n(k) = \bigcup \mathscr{R}_n(k)$. Then $R_n = p_{n+1}^{-1} p_{n+1}(R_n)$ by virtue of (5), and $p_n(R_n)$ is dense in Π_n because $T_n \subseteq R_n$, $n \in N$. Therefore, $R = p^{-1}p(R)$, and $p(R)$ is dense in Π. It suffices to show that $p(x) = p(y)$ implies $f(x) = f(y)$ for any x, y of Π. Obviously, this is true for any $x, y \in R$. To complete the proof, use the fact that p is an open mapping, whence $R = p^{-1}p(R)$ is dense in Π. □

Corollary 5.7. *Suppose that a space Π has a σ-lattice $\mathscr{L}$ of d-open (open) retractions onto its Lindelöf Σ-subspaces. Then $\mathscr{L}$ is factorizative and every continuous image of Π with strict G_δ-diagonal (G_δ-diagonal) has countable network.*

Proof. Every retraction is a quotient mapping, and it remains to apply Lemma 5.5 and Remark 5.6. □

Corrollary 5.8. *Let S be a subset of the product $\Pi = \prod_{\alpha \in A} X_\alpha$ of Lindelöf Σ-spaces X_α, $\alpha \in A$. Suppose that for each countable set $B \subseteq A$, $p_B(S) = \Pi_B$ and there exists $T_B \subseteq S$ such that $wl(T_B, S) \leqslant \aleph_0$ and $p_B(T_B)$ is dense in Π_B (here p_B is the projection of Π onto $\Pi_B = \prod_{\alpha \in B} X_\alpha$). Then every continuous image of S with strict G_δ-diagonal has countable network.*

Proof. The condition "$p_B(S) = \Pi_B$ for each countable $B \subseteq A$" implies that the restriction of p_B to S is open whenever B is countable. Hence the corollary follows from Lemma 5.5. □

Corollary 5.9. *Let Π be a product of Lindelöf Σ-spaces, $a \in \Pi$ and $\Sigma(a)$ be the Σ-product with the base point a. If $\Sigma(a) \subseteq S \subseteq \Pi$ and a space X with G_δ-diagonal is a continuous image of S, then X has countable network.*

Proof. The family $\{p_B|_S \colon B \subseteq A,\ |B| \leqslant \aleph_0\}$ consists of open "retractions" of S. Corollary 5.7 implies $\mathrm{nw}(X) \leqslant \aleph_0$. □

Now we give a reduction of Theorem 3.3 to Theorem 3.4.

Proof of Theorem 3.3. Suppose that spaces Π and X satisfy the conditions of Theorem 3.3, and lattices $\mathscr{L}_\Pi$, $\mathscr{L}_X$ for Π and X witness this. It suffices to show that $\mathscr{L}_\Pi$ is factorizative and $\phi(X)$ has countable network for each $\phi \in \mathscr{L}_X$, i.e., that $\mathscr{L}_\Pi$ and $\mathscr{L}_X$ satisfy the conditions of Theorem 3.4. Clearly, the necessary properties of $\mathscr{L}_\Pi$ and $\mathscr{L}_X$ follow from Corollary 5.7. □

Let us return to the first result of this paper, Theorem 2.2. We begin with the following simple lemma.

Lemma 5.10. *Let $X \xrightarrow{g} Y \xrightarrow{f} Z \xrightarrow{h} T$ be continuous mappings. If f is an M-mapping, then so are fg and hf.*

Proof. Choose a continuous mapping $F : Y^3 \to Z$ witnessing that f is an M-mapping. Define continuous mappings $G : X^3 \to Z$ and $H : Y^3 \to T$ by $G(x_1, x_2, x_3) = F(g(x_1), g(x_2), g(x_3))$ and $H(y_1, y_2, y_3) = h(F(y_1, y_2, y_3))$ for all $x_i \in X$ and $y_i \in Y$, $1 \leqslant i \leqslant 3$. Clearly, G and H witness that fg and hf are M-mappings. □

Suppose we are given the following commutative diagram

$$\begin{array}{ccc} \Pi & \xrightarrow{\ f\ } & X \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ \Pi_0 & \xrightarrow{\ h\ } & X_0 \end{array} \tag{5}$$

where f and h are M-mappings. Let the mappings $F : \Pi^3 \to X$ and $H : \Pi_0^3 \to X_0$ witness this.

Definition 5.11. We call f and h *parallel M-mappings* if $\phi F = Hp^3$. Similarly, the mappings p and ϕ are also called *parallel* in this case.

The following generalization of Theorem 2.2 is valid.

Theorem 5.12. *Let Π be a space with a strong σ-lattice $\mathscr{L}_\Pi$ of open retractions onto its Lindelöf Σ-subspaces and suppose that X is an image of Π under an M-mapping f. Then X is $\aleph_0$-cellular.*

Proof. Denote by $\mathscr{L}_X$ the σ-lattice of all continuous mappings of X onto second-countable spaces. Let $F : \Pi^3 \to X$ be a continuous mapping witnessing that f is an M-mapping. Assume that the space X is not $\aleph_0$-cellular. Then there exists a sequence of pairs (K_α, V_α), $\alpha < \omega_1$, such that $\emptyset \neq K_\alpha \subseteq V_\alpha \subseteq X$, K_α is a nonempty G_δ, V_α is open, and $K_\alpha \cap V_\beta = \emptyset$ whenever $\alpha < \beta < \omega_1$. Diminishing K_α and V_α if necessary, one can assume that for every $\alpha < \omega_1$ there exists a continuous real-valued function h_α on X such that $K_\alpha = h_\alpha^{-1}(0)$ and $X \setminus V_\alpha = h_\alpha^{-1}(1)$.

Now for every integer n, define spaces X_n, Y_n, Z_n, Π_n, continuous mappings $\phi_n, f_n, g_n, \ldots, v_n$ and an ordinal $\alpha_n < \omega_1$ in a manner analogous to that of Lemma 5.2. We put $\mathscr{F} = \{K_\alpha: \alpha < \omega_1\}$, $\Gamma = \bigcup \mathscr{F}$ and $\mathscr{F}_\beta = \{K_\alpha: \alpha < \beta\}$ for each $\beta < \omega_1$. The ordinal α_n determines the countable family $\gamma_n = \mathscr{F}_{\alpha_n}$ as in the proof of Lemma 5.2. Furthermore, we will define for each $n \in N$ continuous mappings $F_n : \Pi_n^3 \to X_n$ and $U_n : Z_n^3 \to X_n$ satisfying the following additional conditions:

(6) f_n and u_n are M-mappings, and F_n, U_n witness this;

(7) f and f_n are parallel M-mappings;

(8) $F_n g_n^3 = U_n v_n$.

