Local scales on curves and surfaces

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ABSTRACT

In this paper, we extend a previous work on the study of local scales of a function to studying local scales of a d-dimensional surface. In the case of a function, the scale functions are computed by convolving the function with a symmetric kernel of zero mean and zero first moments of various scales. From the goodness of fit point of view, this convolution can be viewed as measuring locally the deviations from a linear function at various scales. The local scales are defined as points where deviation from a linear function reaches a local maximum. In the case of a d-dimensional surface, the analogy of the scale functions is to compute local deviations from a d-plane at various scales (this is related to Jones beta number). This analogy is realized through convolving the (surface) measure with a symmetric kernel of zero mean and zero first moments. We then apply the theory of singular integral operators on d-dimensional surfaces to show useful properties of local scales. We also show that the defined local scales are consistent in the sense that the number of local scales are invariant under dilation, and that one can relate the local scales of the original object with its dilated version via the dilating factor. In addition, with the assumption that the d-dimensional surface enjoys a certain degree of smoothness, we prove that our local scales are related to curvatures. Furthermore, this connection makes apparent that our local scales are intimately related to the change in deviation from flatness. Computational examples are also presented. In shape analysis, the local scales and the scale functions on the boundary can be used as local signatures or descriptors.

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1. Introduction

Given a bounded function f (an image) defined on $\mathbb{R}^n$, an important task in image analysis is the extraction of local features and information contained in f. The knowledge of local features is then used for tasks like image matching, texture segmentation, object recognition, and image and domain decompositions. In an image diffusion approach for extracting local features in images [27,28,39,29,3] one assumes that one has a multiscale representation $\{u(x,t)\}_{t \geqslant 0}$ (linear or nonlinear) of f. In particular in [29], D. Lowe developed the SIFT detector for feature extraction which has proven to be very useful in computer vision. This detector uses the vector $\{t\frac{\partial^2 u}{\partial x_i dx_j}(x,t)\}$ at locations (x,t) which are local maxima and minima of $t\frac{\partial u}{\partial t}(x,t)$. Here $u(t) = K_t * f$, where K_t is the Gaussian kernel on $\mathbb{R}^n$. In [30], $\{u(x,t)\}$ is replaced by a set of shapes $\{S(x,t)\}$ that contain x. These methods compute a single meaningful scale at each location x. We refer the readers to [24] and [23]

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for an overview and analysis of local scales in images. See also [43,1,45,23] for different approaches in obtaining the global scales from the point of view of image decompositions.

Motivated from the theory of function spaces, Jones and the first author propose in [20] a method for extracting a vector of scales at each location x. This notion of (multi) local scales is further characterized based on the visibility level of the scales and their separations from other scales. This multiscale analysis of a function f is related to diffusions (linear or nonlinear), and the theory of wavelets [10] and square functions applied to f. In particular, for each $t > 0$, define

$$G_t(x) := t^{-n/2} e^{-\pi \frac{\|x\|^2}{t}}.$$

$G_t(x)$ is the Gaussian kernel on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} G_t(x)\,dx = 1$ for all $t > 0$, and

$$\partial_t G_t(x) = (4\pi)^{-1} \Delta G_t(x). \tag{1}$$

Moreover, the Fourier transform of $G_t(x)$ is given by $\widehat{G_t}(\xi) = e^{-\pi t \|\xi\|^2}$ [37], and for all integer $k \geqslant 0$,

$$\left\| \frac{\partial^k}{\partial t^k} G_t \right\|_{L^1(\mathbb{R}^n)} \leqslant \frac{C_k}{t^k}. \tag{2}$$

For each $t > 0$ and $x \in \mathbb{R}^n$, let $\varphi_t(x) = t \frac{\partial}{\partial t} G_t(x)$. Given an image $f \in L^\infty(\mathbb{R}^n)$, define the *scale function* of f as

$$\mathbf{S}f(x,t) := \varphi_t * f(x) = \int_{\mathbb{R}^n} \varphi_t(x - y) f(y)\,dy. \tag{3}$$

One obtains that

$$\left\| \frac{\partial^k}{\partial t^k} \mathbf{S}f \right\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{C_k}{t^k} \|f\|_{L^\infty(\mathbb{R}^n)}. \tag{4}$$

The following notions of local scales of f are introduced in [20].

