Codes associated to the zero-schemes of sections of vector bundles

Tohru Nakashima

Department of Mathematical and Physical Sciences, Faculty of Science, Japan Women's University, Mejirodai 2-8-1, Bunkyoku, Tokyo 112-8681, Japan

A R T I C L E   I N F O

Article history:
Received 16 January 2009
Revised 24 June 2009
Available online 25 July 2009
Communicated by H. Stichtenoth

Keywords:
Algebraic geometric codes
Vector bundles
Cayley–Bacharach property

A B S T R A C T

We consider the algebraic geometric codes associated to the zero-schemes of sections of vector bundles on a smooth projective variety. We give lower bounds for the minimum distances of the codes exploiting the Cayley–Bacharach property of zero-dimensional subschemes.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

Let $X$ be a smooth projective variety defined over the finite field $\mathbb{F}_q$ with $q$ elements. Let $X(\mathbb{F}_q)$ denote the set of $\mathbb{F}_q$-rational points. For a subset $P = \{P_1, P_2, \ldots, P_n\} \subset X(\mathbb{F}_q)$ and a line bundle $L$ on $X$ defined over $\mathbb{F}_q$, we define an algebraic geometric code $C(X, Z, L)$ to be the image of the evaluation map

$$\varphi_Z : H^0(X, L) \rightarrow \bigoplus_{i=1}^{n} L_{P_i} \cong \mathbb{F}_q^n.$$

Although the most natural choice of $P$ is $P = X(\mathbb{F}_q)$, different choices of $P$ may lead to new codes for which we can determine lower bounds for their minimum distance. For example, the case $Z$ is a complete intersection of two curves in the projective plane $\mathbb{P}^2$ has been considered in [3] and a lower bound for the minimum distance has been given. As another example, the code from the singular set of an algebraic foliation on $\mathbb{P}^2$ has been investigated in [1]. A key ingredient in these
works is the Cayley-Bacharach property of the zero-schemes of sections of rank two vector bundles.

In this note we consider the codes associated to the zero-schemes of sections of a vector bundle on a smooth projective variety. We shall give a lower bound for the minimum distance of \(C(X, Z, L)\) under the assumption that \(L\) satisfies some higher order embedding property called the \(k\)-very ampleness. The proof of our result is based on the generalization of the Griffith–Harris theorem relating the Cayley–Bacharach property of \(Z\) and the \(k\)-very ampleness of the adjoint bundle \(K_X + L\) as established in [7]. Since the line bundle \(O_{\mathbb{P}^m}(k)\) on the projective space \(\mathbb{P}^m\) is \(k\)-very ample, our result reproves the result on the codes from complete intersections in [2].

We also investigate the codes defined by the singular sets of an algebraic foliation, that is, the zero-schemes of a section of the twisted tangent bundle. In addition to the foliations on \(\mathbb{P}^m\), we also consider the codes defined by foliations on surfaces exploiting an analogue of Bogomolov instability theorem for vector bundles in positive characteristic.

We comment on the length \(n\) of our code \(C(X, Z, L)\). For general zero-schemes \(Z\), the individual points of \(Z\) are not necessarily defined over \(\mathbb{F}_q\) even if \(Z\) is defined over \(\mathbb{F}_q\), hence it is difficult to determine \(n\) explicitly. In this paper we assume that the zero-scheme \(Z\) is reduced and all of its points are defined over \(\mathbb{F}_q\). Thus we have \(n = \deg Z\), the degree of \(Z\). If we further assume that \(Z\) is defined by a regular section of \(E\), i.e. the associated Koszul complex is exact, then we have \(\deg Z = c_m(E)\), the \(m\)-th Chern class of \(E\). For example, for the singular set \(Z\) of a foliation on \(\mathbb{P}^2\) by curves of degrees \(r \geq 2\), we have \(\deg Z = r^2 + r + 1\). In [1, p. 202], explicit examples of the codes with \(n = 5\) are constructed in case \(q = 3\), \(r = 2\) from \(Z \setminus I_{\infty}\), the singular points in the affine plane \(\mathbb{A}^2\). For other cases when \(n\) is given explicitly, we refer to [2] where the codes with \(n = q\), \(q^3 - q\) are obtained from certain complete intersections in \(\mathbb{P}^m\).