Only the definition of α_n, F_n and U_n need be clarified. This also requires slight modifications to the definitions of spaces Π_n, Z_n and mappings f_n, p_n, u_n, v_n. Suppose we have already defined the ordinal $\alpha_n < \omega_1$. Put $\phi_n = \Delta\{h_\alpha: \alpha < \alpha_n\}$ and $X_n = \phi_n(X)$. Then $K_\alpha = \phi_n^{-1}\phi_n(K_\alpha)$ and $V_\alpha = \phi_n^{-1}\phi_n(V_\alpha)$ for each $\alpha < \alpha_n$. Consider the strong σ-lattice $\mathscr{L}_\Pi^3 = \{p^3: p \in \mathscr{L}_\Pi\}$ of open retractions of Π^3 onto its Lindelöf Σ-subspaces. By Corollary 5.7, the lattices $\mathscr{L}_\Pi$ and $\mathscr{L}_\Pi^3$ are factorizative. So there exists $p_n \in \mathscr{L}_\Pi$ such that $p_n \prec \phi_n f$, $p_n \prec p_{n-1}$ (if $n \geqslant 1$) and $p_n \prec \phi_n F$. Put $\Pi_n = p_n(\Pi)$ and denote by F_n the unique continuous mapping of Π_n to X_n such that $F_n p_n^3 = \phi_n F$. In turn, since $p_n \prec \phi_n f$, there exists a continuous mapping $f_n : \Pi_n \to X$ with $f_n p_n = \phi_n f$. Then F_n witnesses that f_n is an M-mapping, and f, f_n are parallel. This implies the first part of (6) and (7).

Define spaces Y_n, Z_n and mappings g_n, v_n and u_n as in the proof of Lemma 5.2 and then "correct" them in the following way. Let $\mathscr{L}_n$ be the family of all continuous mappings w of Y_n onto second-countable spaces, $w \prec v_n$. All mappings of $\mathscr{L}_n$ are perfect because v_n is. The space Y_n is Lindelöf and $\mathscr{L}_n^3$ is a σ-lattice for Y_n; hence Lemma 1 of [22] implies that there exists $v_n^* \in \mathscr{L}_n$ such that $(v_n^*)^3 \prec F_n g_n^3$ and $v_n^* \prec v_n$. Put $Z_n^* = v_n^*(Y_n)$ and denote by u_n^* the continuous mapping of Z_n^* to X_n such that $u_n^* v_n^* = f_n g_n$. Since $(v_n^*)^3 \prec F_n g_n^3$, there exists a continuous mapping $U_n : (Z_n^*)^3 \to X_n$ that satisfies $U_n (v_n^*)^3 = F_n g_n^3$. Obviously, u_n is an M-mapping, and U_n witnesses this. Now one can replace Z_n and u_n, v_n by Z_n^* and u_n^*, v_n^* to satisfy the conditions (6)–(8). In the sequel we use the earlier denotations u_n, v_n, Z_n instead of u_n^*, v_n^*, Z_n^*. Thus, we have $U_n v_n = F_n g_n$. It remains to choose α_{n+1}, $\alpha_n < \alpha_{n+1} < \omega_1$, to satisfy the condition (4) at step $n+1$ with $\Gamma_{n+1} = \bigcup \mathscr{F}_{\alpha_{n+1}}$. This completes our construction.

Similarly to that in the proof of Lemma 5.2, define the limit spaces X^*, Π^*, Y, Z and the limit mappings ϕ^*, f^*, p^*, g, v, u^* ($X_0^* = X^*$ and $f_0^* = f^*$, $u_0^* = u^*$ in this case). In the same way define limit mappings $F^* : (\Pi^*)^3 \to X^*$ and $U^* : Z^3 \to X^*$. Now (7) implies that f and f^* are parallel M-mappings, i.e.,

(a) $\phi^* F = F^* \circ (p^*)^3$.

In the same way (8) implies

(b) $F^* \circ g^3 = U^* \circ v^3$.

Put $\beta = \sup\{\alpha_n : n \in N\}$ and $\Gamma^* = \bigcup \mathscr{F}_\beta$. Denote $R = f^{-1}(\Gamma)$, $R^* = f^{-1}(\Gamma^*)$ and $S = vg^{-1}p^*(R)$, $S^* = vg^{-1}p^*(R^*)$. By (4), we have

(c) S^* is dense in S;

and (5) implies

(d) $K_\alpha = (\phi^*)^{-1}\phi^*(K_\alpha)$ and $V_\alpha = (\phi^*)^{-1}\phi^*(V_\alpha)$ for all $\alpha < \beta$.

In particular, $\Gamma^* = (\phi^*)^{-1}\phi^*(\Gamma^*)$.

Pick some points $x_\beta \in X$ and $y_\beta \in \Pi$ so that $x_\beta \in K_\beta$ and $f(y_\beta) = x_\beta$. Let $\bar{y}_\beta \in Y$ and $y_\beta^* \in \Pi^*$ be points satisfying $g(\bar{y}_\beta) = p^*(y_\beta) = y_\beta^*$. Since v is perfect, there exists a point $\bar{y} \in Y$ such that $\bar{y} \in \text{cl } g^{-1}p^*(R^*)$ and $v(\bar{y}) = v(\bar{y}_\beta)$. Put $y^* = g(\bar{y})$ and pick a point $y \in \Pi$ so that $p^*(y) = y^*$. This is possible because $\mathscr{L}_\Pi$ is a strong σ-lattice. We claim that $y \in \text{cl } R^*$. Indeed, from $\Gamma^* = (\phi^*)^{-1}\phi^*(\Gamma^*)$ (see (d)) and $f^*p^* = \phi^* f$ follows $R^* = (p^*)^{-1}p^*(R^*)$. It remains to note that $y^* \in \text{cl } p^*(R^*)$ and p^* is an open mapping.

We have $F(t, t, y_\beta) = f(y_\beta) = x_\beta \in V_\beta$ for all $t \in \Pi$, and in particular, for $t = y$. Since F is continuous and $y \in \text{cl } R^*$, there exists a point $z \in R^*$ such that $F(z, y, y_\beta) \in V_\beta$. Clearly, $f(z) \in K_\alpha$ for some $\alpha < \beta$, and we claim that $F(z, y, y_\beta) \in K_\alpha$.

Indeed, the following equalities are valid:

$$\phi^* F(z, y, y_\beta) \overset{(a)}{=} F^*(p^*)^3(\ldots) = F^*(z^*, y^*, y_\beta^*) = F^* g^3(\bar{z}, \bar{y}, \bar{y}_\beta),$$

where $z^* = p^*(z)$ and $\bar{z} \in Y$, $g(\bar{z}) = z^*$. Then, by virtue of diagram (4),

$$F^* g^3(\bar{z}, \bar{y}, \bar{y}_\beta) \overset{(b)}{=} U^* v^3(\ldots) = U^*\big(v(\bar{z}), v(\bar{y}), v(\bar{y}_\beta)\big).$$

The second and the third arguments of the function U^* coincide, so we have

$$\phi^* F(z, y, y_\beta) = u^* v(\bar{z}) = f^* g(\bar{z}) = f^* p^*(z) = \phi^* f(z) \in \phi^*(K_\alpha),$$

for $f(z) \in K_\alpha$. Since $K_\alpha = (\phi^*)^{-1}\phi^*(K_\alpha)$ by (d), the point $x = F(z, y, y_\beta)$ is in K_α.

Thus, $x \in K_\alpha \cap V_\beta \neq \emptyset$, which contradicts the choice of the sets K_α and V_β. Therefore, X is $\aleph_0$-cellular. □

To prove Theorem 2.3 we need three auxiliary results. A mapping $F : \Pi \times \Pi \to Z$ is said to be *separately continuous* if the functions $f_a(x) = F(x, a)$ and $g_a(x) = F(a, x)$ are continuous for each $a \in \Pi$. The Čech–Stone compactification of Π is denoted by $\beta\Pi$. We omit the proof of the following result.

