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On the domination of hypergraphs by their edges

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Abstract

Given a hypergraph $\mathscr{H}=(E_1,\dots,E_m)$ with vertex set V, let n_o be the number of different possibilities for covering V by an odd number of E's and n_e the number of different possibilities for covering V when selecting an even number of E's. The quantity $d(V,\mathscr{H})=n_o-n_e$ is known as the (reliability) domination of $\mathscr{H}$ and a combinatorial invariant of considerable practical relevance. The present paper addresses the problem to determine this domination. After reviewing the current theory in the area we present some new relationships for $d(V,\mathscr{H})$ with respect to dual and interval hypergraphs. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

Let $V=\{1,\dots,n\}$ be a finite set. A *hypergraph* is a family $\mathscr{H}=(E_1,\dots,E_m)$ of non-empty subsets of V. The E's are called the *edges* and the elements of V the *vertices* of $\mathscr{H}$. A hypergraph $\mathscr{H}$ is called *simple* if no element of $\mathscr{H}$ is contained in another element of $\mathscr{H}$. We assume $\mathscr{H}$ and V to be both non-empty and finite. We say that $\mathscr{H}$ *covers* V if and only if every element of V is contained in at least one element of $\mathscr{H}$. A simple hypergraph presents a generalized form of a graph which in turn is a hypergraph with $|E_i|\leqslant 2$ for all $E_i\in\mathscr{H}$. Simple hypergraphs are also known as clutters, antichains or coherent, binary systems.

The *signed domination* is a combinatorial invariant that is widely examined for graphs. It gained considerable practical relevance when its usefulness in network reliability theory has been discovered [9]. It is therefore sometimes called reliability domination [4]. For hypergraphs we define the signed domination as follows: Let n_o be the number of different subsets of $\mathscr{H}$ with odd cardinality covering V and n_e the

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number of subsets of $\mathscr{H}$ with even cardinality covering V, then the signed domination of the hypergraph $\mathscr{H}$ is defined as

$$d(V,\mathscr{H}) = n_o - n_e. \tag{1}$$

This definition follows [9] and it follows the few studies of the domination invariant for some conceptual counterparts of hypergraphs, namely for coherent binary systems [2] and clutters [5–7].

In the present paper we study $d(V,\mathscr{H})$ for several distinct types of hypergraphs. In fact, the setting in terms of hypergraphs is new. As it will be seen, it makes not only sense due to the generalizing nature of hypergraphs (in relation to graphs), but it allows to study $d(V,\mathscr{H})$ within in comfortably structured theoretical framework. Thus, after reviewing some known results about the domination of graphs, hypergraphs and transversals, we present relationships for dual and interval hypergraphs. Herein, we make use of the well-defined conceptual basis offered by hypergraph theory, as it is defined in [1]. Following this work, hypergraph terminology differs generally from the common terminology in studying for example clutters. Whenever possible, we will therefore give a reference between a hypergraph concept and its clutter equivalent. The same will be done with respect to coherent binary systems used in reliability theory. An additional paragraph is dedicated to elucidate the relevance of the presented results to this particular research area.

2. Simple, transversal, dual and interval hypergraphs

We introduce some further notations: Let $\mathscr{H}$ be a hypergraph with vertex set V. If $E \subseteq V$ is an edge of $\mathscr{H}$, we write $E \in \mathscr{H}$. By $\mathscr{H} - E$ we denote the hypergraph that has the edge set of $\mathscr{H}$ without edge E, and by $\mathscr{H} \cup E$ we denote the hypergraph that has the edge set of $\mathscr{H}$ including a new edge E. If $\mathscr{H}$ and $\mathscr{H}'$ are two hypergraphs on V and V', respectively, $\mathscr{H} \cup \mathscr{H}', \mathscr{H} \cap \mathscr{H}'$ and $\mathscr{H} - \mathscr{H}'$ denote the hypergraphs rendered by the union, the intersection and the difference of the edge sets of $\mathscr{H}$ and $\mathscr{H}'$. The vertex sets of these hypergraphs are given by the subsets of $V \cup V'$ which are covered by the corresponding edge set.

A convenient way to define $d(V,\mathscr{H})$ more formally makes use of the concept of formation (see, e.g., [9]). A *formation of V by $\mathscr{H}$* is defined as a subset of the edges of $\mathscr{H}$ covering V. Let $F(V,\mathscr{H})$ be the set of all possible formations of V by $\mathscr{H}$, and let F_o (F_e) be the set of all possible formations of V by $\mathscr{H}$ by selecting and odd (even) number of subsets of $\mathscr{H}$. Note that $F(V,\mathscr{H}) = F_o \cup F_e$, and, in accordance with definition (1), $n_o = |F_o|$ and $n_e = |F_e|$.