Definition 1 (*Local scales of functions*). Let $f \in L^\infty(\mathbb{R}^n)$. For each $x \in \mathbb{R}^n$, the local scales of f at x is defined as the set

$$T_f(x) := \left\{ t > 0: \left| \mathbf{S}f(x,t) \right| \text{ is a local maximum} \right\}. \tag{5}$$

By a change of variable, let $\tau = \log_a(t)$ for some $a > 1$ and $\mathbf{S}f(x,\tau) := \mathbf{S}f(x, a^\tau) = \mathbf{S}f(x,t)$. Denote

$$\mathcal{T}_f(x) := \left\{ \tau \in \mathbb{R}: t = a^\tau \in T_f(x) \right\}.$$

- For each $\beta > 0$, we say $\tau \in \mathcal{T}_f(x)$ is β-visible if $|\mathbf{S}f(x,\tau)| > \beta$.
- For each $\delta > 0$, we say $\tau \in \mathcal{T}_f(x)$ is δ-separated if $|\frac{\partial^2}{\partial \tau^2} \mathbf{S}f(x,\tau)| > \delta$.

The following characterizations of β-visible and δ-separated local scales are given in [20]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For each $x \in \Omega$ and $\delta > 0$, let

$$\tau_\delta(x) := \left\{ \tau \in \mathcal{T}_f(x): \left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}f(x,\tau) \right| > \delta \right\},$$

and for all $N > 0$, let

$$\Omega_{\delta, N} := \left\{ x \in \Omega: \#\tau_\delta(x) \geqslant N \right\}. \tag{6}$$

Denote by $|E|$ the Lebesgue measure of the set $E \subset \mathbb{R}^n$. The following corollary provides a characterization of $|\Omega_{\delta,N}|$ and it is an application to the John–Nirenberg theorem [17]. It tells us that the Lebesgue measure of the set of points which are embedded in many scales is small, and decays exponentially as a function of the number of scales.

Corollary 1. *Suppose $f \in L^\infty(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ be bounded. Let $\Omega_{\delta,N}$ be defined as in* (6)*. Then there exist constants C_1 and C_2 which depend on $\|f\|_{L^\infty}$ and $|\Omega|$ such that*

$$|\Omega_{\delta,N}| \leqslant C_1 e^{-C_2 N \delta^3}.$$

The proof of this corollary essentially uses (4) and the fact that for any integer $k \geqslant 0$, the Littlewood–Paley g_k-function (a vector-valued singular integral) defined as

$$g_k(f)(x) := \left[\int_0^\infty \left| t^k \frac{\partial^k \varphi_t}{\partial t^k} * f(x) \right|^2 \frac{dt}{t} \right]^{1/2}, \quad x \in \mathbb{R}^n,$$

is bounded on $L^2(\mathbb{R}^n)$. I.e.

$$\left\| g_k(f) \right\|_{L^2} \leqslant A_k \| f \|_{L^2},$$

for some constant $A_k > 0$ (see Chapter IV, Section 1 in [37]). This inequality can be easily verified using Fourier analysis (one can also show the converse inequality using the same techniques).

Another characterization is the following. For $x \in \Omega$ and $\beta, \delta > 0$, define

$$\tau_{\beta,\delta}(x) := \left\{ \tau \in \mathcal{T}_f(x) \colon \left| \mathbf{S}f(x, \tau) \right| > \beta, \ \left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}f(x, \tau) \right| > \delta \right\},$$

and

$$\Omega_{\beta,\delta,N} := \left\{ x \in \Omega \colon \#\tau_{\beta,\delta}(x) \geqslant N \right\}. \tag{7}$$

Corollary 2. *Suppose $f \in L^\infty(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ be bounded. Let $\Omega_{\beta,\delta,N}$ be defined as in* (7). *Then there exist constants C_1 and C_2 which depend on $\| f \|_{L^\infty}$ and $|\Omega|$ such that*

$$|\Omega_{\beta,\delta,N}| \leqslant C_1 e^{-C_2 N \delta^2 \alpha},$$

where $\alpha = \min(\beta, \delta)$.

In this paper, we would like to replace the function f with a d-dimensional surface $\Gamma \subset \mathbb{R}^n$. We note that multiscale and geometric analysis on curves and surfaces has been a wide subject of study since the early 1980's ([7,8,18,33,6], among others). Methods of denoising and reconstructing parametric curves are proposed by L.-M. Reissell [34] using wavelets, and recently by M. Feiszli and P. Jones [15] to denoise piecewise smooth curves while preserving singularity. See also P.L. Rosin [35] for a study of local scales on parametric curves. Our motivation comes from the work of G. David and S. Semmes [12]. In shape analysis, Γ can be viewed as the boundary of a shape and the knowledge of local scales of the boundary is useful for shape matching and comparison [41,31], among others. Just as in images, the local scales of Γ at x can be used to build distinctive feature detector for Γ. This paper is an extension of the ideas contained in the preprint by the first author [22].

