Some further results on weighted sharing three values and Brosch’s theorem

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1. Introduction and main results

In this paper, by meromorphic functions we shall always mean meromorphic functions in the complex plane. Let \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions and let \( a \) be a complex number. We say that \( f \) and \( g \) share the value \( a \) CM (IM) provided that \( f- a \) and \( g- a \) have the same zeros counting multiplicities (ignoring multiplicities). Let \( a, b \in \mathbb{C} \cup \{ \infty \} \), if \( f = a \) when \( g = b \), then we denote it by \( g = b \Rightarrow f = a \). We write \( f = a \iff g = b \) to mean that \( f = a \) if and only if \( g = b \). For standard notations and definitions of value distribution theory we refer to [1].

Let us introduce some definitions (see [2,3]).

Definition 1. Let \( p \) be a positive integer, we denote by \( N_p(f) \) (or \( \overline{N}_p(f) \)) the counting function of poles of \( f \) with multiplicities \( \leq p \) (ignoring multiplicities). We further define

\[
N_{(p+1)} = N(f) - N_p(f), \quad \overline{N}_{(p+1)} = \overline{N}(f) - \overline{N}_p(f).
\]

Definition 2. Let \( k \) be a nonnegative integer or infinity. For any \( a \in \mathbb{C} \cup \{ \infty \} \), we denote by \( E(a, f) \) the set of all zeros of \( f - a \) (ignoring multiplicities), and by \( \overline{E}_k(a, f) \) the set of the zeros of \( f - a \) with multiplicity \( \leq k \) (ignoring multiplicities).

Definition 3. Let \( k \) be a nonnegative integer or infinity. For any \( a \in \mathbb{C} \cup \{ \infty \} \), we denote by \( E_k(a, f) \) the set of all \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a, f) = E_k(a, g) \), we say that \( f, g \) share the value \( a \) with weight \( k \), we also say \( f, g \) share \((a,k)\).
Clearly, if \( f \) and \( g \) share \((a, k)\), then \( f \) and \( g \) share \((a, p)\) for all integers \( p \), \(0 \leq p < k\). Also, we note that \( f \) and \( g \) share a value \( a \) IM or CM if and only if \( f \) and \( g \) share \((a, 0)\) or \((a, \infty)\), respectively.

In 1989, Brosch [4] improved the four value theorem in another direction and proved the following result.

**Theorem A.** Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing \( 0, 1, \infty \) CM. Let \( a, b \) be two complex numbers such that \( a, b \not\in \{0, 1, \infty\} \). If \( f - a \) and \( g - b \) share \( 0 \) IM, then \( f \) is a bilinear transformation of \( g \).

In 2006, Alzahary [5] proved the following result:

**Theorem B.** Let \( f \) and \( g \) be two nonconstant meromorphic functions sharing \((a_1, 1), (a_2, \infty), (a_3, \infty)\), where \( \{a_1, a_2, a_3\} = \{0, 1, \infty\} \), and let \( a \) and \( b \) be two finite complex numbers such that \( a, b \not\in \{0, 1\} \) and \( E_2(a, f) \leq E(b, g) \). If \( f \) is not a fractional linear transformation of \( g \), then

\[
N_{12} \left( r, \frac{1}{f - a} \right) = 0, \quad N_{12} \left( r, \frac{1}{f - b} \right) + N_{12} \left( r, \frac{1}{g - b} \right) = S(r),
\]

\[
f^*(f - b) = g^*(g - b), \quad f^*(f - 1) = g^*(g - 1),
\]

and \( f \) and \( g \) assume one of the following forms:

1. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( a = 3/4 \) and \( b = 3/2 \);
2. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( a = -3 \) and \( b = 3/2 \);
3. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( a = 4/3 \) and \( b = 1/3 \);
4. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{2y - 1}/e^{e - 1} \), with \( a = -1/3 \) and \( b = 2/3 \);
5. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( a = 1/4 \) and \( b = -2 \);
6. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( a = 4 \) and \( b = -1/2 \);
7. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{2y - 1}/e^{e - 1} \), with \( \lambda \neq 1 \) \((\alpha \lambda)^2 = 4(a - 1)\) and \( b = 2 \);
8. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{2y - 1}/e^{e - 1} \), with \( \lambda \neq 1 \) \((4a \lambda)(1 - a) = 1\) and \( b = 1/2 \);
9. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{2y - 1}/e^{e - 1} \), with \( \lambda \neq 1 \) \((1 - a)^2 + 4a \lambda = 0\) and \( b = -1 \);

where \( \gamma \) is a nonconstant entire function.

**Remark 1.** From the conclusion of **Theorem B**, we get that \( a \neq b \), that is to say, if \( a = b \), then \( f \) is a fractional linear transformation of \( g \).

Recently, Han and Yi [6] improved a result of Li and Yi [7]. In fact, they proved the following theorem.

**Theorem C.** Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing \((0, k_1), (1, k_2), (\infty, k_3)\), with \( k_1k_2k_3 \geq k_1 + k_2 + k_3 + 2\), if for some \( a \in C \setminus \{0, 1\} \), we have

\[
N_{11} \left( r, \frac{1}{f - a} \right) \neq T(r, f) + S(r, f),
\]

then \( f \) and \( g \) share \( 0, 1, \infty \) CM,

\[
N_{11} \left( r, \frac{1}{f - a} \right) = \frac{k - 2}{k} T(r, f) + S(r, f),
\]

and one of the following cases will hold:

1. \( f = e^{(a + 1)y - 1}/e^{e - 1} \), \( g = e^{(a + 1)y - 1}/e^{e - 1} \), with \( \frac{(a + 1) k + 1 + s}{(a + 1) k + 1 - s} = \frac{s^2(k + 1 - a)k + 1 - s}{(k + 1) k + 1} \) and \( a \neq \frac{k + 1}{2} \);
2. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( a^2(1 - a)k + 1 - s = \frac{s^2(k + 1 - a)k + 1 - s}{(k + 1) k + 1} \) and \( a \neq \frac{k}{2} \);
3. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( (-a)^{k + 1} \) \((a + 1)^{k + 1} = \frac{s^2(k + 1 - a)k + 1 - s}{(k + 1) k + 1} \) and \( a \neq \frac{k - 1}{2} \);
4. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( \frac{(a - 1)^{k - 1}}{a^{k - 1}} = \frac{s^2(k - a)k + s}{k^2} \) and \( \lambda \neq 0, 1 \);
5. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( \lambda^2(a - 1)^{k + 1} = \frac{s^2(k - a)k + s}{k^2} \) and \( \lambda \neq 0, 1 \);
6. \( f = e^{3y - 1}/e^{e - 1} \), \( g = e^{3y - 1}/e^{e - 1} \), with \( \frac{(a - 1)^{k - 1}}{a^{k - 1}} = \frac{s^2(k - a)k + s}{k^2} \) and \( \lambda \neq 0, 1 \);
where \( \gamma \) is a nonconstant entire function, \( s \) and \( k \geq 2 \) are positive integers such that \( s \) and \( k + 1 \) are mutually prime and \( 1 \leq s \leq k \) in (1), (2) and (3), \( s \) and \( k \) are mutually prime and \( 1 \leq s \leq k - 1 \) in (4), (5) and (6).