In the following Proposition 1 we relate the signed domination of a hypergraph to the signed domination of a certain simple hypergraph that is defined as follows: Let $\mathscr{H}$ be hypergraph with vertex set V. We define *the maximal simple subhypergraph of $\mathscr{H}$* to be the hypergraph $\mathscr{H}'$ that is obtained by deleting all non-minimal or replicated edges from $\mathscr{H}$. If $\mathscr{H}$ is simple then $\mathscr{H} = \mathscr{H}'$. If $\mathscr{H}$ is not simple then $\mathscr{H}'$ is given by a proper subset of edges of $\mathscr{H}$. Note, that in this case the vertex set covered by $\mathscr{H}'$ may be a proper subset of V.

Proposition 1. *Let $\mathscr{H}$ be a hypergraph with vertex set V and $\mathscr{H}'$ its maximal simple subhypergraph. Then,*

$$d(V,\mathscr{H})=d(V,\mathscr{H}'). \tag{2}$$

Proof. If $\mathscr{H}=\mathscr{H}'$ then (2) is trivial. If not, assume E_j to be an edge of $\mathscr{H}$ that is not an edge of $\mathscr{H}'$. Let $F^1=\{f\in F(V,H)\,|\,E_j\in f\}$ denote the set of formations of V by $\mathscr{H}$ containing E_j and $F^2=\{f\in F(V,H)\,|\,E_j\notin f\}$ the set of formations of V not containing E_j. F^1 consists of a set of formations with odd cardinality F_o^1 and a set of formations with even cardinality F_e^1 such that $F^1=F_o^1\cup F_e^1$. Analogously, $F^2=F_o^2\cup F_e^2$. With (1), the signed domination of $\mathscr{H}$ can be expressed in the form

$$d(V,\mathscr{H})=n_o-n_e=(|F_o^1|-|F_e^1|)+(|F_o^2|-|F_e^2|). \tag{3}$$

Since $E_j\notin\mathscr{H}'$ there exists another edge $E_i\in\mathscr{H}$ such that $E_i\subsetneq E_j$. For F^1, if a formation $f\in F^1$ does not contain E_i, we can add E_i. Likewise, if E_i is contained in a formation $f\in F^1$, we can delete it and f remains to be a formation. Thus, $|F_o^1|$ and $|F_e^1|$ are equal and $|F_o^1|-|F_e^1|=0$. Insertion in (3) leads to $d(V,\mathscr{H})=(|F_o^2|-|F_e^2|)=d(V,\mathscr{H}-E_j)$. By applying these considerations repetitively to all edges of $\mathscr{H}$ that are not edges of $\mathscr{H}'$, we obtain the proof. □

The benefit of Proposition 1 is that, if we are interested in the signed domination of hypergraphs, one may concentrate on analyzing simple hypergraphs only.

Perhaps the most detailed study of the signed domination invariant was performed in [7]. A main result of this work consists in a general set theoretic formula for $d(V,\mathscr{H})$ which, in the context of hypergraphs, can be given as follows:

Corollary 1 (Huseby [6,7]). *Let $U(V,\mathscr{H})$ to be defined as the set of all supersets (up to V) of edges of a simple hypergraph $\mathscr{H}$, i.e. $U(V,\mathscr{H})=\{S\,|\,\exists E_i\in\mathscr{H}$ such that $E_i\subseteq S\subseteq V\}$, then $d(V,\mathscr{H})$ can be given as*

$$d(V,\mathscr{H})=\sum_{S\in U(V,\mathscr{H})}(-1)^{|V|-|S|}=\sum_{S\subseteq V}(-1)^{|V|-|S|}I(S), \tag{4}$$

where $I(S)$ is an indicator variable defined as

$$I(S)=\begin{cases}1 & \text{if } S\in U(V,\mathscr{H}),\\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

Proof. See [7]. □

In addition to Corollary 1 we recall another known result that concerns the signed domination of transversal hypergraphs [1]. A *transversal* of a hypergraph $\mathscr{H}$ is a set

$T \subseteq V$ that meets all edges of $\mathscr{H}$, i.e. $T \cap E_j \neq \emptyset$ for every $E_j \in \mathscr{H}$. *The transversal hypergraph* $\mathrm{Tr}(H)$ of a hypergraph $\mathscr{H}$ is defined as the family of minimal transversals of $\mathscr{H}$. In clutter terminology a transversal hypergraph is called the blocking or dual clutter.