Theorem 5.13 (E. Reznichenko). *Let $F : \Pi \times \Pi \to Z$ be a continuous mapping, where Π is pseudocompact and Z is second-countable. Then F extends to a separately continuous function $F : \beta\Pi \times \beta\Pi \to Z$.*

Lemma 5.14. *Suppose that Π is a countably protocompact space, $b\Pi$ is a compactification of Π and $F: b\Pi \times b\Pi \to R$ a separately continuous function, $F(x, x) = 0$ for each $x \in \Pi$. Then $F(y, y) = 0$ for each $y \in b\Pi$.*

Proof. Assume for contradiction that $F(y, y) \neq 0$ for some $y \in b\Pi$. Let for convenience, $F(y, y) = 1$. By the assumption, there exists a dense set S in Π, every infinite subset of which has a cluster point in Π. Put $U_1 = \{x \in b\Pi: F(x, y) > 1/2\}$. Then U_1 is open in Π and $y \in U_1$. Pick a point $x_1 \in U_1 \cap S$. Since F is separately continuous and $F(x_1, y) > 1/2$, there exists an open neighborhood W_1 of y in $b\Pi$ such that $F(\{x_1\} \times W_1) \subseteq (1/2, \infty)$. Let U_2 be an open neighborhood of y in $b\Pi$ such that $\mathrm{cl}_{b\Pi} U_2 \subseteq W_1 \cap U_1$. Pick a point $x_2 \in U_2 \cap S$, find an open set W_2 in $b\Pi$ with $y \in W_2$ and $F(\{x_2\} \times W_2) \subseteq (1/2, \infty)$, and so on.

We have defined the sequences $\{x_n: n \in N\}$, $\{U_n: n \in N\}$ and $\{W_n: n \in N\}$. Put $\Phi = \bigcap_{n=0}^{\infty} U_n$; Φ is a nonempty closed subset of $b\Pi$. Since $x_n \in S \cap U_n$ for each $n \in N$, the sequence $\{x_n: n \in N\}$ has a cluster point x^* in $\Phi \cap \Pi$. On the one hand, $F(x^*, x^*) = 0$, for $x^* \in \Pi$. On the other hand, $F(x_n, x^*) > 1/2$ for each $n \in N$, which contradicts the continuity of $(\cdot, x^*)$. □

Lemma 5.15. *Let $\lambda: \Pi \to E$ be a continuous mapping of a pseudocompact space Π to a second-countable space E. Suppose that $\Phi: \Pi^2 \to Z$ is a separately continuous mapping and a subset A of Π satisfies the condition; $\lambda \prec \Phi(a, \cdot)$ for each $a \in A$. Then $\lambda \prec \Phi(b, \cdot)$ whenever $b \in \mathrm{cl}\ A$.*

Proof. Assume that $\Phi(b, x) \neq \Phi(b, y)$ for some points $b \in \mathrm{cl}\ A$ and $x, y \in \Pi$. Since Φ is continuous in the first argument, there exists an open subset U of Π such that $b \in U$ and $\Phi(U \times \{x\}) \cap \Phi(U \times \{y\}) = \emptyset$. Pick a point $a \in U \cap A$. Then $\Phi(a, x) \neq \Phi(a, y)$, whence $\lambda(x) \neq \lambda(y)$. Therefore, one can find a mapping ϕ of $\lambda(\Pi)$ to Z such that $\Phi(b, y) = \phi\lambda(y)$ for a given point $b \in \mathrm{cl}\ A$ and all $y \in \Pi$. It remains to note that ϕ is continuous: apply the fact that λ is z-closed [7, Lemma 7] and thereby is R-quotient [16] as a continuous mapping of a pseudocompact space to a second countable space. □

Proof of Theorem 2.3. There exists a continuous mapping F of Π^3 to X that witnesses that f is an M-mapping. Consider an arbitrary family of pairs (K_α, V_α), $\alpha < \omega_1$, where K_α is a nonempty G_δ-set in X and V_α is an open neighborhood of K_α in X. It suffices to show that $K_\alpha \cap V_\beta \neq \emptyset$ for some $\alpha, \beta < \omega_1$, $\alpha < \beta$. Diminishing K_α and V_α, one can assume that, for each $\alpha < \omega_1$, there exists a continuous real-valued function h_α on X such that $K_\alpha = h^{-1}(0)$ and $X \setminus V_\alpha = h^{-1}(1)$. For every $\alpha < \omega_1$ pick a point $x_\alpha \in \Pi$ so that $f(x_\alpha) \in K_\alpha$, and put $T = \{x_\alpha: \alpha < \omega_1\}$ and $T_\beta = \{x_\alpha: \alpha < \beta\}$, $\beta < \omega_1$. For every α, $\beta < \omega_1$ with $\alpha < \beta$ put $\psi_\alpha = \{h_\nu: \nu \leqslant \alpha\}$, $X_\alpha = \psi_\alpha(X)$ and $\psi_{\beta,\alpha} = \psi_\alpha \circ \psi_\beta^{-1}$. All spaces X_α are second-countable, and since Π and X are countably protocompact, the X_α are compact. By the definition, we have $\psi_\alpha \prec h_\alpha$; hence $K_\alpha = \psi_\alpha^{-1}\psi_\alpha(K_\alpha)$ and $X \setminus V_\alpha = \psi_\alpha^{-1}\psi_\alpha(X \setminus V_\alpha)$. Since X is pseudocompact, ψ_α takes zero sets to closed sets [7]. Therefore, $U_\alpha = \psi_\alpha(V_\alpha) = X_\alpha$

$\setminus\psi_\alpha(X\setminus V_\alpha)$ is open in X_α, and $V_\alpha=\psi_\alpha^{-1}(U_\alpha)$ for each $\alpha<\omega_1$. For every $\alpha,\nu<\omega_1$ define the continuous mapping $\phi_{\alpha,\nu}$ of Π to X_α by $\phi_{\alpha,\nu}(x)=\psi_\alpha F(x_\alpha, x_\nu, x)$, $x\in\Pi$.

Now we proceed to the following inductive construction. Let λ_0 be a mapping of Π to a one-point space E_0, and $\alpha(0)=0$. Suppose we have already defined a continuous mapping λ_n of Π onto a metrizable compact space E_n and an ordinal $\alpha(n)<\omega_1$ for some $n\in N$. Denote by λ_{n+1} the diagonal product of the mapping λ_n and the family of mappings $\phi_{\alpha,\nu}$ for $\alpha,\nu\leqslant\alpha(n)$. Put $E_{n+1}=\lambda_{n+1}(\Pi)$; E_{n+1} is a metrizable compact space. There exists an ordinal $\alpha(n+1)$, $\alpha(n)<\alpha(n+1)<\omega_1$, such that $\lambda_{n+1}(T_{\alpha(n+1)})$ is dense in $\lambda_{n+1}(T)$.

Now put $\lambda=\Delta_{n=0}^{\infty}\lambda_n$, $E=\lambda(\Pi)$ and $\beta=\sup_n\alpha(n)$. Then E is compact metrizable and $\beta<\omega_1$. By the construction, $\lambda(T_\beta)$ is dense in $\lambda(T)$ and $\lambda\prec\phi_{\alpha,\nu}$ for all $\alpha,\nu<\beta$. Extend λ to a continuous mapping $\tilde\lambda:\beta\Pi\to E$ where $\beta\Pi$ is the Čech–Stone compactification of Π. Since $\tilde\lambda$ is a closed mapping, there exists a point $x^*\in\beta\Pi$ such that $x^*\in \mathrm{cl}_{\beta\Pi}T_\beta$ and $\tilde\lambda(x^*)=\tilde\lambda(x_\beta)$.