2. Codes and the Cayley–Bacharach property

Let \(X\) be a smooth projective variety defined over \(\mathbb{F}_q\). For a zero-dimensional subscheme \(Z\) of \(X\), let \(I_Z\) denote its ideal sheaf. We assume that \(Z\) is reduced, of degree \(n\) such that all of its points are defined over \(\mathbb{F}_q\). For a line bundle \(L\) on \(X\) defined over \(\mathbb{F}_q\), let

\[
\varphi_Z : H^0(X, L) \to H^0(Z, L \otimes O_Z) \cong \mathbb{F}_q^n
\]

be the evaluation (or the restriction) map. We define the algebraic geometric code \(C(X, Z, L)\) to be the image of \(\varphi_Z\), which is a linear code over \(\mathbb{F}_q\) of length \(n\). Thus there exists an exact sequence

\[
0 \to H^0(I_Z(L)) \to H^0(X, L) \xrightarrow{\varphi_Z} C(X, Z, L) \to 0.
\]

In this paper we will be interested in the minimum distance of \(C(X, Z, L)\).

A zero-dimensional subscheme \(Z\) is said to satisfy the Cayley–Bacharach property of order \(k\) with respect to \(L\) if, for any subscheme \(Z' \subset Z\) of degree \(k\), we have \(h^0(I_{Z'}(L)) = h^0(I_Z(L))\). Since \(H^0(I_Z(L))\) is the kernel of \(\varphi_Z\), this condition implies that for any \(f \in H^0(X, L)\) satisfying \(\varphi_Z(f) = 0\) for a subscheme \(Z' \subset Z\) of degree \(k\), we have \(\varphi_Z(f) = 0\).

Let \(Z' \subset Z\) be a subscheme. The residual subscheme of \(Z'\) in \(Z\) is the subscheme \(Z''\) such that

\[
I_{Z''}/I_Z = \mathcal{H}om(O_{Z'}, O_Z).
\]

We use the notation \(Z'' = Z - Z'\). For \(f \neq 0 \in H^0(X, L)\), let \(F = (f)_0\) denote the divisor defined by \(f\), we let \(ZF\) denote their intersection scheme and define

\[
Z_f = ZF, \quad \Delta_f = Z - Z_f.
\]

If \(Z', Z'' \subset Z\) are subschemes residual to one another, we define
Let \( X \) be a smooth projective variety defined over \( \mathbb{F}_q \). For a line bundle \( L \) on \( X \) and a reduced zero-dimensional subscheme \( Z \) of degree \( d \) such that all of its points are defined over \( \mathbb{F}_q \), let \( C = C(X, Z, L) \). If \( Z \) satisfies the Cayley–Bacharach property of order \( k \), then we have

\[
d_{\min}(C) \geq d - k + 1.
\]

**Proof.** We notice that \( d_{\min}(C) \) is given by

\[
d_{\min}(C) = \min_{f \in H^0(X, L), \varphi_Z(f) \neq 0} \deg \Delta_f.
\]

Assume that \( f \in H^0(X, L) \) satisfies \( \varphi_Z(f) \neq 0 \). If there exists a subscheme \( Z' \subset Z \) of degree \( k \) such that \( \deg \Delta' \geq d - k + 1 \), then the claim for \( d_{\min}(C) \) is clear. Hence we may assume that for all subschemes \( Z' \subset Z \) of degree \( k \), we have \( \deg \Delta' \leq d - k \).

We claim the following inequality holds:

\[
\deg \Delta'' \geq d - k + 1 - \deg \Delta'.
\]

Since \( \deg \Delta'' = \deg \Delta - \deg \Delta' \), the above inequality may be rewritten as \( \deg \Delta \geq d - k \), which implies \( d_{\min}(C) \geq d - k + 1 \) as desired.

Assume that the claimed inequality does not hold, that is, \( \deg \Delta' \leq d - k - \deg \Delta' \). Since \( \deg Z'' = \deg Z' - \deg \Delta'' \), we have

\[
\deg Z'' \geq d - k - (d - k - \deg \Delta') = \deg \Delta'.
\]

Hence we can find a subscheme \( W'' \subset Z'' \) with \( \deg W'' = \deg \Delta' \). Let \( W := Z'_f \cup W'' \subset Z_f \). Then \( W \) is a subscheme of \( Z \) such that \( \varphi_W(f) = 0 \) and \( \deg W = \deg Z'_f + \deg W'' = \deg Z'' = k \). Since \( Z \) satisfies the Cayley–Bacharach property of order \( k \) by assumption, we must have \( \varphi_Z(f) = 0 \), which is a contradiction. This completes the proof. \( \square \)

Let \( X \) be a smooth projective variety defined over an algebraically closed field. A line bundle \( L \) on \( X \) (or the linear system \(|L|\)) is said to be \( k \)-very ample if, for any zero-dimensional subscheme \( Z \) of degree \( k \), the restriction map \( \varphi_Z : H^0(X, L) \rightarrow H^0(Z, L \otimes \mathcal{O}_Z) \) is surjective. By means of the notion of \( k \)-very ampleness, we have the following generalization of the Griffith–Harris theorem [7, Theorem 7].