It is natural to ask whether the value-sharing assumptions of Theorem 1 are weakened anymore? Is it still true if we replace the assumption \( E_2(a, f) \subseteq E(b, g) \) by \( E_1(a, f) \subseteq E(b, g) \) in Theorem A? In general, the answer is negative. The following counterexample shows this:

**Example.** Let

\[
 f = \frac{e^{3\gamma} - 1}{e^{\gamma} - 1}, \quad g = \frac{e^{-3\gamma} - 1}{e^{-\gamma} - 1}, \quad a = 3/4 \text{ and } b \neq 3.
\]

Then \( f \) and \( g \) share 0, 1, \( \infty \) CM, and

\[
f - a = (e^{\gamma} - 1/2)^2,
\]

thus \( f - a \) only has multiple zeros and \( E_1(a, f) \subseteq E(b, g) \), but we cannot get \( b = 3 \) in the first case of Theorem A.

In this paper, we study the problem and get the following theorem.

**Theorem 1.1.** Let \( f \) and \( g \) be two nonconstant meromorphic functions sharing \((a_1, k_1), (a_2, k_2), (a_3, k_3)\), where \( a_1, a_2, a_3 \in \{0, 1, \infty\} \) and \( k_1k_2k_3 \geq k_1 + k_2 + k_3 + 2 \), and let \( a \) and \( b \) be two finite complex numbers such that \( a, b \notin \{0, 1\} \) and \( E_1(a, f) \subseteq E(b, g) \). If \( f \) is not a fractional linear transformation of \( g \), then \( f \) and \( g \) assume one of the following forms:

1. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = 3/4 \);  
2. \( f = 2^a \), \( g = 2^{-a} \), with \( a = -3 \);  
3. \( f = 2^a \), \( g = 2^{-a} \), with \( a = 4 \);  
4. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = -1 \);  
5. \( f = 2^a \), \( g = 2^{-a} \), with \( a = 1 \);  
6. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = 4 \);  
7. \( f = 2^a \), \( g = 2^{-a} \), with \( a = 1 \);  
8. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = 4 \);  
9. \( f = 2^a \), \( g = 2^{-a} \), with \( a = 1 \);  
10. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = 4 \);  
11. \( f = 2^a \), \( g = 2^{-a} \), with \( a = -1 \);  
12. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = -3 \);  
13. \( f = 2^a \), \( g = 2^{-a} \), with \( a = -1 \);  
14. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = -3 \);  
15. \( f = 2^a \), \( g = 2^{-a} \), with \( a = -1 \);  
16. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = -3 \);  
17. \( f = 2^a \), \( g = 2^{-a} \), with \( a = -1 \);  
18. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = -3 \);  
19. \( f = 2^a \), \( g = 2^{-a} \), with \( a = -1 \);  
20. \( f = 2^a \), \( g = 2^{a/3} \), with \( a = -3 \);  
21. \( f = 2^a \), \( g = 2^{-a} \), with \( a = -1 \);

where \( \gamma \) is a nonconstant entire function and \( A = \frac{3^3}{4^4}, B = \frac{2^2}{3^2} \).

**Remark 2.** From the conclusion of Theorem 1.1, we can easily get the values of \( a \) and \( b \). And the value \( a \) can equal value \( b \), which is different from Theorem B.

In 2005, Alzahary [6] discussed the small meromorphic functions and proved the following result:
Let $f$ and $g$ be two nonconstant distinct meromorphic functions sharing 0, 1, $\infty$ CM, and let $a (\neq 0, 1)$ be a small meromorphic function of $f$ and $g$. If

$$N \left( r, \frac{1}{f - a} \right) \neq T(r, f) + S(r, f),$$

then $N \left( r, \frac{1}{f - a} \right) = S(r, f)$, and $f$ and $g$ satisfy one of the following three relations:

(i) $(f - a)(g + a - 1) = a(1 - a)$;
(ii) $f + (a - 1)g = a$;
(iii) $f = ag$.


**Theorem E.** Let $f$ and $g$ be two nonconstant distinct meromorphic functions sharing 0, 1, $\infty$ CM, and let $a (\neq 0, 1)$ be a nonconstant small meromorphic function of $f$ and $g$ such that $E_1(a, f) \subseteq \overline{E}(a, g)$, then $f$ and $g$ satisfy one of the relations (i)–(iii) in Theorem D.

Naturally, we will ask what will happen when the assumption $E_1(a, f) \subseteq \overline{E}(a, g)$ is replaced by $E_1(a, f) \subseteq \overline{E}(b, g)$ in Theorem D, where $b \neq a$ is also a small meromorphic function? In this paper we solve the problem and prove

**Theorem 1.2.** Let $f$ and $g$ be two nonconstant distinct meromorphic functions sharing 0, 1, $\infty$ CM, and let $a, b (\neq a, \infty)$ be a nonconstant small meromorphic function of $f$ and $g$ such that $E_1(a, f) \subseteq \overline{E}(b, g)$.

If

$$N \left( r, \frac{1}{f - a} \right) \neq T(r, f) + S(r, f),$$

then $N \left( r, \frac{1}{f - a} \right) = S(r, f)$, and $f, g$ satisfy one of the relations (i)–(iii) in Theorem C.

If

$$N \left( r, \frac{1}{f - a} \right) = T(r, f) + S(r, f),$$

then $f$ is a fractional linear transformation of $g$ and $f, g$ satisfy one of the following relations:

1. $(f - 1)(b - 1) = (g - 1)(a - 1)$;
2. $\frac{g(1 - a)}{f(1 - b)} = \frac{(g - 1)b}{(f - 1)a}$;
3. $bf = ag$;
4. $fg = ab$ and $ab$ is a constant;
5. $f + g = 1$ and $a + b = 1$;
6. $f = \frac{a - 1}{b - 1}g + \frac{b - a}{b - 1}$, where $\frac{a - 1}{b - 1}$ and $\frac{b - a}{b - 1}$ are constants;
7. $f = \frac{a}{g - 1}$ and $a = \frac{b}{b - 1}$;
8. $f = \frac{a(1 - b)g}{(a - b)(g(1 - b))}$, where $\frac{a(1 - b)}{a - b}$ and $\frac{b(a - 1)}{a - b}$ are two constants.

From Theorem 1.2, we can easily get the corollary.

**Corollary 1.3.** Let $f$ and $g$ be two nonconstant meromorphic functions sharing 0, 1, $\infty$ CM, and let $a, b (\neq a, \infty)$ be a nonconstant small meromorphic function of $f$ and $g$ such that $E_1(a, f) \subseteq \overline{E}(b, g)$. If

$$\lim_{\substack{a \to \infty \\ r \notin E}} N \left( r, \frac{1}{f - a} \right) \neq 0, 1,$$

then $f \equiv g$.

Since $f$ and $g$ share the values $a_1, a_2, a_3$ IM, then we have $S(r, f) = S(r, g)$. In what follows, we denote this term by $S(r)$ for the sake of brevity.
2. Some lemmas

Lemma 2.1 ([6]). Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing \( (a_1, k_1), (a_2, k_2), (a_3, k_3) \), where \( \{a_1, a_2, a_3\} = \{0, 1, \infty\} \) and \( k_1 k_2 k_3 \geq k_1 + k_2 + k_3 + 2 \). If \( f \) is not a fractional linear transformation of \( g \), then for any \( a \in \mathbb{C} \setminus \{0, 1\} \),

\[
T(r, f) + T(r, g) = N_{12} \left( r, \frac{1}{f - 1} \right) + N_{12} \left( r, \frac{1}{f} \right) + N_{13}(r, f) + \overline{N}_0(r) + S(r)
\]

and

\[
T(r, f) = N \left( r, \frac{1}{f - a} \right) + S(r, f).
\]

Here, \( \overline{N}_0(r) \) denotes the reduced counting function of the zeros of \( f - g \) that are not zeros of \( f(f - 1), 1/f \).