Define the continuous mapping $\Psi_\beta:\Pi^2\to X_\beta$ by the rule: $\Psi_\beta(x, y)=\psi_\beta F(x, y, x_\beta)$, $x,y\in\Pi$. Since X_β is compact metrizable, Theorem 5.13 implies that Ψ_β extends to a separately continuous mapping $\tilde\Psi_\beta:(\beta\Pi)^2\to X_\beta$. We have $\Psi_\beta(x, x)=\psi_\beta F(x, x, x_\beta)=\psi_\beta f(x_\beta)$ for all $x\in\Pi$; therefore Lemma 5.14 implies $\tilde\Psi_\beta(x^*, x^*)=\psi_\beta f(x_\beta)$. Obviously, $f(x_\beta)\in K_\beta\subseteq V_\beta$ and $\psi_\beta f(x_\beta)\in U_\beta$. Hence $\tilde\Psi_\beta(x^*, x^*)\in U_\beta$ and the continuity of $\tilde\Psi_\beta$ in the first argument implies that there exists an ordinal $\alpha<\beta$ such that $y_\beta=\tilde\Psi_\beta(x_\alpha, x^*)\in U_\beta$.

We claim that the point $y_\alpha=\psi_{\beta,\alpha}(y_\beta)$ belongs to $\psi_\alpha(K_\alpha)$. Indeed, it follows from the construction that $\lambda\prec\phi_{\alpha,\nu}$ for each $\nu<\beta$; hence $\tilde\lambda\prec\tilde\phi_{\alpha,\nu}$ where $\tilde\phi_{\alpha,\nu}$ is the continuous extension of $\phi_{\alpha,\nu}$ to $\beta\Pi$, $\nu<\beta$. Define the continuous mapping $\Lambda_\alpha:\Pi^2\to X_\alpha$ by the rule: $\Lambda_\alpha(x, y)=\psi_\alpha F(x_\alpha, x, y)$, $x,y\in\Pi$. By Theorem 5.13, Λ_α extends to a separately continuous mapping $\tilde\Lambda_\alpha:\beta\Pi\times\beta\Pi\to X_\alpha$. Note that $\Lambda_\alpha(x_\nu, y)=\phi_{\alpha,\nu}(y)$ for all $\nu<\beta$ and $y\in\Pi$; hence $\tilde\Lambda_\alpha(x_\nu,\cdot)=\tilde\phi_{\alpha,\nu}$ for each $\nu<\beta$. Since $x^*\in\mathrm{cl}\ T_\beta$, Lemma 5.15 implies $\tilde\lambda\prec\tilde\Lambda_\alpha(x^*,\cdot)$. By the choice of x^*, we have $\tilde\lambda(x^*)=\tilde\lambda(x_\beta)$, whence $\tilde\Lambda_\alpha(x^*, x^*)=\tilde\Lambda_\alpha(x^*, x_\beta)$. By Lemma 5.14, $\tilde\Lambda_\alpha(x^*, x^*)=\psi_\alpha f(x_\alpha)$, and it is easy to verify that $\tilde\Lambda_\alpha(x^*, x_\beta)=\psi_{\beta,\alpha}\tilde\Psi_\beta(x_\alpha, x^*)=\psi_{\beta,\alpha}(y_\beta)=y_\alpha$. Consequently, $y_\alpha=\psi_\alpha f(x_\alpha)\in\psi_\alpha(K_\alpha)$.

Pick a point $y\in X$ so that $\psi_\beta(y)=y_\beta$. Then $y\in K_\alpha\cap V_\beta\neq\emptyset$. Indeed, from $\psi_\beta(y)=y_\beta\in U_\beta$ and $V_\beta=\psi_\beta^{-1}(U_\beta)$ follows $y\in V_\beta$. Furthermore, $\psi_\alpha(y)=\psi_{\beta,\alpha}\psi_\beta(y)=\psi_{\beta,\alpha}(y_\beta)=y_\alpha$. Since $K_\alpha=\psi_\alpha^{-1}\psi_\alpha(K_\alpha)$, the inclusion $y_\alpha\in\psi_\alpha(K_\alpha)$ implies $y\in K_\alpha$. Hence $y\in K_\alpha\cap U_\beta$, as is required. Thus the theorem is proved. □

We conclude this section with two remarks concerning Theorems 3.3 and 3.4.

Remark 5.16. Suppose that the space X in Theorems 3.3 and 3.4 satisfies the following weaker condition: there exists a weak σ-lattice $\mathscr{L}_X$ for X that consists of *skeletal* mappings onto the corresponding spaces. Then the cellularity of X is countable (even if the σ-lattice $\mathscr{L}_\Pi$ is not assumed to be *strong*).

Recall that a mapping $\phi : X \to Y$ is *skeletal* if $\phi^{-1}(N)$ is nowhere dense in X whenever N is a nowhere dense subset of Y. To prove the above assertion consider a disjoint family γ of open sets in X. One can assume that every element $V \in \gamma$ is of the form $V = \phi^{-1}(U_V)$ for some $\phi_V \in \mathscr{L}_X$ and an open set $U_V \subseteq \phi_V(X)$. Similarly to that in the proof of Lemma 5.2, define a countable subfamily $\gamma^* \subseteq \gamma$ and the following commutative diagram,

$$\begin{array}{ccc} \Pi & \xrightarrow{\ f\ } & X \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle \phi^*} \\ \Pi^* & \xrightarrow{\ f^*\ } & X^* \end{array} \tag{6}$$

in which $p^* \in \mathscr{L}_\Pi$, $\phi^* \in \mathscr{L}_X$ and the conditions (1)–(3) below are satisfied:

(1) $\phi^* \prec \phi_V$ for all $V \in \gamma^*$;

(2) $V = (\phi^*)^{-1}\phi^*(V)$ for all $V \in \gamma^*$;

(3) $\phi^*(\Gamma) \subseteq f^*(\text{cl}\, p^* f^{-1}(\Gamma^*))$, where $\Gamma = \bigcup \gamma$ and $\Gamma^* = \bigcup \gamma^*$.

Then $\Gamma^* = (\phi^*)^{-1}\phi^*(\Gamma^*)$ by virtue of (2), so $\phi^*(\Gamma^*)$ is open in X^* (use (1)) and dense in $\phi^*(\Gamma)$ by (3). Consequently, $\phi^*(\Gamma \backslash \Gamma^*)$ is a nowhere dense subset of X^* lying in cl $\phi^*(\Gamma^*) \backslash \phi^*(\Gamma^*)$ (apply (2) and (3)). Since γ is a disjoint family, the set $\Gamma \backslash \Gamma^*$ is open in X, and the fact that ϕ^* is skeletal implies $\Gamma \backslash \Gamma^* = \emptyset$. Thus, $\gamma = \gamma^*$, i.e., γ is countable.

Remark 5.17. Now weaken the conditions of Theorem 3.3 and 3.4 on X by assuming that X is a D-space, i.e., suppose that an appropriate weak σ-lattice $\mathscr{L}_X$ consists of d-open mappings. Then $\mathscr{L}_X$ is factorizative (even if the σ-lattice $\mathscr{L}_\Pi$ for Π is not assumed to be *strong*), and X is *perfectly k-normal*, i.e., the closures in X of open sets are zero sets.