We begin by defining different notions of regularity and rectifiability on Γ [11].

Definition 2. Let $\Gamma \subset \mathbb{R}^n$ with Hausdorff dimension d, and μ be a non-negative Radon measure in $\mathbb{R}^n$ restricted to Γ (for instance, take μ to be $\mathcal{H}^d$, the d-dimensional Hausdorff measure).

1. We say Γ is a d-dimensional **Lipschitz graph** (with constant C) if there is a d-plane P, and an $(n - d)$-plane $P^\perp$ orthogonal to P, and a Lipschitz function $A : P \to P^\perp$ (with Lipschitz norm C) such that

$$\Gamma = \left\{ p + A(p) \colon p \in P \right\}.$$

By a change of coordinate system, we can view $P \subset \mathbb{R}^d$ and write Γ as

$$\Gamma = \left\{ \left(p, A(p) \right) \colon p \in P \right\}.$$

2. We say Γ is **regular** if it is closed and if there exists a constant C such that

$$C^{-1} R^d \leqslant \mu \left(\Gamma \cap B_R(x) \right) \leqslant C R^d, \tag{8}$$

for all $x \in \Gamma$ and $0 < R < \operatorname{diam}(\Gamma)$. Here

$$B_R(x) = \left\{ y \in \mathbb{R}^n \colon \| x - y \| < R \right\},$$

and $\operatorname{diam}(\Gamma)$ is the diameter of Γ.

3. We say Γ is **CBPLG** (Contains Big Pieces of Lipschitz Graphs) if it is regular and if there exist $C > 0$, and $\epsilon > 0$ so that for every $x \in \Gamma$ and $0 < R < \operatorname{diam}(\Gamma)$, there is a d-dimensional Lipschitz graph E (with constant $\leqslant C$) such that

$$\mu \left(\Gamma \cap B_R(x) \cap E \right) \geqslant \epsilon R^d.$$

The family of kernels that we are using for the case of a d-dimensional surface is the following: Define

$$K_t(x) := t^{-d/2} e^{-\pi \frac{\|x\|^2}{t}},$$

and $\psi_t(x) = t \frac{\partial}{\partial t} K_t(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. Let μ be the d-dimensional surface measure restricted to Γ. The *scale function* of Γ is defined as

$$\mathbf{S}\Gamma(x, t) := \psi_t * \mu(x) := \int\limits_\Gamma \psi_t(x - y)\, d\mu(y). \tag{9}$$

Note that in defining the scale function $\mathbf{S}\Gamma$ in (9), we compute local measurements by convolving the measure μ with a symmetric kernel ψ_t of mean zero. We mention that there are other types of local measurements. For instance, in [19, 14], and [25], local deviations of Γ from a d-plane (known as Jones beta number) are computed. In [40] and [26], local (multiscale) Menger curvatures are used to quantify the regularity of Γ. In [44], regularity of Γ is characterized in terms of the Wasserstein L^2-distance from optimal mass transport. The area (or volume) of an ϵ-neighborhood of Γ is used as a regularization term in [2] for image segmentation as supposed to the length (or area) of Γ.

Remark 1. Let f be a function defined on $\mathbb{R}$ (assume f is Lipschitz). The graph of f is the curve $\Gamma = \{(x, f(x)): x \in \mathbb{R}\}$ embedded in $\mathbb{R}^2$ with $d\mu(x) = \sqrt{1 + |f'(x)|^2}\, dx$. The scale function of f can be rewritten as

$$\mathbf{S}f(x, t) = \int\limits_\mathbb{R} t \frac{\partial}{\partial t} \big[k_t\big(\rho(x, y)\big) \big] f(y)\, dy,$$

where $\rho(x, y) = |x - y|$ and $k_t(\rho) = t^{-1/2} e^{-\pi \rho^2 / t}$. For the same kernel k_t, the scale function of Γ can also be rewritten as

$$\mathbf{S}\Gamma(x, t) = \int\limits_\mathbb{R} t \frac{\partial}{\partial t} \big[k_t\big(\rho_f(x, y)\big) \big] \sqrt{1 + \big|f'(y)\big|^2}\, dy,$$

where $\rho_f(x, y) = \sqrt{|x - y|^2 + |f(x) - f(y)|^2}$. The main difference between $\mathbf{S}f$ and $\mathbf{S}\Gamma$ is in the choice of the metric ρ and ρ_f, respectively. Note that for an oscillatory Lipschitz function f, the kernel $t \frac{\partial}{\partial t} [k_t(\rho_f(x, y))]$ is much more oscillatory than the kernel $t \frac{\partial}{\partial t} [k_t(\rho(x, y))]$. We note that metric of the type ρ_f (nonlocal) has been used recently for image denoising [5,42,21], and image colorization [16].