Proposition 2 (Barlow and Iyer [2]). *Let $\mathscr{H}$ be a hypergraph with vertex set V and $\mathrm{Tr}(\mathscr{H})$ the transversal hypergraph of $\mathscr{H}$, then*

$$d(V,\mathscr{H}) = (-1)^{|V|+1} d(V, \mathrm{Tr}(\mathscr{H})). \tag{6}$$

Proof. See [2,7]. □

We use this proposition to prove a similar general result for dual hypergraphs [1] which should not be confused with dual clutters (see above): The *dual* of a hypergraph $\mathscr{H} = (E_1, \ldots, E_m)$ with vertex set $V = \{1, \ldots, n\}$ is the hypergraph $\mathrm{Dl}(\mathscr{H}) = (E'_1, \ldots, E'_n)$ which covers the vertex set $V' = \{1, \ldots, m\}$ and whose edges are given by $E'_i = \{j \in V' \mid i \in E_j\}$.

Proposition 3. *Let $\mathscr{H}$ be a hypergraph and $\mathrm{Dl}(\mathscr{H})$ the dual hypergraph of $\mathscr{H}$, then*

$$d(V,\mathscr{H}) = d(V', \mathrm{Dl}(\mathscr{H})), \tag{7}$$

where V and V' are the vertex sets of $\mathscr{H}$ and $\mathrm{Dl}(\mathscr{H})$, respectively.

Proof. Let $f \in F(V,\mathscr{H})$. Since for every element of f there exists exactly one vertex of V', f corresponds to some vertex subset $S \subseteq V'$. From the fact that f covers V and $|V| = |\mathrm{Dl}(\mathscr{H})|$ it follows that each edge $E'_i \in \mathrm{Dl}(\mathscr{H})$ contains at least one vertex of S. In other words, S is a (proper or improper) superset of some element of the transversal $\mathrm{Tr}(\mathrm{Dl}(\mathscr{H}))$ and there exists a one-to-one correspondence between the elements of $F(V,\mathscr{H})$ and $U(V', \mathrm{Tr}(\mathrm{Dl}(\mathscr{H})))$. Since $|S|$ is even iff $|f|$ is even and $|S|$ is odd iff $|f|$ is odd, it is

$$d(V,\mathscr{H}) = n_o - n_e = \sum_{S \in U(V', \mathrm{Tr}(\mathrm{Dl}(\mathscr{H})))} (-1)^{|S|-1} \tag{8}$$

and, using Corollary 1 and Proposition 2,

$$\begin{aligned} d(V,\mathscr{H}) &= (-1)^{|V'|+1} \left(\sum_{S \in U(V', \mathrm{Tr}(\mathrm{Dl}(\mathscr{H})))} (-1)^{|V'|-|S|} \right) \\ &= (-1)^{|V'|+1} d(V', \mathrm{Tr}(\mathrm{Dl}(\mathscr{H}))) = d(V', \mathrm{Dl}(\mathscr{H})). \quad \square \end{aligned} \tag{9}$$

It should be noted that the dual hypergraph of a simple hypergraph is not necessarily simple. However, if we are interested in the signed domination of the maximal simple subhypergraph of a dual hypergraph, we apply Proposition 1 and obtain the result at once.

In the rest of this paragraph we study the signed domination of a special hypergraph class, the so-called interval hypergraphs [1]. Interval hypergraphs exhibit several interesting combinatorial properties, e.g. the coloured edge property, the Helly property and the König property (see [1]).

A hypergraph $\mathscr{H}$ is an *interval hypergraph* if there exists an ordering of the vertex set V such that for every two vertices $i,k \in V$ with $i<k$, which are contained in an edge $E \in \mathscr{H}$, all vertices j in $i<j<k$ are also contained in E. An ordering of V which satisfies this condition is called an *interval preserving ordering.*

It is sometimes helpful to imagine oneself the vertices V of an interval hypergraph to be arranged on a line according to the interval preserving ordering. Then, an edge family of an interval hypergraph is a set of subsets of V such that each edge consists of a set of all vertices lying in an interval on the line.

For a simple interval hypergraph an interval preserving ordering of the hypergraph vertices may be used to order the edges of the hypergraph. In the proof of the following theorem we make use of this fact. In ordering the edges of a given simple interval hypergraph $\mathscr{H}$ such that $E<E'$ for $E,E' \in \mathscr{H}$ if and only if E contains a vertex i with $i<j$ for every $j \in E'$, we obtain an ordering of the edges of $\mathscr{H}$ that depends on the interval preserving ordering of the hypergraph vertices.