Indeed, every d-open mapping is skeletal, so $c(X) \leqslant \aleph_0$ by Remark 5.16. Now apply an argument of [23] as follows. Let $h : X \to Y$ be a continuous mapping of X to a second-countable space Y. Choose a countable base $\mathscr{B}$ for Y. Since $c(X) \leqslant \aleph_0$, one can find for every $U \in \mathscr{B}$ a mapping $\phi_U \in \mathscr{L}_X$ and an open subset V_U of $\phi_U(X)$ so that $\phi_U^{-1}(V_U) \subseteq h^{-1}(U) \subseteq \text{cl } \phi^{-1}(V_U)$. Apply the fact that ϕ_U is d-open to deduce the equalities cl $h^{-1}(U) = \text{cl } \phi^{-1}(V_U) = \phi^{-1}(\text{cl } V_U)$. Therefore, the set $K_U = \text{cl } h^{-1}(U)$ satisfies the condition $K_U = \phi_U^{-1}\phi_U(K_U)$, and $\phi_U(K_U) = \text{cl } V_U$ is closed in $\phi_U(X)$. Since $|\mathscr{B}| \leqslant \aleph_0$, there exists $\phi \in \mathscr{L}_X$ such that $\phi \prec \phi_U$ for all $U \in \mathscr{B}$. One easily verifies that $\phi \prec h$.

To conclude that X is perfectly k-normal, consider an open subset O of X and choose $\phi \in \mathscr{L}_X$ so that cl $O = \phi^{-1}\text{cl } \phi(O)$. Since $\phi(X)$ has countable network, cl $\phi(O)$ is a zero set in $\phi(X)$, whence follows that the set cl O is so in X.

6. Proofs of Theorems 4.2 and 4.3

It is clear that Theorem 4.3 and Corollary 5.7 together imply Theorem 4.2. Therefore, only Theorem 4.3 requires a proof. Furthermore, the proof of Theorem

2.18 of [27] gives an approach to a proof of Theorem 4.3(b); all necessary supplementary results may be found in Sections 5 and 6 here. So we focus our attention on item (a) of Theorem 4.3.

It is not known whether a space X that satisfies the conditions of Theorem 4.3 (or Theorem 4.2) must be $\aleph_0$-cellular. However, such spaces have a somewhat weaker property.

Lemma 6.1. *Suppose two spaces Π and X and their lattices of continuous mappings $\mathscr{L}_\Pi$ and $\mathscr{L}_X$ satisfy the conditions of Theorem 4.3. Also, let $\{\phi_\alpha: \alpha < \omega_1\}$ be a sequence in $\mathscr{L}_X$ and $\{F_\alpha: \alpha < \omega_1\}$ be a sequence of closed sets in X that satisfy the following conditions*:

(1) *$\phi_\beta \prec \phi_\alpha$ and $F_\alpha \subseteq F_\beta$ whenever $\alpha < \beta < \omega_1$;*

(2) *$F_\alpha = \phi_\alpha^{-1}\phi_\alpha(F_\alpha)$, and $\phi_\alpha(F_\alpha)$ is closed in $\phi_\alpha(X)$ for each $\alpha < \omega_1$;*

(3) *$\phi_\beta = w\ \lim_{\alpha<\beta}\phi_\alpha$ for each limit ordinal $\beta < \omega_1$ (see Definition 3.1).*

Then the sequence $\{F_\alpha: \alpha < \omega_1\}$ stabilizes at some step $\beta < \omega_1$.

Proof. Let j be the diagonal product of mappings ϕ_α, $\alpha < \omega_1$. Put $\tilde{X} = j(X)$ and $X_\alpha = \phi_\alpha(X)$, $\alpha < \omega_1$. Then for any $\alpha < \omega_1$ there exists a continuous mapping $\tilde{\phi}_\alpha: \tilde{X} \to X_\alpha$ such that $\phi_\alpha = \tilde{\phi}_\alpha j$. Clearly, $\tilde{X}$ is a continuous image of Π and the family $\mathscr{L}_{\tilde{X}} = \{\tilde{\phi}_\alpha: \alpha < \omega_1\}$ is a weak σ-lattice for $\tilde{X}$. Therefore, one can assume $\tilde{X} = X$ and $\mathscr{L}_{\tilde{X}} = \mathscr{L}_X$. Apply Lemma 5.2 to find a dense triple (p, ϕ_β, γ) where $p \in \mathscr{L}_\Pi$, $\beta < \omega_1$ and $\gamma = \{F_\alpha: \alpha < \beta\}$. Since $F_\alpha \subseteq F_\beta$ for all $\alpha < \beta$, $\phi_\beta(F_\beta)$ is dense in $\phi_\beta(\Gamma)$ where $\Gamma = \bigcup\{F_\nu: \nu < \omega_1\}$. However, $F_\beta = \phi_\beta^{-1}\phi_\beta(F_\beta)$, and $\phi_\beta(F_\beta)$ is closed in X_β, whence $\Gamma = F_\beta$. □

Remark 6.2. The proof of Lemma 6.1 did not use the assumption that the mappings in $\mathscr{L}_X$ are d-open.

Let βZ be the Čech–Stone compactification of Z, and $\mathscr{F}$ a family of closed sets in βZ. We say that *$\mathscr{F}$ separates points of Z from the points of $\beta Z \setminus Z$* if for any $z \in Z$ and $x \in \beta Z \setminus Z$ there is an $F \in \mathscr{F}$ with $z \in F$ and $x \notin F$. We omit the proof of the following simple lemma.

Lemma 6.3. *Suppose a space Π and its strong σ-lattice $\mathscr{L}_\Pi$ satisfy the conditions of Theorem 4.3. Let $p_0 \succ p_1 \succ \cdots$ be a sequence in $\mathscr{L}_\Pi$ and suppose a family $\mathscr{F}_n$ of closed sets in $\Pi_n = p_n(\Pi)$ separates points of Π_n from the points of $\beta\Pi_n \setminus \Pi_n$, $n \in N$. Then the family $\mathscr{F}^* = \{\hat{p}(\hat{p}_n)^{-1}(F): F \in \mathscr{F}_n, n \in N\}$ separates points of $\Pi^* = p(\Pi)$ from the points of $\beta\Pi^* \setminus \Pi^*$, where $p = \Delta_{n=0}^{\infty} p_n$, and $\hat{p}$, $\hat{p}_n$ are continuous extensions of p, p_n over $\beta\Pi$.*

Lemma 6.4. *Suppose R is a $G_{\delta,\Sigma}$-set in a space Y of pointwise-countable type and $y \in \mathrm{cl}_Y R = B$. If $\pi\chi(y, B) \leqslant \aleph_0$ then there exists a countable π-base μ for Y at y such that $U \cap B \neq \emptyset$ for all $U \in \mu$.*

Proof. By assumption, there exists a countable π-base λ for B at y. Then for any $U \in \lambda$, $U \cap R$ is a nonempty $G_{\delta,\Sigma}$-set in Y. Since Y is of pointwise-countable type, one can find a nonempty compact set $K_U \subseteq U \cap R$ so that $\chi(K_U, Y) \leqslant \aleph_0$. Let γ_U be a countable base for Y at K_U. It is easy to see that the family $\mu = \bigcup\{\gamma_U: U \in \lambda\}$ is as required. □

The following lemma is a generalization of [27, Lemma 2.15] to d-open mappings.