Following [20], we define the following notions of local scales on Γ.

Definition 3 (*Local scales on surfaces*). Let Γ be a d-dimensional surface embedded in $\mathbb{R}^n$ with μ being its surface measure. For each $x \in \Gamma$, the local scales on Γ at x is defined as the set

$$T_\Gamma(x) := \big\{ t > 0: \ \big| \mathbf{S}\Gamma(x, t) \big| \text{ is a local maximum} \big\}. \tag{10}$$

By a change of variable, let $\tau = \log_a(t)$ for some $a > 1$ and $\mathbf{S}\Gamma(x, \tau) := \mathbf{S}\Gamma(x, a^\tau) = \mathbf{S}\Gamma(x, t)$. Denote

$$\mathcal{T}_\Gamma(x) := \big\{ \tau \in \mathbb{R}: \ t = a^\tau \in T_\Gamma(x) \big\}.$$

- For each $\beta > 0$, we say $\tau \in \mathcal{T}_\Gamma(x)$ is β-visible if $|\mathbf{S}\Gamma(x, \tau)| > \beta$.
- For each $\delta > 0$, we say $\tau \in \mathcal{T}_\Gamma(x)$ is δ-separated if $|\frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(x, \tau)| > \delta$.

We wish to extend the results from Corollaries 1 and 2 for local scales on Γ. Thus, we are interested in the following questions.

Question 1. What are sufficient conditions on Γ so that for all $k \geqslant 0$,

$$\sup_{x \in \Gamma} \left| \frac{\partial^k}{\partial t^k} \psi_t * \mu(x) \right| \leqslant \frac{C_{k,\Gamma}}{t^k}, \tag{11}$$

where $C_{k,\Gamma}$ depends on k and Γ.

Question 2. What are sufficient conditions on Γ so that the Littlewood–Paley function (still denoted by g_k) defined by

$$g_k(f)(x) := \left[\int\limits_0^\infty \left| t^k \frac{\partial^k}{\partial t^k} \psi_t * f(x) \right|^2 \frac{dt}{t} \right]^{1/2}, \quad x \in \Gamma, \tag{12}$$

is bounded on $L^2(\Gamma, \mu)$ for all integer $k \geqslant 0$. In particular, we would like to know whether the following inequality holds,

$$\left[\int_{\Gamma} \left| g_k(f)(x) \right|^2 d\mu(x) \right]^{1/2} \leqslant C \|f\|_{L^2(\Gamma)}. \tag{13}$$

Take f to be the characteristic function of $\Gamma \cap B_r(x)$, $x \in \Gamma$, then the above condition implies that $|t^k \frac{\partial^k}{\partial t^k} (\psi_t * \mu(x))|^2 \frac{dt\, d\mu(x)}{t}$ is a Carleson measure (see [11] for a definition).

In Section 2, we go over a framework for studying local scales on Γ using kernel methods. In Section 2.1, we show that under the assumption that Γ is a Lipschitz graph (or Γ is CBPLG), we have positive answers for Questions 1 and 2. The proof for the L^2-boundedness of vector-valued singular integral operators on Γ, of which the bound (13) on g_k is a special case, is outlined in Appendix A. In Section 2.2, we extend the characterizations of local scales of f from Corollaries 1 and 2 to local scales on Γ. Some useful properties of local scales are also presented.

In Section 3, to better understand the scale function of Γ and its local scales we provide examples, numerical experiments and geometric interpretations. In particular, when Γ is a smooth curve in $\mathbb{R}^2$, in Theorem 1 we obtain that for all $t > 0$ sufficiently small,

$$\mathbf{S}\Gamma(z, t) = \frac{1}{16\pi} \cdot \left(\kappa(z) \right)^2 t + O\left(t^2 \right). \tag{14}$$

Here $\kappa(z)$ stands for the curvature of Γ at point the z. Flat curves have zero curvature and hence this provides a quantitative restatement of the claim that the scale function measures "curviness" at scale $t > 0$. Furthermore, this connection makes apparent that the scale function exactly reflects the change in deviation from flatness. In the case when Γ is a smooth 2-dimensional surface in $\mathbb{R}^3$, in Theorem 2 we prove that for all $z \in \Gamma$

$$\mathbf{S}\Gamma(z, t) = \frac{1}{16\pi} \cdot \left(\kappa_1(z) - \kappa_2(z) \right)^2 t + O\left(t^{3/2} \right), \tag{15}$$

where $\kappa_1(z)$ and $\kappa_2(z)$ stand for the *principal curvatures* of Γ at the point z.