Lemma 2.2 ([10]). Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing \( 0, 1, \infty \) CM. If

\[
\lim_{r \to \infty} \frac{N_0(r)}{T(r, f)} > 1/2,
\]

then \( f \) is a fractional linear transformation of \( g \). Here, \( N_0(r) \) denotes the counting function of the zeros of \( f - g \) that are not zeros of \( f(f - 1), 1/f \).

Lemma 2.3 ([11]). Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing \( (a_1, k_1), (a_2, k_2), (a_3, k_3) \), where \( \{a_1, a_2, a_3\} = \{0, 1, \infty\} \) and \( k_1 k_2 k_3 \geq k_1 + k_2 + k_3 + 2 \). Then, for \( h \in \{f, g\} \), we have

\[
\overline{N}_2(r, h) + \overline{N}_2 \left( r, \frac{1}{h} \right) + \overline{N}_2 \left( r, \frac{1}{h - 1} \right) = S(r).
\]

Lemma 2.4 ([1]). Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing \( 0, 1, \infty \) CM. Then

\[
f = \frac{e^\alpha - 1}{e^\beta - 1}, \quad g = \frac{e^{-\alpha} - 1}{e^{-\beta} - 1}
\]

where \( e^\alpha \neq 1, e^\beta \neq 1 \) and \( e^{\alpha - \beta} \neq 1 \), and

\[
T(r, g) + T(r, e^\alpha) + T(r, e^\beta) = O(T(r, f)) \quad (r \not\in E).
\]

Lemma 2.5 ([10]). Let \( f_1 \) and \( f_2 \) be two nonconstant meromorphic functions satisfying

\[
\overline{N}(r, f_i) + \overline{N} \left( r, \frac{1}{f_i} \right) = S(r), \quad i = 1, 2.
\]

Then either

\[N_0(r, 1, f_1, f_2) = S(r),\]

or there exist two integers \( s, t (|s| + |t| > 0) \) such that

\[f_1^s f_2^t = 1,\]

where \( N_0(r, 1, f_1, f_2) \) denotes the reduced counting function of \( f_1 \) and \( f_2 \) related to the common 1-points, \( T(r) = T(r, f_1) + T(r, f_2) \) and \( S(r) = o(T(r)) \) \((r \to \infty, r \not\in E)\) only depending on \( f_1 \) and \( f_2 \).

Lemma 2.6 ([12]). Let \( f \) be a nonconstant meromorphic function that satisfies the following Riccati differential equation

\[
f' = a_0 + a_1 f + a_2 f^2,
\]

where \( a_0, a_1 \) and \( a_2 \neq 0 \) are small meromorphic functions related to \( f \), which means that \( T(r, a_j) = S(r, f) \) for \( j = 0, 1, 2 \). Then, for any small meromorphic function \( \omega \) of \( f \), which obviously could be some constant anyway, if it is a solution of equation (2.2), then we have \( \overline{N}(r, \frac{1}{\omega}) = S(r, f) \), while if it is not a solution of equation (2.2), then we have \( T(r, f) = \overline{N}(r, \frac{1}{\omega}) + S(r, f) \).

Lemma 2.7 ([19]). Let \( f \) and \( g \) be two nonconstant distinct meromorphic functions sharing \( 0, 1, \infty \) CM, and let \( a (\neq \infty) \) be a nonconstant small meromorphic function of \( f \) and \( g \), then \( N_{12} \left( r, \frac{1}{r - a} \right) = S(r, f) \) and \( N_{12} \left( r, \frac{1}{g - a} \right) = S(r, f) \).
3. Proof of Theorem 1.1

From the assumption of Theorem 1.1, without loss of generality, we assume that \( f \) and \( g \) share \((0, k_1), (1, k_2)\) and \((\infty, k_3)\). In the following, we distinguish two cases.

First case. Suppose that \( a \) is not a Picard value of \( f \) and

\[
N_1 \left( r, \frac{1}{f - a} \right) \neq T(r, f) + S(r, f). \tag{3.1}
\]

From Theorem C, we can get (1.1) and six cases. Next, we discuss the six cases.

Case 1.1.

\[
f = \frac{e^{(k+1)\gamma} - 1}{e^{\gamma} - 1}, \quad g = \frac{e^{-(k+1)\gamma} - 1}{e^{-\gamma} - 1}, \tag{3.2}
\]

and

\[
\left( \frac{a - 1}{a^{k+1}} \right)^{k+1-s} = \frac{s(k + 1 - s)^{k+1-s}}{(k+1)^{k+1}} \quad \text{and} \quad a \neq \frac{k + 1}{s}. \tag{3.3}
\]

By Lemma 2.7 of [7] we can set,

\[
f - a = \frac{(e^{\gamma} - a_1)(e^{\gamma} - a_2) \cdots (e^{\gamma} - a_l)(e^{\gamma} - a_0)^2}{1 + e^{\gamma} + \cdots + e^{(s-1)\gamma}} \quad (l = k - 2) \tag{3.4}
\]

where \( a_j (j = 0, 1, \ldots, l) \) is constant and \( a_i \neq a_j \) \((i \neq j)\).

Let \( z_j \) \((1 \leq j \leq l)\) be a zero of \( e^{\gamma} - a_j \), which is not a zero of \( e^{\gamma} \), then \( z_j \) is a simple zero of \( f \). Then from the assumption \( E_1(a, f) \subseteq \tilde{E}(b, g) \), we get \( g(z_j) = b, (1 \leq j \leq l) \). Substitute \( z_j \) into (3.2), we get

\[a = \frac{a_j^{(k+1) - s}}{a_j^{k+1} - 1}, \quad b = \frac{b(a - 1)}{a(b - 1)}, \quad (1 \leq j \leq l).\]

Thus we can get

\[a_j^{(k+1) - s} = a/b, \quad a_j^2 = \frac{b(a - 1)}{a(b - 1)}, \quad (1 \leq j \leq l). \tag{3.5}\]

By (3.5), we know \( a_j \) \((1 \leq j \leq l)\) is the common root of the equations \( z^{k+1-s} = a/b \) and \( z^2 = \frac{b(a - 1)}{a(b - 1)} \), while the equations \( z^{k+1-s} = a/b \) and \( z^2 = \frac{b(a - 1)}{a(b - 1)} \) have \( k + 1 - s \) and \( s \) roots respectively. Thus, we have \( l = k - 2 \leq k + 1 - s \) and \( l = k - 2 \leq s \).

Hence we get

\[s \leq 3, \quad k \leq s + 2 \leq 5. \tag{3.6}\]

By Theorem C, we get \( s, k + 1 = 1, k + 2 \geq 2 \) and \( 1 \leq s \leq k \). Now, we deal with the division into those subcases.

Case 1.1.1. \( k = 2, s = 1 \). Then \( l = k - 2 = 0 \), which implies that \( f - a \) only has multiple zeros. Hence

\[f - a = e^{2\gamma} - e^{\gamma} + 1 - a = (e^{\gamma} - 1/2)^2,\]

we have \( a = 3/4 \), we get (1) of Theorem 1.1.

Case 1.1.2. \( k = 2, s = 2 \). Then \( l = k - 2 = 0 \), similarly as Case 1.1.1, we get \( a = -3 \), which implies (2).

Case 1.1.3. \( k = 3, s = 1 \). Then \( l = k - 2 = 1 \). From (3.5), we get

\[a_j = a/b, \quad a_j = \frac{b(a - 1)}{a(b - 1)}. \]

From this and (3.3) we get

\[\frac{(a - 1)^3}{a^4} = \frac{(b - 1)^3}{b^4} = A \quad \text{and} \quad a \neq 4,\]

where \( A = \frac{3}{2} \). Thus we get (3).