Lemma 6.5. *Let $\psi: X \to Z$ be a d-open mapping of X onto a space Z with countable network, and η be a family of open sets in X such that $\mathrm{cl}_X U = \psi^{-1}(\mathrm{cl}_Z \psi(U))$ for all $U \in \eta$. Then for any filter $\mathscr{R}$ on the set η, the upper limit set $F = \bigcap_{R \in \mathscr{R}} \mathrm{cl}_X(\bigcup\{U \in \eta: U \in R\})$ is a G_δ-set in X; moreover, $F = \psi^{-1}\psi(F)$, and $\psi(F)$ is closed in Z.*

Proof. Apply the argument of [27, Lemma 2.15] along with the fact that $\mathrm{cl}\ \psi^{-1}(V) = \psi^{-1}(\mathrm{cl}\ V)$ for any open subset V of Z. □

Thereafter to the end of the proof of Theorem 4.3 we use the following notation. The spaces Π, X, Y and lattices $\mathscr{L}_\Pi$, $\mathscr{L}_X$ for Π and X are assumed to satisfy the conditions of Theorem 4.3. Let $\beta\Pi$ and βX be the Čech–Stone compactifications of Π and X. Denote by $\hat{f}$ the continuous extension of f over $\beta\Pi$. Let e be a lifting operator witnessing that $g: S \to Y$ is an M-mapping. One can assume that e is *monotone*, i.e., $e(U) \subseteq e(O)$ whenever $U \subseteq O$ (see [27, Section 2]). Denote by $\mathscr{T}(y)$ the family of all open neighborhoods of a point $y \in Y$. Put $F_y = \bigcap\{\mathrm{cl}_X e(O): O \in \mathscr{T}(y)\}$. Note that $S \cap F_y = g^{-1}(y)$. Indeed, pick a point $z \in Y$, $z \neq y$. There exist disjoint open sets U and O in Y such that $z \in U$ and $y \in O$. Then $e(U) \cap e(O) = \emptyset$ and $g^{-1}(y) \subseteq g^{-1}(O) \subseteq e(O) \cap S$; hence $F_y \subseteq \mathrm{cl}_X e(O) \subseteq X \setminus g^{-1}(z)$. Consequently, $S \cap F_y = g^{-1}(y)$.

For a subset P of Y, denote by $\tilde{P}$ a union of all G_δ-sets in Y lying in P. If $\mathscr{K}$ is a family of sets in $\beta\Pi$, put $\mathscr{K}(x) = \{K \in \mathscr{K}: x \in K\}$ for each $x \in \beta\Pi$; put also $S_y = S \setminus F_y$.

Lemma 6.6. *Let $\mathscr{K}$ be a countable family of sets in $\beta\Pi$ such that $f^{-1}(S_y) \subseteq \bigcup \mathscr{K}$. If $\chi(y, Y) > \aleph_0$ then there exists a point $x \in f^{-1}(S_y)$ such that $y \in \mathrm{cl}\ \tilde{P}_K$ for all $K \in \mathscr{K}(x)$ where $p_K = \mathrm{cl}\ g(f(K) \cap S)$.*

Proof. Assume for contradiction that the lemma is false for some family $\mathscr{K}$. Then for every $x \in f^{-1}(S_y)$ there exists $K(x) \in \mathscr{K}(x)$ with $y \notin \tilde{P}_{K(x)}$. Put $\xi = \{P_{K(x)}: x \in f^{-1}(S_y)\}$; clearly, $|\xi| \leqslant |\mathscr{K}| \leqslant \aleph_0$. Since $Y \setminus \{y\} = g(S_y)$ and $gf(x) \in P_{K(x)}$ for each $x \in f^{-1}(S)$, we have $Y \setminus \{y\} \subseteq \bigcup \xi$. If $y \notin \bigcup \xi$ then $\psi(y, Y) \leqslant |\xi| \leqslant \aleph_0$; since Y is of pointwise-countable type, $\chi(y, Y) \leqslant \aleph_0$, a contradiction. So $Y = \bigcup \xi$. For every $P \in \xi$ choose $V_P \in \mathscr{T}(y)$ so that $V_P \cap \tilde{P} = \emptyset$. Then $G = \bigcap\{V_P: P \in \xi\}$ is a G_δ-set in Y. Since Y is of pointwise-countable type, one can find a compact G_δ-set G' in Y so that $y \in G' \subseteq G$. By Lemma 2.16 of [27], there exists a nonempty G_δ-set H in Y such that $H \subseteq G'$ and for every $P \in \xi$ either $H \cap P = \emptyset$ or $H \subseteq P$. Since ξ is cover of Y, $P^* \cap H \neq \emptyset$ for some $P^* \in \xi$. Now the definition of H implies $H \subseteq P^*$, whence $H \subseteq \tilde{P}^*$. The last inclusion contradicts the facts that $H \subseteq G \subseteq V_{P^*}$ and $V_{P^*} \cap \tilde{P}^* = \emptyset$. □

If μ is a family of open sets in Y, we put $V_O(\mu) = \bigcup\{e(U): U \in \mu, U \subseteq O\}$ for each $O \in \mathscr{T}(y)$ and then define the set $F_y(\mu) = \bigcap\{\text{cl}_X V_O(\mu): O \in \mathscr{T}(y)\}$. Clearly, $F_y(\mu) \subseteq F_y$ for any family μ of open sets in Y. The following key lemma incorporates the most difficult technical details.

Lemma 6.7. *Suppose $\mathscr{B}$ is a base for Y, $y \in Y$, and $\aleph_0 = t(y, Y) < \chi(y, Y)$. Then for any countable family $\mu \subseteq \mathscr{B}$ there exists a countable family $\mu^* \subseteq \mathscr{B}$ and a mapping $\phi \in \mathscr{L}_X$ such that*

(a) *$\mu \subseteq \mu^*$ and $F_y(\mu)$ is a proper subset of $F_y(\mu^*)$;*

(b) *$F_y(\mu^*) = \phi^{-1}\phi(F_y(\mu^*))$, and $\phi(F_y(\mu^*))$ is closed in $\phi(X)$.*

Proof. Let B be an arbitrary closed subset of Y, and $y \in B \cap M$. Then B is of pointwise-countable type, so one can find a compact set B_0 with $y \in B_0 \subseteq B$ and $\chi(B_0, B) \leqslant \aleph_0$. By a theorem in [15], $\pi\chi(y, B_0) \leqslant t(y, B_0)$. Lemma 1 of [2] implies $\pi\chi(y, B) \leqslant \pi\chi(y, B_0) \cdot \chi(B_0, B)$. Since $t(y, B_0) \leqslant t(y, Y) \leqslant \aleph_0$, we have $\pi\chi(y, B) \leqslant \aleph_0$. Thus, the hereditary π-character of Y at any point y of M is countable.