In Section 4 we list a collection of topics of future interest.

2. Local scales on *d*-dimensional surfaces

Let $P_d = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0, \forall i > d\} \subset \mathbb{R}^n$ be the d-plane, and for $x \in \mathbb{R}^n$, let $\phi(x) = e^{-\pi \|x\|^2}$. We have $\int_{P_d} \phi(x)\, d\mathcal{H}^d(x) = 1$. Note that since ϕ is radially symmetric, we have that for any P which is a rotation of P_d at the origin, $\int_P \phi(x)\, d\mathcal{H}^d(x) = 1$. For each $t > 0$, define

$$K_t(x) = t^{-d/2} \phi\left(\frac{x}{\sqrt{t}} \right).$$

Then we have

$$\int_{P_d} K_t(x)\, d\mathcal{H}^d(x) = 1, \quad \text{for all } t > 0,$$

and, for all $k \geqslant 1$,

$$\int_{P_d} \left| \frac{\partial^k}{\partial t^k} K_t(x) \right| d\mathcal{H}^d(x) \leqslant \frac{c_k}{t^k}. \tag{16}$$

In other words, K_t restricted to P_d which we can think as $\mathbb{R}^d$ is the heat kernel on $\mathbb{R}^d$. Define $\psi_t(x) = t \frac{\partial}{\partial t} K_t(x)$ for $x \in \mathbb{R}^n$ and $t > 0$, then it is easy to see that

$$\int_{P_d} \psi_t(x)\, d\mathcal{H}^d(x) = 0,$$

and ψ_t also has zero first moments on P_d.

Let $\Gamma \subset \mathbb{R}^n$ be a d-dimensional subset, and let μ be the d-dimensional surface or Hausdorff measure restricted to Γ. Recall the *scale function* $\mathbf{S}\Gamma : \Gamma \times (0, \infty) \to \mathbb{R}$ defined as

$$\mathbf{S}\Gamma(x, t) = \psi_t * \mu(x) := \int_{\Gamma} \psi_t(x - y)\, d\mu(y). \tag{17}$$
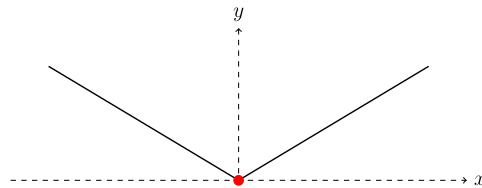
Fig. 1. A Lipschitz curve in $\mathbb{R}^2$ exhibiting a point with vanishing scale function.

For each $x \in \Gamma$ and $t > 0$, we define

$$\mathbf{K}\Gamma(x,t) = K_t * \mu(x) := \int_\Gamma K_t(x-y)\,d\mu(y). \tag{18}$$

By the property of ψ_t, we see that if Γ is a d-plane, then $\mathbf{S}\Gamma(x,t) = 0$ for all $t > 0$. Locally, the quantity $|\mathbf{S}\Gamma(x,t)|$ provides a measurement of how well Γ can be approximated by a d-plane near $x \in \Gamma$ at scale t. If we accept that $|\mathbf{S}\Gamma(x,t)|$ measures the deviation of Γ at x from a d-plane at scale t (which we can think of as the time variable), then the local scales (defined in Definition 3) are the points in time where deviations start to decrease, that is when $|\mathbf{S}\Gamma(x,t)|$ is a local maximum. Thus, as far as local scales are concerned, we are only interested in local changes in deviations from d-planes. In this respect, we note that there exists Γ such that $\mathbf{S}\Gamma(x,t) = 0$ for some $x \in \Gamma$ even though Γ is not linear at x.

Example 1. Let Γ be a curve in $\mathbb{R}^2$ consisting of two infinite straight lines emanating from the origin, see Fig. 1. Invoking the symmetry of ψ_t, we deduce that $\mathbf{S}\Gamma((0,0),t) = 0$ for all $t > 0$. More examples are given in Section 3.