Theorem 1. *Let $\mathscr{H}$ be an interval hypergraph with vertex set V. Then, the signed domination $d(V,\mathscr{H})$ takes the value $-1,0$ or $+1$.*

Proof. The proof is performed by induction on $|\mathscr{H}|$. Obviously, for $|\mathscr{H}|=1$ it is $d(V,\mathscr{H})=1$. For any hypergraph $\mathscr{H}$ with $|\mathscr{H}|>1$, we first use Proposition 1 to pass to the maximal simple subhypergraph $\mathscr{H}^{(1)}$. Assume that the theorem holds for $|\mathscr{H}^{(1)}|=m$ and should be proven for $|\mathscr{H}^{(1)}|=m+1$. We order the $m+1$ edges of $\mathscr{H}^{(1)}$ according to an interval preserving ordering of their vertices, that is to say, $E<E'$ for $E,E' \in \mathscr{H}^{(1)}$ if and only if E contains a vertex i such that $i<j$ for every $j \in E'$. If with respect to the obtained ordering $E_1,\ldots,E_{m+1}$ the edge E_{m+1} does not cover vertex n, then $d(V,\mathscr{H})=0$. Otherwise $E_{m+1}=\{r,\ldots,n\}$, $r \leqslant n$, is the only edge covering vertex n. It follows that every formation of V by $\mathscr{H}^{(1)}$ is comprised of E_{m+1} together with a formation of $V^{(2)}=\{1,\ldots,r-1\}$ by the edges $E_1-\{r,\ldots,n\},\ldots,E_m-\{r,\ldots,n\}$. Together with $V^{(2)}$ these edges form an interval hypergraph $\mathscr{H}^{(2)}$. Note that the formations of $V^{(2)}$ by $\mathscr{H}^{(2)}$ are in one-to-one correspondence with those of V by $\mathscr{H}^{(1)}$ with the parity switched. Now apply the inductive hypothesis. □

The inductive progressing that is pursued in the proof forms a series of interval subhypergraphs $\mathscr{H}^{(1)},\mathscr{H}^{(2)},\ldots,\mathscr{H}^{(i)},\ldots,\mathscr{H}^{(k)}$ where k is the number of passed induction stages. Since each stage contributes to the signed domination by switching the sign, $d(V,\mathscr{H})$ is equivalently given by

$$d(V,\mathscr{H})=(-1)^{k-1} \tag{10}$$

unless $d(V,\mathscr{H})=0$.

Further, at each stage i of the induction the edges of the corresponding interval hypergraph $\mathscr{H}^{(i)}$ are truncated by some vertex subset $S_i \subseteq V$. Thus, the induction leads to disjoint sets $S_1, \ldots, S_k$ which, in the case of $d(V, \mathscr{H}) \neq 0$, are a partitioning of V and, in the case of $d(V, \mathscr{H}) = 0$, are not a partitioning of V.

3. Applications

The signed domination invariant has been originally studied in reliability theory [9]. Translating the hypergraph terminology to notions used in this research field, a hypergraph $\mathscr{H}$ corresponds to a *coherent binary system* or *reliability system* [3], the hypergraph vertices are the *components* and the hypergraph edges are the *path sets* of the reliability system. The *minimal path sets* of a reliability system are the edges of the maximal simple subhypergraph of a hypergraph under consideration. Its *minimal cutsets* are the edges of the transversal hypergraph $\mathrm{Tr}(\mathscr{H})$. They form the *dual* reliability system [3]. Moreover, $\mathrm{Dl}(\mathscr{H})$ is sometimes called the *family of transposed minimal path sets* of a reliability system [7].

The actual relevance of domination theory in reliability analysis renders from the fact that for certain classes of reliability systems — or, equivalently, hypergraphs — algorithms are known which use the signed domination invariant for a fast reliability computation of these systems. In addition to *k-out-of-n-systems* [2,5] and *consecutive k-out-of-n-systems* [8], such a fast reliability computing algorithm exists in particular for certain *directed network systems* which (in their simplest form) are to be described as hypergraphs whose vertices correspond to the edges of a directed graph and whose edges are representable by the paths between two fixed vertices in the directed graph [7,9]. It was shown that for such a directed network system — or hypergraph $\mathscr{H}$ — $d(S, \mathscr{H}(S))$, where $\mathscr{H}(S)$ denotes the subhypergraph of $\mathscr{H}$ having exactly those edges $E \in \mathscr{H}$ with $E \subseteq S$ for $S \subseteq V$, takes either the value 1, 0 or -1 for every $S \subseteq V$ (see [7,9]).