By Remark 5.17, X is perfectly k-normal and the lattice $\mathscr{L}_X$ is factorizative. Hence for each $U \in \mu$ there exists ϕ_U in $\mathscr{L}_X$ such that $\text{cl}_X e(U) = \phi^{-1}(\text{cl}\, \phi_U e(U))$. Choose $\phi_0 \in \mathscr{L}_X$ so that $\phi_0 \prec \phi_U$ for all $U \in \mu$. Then the above equality is valid for ϕ_0 instead of ϕ_U, so Lemma 6.5 implies that the set $F = F_y(\mu)$ satisfies $F = \phi_0^{-1}\phi_0(F)$ and that $\phi_0(F)$ is closed in $X_0 = \phi_0(X)$. Since $\text{nw}(X_0) \leqslant \aleph_0$, there exists a countable family λ of closed sets in X_0 such that $X_0 \setminus \phi_0(F) = \bigcup \lambda$. Furthermore, X_0 admits a continuous bijection onto a second-countable space. This fact and the factorizative property of the lattice $\mathscr{L}_\Pi$ imply that there exists $p_0 \in \mathscr{L}_\Pi$ with $p_0 \prec \phi_0 f$; now one can find a continuous mapping f_0 of $\Pi_0 = p_0(\Pi)$ onto X_0 such that $f_0 p_0 = \phi_0 f$. Let $\hat{f}_0 : \beta\Pi_0 \to \beta X_0$ and $\hat{p}_0 : \beta\Pi \to \beta\Pi_0$ be continuous extensions of f_0 and p_0. Put $\hat{\lambda} = \{\text{cl}_{\beta X_0} Q: Q \in \lambda\}$ and $\theta = \{(\hat{f}_0 \hat{p}_0)^{-1}(L): L \in \hat{\lambda}\}$. Then the equalities $F = \phi_0^{-1}\phi_0(F)$ and $f_0 p_0 = \phi_0 f$ imply

(0) $\Pi \cap (\bigcup \theta) = \Pi \setminus f^{-1}(F)$ and $K = (\hat{p}_0)^{-1}\hat{p}_0(K)$ for any $K \in \theta$.

Since $p_0 \in \mathscr{L}_\Pi$, Π_0 is a Lindelöf Σ-space. Therefore [27,30], one can find a countable family $\mathscr{L}_0$ of closed sets in $\beta\Pi_0$ which separates points of Π_0 from the points of $\beta\Pi_0 \setminus \Pi_0$. Denote by $\mathscr{K}_0$ the minimal family of closed sets in $\beta\Pi$ containing the family $\theta \cup \{(\hat{p}_0)^{-1}(L): L \in \mathscr{L}_0)$ and closed under finite intersections. Put $\mu_0 = \mu$.

Let $n \in N$ and suppose we have already defined for all $k < n$ the mappings $\phi_k \in \mathscr{L}_X$, $p_k \in \mathscr{L}_\Pi$ and the countable families μ_k, $\mathscr{L}_k$, $\mathscr{K}_k$ satisfying the following conditions:

(1) $p_k \prec \phi_k f$;

(2) $\mathscr{L}_k$ separates points of $\Pi_k = p_k(\Pi)$ from the points of $\beta\Pi_k \setminus \Pi_k$;

(3) $\mathscr{K}_k = \{(\hat{p}_k)^{-1}(L): L \in \mathscr{L}_k\}$;

(4) $\text{cl}_X e(U) = \phi_k^{-1}(\text{cl}_{X_k} \phi_k e(U))$ for all $U \in \mu_k$, where $X_k = \phi_k(X)$;

(5) if $K \in \mathscr{K}_{k-1}$ and $y \in \mathrm{cl}\,\tilde{P}_k$, then, for every $O \in \mathscr{T}(y)$, there exists $U \in \mu_k$ such that $U \subseteq O$ and $U \cap P_K \neq \emptyset$.

Obviously, the conditions (1)–(5) for $k = 0$ are fulfilled. Put $\mathscr{F}_n = \{f(K): K \in \mathscr{K}_n\}$ and $\xi_n = \{\mathrm{cl}_Y g(F \cap S): F \in \mathscr{F}_n\}$. Then the families $\mathscr{F}_n$ and ξ_n are countable and the inclusion $\Pi \subseteq \bigcup \mathscr{K}_n$ implies $X \subseteq \bigcup \xi_n$. Put $\tilde{\xi}_n = \{P \in \xi_n: y \in \mathrm{cl}\,\tilde{P}\}$. Making use of Lemma 6.4, choose for every $P \in \tilde{\xi}_n$ a countable π-base ν_P for Y at y so that $U \cap P \neq \emptyset$ for each $U \in \nu_P$. Then the countable family $\mu_{n+1} = \mu_n \cup (\bigcup\{\nu_P: P \in \tilde{\xi}_n\})$ satisfies the condition (5) for $k = n + 1$. An argument similar to that for $n = 0$ is applied to define a mapping $\phi_{n+1} \in \mathscr{L}_X$ satisfying (4) and a mapping $p_{n+1} \in \mathscr{L}_\Pi$ satisfying (1). One can choose these mappings to satisfy the natural conditions $\phi_{n+1} \prec \phi_n$ and $p_{n+1} \prec p_n$. Now define countable families $\mathscr{L}_{n+1}$ and $\mathscr{K}_{n+1}$ satisfying (2) and (3) such that $\mathscr{K}_n \subseteq \mathscr{K}_{n+1}$ and $\mathscr{K}_{n+1}$ is closed under finite intersections.

Put $\phi = w\ \lim_n \phi_n$, $p = \Delta_{n=0}^{\infty} p_n$, $\mu^* = \bigcup_{n=0}^{\infty} \mu_n$ and $\mathscr{K} = \bigcup_{n=0}^{\infty} \mathscr{K}_n$. We claim that μ^*, ϕ and $F_y(\mu^*) = F^*$ are as required. Note that $\phi \in \mathscr{L}_X$ and $p \in \mathscr{L}_\Pi$. Put $\Pi^* = p(\Pi)$ and $X^* = \phi(X)$. We have $p \prec p_n \prec \phi_n f$ for all $n \in N$, and since p is quotient, $p \prec \phi f$. Consequently, there exists a continuous mapping $f_* : \Pi^* \to X$ such that $f_* p = \phi f$. Since $\phi \prec \phi_n$ for each $n \in N$, (4) implies

(6) $\mathrm{cl}_X e(U) = \phi^{-1}(\mathrm{cl}_{X^*} \phi e(U))$ for all $U \in \mu^*$.

Now apply (6) and Lemma 6.5 to conclude that $F = \phi^{-1}\phi(F^*)$, and $\phi(F^*)$ is closed in X^*. The inclusion $F = F_y(\mu) \subseteq F_y(\mu^*) = F^*$ follows from $\mu \subseteq \mu^*$. We are left to verify $F^* \setminus F \neq \emptyset$.

Extend p to a continuous mapping $\hat{p} : \beta\Pi \to \beta\Pi^*$. Then (3) and the fact that $p \prec p_n$ for all n together imply

(7) $K = (\hat{p})^{-1}\hat{p}(K)$ for each $K \in \mathscr{K}$.

Furthermore, (2), (3) and Lemma 6.3 give

(8) the family $\mathscr{L} = \{\hat{p}(K): K \in \mathscr{K}\}$ separates points of Π^* from the points of $\beta\Pi^* \setminus \Pi^*$.