Remark 2. Since ψ_t is radially symmetric, $\mathbf{S}\Gamma(x,t)$ transforms properly under Euclidean isometries. In other words, let $T \in E(n)$ be a Euclidean isometry and denote by $T\Gamma = \{Tx: x \in \Gamma\}$. Then, for all $t > 0$

$$\mathbf{S}(T\Gamma)(Tx,t) = \mathbf{S}\Gamma(x,t).$$

2.1. Smoothness of $\mathbf{S}\Gamma$ and L^2-boundedness of g_k

To address Questions 1 and 2, we begin by considering Γ being a d-dimensional Lipschitz graph. Let $\Gamma = \{z(r) = (r, A(r)): r \in F\}$ be a Lipschitz graph for some closed subset F in $\mathbb{R}^d$. In this situation, we see that the surface measure μ and the d-dimensional Hausdorff measure $\mathcal{H}^d$ restricted to Γ are equivalent. Denote by

$$\|z'(r)\| = \sqrt{\det\big[(g_{i,j})_{i,j=1,\ldots,d}\big]}, \quad \text{where } g_{ij} = \left\langle \frac{\partial z}{\partial r_i}, \frac{\partial z}{\partial r_j} \right\rangle, \tag{19}$$

and

$$\|\Gamma\|_* = \sup_{r \in F} \|z'(r)\|. \tag{20}$$

We have that for a measurable $E \subset \Gamma$,

$$\mu(E) = \int_\Gamma \chi_E(x)\,d\mu(x) = \int_F \chi_E\big(z(r)\big)\|z'(r)\|\,dr,$$

and

$$\mathcal{H}^d(E) = \int_F \chi_E\big(z(r)\big)\,dr.$$

But $1 \leqslant \|z'(r)\| \leqslant \|z\|_{\mathrm{Lip}} = C$. This implies

$$\mathcal{H}^d(E) \leqslant \mu(E) \leqslant C\mathcal{H}^d(E).$$

Thus it is equivalent to consider either $d\mu$ or $d\mathcal{H}^d = dr$ on Γ that is a Lipschitz graph. From now on, we assume μ to be either the Hausdorff or surface measure on Γ. The following proposition addresses Question 1.

Proposition 1. *Let Γ be a d-dimensional Lipschitz graph. Then, for all $x \in \Gamma$, $\mathbf{K}\Gamma(x,t)$ and $\mathbf{S}\Gamma(x,t)$ are infinitely differentiable in t, and for all $k \geqslant 0$, we have*

$$\sup_{x \in \Gamma} \left| \frac{\partial^k}{\partial t^k} \mathbf{S}\Gamma(x, t) \right| \leqslant \frac{C_k}{t^k} \|\Gamma\|_*, \tag{21}$$

where μ is the d-dimensional surface measure, and $\|\Gamma\|_*$ is defined in (20) and C_k depends only on k and d. If μ is the d-dimensional Hausdorff measure $\mathcal{H}^d$, then $\|\Gamma\|_*$ can be removed from (21).

Remark 3. Let P be any closed subset of $\mathbb{R}^d$ and a Lipschitz function $A : P \to \mathbb{R}^{n-d}$. The function A can be extended to $\mathbb{R}^d$ with the same Lipschitz constant using an extension theorem of Whitney (see Theorem 3 of Chapter VI, Section 2 of A. Stein [37]). Thus without lost of generality, we may assume that $P = \mathbb{R}^d$, that is

$$\Gamma = \{(r, A(r)) : r \in \mathbb{R}^d\}.$$

Before proving Proposition 1, we need the following result, where the proof is given in Appendix B.

Proposition 2. Let Γ be as in Proposition 1 with $F = \mathbb{R}^d$. For all $k \geqslant 0$, let

$$\psi_{t,k}(x - y) = t^{-d/2} \left[\pi \frac{\|x - y\|^2}{t} \right]^k e^{-\pi \frac{\|x-y\|^2}{t}}, \tag{22}$$

for all $x = (r, A(r)), y = (s, A(s)) \in \Gamma$. Then

$$\int_{\mathbb{R}^d} \psi_{t,k}(x - y) \, ds \leqslant C_k, \tag{23}$$

where C_k is a constant that depends only on k and d. Here $ds = d\mathcal{H}^d(y)$.