Case 1.1.4. \( k = 3, s = 2 \). Then \( (k + 1, s) = 2 \neq 1 \), a contradiction.

Case 1.1.5. \( k = 3, s = 3 \). Then \( l = k - 2 = 1 \), similarly as Case 1.1.3, we get (4) of Theorem 1.1.

Case 1.1.6. \( k = 4, s = 1 \). Then \( 1 \leq s \leq l = k - 2 = 2 \), a contradiction.

Case 1.1.7. \( k = 4, s = 2 \). Then \( l = k - 2 = 2 \). From (3.5), we get

\[a_j = a/b, \quad a_j^2 = \frac{b(a - 1)}{a(b - 1)}, \quad j = 1, 2.\]
Thus, we get $a_j = \frac{a^2(b-1)}{b(a-1)}$, $j = 1, 2$. We know $a_1 \neq a_2$, thus we get a contradiction.

Case 1.1.8. $k = 4, s = 3$. Similarly as Case 1.1.7, we get a contradiction.

Case 1.1.9. $k = 4, s = 4$. We know $s \leq 3$, a contradiction.

Case 1.1.10. $k = 5$. Then $l = k - 2 = 3$, then $s \leq l = 3$, by (3.6), we get $s = 3$. Hence $(k + 1, s) = 3 \neq 1$, a contradiction.

Thus we complete Case 1.1. In the same way we can discuss the other five cases and get (5)-(21) of Theorem 1.1. Hence we prove the first case.

Second case. $a$ is a Picard value of $f$ or $N_1 (\frac{r}{f-a}) = T(r, f) + S(r, f)$.

We claim that in this case $f$ is a fractional linear transformation of $g$. Now, we prove this claim.

If $a$ is a Picard value of $f$, we get $N (\frac{r}{f-a}) = 0$, by Lemma 2.1, we get $f$ is a fractional linear transformation of $g$.

If $N_1 (\frac{r}{f-a}) = T(r, f) + S(r, f)$, we get

$$N_2 (\frac{r}{f-a}) = S(r, f).$$

We suppose that $a = b$. Define

$$\varphi = \frac{f'(f-a)}{f(f-1)} \frac{g'(g-a)}{g(g-1)}. \quad (3.7)$$

If $\varphi \neq 0$, then $T(r, \varphi) = S(r)$ by the lemma of logarithmic derivative and the conclusion of Lemma 2.3, since the poles of $\varphi$ are all simple and derive from the zeros, 1-points and poles of $f$ and $g$ with different multiplicities. Now, since we assume that $E_1(a, f) \subseteq E(a, g)$, then

$$N_1 (\frac{r}{f-a}) \leq N (\frac{r}{\varphi}) \leq T(r, \varphi) + O(1) = S(r).$$

This is a contradiction.

If $\varphi \equiv 0$, then it is easy to see that $f$ and $g$ share $0, 1, \infty$ CM, we know that

$$N_0 (r) \geq N_1 (\frac{r}{f-a}) = T(r, f) + S(r).$$

Thus,

$$\lim_{r \to \infty} \frac{N_0 (r)}{T(r, f)} = 1 > 1/2,$$

by Lemma 2.2, we get, $f$ is a fractional linear transformation of $g$.

Now, we assume $a \neq b$, we can easily get that $f$ is a fractional linear transformation of $g$. In fact, this case has been proved by Qi Han, Seiki Mori and Kazuya Tohge [13].

Thus we prove the second case.

Hence we complete Theorem 1.1.

4. Proof of Theorem 1.2

We consider two cases.

**Case 1:** If $N (\frac{r}{f-a}) \neq T(r, f) + S(r, f)$, then by Theorem D we get

$$N (\frac{r}{f-a}) = S(r, f)$$

and $f, g$ satisfy one of the relations (i)-(iii) in Theorem D.

**Case 2:** If $N (\frac{r}{f-a}) = T(r, f) + S(r, f)$, by Lemma 2.7, we get

$$N_2 (\frac{r}{f-a}) = S(r, f) \quad \text{and} \quad N_1 (\frac{r}{f-a}) = T(r, f) + S(r, f).$$

From Lemma 2.1, we get

$$\frac{f}{g} = e^{\alpha - \beta}, \quad \frac{f - 1}{g - 1} = e^\alpha. \quad (4.1)$$
Let \( z_0 \) be a simple zero of \( f - a \) such that \( a(z_0) \neq 0, \infty, 1 \) and \( b(z_0) \neq 0, \infty, 1 \), then by the assumption of Theorem 1.2, we get \( g(z_0) = b(z_0) \). Substitute \( z_0 \) into (2.1), we get
\[
a(z_0) = \frac{e^{a(z_0)} - 1}{e^\beta(z_0) - 1}, \quad b(z_0) = \frac{e^{-a(z_0)} - 1}{e^{-\beta(z_0)} - 1}.
\]
Then we get
\[
e^\beta(z_0) = \frac{e^{a(z_0)} + a(z_0) - 1}{a(z_0)}, \quad e^{-\beta(z_0)} = \frac{e^{-a(z_0)} + b(z_0) - 1}{b(z_0)}.
\]
From (4.2), we get
\[
[(b(z_0) - 1)e^{a(z_0)} - (a(z_0) - 1)]e^{a(z_0)} - 1 = 0.
\]
Now, we distinguish four subcases.

Subcase 2.1. \( T(r, e^\alpha) = S(r) \).

If \( e^\alpha - \frac{a-1}{b-1} \neq 0 \), by (4.3) and \( e^\alpha \neq 1 \) we get
\[
T(r, f) + S(r) \leq N_1 \left( r, \frac{1}{f-a} \right) \leq \overline{N} \left( r, \frac{1}{e^\alpha - \frac{a-1}{b-1}} \right) + \overline{N} \left( r, \frac{1}{e^{-\alpha} - 1} \right) = S(r),
\]
which is a contradiction. Then we get \( e^\alpha \equiv \frac{a-1}{b-1} \), from (4.1), we get
\[
(f-1)(b-1) \equiv (g-1)(a-1),
\]
which is (1) of Theorem 1.2.

Subcase 2.2. \( T(r, e^\beta) = S(r) \). Similarly as subcase 2.1, we get \( e^\beta \equiv \frac{(a-1)b}{(b-1)a} \) and
\[
\frac{(f-1)g}{(g-1)^f} \equiv \frac{(a-1)b}{(b-1)a}.
\]
Thus we get (2).

Subcase 2.3. \( T(r, e^{a-\beta}) = S(r) \). Similarly as subcase 2.1, we get \( e^{a-\beta} \equiv \frac{a}{b} \) and
\[
f - b = ag.
\]
We deduce (3).

Subcase 2.4. \( T(r, e^\alpha) \neq S(r) \), \( T(r, e^\beta) \neq S(r) \), \( T(r, e^{a-\beta}) \neq S(r) \). In the following, we will prove that \( f \) is a fractional linear transformation of \( g \).

Suppose to the contrary, \( f \) is not a fractional linear transformation of \( g \). From (4.3), we get
\[
T(r, f) + S(r) \leq N_1 \left( r, \frac{1}{f-a} \right) \leq \overline{N} \left( r, \frac{1}{e^\alpha - \frac{a-1}{b-1}} \right) + \overline{N} \left( r, \frac{1}{e^{-\alpha} - 1} \right).
\]
From (4.3) and (4.4), we get either \( \overline{N}(r, a; f | e^\alpha = 1) \neq S(r) \) or \( \overline{N}(r, a; f | e^\alpha = \frac{a-1}{b-1}) \neq S(r) \).