A common property of the domination-based algorithms for a given hypergraph $\mathscr{H}$ of these classes is that they, first, generate efficiently all subhypergraphs $\mathscr{H}(S)$ with $d(S, \mathscr{H}(S)) \neq 0$, $S \subseteq V$, and, secondly, compute efficiently $d(S, \mathscr{H}(S))$. This means that following these algorithms the signed domination has to be determined not only for $\mathscr{H}$ but for all subhypergraphs $\mathscr{H}(S)$.

Interval hypergraphs or, as they are sometimes called in reliability theory, *consecutively connected systems* [10] represent another reliability system class for which the signed domination is limited by 1, 0 or -1 (see Theorem 1). It can be easily seen that if V is the vertex set of an interval hypergraph $\mathscr{H}$, then this holds not only for $d(V, \mathscr{H})$ but also for every $d(S, \mathscr{H}(S))$, $S \subseteq V$, since every subhypergraph of $\mathscr{H}$ which is obtained by deleting some edges from $\mathscr{H}$ is an interval hypergraph, too.

It is an interesting question whether this result can be used for constructing a fast reliability computing algorithm of interval hypergraphs, possibly in analogy to the ex-

```
procedure INT_HYP_DOM ( simple interval hypergraph H(S) with ordered edge set
                                   ( E1, ..., Em ) );
begin
        s_partition := {∅};
        minimizing_set := ∅;
        for i := 1 to |H(S)| do
        begin
                if minimizing_set = ∅ or minimizing_set ⊄ Ei then
                begin
                        T := Ei - || s_partition || ;
                        minimizing_set := T ;
                        s_partition := s_partition ∪ {T};
                end
        end;
        if || s_partition || ⊂ S then d(S,H(S)) := 0;
        else
                begin
                        if | s_partition | is odd then d(S,H(S)) := 1;
                        if | s_partition | is even then d(S,H(S)) := -1;
                end
        end
end INT_HYP_DOM;
```

Fig. 1. Algorithm for computing $d(S,\mathscr{H}(S))$ of an interval hypergraph.

isting algorithms for directed network systems. If so, one must find efficient algorithms which, first, generate all subhypergraphs $\mathscr{H}(S)$, $S\subseteq V$, with $d(S,\mathscr{H}(S))\neq 0$ of an interval hypergraph $\mathscr{H}$ and, secondly, compute $d(S,\mathscr{H}(S))$ efficiently. While a solution to the first task seems not to be found so easily and have to be left as an open problem a solution to the second task, i.e. the efficient computability of $d(S,\mathscr{H}(S))$, is evident, as a short algorithm shows which 'implements' the proof of Theorem 1 (see Fig. 1).

To make the relationship between this algorithm and the proof of Theorem 1 clear, we add that, for sake of a short presentation, the interval hypergraph provided as the input to the procedure INT_HYP_DOM is assumed to be simple and to possess an ordered edge set in accordance with a given interval preserving ordering of its vertices. Further, for obtaining a simple interval hypergraph at each induction stage of the proof a minimizing set is used in the algorithm that allows us to neglect non-minimal edges. Finally, s_partition is the set family $S_1,\ldots,S_k$ that, along with the comments after Theorem 1, decides upon the value of $d(S,\mathscr{H}(S))$.

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References

[1] C. Berge, Hypergraphs, vol. 45, North-Holland, Mathematical Library, Amsterdam, 1989.
[2] R.E. Barlow, S. Iyer, Computational complexity of coherent systems and the reliability polynomial, ORC-Report 85-6, Operations Research Center, Univ. of California, Berkeley, 1985.
[3] R.E. Barlow, E. Proschan, Statistical Theory of Reliability and Life Testing, Holt Rinehart, and Wilson, New York, 1975.
[4] F.T. Boesch, A. Satyanarayana, C.L. Suffel, Some alternate characterizations of reliability domination, Probab. Eng. Inform. Sci. 4 (1990) 257–276.
[5] A. Behr, L. Camarinopoulos, G. Pampoukis, On the domination of k-out-of-n systems, IEEE Trans. Reliability 44 (1995) 705–708.
[6] A.B. Huseby, Domination theory and the Crapo beta-invariant, Networks 19 (1989) 135–149.
[7] A.B. Huseby, A unified theory of domination and signed domination, Statistical Research Report 3, University of Oslo, 1984.
[8] A. Kossow, W. Preuss, Reliability of consecutive-k-out-of-n: F systems with non-identical component reliabilities, IEEE Trans. Reliability R-38 (1989) 229–233.
[9] A. Satyanarayana, A. Prabhakar, New topological formula and rapid algorithm for reliability analysis of complex networks, IEEE Trans. Reliability R-27 (1978) 82–100.
[10] J.G. Shantikumar, Reliability systems with consecutive minimal cutsets, IEEE Trans. Reliability R-36 (1987) 546–549.