Remark 4. Denote by

$$\|A'\|_{L^\infty} := \|\nabla A\|_{L^\infty} = \sup_{r \in \mathbb{R}^d} \left[\sum_{i=1}^d \left| \frac{\partial A}{\partial r_i}(r) \right|^2 \right]^{1/2}.$$

The following is another estimate for $\int_{\mathbb{R}^d} \psi_{t,k}(x - y) \, ds$.

$$\int_{\mathbb{R}^d} \psi_{t,k}(x - y) \, ds \leqslant \int_{\mathbb{R}^d} t^{-d/2} \left(1 + \|A'\|_{L^\infty}^2\right)^k \left[\pi \frac{\|r - s\|^2}{t} \right]^k e^{-\pi \frac{\|r-s\|^2}{t}} \, ds = A_k \left(1 + \|A'\|_{L^\infty}^2\right)^k, \tag{24}$$

where

$$A_k = \int_{\mathbb{R}^d} \left(\pi \|s\|^2\right)^k e^{-\pi \|s\|^2} \, ds.$$

Note that the first estimate (23) does not depend on $\|A'\|_{L^\infty}$, while the second estimate (24) does. Thus we see that if $\|A'\|_{L^\infty}$ is large then (23) is a better estimate than (24).

Proof of Proposition 1. From Remark 2, we may assume $P = \mathbb{R}^d$, and $P^\perp = \mathbb{R}^{n-d}$ and without loss of generality, we may also assume

$$\Gamma = \{(r, A(r)) : r \in \mathbb{R}^d\},$$

where $\|A\|_{\text{Lip}} < \infty$. Fix $x \in \Gamma$. Observe that for all $y \in \Gamma$,

$$\frac{\partial^k}{\partial t^k} \psi_t(x - y) = \frac{1}{t^k} \sum_{i=0}^{k+1} c_i \psi_{t,i}(x - y),$$

where $\psi_{t,i}(x - y)$ is defined in (22), and c_i are constants that depend only on d and k. Using the notation $x = (r, A(r))$ and $y = (s, A(s))$, we have

$$\int_{\Gamma} \left| \frac{\partial^k}{\partial t^k} \psi_t(x - y) \right| d\mu(y) \leqslant \int_{\mathbb{R}^d} \left| \frac{\partial^k}{\partial t^k} \psi_t\big((r, A(r)) - (s, A(s))\big) \right| \|\Gamma\|_* \, ds$$

$$\leqslant \frac{\|\Gamma\|_*}{t^k} \sum_{i=0}^{k+1} |c_i| \int_{\mathbb{R}^d} \psi_{t,i}\big((r, A(r)) - (s, A(s))\big) \, ds \leqslant \frac{\|\Gamma\|_*}{t^k} \sum_{i=0}^{k+1} |c_i| C_i, \tag{25}$$

where the last inequality follows from (23). Using standard arguments involving the Dominated Convergence Theorem, we have

$$\frac{\partial^k}{\partial t^k}\mathbf{S}\Gamma(x,t) = \int_\Gamma \frac{\partial^k}{\partial t^k}\psi_t(x-y)\,d\mu(y), \tag{26}$$

which shows that

$$\left|\frac{\partial^k}{\partial t^k}\mathbf{S}\Gamma(x,t)\right| \leqslant \int_\Gamma \left|\frac{\partial^k}{\partial t^k}\psi_t(x-y)\right| d\mu(y) \leqslant \frac{C_k}{t^k}\|\Gamma\|_*,$$

where C_k is a new constant that depends only on d and k. Since $x \in \Gamma$ is arbitrary, we have that (21) holds. □

Remark 5. Note that as a consequence of the Dominated Convergence Theorem, we have that for all $x \in \Gamma$ and $t > 0$,

$$\mathbf{S}\Gamma(x,t) = t\frac{\partial\mathbf{K}\Gamma}{\partial t}(x,t).$$

Let $z : \mathbb{R}^d \to \mathbb{R}^n$ be a continuous function (not necessarily Lipschitz or differentiable), and suppose

$$\Gamma = \{z(r)\colon r \in \mathbb{R}^d\}.$$

For $x = z(r)$, let the measure in μ in this case be the d-dimensional Hausdorff measure restricted to Γ, then

$$\mathbf{S}\Gamma(x,t) = \psi_t * \mu(x) = \int_{\mathbb{R}^d} \psi_t\big(z(r) - z(s)\big)\,ds.$$

Then as a consequence to Proposition 1, we have

$$\left|\frac{\partial^k}{\partial t^k}\mathbf{S}\Gamma(x,t)\right| \leqslant \frac{C_k}{t^k}, \tag{27}$$

where C_k does not depend on Γ. Note that in this case, we get the same bound as in (21) but the quantity $\|\Gamma\|_*$ is removed. Thus by restricting to a d-dimensional Hausdorff measure, the condition on Γ can be weakened. Note that this is the worst possible bound and it does not take into account the regularity of Γ. When Γ is smooth, the bound can be strengthen as in (14).