Apply Lemma 6.6 to find a point $x \in f^{-1}(S)$ so that $x \in \mathrm{cl}\,\tilde{P}_k$ for all $K \in \mathscr{K}(x)$. By (5), we have $\hat{f}(K) \cap V_O(\mu^*) \neq \emptyset$ for all $O \in \mathscr{T}(y)$ and $K \in \mathscr{K}(x)$. Therefore, $\hat{f}(R) \cap \mathrm{cl}_{\beta X} F^* \neq \emptyset$ where $R = \bigcap \mathscr{K}(x)$ (use the compactness of $\beta\Pi$ and the closedness of $\mathscr{K}(x)$ under finite intersections). This in turn implies $\phi f(R) \cap \phi(F_0^*) \neq \emptyset$ or equivalently, $\hat{f}_* \hat{p}(R) \cap \hat{\phi}(F_0^*) \neq \emptyset$ where $F_0^* = \mathrm{cl}_{\beta X} F^*$ and $\hat{\phi}$ and $\hat{f}_*$ are continuous extensions of ϕ and f_* over βX and $\beta\Pi$ respectively. Put $x_0 = p(x)$ and $T = \bigcap \mathscr{L}(x_0)$. Then (7) and (8) imply that $R = (\hat{p})^{-1}(T)$ and that T is a nonempty compact subset of Π^*. Thus, $\hat{f}_*(T) \cap \hat{\phi}(F_0^*) \neq \emptyset$ and $\hat{f}_*(T) = f_*(T) \subseteq X^*$. Since $F^* = \phi^{-1}\phi(F^*)$ and $\phi(F^*)$ is closed in X^*, we have $\hat{\phi}(F_0^*) \cap X^* = \phi(F^*)$. Pick points $t \in T$ and $r \in R \cap \Pi$ so that $f_*(t) \in \phi(F^*)$ and $p(r) = t$. Then $\phi f(r) = f_* p(r) = f_*(t) \in \phi(F^*)$, and the equality $F^* = \phi^{-1}\phi(F^*)$ implies

(9) $f(r) \in F^* \cap f(R \cap \Pi)$.

We claim that $f(R \cap \Pi) \cap F = \emptyset$. Indeed, $F \subseteq F_y$ by the monotonicity of e. From the choice of x follows $f(x) \in S \setminus F_y \subseteq S \setminus F$, i.e., $x \in \Pi \setminus f^{-1}(F)$. Apply (0) to find $K \in \theta$ with $x \in K$. Since $\theta \subseteq \mathscr{K}_0 \subseteq \mathscr{K}$, the definition of R implies $R \subseteq K$ and (0) in turn implies $f(R \cap \Pi) \subseteq f(K \cap \Pi) \subseteq X \setminus F$. Apply (9) to conclude that $f(r) \in F^* \setminus F$. □

Proof of Theorem 4.3(a). We divide the proof into two steps.

Step 1. $\chi(y, Y) \leqslant \aleph_0$ for each $y \in M$.

Assume for contradiction that $\chi(y, Y) > \aleph_0$ for some $y \in M$. Making use of Lemma 6.7, define sequences $\{\mu_\alpha\colon \alpha < \omega_1\}$ and $\{\phi_\alpha\colon \alpha < \omega_1\} \subseteq \mathscr{L}_X$ so that the following conditions hold for all $\alpha < \omega_1$:

(a) μ_α is a countable family of closed sets in Y;

(b) $\mu_\alpha \subseteq \mu_\beta$ and $\phi_\beta \prec \phi_\alpha$ whenever $\alpha < \beta < \omega_1$;

(c) $\mu_\beta = \bigcup_{\alpha<\beta}\mu_\alpha$ and $\phi_\beta = w\ \lim_{\alpha<\beta}\phi_\alpha$ for each limit ordinal $\beta < \omega_1$;

(d) $F_y(\mu_\alpha) = \phi_\alpha^{-1}\phi_\alpha(F_y(\mu_\alpha))$, and $\phi_\alpha(F_y(\mu_\alpha))$ is closed in $\phi_\alpha(X)$;

(e) $F_y(\mu_\alpha)$ is a proper subset of $F_y(\mu_{\alpha+1})$.

Put $F_\alpha = F_y(\mu_\alpha)$ for each $\alpha < \omega_1$. Clearly, the sequences $\{F_\alpha\colon \alpha < \omega_1\}$ and $\{\phi_\alpha\colon \alpha < \omega_1\}$ satisfy the conditions (1)–(3) of Lemma 6.1. However, (e) contradicts the conclusion of Lemma 6.1.

Step 2. M has countable network.

Assume the contrary and define sequences $\{\phi_\alpha\colon \alpha < \omega_1\} \subseteq \mathscr{L}_X$ and $\{F_\alpha\colon \alpha < \omega_1\}$ satisfying the conditions (1)–(3) of Lemma 6.1. Following [30], define for every $\phi \in \mathscr{L}_X$ a closed subset Z_ϕ of $\phi(X)$ as follows. Let $\Gamma = \{U_1, U_2\}$ be an open cover of Y. Denote by V_i the maximal open subset of $\phi(X)$ with $\phi^{-1}(V_i) \subseteq \mathrm{cl}_X e(U_i)$, $i = 1, 2$. Put $Z_\Gamma = \mathrm{cl}_{\phi(X)}(V_1 \cup V_2)$ and $Z_\phi = \bigcap_\Gamma Z_\Gamma$, where Γ runs through all two-element open covers of X. By [30, Lemma 13], there exists the unique continuous mapping g_ϕ of $Z_\phi \cap \phi(S)$ to Y such that $g(x) = g_\phi(\phi(x))$ for all $x \in S \cap \phi^{-1}(Z_\phi)$.

Let $\beta < \omega_1$ and suppose we have already defined for all $\alpha < \beta$ the mappings $\phi_\alpha \in \mathscr{L}_X$ and the closed sets $Z_\alpha = Z_{\phi_\alpha} \subseteq \phi_\alpha(X)$ and $F_\alpha = \phi^{-1}(Z_\alpha)$. Put $g_\alpha = g_{\phi_\alpha}$; $g_\alpha : Z_\alpha \cap \phi_\alpha(S) \to Y$. If β is limit, put $\phi_\beta = w\ \lim_{\alpha<\beta}\phi_\alpha$ and define Z_β and F_β as above.

Now suppose $\beta = \alpha + 1$. Then we have

$$\mathrm{nw}(g(F_\alpha \cap S)) = \mathrm{nw}(g_\alpha(Z_\alpha \cap \phi_\alpha(S))) \leqslant \mathrm{nw}(Z_\alpha) \leqslant \mathrm{nw}(\phi_\alpha(X)) \leqslant \aleph_0,$$

so the assumption $\mathrm{nw}(M) > \aleph_0$ implies $M \setminus g_\alpha(F_\alpha \cap S) \neq \emptyset$. Pick a point y in this set. By Step 1 of the proof, there exists a countable base γ for Y at y. Similarly to that in Step 1, for every $U \in \gamma$ there exists $\phi_U \in \mathscr{L}_X$ such that $\mathrm{cl}\ e(U) = \phi^{-1}\mathrm{cl}\ \phi_U(e(U))$. Choose $\phi_\beta \in \mathscr{L}_X$ so that $\phi_\beta \prec \phi_\alpha$ and $\phi_\beta \prec \phi_U$ for all $U \in \gamma$. The definition of Z_β and F_β is clear. The inclusion $F_\alpha \subseteq F_\beta$ follows from $\phi_\beta \prec \phi_\alpha$. The definition of F_β implies $y \in f(F_\beta \cap S)$, so $F_\beta \setminus F_\alpha \neq \emptyset$.

Thus, the mappings ϕ_α and the sets F_α, $\alpha < \omega_1$, satisfy the conditions (1)–(3) of Lemma 6.1. By the construction, $F_{\alpha+1} \setminus F_\alpha \neq \emptyset$ for each $\alpha < \omega_1$, which contradicts Lemma 6.1. □

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