**Proposition 2.2.** Let \( X \) be a smooth projective variety of dimension \( m \) defined over \( \mathbb{F}_q \) and let \( L \) be a line bundle on \( X \). Let \( E \) be a rank \( m \) vector bundle on \( X \) and \( Z \) the reduced zero-scheme of a section of \( E \) with \( \deg Z = d \). If \(|K_X + L|\) is \((d - k - 1)\)-very ample, then \( Z \) satisfies the Cayley–Bacharach property of order \( k \) with respect to \( \det E - L \).
Let $s$ be a non-zero section of a vector bundle $E$ of rank $m$ and let $Z$ denote its zero-scheme. $s$ is said to be regular if the associated Koszul complex

$$0 \to \bigwedge^m E^\vee \to \cdots \to E^\vee \to O_X \to O_Z \to 0$$

is exact.

We have the following lower bound for the minimum distance of the code defined from the zero-schemes of regular sections of vector bundles.

**Theorem 2.3.** Let $X$ be a smooth projective variety of dimension $m \geq 2$ defined over $\mathbb{F}_q$. Let $L$ be a line bundle on $X$ and $E$ a vector bundle of rank $m$ on $X$ defined over $\mathbb{F}_q$. Let $Z$ be the reduced zero-scheme of a regular section of $E$ such that all of its points are defined over $\mathbb{F}_q$. If $|K_X + \det E - L|$ is $k$-very ample, then the length $n(C) = C(X, Z, L)$ is equal to $c_m(E)$ and the minimum distance satisfies

$$d_{\min}(C) \geq k + 2.$$

**Proof.** Let $\mathcal{L} := \det E - L$. By Proposition 2.2, if $|K_X + \mathcal{L}|$ is $k$-very ample, then $Z$ satisfies the Cayley–Bacharach property of order $d - k - 1$ with respect to $L = \det E - \mathcal{L}$. It follows from Proposition 2.1 that

$$d_{\min}(C) \geq d - (d - k - 1) + 1 = k + 2. \quad \square$$

We may apply the above theorem to bounding the minimum distance of the codes on the projective space $\mathbb{P}^m$ of dimension $m$. We obtain the following result concerning the Reed–Muller codes from the complete intersections proved in [2].

**Corollary 2.4.** Let $Z$ be a reduced complete intersection of $m$ hypersurfaces of degree $d_i$ ($1 \leq i \leq m$) in $\mathbb{P}^m$ such that all of its points are defined over $\mathbb{F}_q$. Then, for $C = C(\mathbb{P}^m, Z, O_{\mathbb{P}^m}(l))$, we have

$$d_{\min}(C) \geq \sum_{i=1}^m d_i - n - l + 1.$$

**Proof.** We notice that $Z$ is the zero-schemes of a section of the rank $m$ decomposable bundle $E = \bigoplus_{i=1}^m O_{\mathbb{P}^m}(d_i)$. For an integer $l$ with $\sum_{i=1}^m d_i - m - l - 1 > 0$ and $L = O_{\mathbb{P}^m}(l)$, $K_{\mathbb{P}^m} + \det E - L = O_{\mathbb{P}^m}(\sum_{i=1}^m d_i - m - l - 1)$ is $(\sum_{i=1}^m d_i - m - l - 1)$-very ample. Indeed, we know that the line bundle $O_{\mathbb{P}^m}(k)$ is $k$-very ample [5]. Thus the claim follows from Theorem 2.3. \square

As another example, we consider the codes defined by the singular sets of algebraic foliations. Let $X$ be a smooth projective variety of dimension $m \geq 2$ defined over $\mathbb{F}_q$ and let $T_X$ denote its tangent bundle. For a line bundle $H$ on $X$ defined over $\mathbb{F}_q$, a regular section of $T_X \otimes H$ is said to define an algebraic foliation by curves $\mathcal{F}$ on $X$ with tangent line bundle $H$. The zero-scheme $Z$ of a section $s \in H^0(X, T_X \otimes H)$ corresponding to $\mathcal{F}$ is called its singular set. When $X = \mathbb{P}^m$, a regular section of the twisted tangent bundle $E = T_{\mathbb{P}^m}(r)$ for an integer $r \geq 0$ is called an algebraic foliation by curves of degree $r$ in $\mathbb{P}^m$. In [1], the codes from the singular sets of foliations on $\mathbb{P}^2$ have been considered. We have the following result for general $m$.