Here, we denote by \( \overline{N}(r, a; f | g = b) \) the reduced counting function of those \( a \)-points of \( f \), which are the \( b \)-points of \( g \). By (4.1), we get that
\[
e^\alpha - 1 = \frac{f - g}{g - 1}.
\]
Let \( z_0 \) be a common zero of \( f - a \) and \( e^\alpha - 1 \). If \( z_0 \) is a simple zero of \( f - a \), from \( E_1(a; f) \subseteq \overline{E}(b; g) \), we have \( g(z_0) = b(z_0) \). Substitute \( z_0 \) into the above equation, we get \( 0 = \frac{a(z_0) - b(z_0)}{b(z_0) - 1} \) and \( a(z_0) - b(z_0) = 0 \). Thus, we have
\[
\overline{N}(r, a; f | e^\alpha = 1) \leq N \left( r, \frac{1}{a - b} \right) + N_2 \left( r, \frac{1}{f - a} \right) = S(r).
\]
That is to say \( \overline{N}(r, a; f | e^\alpha = 1) = S(r) \).

Hence we get \( \overline{N}(r, a; f | e^\alpha = \frac{a-1}{b-1}) \neq S(r) \).

Let \( z_0 \) be a zero of \( f - a \) and \( e^\alpha - \frac{a-1}{b-1} \) such that \( \alpha(z_0) \neq 1, \infty \) and \( \beta(z_0) \neq 1, \infty \). By (4.2), we get \( e^{\beta(z_0)} = \frac{(a(z_0) - 1) (b(z_0))}{(b(z_0)) - (a(z_0))} \).

Let us now define the following two functions:
\[
f_1 = \frac{b - 1}{a - 1} e^a, \quad f_2 = \frac{(b - 1) a}{(a - 1) b} e^\beta.
\]
and consider
\[ T_0(r) = T(r, f_1) + T(r, f_2), \quad S_0(r) = T_0(r) \quad (r \not\in E) \]
where \( E \) is a set of \( r \) of finite linear measure. From this we get \( S_0(r) = S(r) \) and
\[ \mathbb{N}(r, f_i) + \mathbb{N} \left( r, \frac{1}{f_i} \right) = S(r, f_1, f_2), \quad i = 1, 2. \quad (4.6) \]
Then we get \( f_i(z_0) = 1, i = 1, 2 \), thus we deduce that
\[ \mathbb{N} \left( r, a; f \right)e^\alpha = \frac{a - 1}{b - 1} \leq N_0(r, 1, f_1, f_2) + S(r). \]
Hence \( N_0(r, 1, f_1, f_2) \neq S(r) \). By Lemma 2.5, there exist two integers \( s \) and \( t \) \((|s| + |t| > 0)\) such that
\[ f_1^sf_2^t = 1. \quad (4.7) \]
From (4.1), (4.5) and (4.7) we get
\[ e^{s\alpha+t\beta} = \left( \frac{f - 1}{g - 1} \right)^{s+t} \left( \frac{g}{f} \right)^t = \left( \frac{a - 1}{b - 1} \right)^{s+t} \left( \frac{b}{a} \right)^t. \quad (4.8) \]
Thus
\[ \left( \frac{f - 1}{g - 1} \right)^{s+t} = \left( \frac{a - 1}{b - 1} \right)^{s+t} \left( \frac{b}{a} \right)^t \left( \frac{g - 1}{f^t} \right)^{s+t}. \quad (4.9) \]
From (4.9), we have
\[ T(r, f) = T(r, g) + S(r). \quad (4.10) \]
By (4.1), we get
\[ e^\alpha - 1 = \frac{f - g}{g - 1}, \quad e^\beta - 1 = \frac{f - g}{(g - 1)f}, \quad e^{\alpha - \beta} - 1 = \frac{f - g}{g}. \quad (4.11) \]
From the first equation of (4.11), we derive that
\[ T(r, e^\alpha) \leq \mathbb{N} \left( r, \frac{1}{e^\alpha - 1} \right) + \mathbb{N} \left( r, \frac{1}{e^\alpha} \right) + S(r) \]
\[ \leq \mathbb{N} \left( r, \frac{1}{e^\alpha - 1} \right) + S(r) = N_1 \left( r, \frac{1}{e^\alpha - 1} \right) + \mathbb{N} \left( r, \frac{1}{e^\alpha} \right) + S(r) \]
\[ \leq N_1 \left( r, \frac{1}{e^\alpha - 1} \right) + N \left( r, \frac{1}{\alpha} \right) + S(r) = N_1 \left( r, \frac{1}{e^\alpha - 1} \right) + S(r) \]
\[ \leq T(r, e^\alpha) + S(r). \]
Thus, we get
\[ T(r, e^\alpha) = N_1 \left( r, \frac{1}{e^\alpha - 1} \right) + S(r) = \mathbb{N} \left( r, \frac{1}{e^\alpha - 1} \right) + S(r, f). \]
By (4.11), we have
\[ \mathbb{N} \left( r, \frac{1}{e^\alpha - 1} \right) = \mathbb{N} \left( r, \frac{f - g}{g - 1} \right). \]
The possible zeros of \( e^\alpha - 1 \) come from the zeros of \( f - g \) and the poles of \( g \).
If \( z_\infty \) is both a pole of \( g \) and a zero of \( e^\alpha - 1 \), then \( z_\infty \) is also a pole of \( f \). By Lemma 2.4, we have
\[ f = \frac{e^\alpha - 1}{e^\beta - 1}. \]
it is easy to see that \( z_\infty \) is a multiple zero of \( e^\beta - 1 \). Note that the multiple zeros of \( e^\beta - 1 \) is the zeros of \( \beta' \), so
\[ \mathbb{N}_2 \left( r, \frac{1}{e^\beta - 1} \right) \leq N(r, \frac{1}{\beta'}) = S(r). \]
Thus, the reduced counting function of those points such that as \( z_\infty \)'s form a \( S(r) \).
Now, we discuss the zeros of \( f - g \). We know \( f \) and \( g \) share \( 0, 1 \infty CM \) and \( e^\alpha - 1 = \frac{f - g}{g - 1} \).
If \( z_0 \) is a zero of \( f \), then \( z_0 \) is a zero of \( e^\alpha - 1 \).
If \( z_0 \) is a zero of \( f - 1 \), then \( z_0 \) is not a zero of \( e^\alpha - 1 \).
If \(z_0\) is a zero of \(f - g\) and not a zero of \(f, f - 1, \frac{1}{f}\), then \(z_0\) is a zero of \(e^x - 1\).