**Corollary 2.5.** Let $\mathcal{F}$ be an algebraic foliation by curves of degree $r$ in $\mathbb{P}^m$ and let $Z$ be the reduced singular set of $\mathcal{F}$ such that all of its points are defined over $\mathbb{F}_q$. Let $C = C(\mathbb{P}^m, Z, O_{\mathbb{P}^m}(l))$. Then the length $n(C)$ of $C$ is given by $n(C) = m + 1$ if $r = 0$ and
\[ n(C) = \frac{(r+1)^m+1}{r} \quad \text{if } r \geq 1. \]

The minimum distance of \( C \) satisfies
\[ d_{\text{min}}(C) \geq n + nr - l + 3. \]

**Proof.** Since we have \( \deg Z = c_m(T_\pi(r)) \), the first claim is clear. The second claim follows from Theorem 2.3 since \( K_\pi + \det E - L = O_{\pi}(m+1+mr-l) \) is \((m+1+mr-l)\)-very ample. \(\square\)

3. Codes from foliations on surfaces

Let \( S \) be a smooth projective surface defined over an algebraically closed field \( k \). A rank two bundle \( E \) on \( S \) is said to be **unstable** (in the sense of Bogomolov) if \( E \) fits in an extension
\[
0 \to O_S(A) \to E \to I_Z(B) \to 0,
\]
where \( Z \) a zero-dimensional subscheme and \( A, B \) are divisors on \( S \) such that \( A - B \) belongs to the positive cone of \( S \), that is, the following conditions hold:

1. \( (A - B)^2 > 0 \);
2. \( (A - B) \cdot H > 0 \) for any ample divisor \( H \).

A theorem of Bogomolov states that if the characteristic of \( k \) is zero and \( \delta(E) := 4c_2(E) - c_1(E)^2 < 0 \), then \( E \) is unstable. Although this result does not hold in positive characteristic, by [6] we have the following

**Proposition 3.1.** Let \( S \) be a smooth projective surface defined over an algebraically closed field of characteristic \( p > 0 \) which is neither of general type nor a quasi-elliptic surface of Kodaira dimension 1. Let \( E \) be a rank two bundle on \( S \). Then \( E \) is unstable if \( \delta(E) < 0 \).

**Theorem 3.2.** Let \( S \) be a smooth projective surface defined over \( \mathbb{F}_q \) which is neither of general type nor a quasi-elliptic surface of Kodaira dimension 1. Let \( E \) be a rank two bundle on \( S \) defined over \( \mathbb{F}_q \), which admits a regular section whose zero-scheme \( Z \) is reduced and all of its points are defined over \( \mathbb{F}_q \). Let \( H \) be an ample line bundle on \( S \) defined over \( \mathbb{F}_q \) and let \( L = \det E - LH \) for an integer \( l \geq k + 3 \) where \( k > 0 \). Then \( C = C(S, Z, L) \) has the minimum distance
\[ d_{\text{min}}(C) \geq k + 2. \]

**Proof.** We follow the proof of [4, Theorem 3.1]. In view of Theorem 2.3, it suffices to show that \( |K_S + \det E - L| = |K_S + LH| \) is \( k \)-very ample. Since Kodaira vanishing theorem holds for our \( S \) (cf. [6]), we have \( H^1(K_S + LH) = 0 \). Hence, by the proof of [4, Lemma 2.2], if \( |K_S + LH| \) is \((k-1)\)-very ample but not \( k \)-very ample, then there exists a rank two bundle \( E \) which fits in the exact sequence
\[
0 \to O_S \to E \to I_W(LH) \to 0,
\]
where \( W \) is a zero-dimensional subscheme of degree \( k+1 \) which violates \( k \)-very ampleness. We have
\[
\delta(E) = 4 \deg W - l^2 H^2 \leq 4(k+1) - (k+3)^2 = -k^2 - 2k - 3 < 0.
\]

Hence \( E \) is unstable by Proposition 3.1. As in [4, Lemma 2.3], we see that there exists an effective divisor \( D \) containing \( W \) such that
\[ lH \cdot D - (k + 1) \leq D^2 < \frac{lH \cdot D}{2} < k + 1. \]

Since \( l \geq k + 3 \) and \( H \cdot D > 0 \), we have

\[ \frac{(k + 3)H \cdot D}{2} \leq \frac{lH \cdot D}{2} < k + 1. \]

This yields \( H \cdot D = 1 \). Then we have \( D^2H^2 \leq (D \cdot H)^2 = 1 \) by Hodge index theorem. However, this is impossible since \( D^2 \geq lH \cdot D - (k + 1) \geq k + 3 - (k + 1) = 2 \). Hence we are done.

We apply the above theorem to the singular sets of algebraic foliations on a surface.

**Proposition 3.3.** Let \( S \) be a smooth projective surface defined over \( \mathbb{F}_q \) which is neither of general type nor a quasi-elliptic surface of Kodaira dimension 1. Let \( F \) be an algebraic foliation defined by \( TS(rH) \) where \( H \) is an ample line bundle on \( S \) defined over \( \mathbb{F}_q \) and \( r \geq 0 \) is an integer. Let \( Z \) be the reduced singular set of \( F \) such that all of its points are defined over \( \mathbb{F}_q \). Let \( L = -K_S + (2r - l)H \) for an integer \( l \geq k + 3 \), where \( k > 0 \). Then \( C = C(S, Z, L) \) has the length

\[ n(C) = c_2(S) - K_S \cdot H + H^2 \]

and the minimum distance satisfies

\[ d_{\text{min}}(C) \geq k + 2. \]

**Proof.** Since \( \deg Z = c_2(TS \otimes H) = c_2(S) - K_S \cdot H + H^2 \), the first claim follows. The second claim is an immediate consequence of Theorem 3.2. \( \square \)

**Corollary 3.4.** Let \( S \) be a del-Pezzo surface, that is, a surface with ample \( -K_S \) defined over \( \mathbb{F}_q \). Let \( F \) be an algebraic foliation defined by \( TS(-rK_S) \) for an integer \( r \geq 0 \) and let \( Z \) be the reduced singular set of \( F \) such that all of its points are defined over \( \mathbb{F}_q \). Let \( L = -(2r - l + 1)(-K_S) \) for \( l \geq k + 3 \), where \( k > 0 \). Then \( C = C(S, Z, L) \) satisfies

\[ d_{\text{min}}(C) \geq k + 2. \]

**Proof.** Since a del-Pezzo surface \( S \) is neither of general type nor a quasi-elliptic surface of Kodaira dimension 1, we may apply Proposition 3.3 for \( H = -K_S \) and \( L = -K_S + (2r - l)(-K_S) = (2r - l + 1)(-K_S) \). \( \square \)

For the codes from foliations on surfaces of general type, we obtain the following result.

**Proposition 3.5.** Let \( S \) be a minimal surface of general type with ample \( K_S \) defined over \( \mathbb{F}_q \). Let \( Z \) be the reduced singular set of an algebraic foliation defined by \( TS(rK_S) \) for an integer \( r \geq 0 \), such that all of its points are defined over \( \mathbb{F}_q \). Let \( L = (2r - l)K_S \) for an integer \( l \) with \( k + 4 \leq l < 2r \), where \( k \geq 2 \). Then \( C = C(S, Z, L) \) satisfies

\[ d_{\text{min}}(C) \geq k + 2. \]

**Proof.** Since \( K_S + \det TS(rK_S) - L = IK_S \) is \( k \)-very ample for all \( l \geq k + 4 \), where \( k \geq 2 \) by [4, Corollary 3.3], our claim follows from Theorem 2.3. \( \square \)
Finally we recall a construction of rank two bundles on a surface from divisors as given in [7, Theorem 10]. Let $F_1$, $F_2$, $F_3$ be three curves in a surface $S$ defined over $\mathbb{F}_q$ and let $f_i$ denote the corresponding sections of the line bundles $\mathcal{O}_S(F_i)$. Assume that $F_1$ and $F_2$ have no common component and $Z = F_1 F_2$ is reduced of pure codimension two. Let $\mathcal{F}$ denote the syzygy sheaf:

$$
0 \rightarrow \mathcal{F} \rightarrow \bigoplus_{i=1}^{3} \mathcal{O}_S(-F_i) \xrightarrow{f} \mathcal{O}_S \rightarrow 0,
$$

where $f$ is defined by $f(x, y, z) = f_1 x + f_2 y + f_3 z$. Then $Z$ is the zero scheme of a section of the rank two bundle $E = \mathcal{F}(F_1 + F_2)$ and $E$ fits in the extension

$$
0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow \mathcal{I}_Z(F_1 + F_2 - F_3) \rightarrow 0.
$$

Acknowledgment

We would like to thank the referees for their valuable comments on the original manuscript of this paper.

References