We know \(\mathcal{N}_0(r)\) denotes the reduced counting function of the zeros of \(f - g\) but not the zeros of \(f, f - 1, \frac{1}{f}\). Thus, from the above discussion, we deduce that

\[
\mathcal{N}\left( r, \frac{1}{e^\alpha - 1} \right) = \mathcal{N}\left( r, \frac{1}{f} \right) + \mathcal{N}_0(r) + S(r).
\]

Hence,

\[
T(r, e^\alpha) = N_1\left( r, \frac{1}{e^\alpha - 1} \right) + S(r) = \mathcal{N}\left( r, \frac{1}{f} \right) + \mathcal{N}_0(r) + S(r).
\] 

Similarly, we can get

\[
T(r, e^\beta) = N_1\left( r, \frac{1}{e^\beta - 1} \right) + S(r) = \mathcal{N}(r, f) + \mathcal{N}_0(r) + S(r)
\] 

and

\[
T(r, e^{\alpha-\beta}) = N_1\left( r, \frac{1}{e^{\alpha-\beta} - 1} \right) + S(r) = \mathcal{N}\left( r, \frac{1}{e^{\alpha-\beta} - 1} \right) + \mathcal{N}_0(r) + S(r).
\]

We define

\[
\phi = (f - a)(e^\beta - 1) = e^\alpha - ae^\beta + a - 1
\] 

and

\[
\omega = \frac{\phi'}{\phi}.
\]

Then, neither \(\phi\) nor \(\omega\) is a constant. If not, say, \(\omega = c \neq 0\), then \(\phi = Ae^{Bz}\), where \(A\) and \(B\) are nonzero constants. Let \(z\) be a simple zero of \(f - a\); since \(z\) is not a zero of \(\phi\), then from (4.15), \(z\) must be a pole of \(e^\beta\), a contradiction. The discussions hold well for showing that \(\phi\) is not a constant.

Note that \(\phi = (f - a)(e^\beta - 1)\), we have the fact that the multiple zeros of \(\phi\) may be the multiple zeros of \(f - a, e^\beta - 1\) or the common zero of \(f - a\) and \(e^\beta - 1\). But we know \(N_2\left( r, \frac{1}{f-a} \right) = S(r)\) and \(N_2\left( r, \frac{1}{e^\beta-1} \right) = S(r)\). From the middle equation of (4.11), we can get \(N(r, a; f|e^\beta = 1) = S(r)\). Thus \(N_2\left( r, \frac{1}{\phi} \right) = S(r)\) and

\[
N\left( r, \frac{1}{\phi} \right) = \mathcal{N}\left( r, \frac{1}{\phi} \right) + S(r).
\] 

From (4.15), \(\phi = (f - a)(e^\beta - 1)\), the possible zeros of \(\phi\) come from the zeros of \(f - a\) and \(e^\beta - 1\). Noting (4.13), we get

\[
T(r, e^\beta) = N_1\left( r, \frac{1}{e^\beta - 1} \right) + S(r) \leq \mathcal{N}\left( r, \frac{1}{e^\beta - 1} \right) + S(r) \leq T(r, e^\beta) + S(r).
\]

Thus,

\[
\mathcal{N}\left( r, \frac{1}{e^\beta - 1} \right) = N_1\left( r, \frac{1}{e^\beta - 1} \right) + S(r) = \mathcal{N}(r, f) + \mathcal{N}_0(r) + S(r).
\]

Hence, the possible zeros of \(e^\beta - 1\) come from the poles of \(f\) and the zeros of \(f - g\) but not the zeros of \(f, f - 1, \frac{1}{f}\).

If \(z_\infty\) is a pole of \(f\) such that \(\mathcal{e}^{\beta(z_\infty)} = 0\) and \(\phi(z_\infty) = 0\); then from (4.15), we get \(z_\infty\) is a multiple zero of \(e^\beta - 1\). Similarly as above, we deduce the counting function of those points such that \(z_\infty\)'s form a \(S(r)\).

From (4.17) and the above discussion, we obtain

\[
N\left( r, \frac{1}{\phi} \right) = \mathcal{N}\left( r, \frac{1}{\phi} \right) + S(r) = N_1\left( r, \frac{1}{f - a} \right) + \mathcal{N}_0(r) + S(r).
\] 

Then we have

\[
T(r, \omega) = N(r, \omega) + m(r, \omega) = \mathcal{N}(r, \omega) + S(r)
\]

\[
= N(r, \phi) + \mathcal{N}\left( r, \frac{1}{\phi} \right) + S(r)
\]

\[
= N_1\left( r, \frac{1}{f - a} \right) + \mathcal{N}_0(r) + S(r).
\]
and hence
\[ T(r, \omega) = T(r, f) + N_o(r) + S(r). \] (4.19)

We define three meromorphic auxiliary functions
\[ \tau_1 = \frac{a - 1}{b - 1} \left[ \alpha' - (\alpha' + a\beta') \frac{b}{a} \right] + a', \]
\[ \tau_2 = \frac{a - 1}{b - 1} \left[ (\alpha'' + \alpha'^2) - (\alpha'' + 2\alpha' \beta' + a\beta' + a\beta'^2) \frac{b}{a} \right] + a'' \]
and
\[ \tau_3 = \frac{a - 1}{b - 1} \left[ (\alpha''' + 3\alpha' \alpha'' + \alpha'^3) - (\alpha''' + 3\alpha' \beta' + 3\alpha'' \beta' + 3\alpha'' \beta'' + 3\alpha' \beta'' + a\beta'^3) \frac{b}{a} \right] + a'''. \]

If \( \tau_1 \equiv 0 \), let \( z_o \) be a simple zero of \( f - a \), such that \( a(z_o) \neq 1 \) and \( b(z_o) \neq 1 \), then \( g(z_o) = b(z_o) \), \( e^{\phi(z_o)} = \frac{d(z_o) - 1}{b(z_o) - 1} \) and \( e^{\beta(z_o)} = \frac{d(z_o) - 1}{b(z_o) - 1} \).

By (4.15) and (4.16), the Laurent expansion of \( \phi \) around \( z_o \) is
\[ \phi(z) = \tau_2(z_o)(z - z_o)^2 + \tau_3(z_o)(z - z_o)^3 + O((z - z_o)^4). \]

Thus, \( z_o \) is a multiple zero of \( \phi(z) \). Hence
\[ N_{\phi}(r, \frac{1}{a - 1}) \leq N \left( r, \frac{1}{a - 1} \right) + N \left( r, \frac{1}{b - 1} \right) + N(z_o) \left( r, \frac{1}{\phi} \right) \leq N(z_o) \left( r, \frac{1}{\phi} \right) + S(r), \]
which implies that
\[ N_{\phi}(r, \frac{1}{\phi}) \geq N(z_o) \left( r, \frac{1}{\phi} \right) + S(r). \]

This contradicts (4.17). Thus, \( \tau_1 \equiv 0 \), and \( T(r, \tau_j) = S(r) \) (\( j = 1, 2, 3 \)).

Now, take \( z_o \) to be a simple zero of \( f - a \), such that \( \tau_1(z_o) \neq 0 \), \( a(z_o) \neq 1 \) and \( b(z_o) \neq 1 \). Then \( g(z_o) = b(z_o) \), \( e^{\phi(z_o)} = \frac{d(z_o) - 1}{b(z_o) - 1} \) and \( e^{\beta(z_o)} = \frac{d(z_o) - 1}{b(z_o) - 1} \).

Again by (4.15) and (4.16), the Laurent expansions of \( \phi \) and \( \omega \) around \( z_o \), respectively, are
\[ \phi(z) = \tau_2(z_o)(z - z_o)^2 + \tau_3(z_o)(z - z_o)^3 + O((z - z_o)^4) \]
(4.20)
and
\[ \omega(z) = \frac{1}{z - z_o} + \frac{\mu(z_o)}{2} + \mu_1(z_o)(z - z_o) + O((z - z_o)^2), \]
(4.21)
where \( \tau_2, \tau_3 \) are small functions of \( f \) and \( g \), \( \mu = \frac{2\tau_2}{\tau_1} \) and \( \mu_1 = \frac{2\tau_3}{\tau_1} - \left( \frac{\tau_2}{\tau_1} \right)^2 \).

Define the function
\[ R := \omega + \omega^2 - \mu \omega - \mu \]
(4.22)
where \( \mu_2 = 3\mu_1 - \mu^2 / 4 - \mu' \). By using (4.20) and (4.21), we get
\[ w'(z) = \frac{-1}{(z - z_o)^2} + \mu_1(z_o) + O(z - z_o), \]
\[ w'(z) = \frac{1}{(z - z_o)^2} + \frac{\mu(z_o)}{z - z_o} + \frac{\mu(z_o)^2}{4} + 2\mu_1(z_o) + O(z - z_o) \]
and
\[ \mu(z)w(z) = \left[ \mu(z_o) + \mu'(z_o)(z - z_o) + O((z - z_o)^2) \right] \left[ \frac{1}{z - z_o} + \frac{\mu(z_o)}{2} + \mu_1(z_o)(z - z_o) \right] \]
\[ + O((z - z_o)^2) = \frac{\mu(z_o)}{z - z_o} + \frac{\mu(z_o)^2}{2} + \mu'(z_o) + O(z - z_o). \]

Substitute them into (4.22), we have \( R(z) = O(z - z_o) \). Thus, \( z_o \) is a zero of \( R(z) \).

If \( R \neq 0 \), by (4.22), we have
\[ R = \frac{\phi''}{\phi} - \mu \frac{\phi'}{\phi} - \mu_2. \]
By the lemma of logarithmic derivative we get
\[ N_{11}\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{R}\right) + S(r) \leq T(r, R) + S(r) = m(r, R) + N(r, R) + S(r) \]
\[ \leq m\left(r, \frac{\phi'}{\phi}\right) + m\left(r, \frac{\phi'}{\phi_1}\right) + m(r, \mu_2) + N(r, R) + S(r) = N(r, R) + S(r), \]
that is
\[ N_{11}\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{R}\right) + S(r) \leq T(r, R) + S(r) = N(r, R) + S(r). \]  
(4.23)

From the equation \( R = \frac{\phi'}{\phi} - \frac{\mu}{\phi^2} - \mu_2 \), we get almost all the poles of \( R \) coming from the poles of \( \phi \) and the zeros of \( \phi \) but not the zeros of \( f - a \). By (4.18), we have \( N\left(r, \frac{1}{\phi}\right) = N_{11}\left(r, \frac{1}{f-a}\right) + \overline{N}_0(r) + S(r) \), while the simple zeros of \( f - a \) are not poles of \( R(z) \). From (4.15), we have \( N(r, \phi) \leq N(r, \phi) = S(r) \). Thus
\[ \overline{N}(r, R) \leq N(r, \phi) + N\left(r, \frac{1}{\phi}\right) - N_{11}\left(r, \frac{1}{f-a}\right) \leq \overline{N}_0(r) + S(r). \]  
(4.24)

From \( N(r, \phi) \leq N(r, a) = S(r) \) and \( N\left(r, \frac{1}{\phi}\right) = N_{11}\left(r, \frac{1}{f-a}\right) + \overline{N}_0(r) + S(r) \), we know almost all the poles of \( R \) come from the zeros of \( \phi \) such that \( f(z) \neq \alpha(z) \). Suppose that \( z_0 \) is a simple zero of \( \phi \) such that \( f(z_0) \neq \alpha(z_0) \), then analogous discussions as above imply that the Laurent expansion of \( R \) around \( z_0 = O(\frac{1}{z-z_0}) \). This implies that \( z_0 \) is a simple pole of \( R(z) \).

Thus, almost all the poles of \( R(z) \) are simple, that is
\[ N(r, R) = \overline{N}(r, R) + S(r). \]

From this with (4.23) and (4.24) we get
\[ \overline{N}_0(r) \geq N_{11}\left(r, \frac{1}{f-a}\right) + S(r) = T(r, f) + S(r). \]

Then by Lemma 2.2, we get \( f \) is a fractional linear transformation of \( g \), which is a contradiction.

All the foregoing discussions yield \( R \equiv 0 \), which implies that \( \omega \) is a solution of the following Riccati differential equation
\[ \omega' = \mu_2 + \mu \omega - \omega^2, \]  
(4.25)
where \( T(r, \mu_1) = T(r, \mu_2) = S(r) \). By (4.15) we get
\[ \phi' = \alpha'(\omega) - (\alpha' + \alpha')(\omega) + \alpha'. \]  
(4.26)

By (4.15) and (4.26), we have
\[ \phi\left(\frac{\omega - \frac{\alpha' + \omega^2}{a}}{\alpha'}\right) = \xi_1 \omega + \eta_1. \]  
(4.27)
where \( \xi_1 = (\alpha' - \frac{\alpha' + \omega^2}{a}) \) and \( \eta_1 = \alpha' - \frac{\alpha' + \omega^2}{a}(\alpha - 1) \).

If \( \xi_1 \equiv 0 \), we get \( \frac{\alpha'}{\alpha'} = \alpha' - \alpha' \). Then \( a = \alpha' - \beta' \), where \( A = \alpha' - \beta' \), which is a contradiction. Hence \( \xi_1 \neq 0 \).

If \( \eta_1 \equiv 0 \), we get \( \frac{\alpha'}{\alpha'} = \alpha' - \beta' \). Then \( a = \alpha' - \beta' \), where \( A = \alpha' - \beta' \), which is a contradiction. Thus, \( \eta_1 \neq 0 \).

By (4.27), we have
\[ \overline{N}\left(r, \frac{1}{\omega - \frac{\alpha' + \omega^2}{a}}\right) \leq \overline{N}\left(r, \frac{1}{\omega} + \frac{\eta_1}{\omega}\right) + \overline{N}\left(r, \frac{1}{\xi_1}\right) + \overline{N}(r, \phi) + S(r) \]
\[ \leq \overline{N}\left(r, \frac{1}{\omega} + \frac{\eta_1}{\omega}\right) + S(r) \leq \overline{N}\left(r, \frac{1}{\phi} + \frac{\eta_1}{\omega - \frac{\alpha' + \omega^2}{a}}\right) + S(r) \]
\[ \leq \overline{N}\left(r, \frac{1}{\omega - \frac{\alpha' + \omega^2}{a}}\right) + \overline{N}(r, \phi) + S(r) \leq \overline{N}\left(r, \frac{1}{\omega - \frac{\alpha' + \omega^2}{a}}\right) + S(r). \]  
(4.28)
Similarly we obtain

\[ T(r, e^\alpha) = \overline{N} \left( r, \frac{1}{\omega - \frac{\alpha' + \alpha\beta'}{a}} \right) + S(r), \]

which combined with (4.12) yields that

\[ \overline{N} \left( r, \frac{1}{\omega - \frac{\alpha' + \alpha\beta'}{a}} \right) = T(r, e^\alpha) + S(r) = N_{1j} \left( r, \frac{1}{e^\alpha - 1} \right) + S(r) = \overline{N} \left( r, \frac{1}{f} \right) + \overline{N}_0(r) + S(r). \]  

Furthermore, by (4.15) and (4.26), we could also obtain

\[ \phi(\omega - \alpha') = \phi' - \alpha' \phi = \left[ a\omega' - (\alpha' + \alpha\beta') \right] e^\beta + \alpha' (a - 1) = \xi_2 e^\beta + \eta_2, \]

\[ \phi \left( \omega - \frac{\alpha'}{a - 1} \right) = \phi' - \frac{\alpha'}{a - 1} \phi = \left[ \alpha' - \frac{\alpha'}{a - 1} \right] e^\beta + \left[ a - \frac{\alpha'}{a - 1} - (\alpha' + \alpha\beta') \right] e^\beta = e^\beta \left[ \xi_3 e^{-\beta} + \eta_3 \right], \]

where \( \xi_2 = a\omega' - (\alpha' + \alpha\beta') \neq 0, \eta_2 = \alpha' - \alpha'(a - 1) \neq 0, \xi_3 = \alpha' - \frac{\alpha'}{a - 1} \neq 0 \) and \( \eta_3 = \frac{a\alpha'}{a - 1} - (\alpha' + \alpha\beta') \neq 0. \) Apply analogous discussions as those after (4.26) for obtaining

\[ \overline{N} \left( r, \frac{1}{\omega - \alpha'} \right) = T(r, e^\beta) + S(r) = N_{1j} \left( r, \frac{1}{e^\beta - 1} \right) + S(r) = \overline{N}(r, f) + \overline{N}_0(r) + S(r). \]

Now, if one of \( \frac{\alpha' + \alpha\beta'}{a} \), \( \alpha' \) and \( \frac{\alpha'}{a - 1} \) is a solution of the Riccati differential equation (4.25), say, \( \frac{\alpha' + \alpha\beta'}{a} \), then by Lemma 2.6, we get \( \overline{N} \left( r, \frac{1}{\omega - \frac{\alpha' + \alpha\beta'}{a}} \right) = S(r) \), then by (4.30), we get \( T(r, e^\alpha) = S(r) \), a contradiction. Analogously, neither \( \alpha' \) nor \( \frac{\alpha'}{a - 1} \) is a solution of the Riccati differential equation (4.25). Hence by Lemma 2.6 again, together with (4.19), (4.30), (4.32) and (4.33), we have

\[ T(r, f) + \overline{N}_0(r) + S(r) = T(r, \omega) = \overline{N} \left( r, \frac{1}{\omega - \alpha'} \right) + S(r) = \overline{N}(r, f) + \overline{N}_0(r) + S(r), \]

\[ T(r, f) + \overline{N}_0(r) + S(r) = T(r, \omega) = \overline{N} \left( r, \frac{1}{\omega - \frac{\alpha'}{a - 1}} \right) + S(r) = \overline{N} \left( r, \frac{1}{f - 1} \right) + \overline{N}_0(r) + S(r), \]

and

\[ T(r, f) + \overline{N}_0(r) + S(r) = T(r, \omega) = \overline{N} \left( r, \frac{1}{\omega - \frac{\alpha' + \alpha\beta'}{a}} \right) + S(r) = \overline{N} \left( r, \frac{1}{f} \right) + \overline{N}_0(r) + S(r), \]

which imply that

\[ T(r, f) = \overline{N}(r, f) + S(r) = \overline{N} \left( r, \frac{1}{f - 1} \right) + S(r). \]

From Lemma 2.3, we have \( N\left( r, \frac{1}{f - 1} \right) = S(r) \). Thus, by (4.34) we get

\[ T(r, f) = \overline{N} \left( r, \frac{1}{f} \right) + S(r) = N_{1j} \left( r, \frac{1}{f} \right) + \overline{N}_0(r) + S(r) = N_{1j} \left( r, \frac{1}{f} \right) + S(r). \]

Similarly we obtain \( T(r, f) = N_{1j} \left( r, \frac{1}{f - 1} \right) + S(r) = N_{1j}(r, f) + S(r). \)

Then by Lemma 2.1 and (4.10), \( T(r, f) = T(r, g) + S(r) \), we have

\[ 2T(r, f) = T(r, f) + T(r, g) = N_{1j} \left( r, \frac{1}{f - 1} \right) + N_{1j} \left( r, \frac{1}{f} \right) + N_{1j}(r, f) + \overline{N}_0(r) + S(r) \]

\[ = 3T(r, f) + \overline{N}_0(r) + S(r), \]

which implies that \( T(r, f) = S(r) \).
By Lemma 2.1, (4.10) and (4.34), we get $T(r, f) = S(r)$, a contradiction.
Thus we get, $f$ is a fractional linear transformation of $g$. Therefore

$$f = \frac{Ag + B}{Dg + D},$$

(4.35)

where $A, B, C, D$ are constants and $AD - BC \neq 0$. We see $ABCD = 0$, otherwise, $g$ has three Picard exceptional values 0, $\infty$, $-\frac{B}{A}$.

Case I. If $A = 0$, then $D = 0$ (otherwise, 0, $\infty$ and $-D/C$ are Picard exceptional values of $g$). Hence $fg \equiv 1$ and $ab = 1$, which is (4).

Case II. Suppose $C = 0$.
If $B = 0$, then $f = (A/D)g$. Suppose that $p_0$ is a simple zero of $f - a$. Since $E_{\tilde{f}}(a; f) \subseteq \tilde{E}(b; g)$, we get $g(p_0) = b(p_0)$. Substitute $p_0$ into the equation $f = \frac{A}{B}g$, we have $a(p_0) = \frac{A}{B}b(p_0)$. If $a \neq \frac{A}{B}b$, we obtain

$$T(r, f) = N_1 \left( \frac{1}{f - a} \right) + S(r) \leq N \left( \frac{1}{a - \frac{A}{B}b} \right) + S(r) = S(r),$$

a contradiction. Thus, $a \equiv \frac{A}{B}b$, and $fb = ag$, then we have (3).

If $B \neq 0$. Suppose that $B/D = 1$, then $f = (A/D)g + 1$; hence 0 and 1 are Picard exceptional values of $f$ and $g$. From the equation $f = (A/D)g + 1$, we obtain, $\frac{A}{B} + 1$ is also a Picard exceptional value of $f$. We know that $f$ at most has two Picard exceptional values. Since $\frac{A}{B} + 1 \neq 1$, then $\frac{A}{B} + 1 = 0$ and $\frac{A}{B} = -1$. Then $f + g = 1$ and $a + b = 1$, which implies (5).

Now, we assume $B/D \neq 1$, then 0 and $B/D$ are Picard exceptional values of $f$, thus 1 is not a Picard exceptional value of $f$. Hence there is $z_0$ such that $f(z_0) = 1$ and $1 = \frac{A}{B} + \frac{B}{D}$. In a similar way we get, $a \equiv \frac{A}{B}b + \frac{B}{D}$. Hence we deduce that $\frac{A}{B} = \frac{a - 1}{b - 1}$ and $\frac{B}{D} = \frac{b - a}{b - 1}$; from this, we get (6).

Case III. If $D = 0$ and $A \neq 0$, then $ABC \neq 0$, and we have 0, $\infty$, $-B/A$ are Picard exceptional values of $g$, which is impossible. If $D = 0$ and $A = 0$, then $fg \equiv ab = \text{Constant}$, we deduce (4).

Case IV. Suppose that $B = 0$ and $ACD \neq 0$. Therefore, by (4.35), we get

$$f = \frac{A_1 g}{g - D_1},$$

(4.36)

where $A_1, D_1$ are constants. If $A_1 = 1$, then 1 and $\infty$ are Picard exceptional values of $f$ and $g$. By (4.36), we get, $\frac{1}{1 - D_1}$ is also a Picard exceptional value of $f$. We know that $f$ at most has two Picard exceptional values. Since $\frac{1}{1 - D_1} \neq 1$, then $\frac{1}{1 - D_1} = \infty$ and $D_1 = 1$, thus we get (7). Suppose that $A_1 \neq 1$, then 1 is not a Picard exceptional value of $f$ and $g$. Then $A_1 + D_1 = 1$. Again in a similar way as above, we get $a \equiv \frac{bD_1}{b - D_1}$. Hence $A_1 = \frac{a(1 - b)}{a - b}$ and $D_1 = \frac{b(a - 1)}{a - b}$. We get (8).

Thus, we prove Theorem 1.2.

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References