Next, we would like to address Question 2, in particular, condition (13). To this end, we follow [11]. The following lemma is the vector-valued version of the result from [11] (Part II, Section 6, Example 6.7), and it is a general case of (13).

Lemma 1. *Let $0 < d \leqslant n$ be integers, and let $k(x)$ be a C^∞ function, defined on $\mathbb{R}^n \setminus \{0\}$, with values in some Hilbert space $\mathcal{H}$, and such that*

$$\big\|\nabla^j k(x)\big\|_{\mathcal{H}} \leqslant C(j)\|x\|^{-d-j} \quad \text{for all } j \geqslant 0, \tag{28}$$

and for all $M > 0$,

$$\sup_{0 < \epsilon < M} \left\| \int_{\epsilon < |\rho| < M} k(\rho\theta)|\rho|^{d-1}\,d\rho \right\|_{\mathcal{H}} \leqslant C \quad \text{for all } \theta \in S^{n-1}. \tag{29}$$

Let $A : \mathbb{R}^d \to \mathbb{R}^{n-d}$ be a Lipschitz function. Then the kernel

$$K\big((r,s)\big) = k\big(r-s, A(r) - A(s)\big)$$

defines a bounded singular integral operator from $L^2(\mathbb{R}^d, \mu)$ to $L^2(\mathbb{R}^d, \mathcal{H}, \mu)$.

The proof of this lemma in the scalar case is outlined in [11] in a series of exercises. For completeness, we will sketch the proof of this lemma in Appendix A, which is similar to the proof of the L^2-boundedness of the Cauchy integral operator on Lipschitz graphs for the vector-valued case.

Remark 6. Lemma 1 can be extended to Γ being CBPLG. This extension can be done by following the proof of Corollary 3.6 from [11] for the vector-valued case.

For a kernel k that is radially symmetric and decays exponentially as $\|x\| \to \infty$, an equivalent condition to (29) is the following

$$\sup_{\epsilon > 0} \left\| \int_{\{r > \epsilon\}} r^{-d/2} k(\theta/\sqrt{r}) \frac{dr}{r} \right\|_{\mathcal{H}} \leqslant C \quad \text{for all } \theta \in S^{n-1}. \tag{30}$$

Here, the normalization $r^{-d} k(\theta/r)$ is replaced by $r^{-d/2} k(\theta/\sqrt{r})$.

Using Lemma 1, we would like to show that (13) holds.

Corollary 3. *Let $\Gamma = \{x = (r, A(r)), \ r \in \mathbb{R}^d\}$ be a d-dimensional Lipschitz graph, for some Lipschitz function $A : \mathbb{R}^d \to \mathbb{R}^{n-d}$. Then the Littlewood–Paley function g_k defined in (12) is bounded on $L^2(\Gamma, \mu)$. In particular,*

$$\left[\int_{\mathbb{R}^d} |g_k(f)(x)|^2 \, d\mu(x) \right]^{1/2} \leqslant C_k \left[\int_{\mathbb{R}^d} |f(x)|^2 \, d\mu(x) \right]^{1/2}. \tag{31}$$

Proof. We will first prove the claim for g_0. We have

$$g_0(f)(x) = \left[\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right]^{1/2} = \left[\int_0^\infty \left| t \frac{\partial}{\partial t} K_t * f(x) \right|^2 \frac{dt}{t} \right]^{1/2}.$$

We follow Chapter IV, Section 1.3 from [37] by considering $\mathcal{H}$ to be the L^2 space on $(0, \infty)$ with the measure $t \, dt$, i.e.

$$\mathcal{H} = \left\{ f \colon \|f\|_{\mathcal{H}}^2 := \int_0^\infty |f(t)|^2 t \, dt < \infty \right\}.$$

Let $k(x) = \frac{\partial K_t}{\partial t}(x)$. We will show that $k(x)$ satisfies the hypothesis of Lemma 1: That is

$$\left\| \nabla^j k(x) \right\|_{\mathcal{H}} \leqslant C(j) \|x\|^{-d-j} \quad \text{for all } j \geqslant 0, \tag{32}$$

and

$$\sup_{\epsilon > 0} \left\| \int_{\{r > \epsilon\}} r^{-d/2} k(\theta/\sqrt{r}) \frac{dr}{r} \right\|_{\mathcal{H}} \leqslant C \quad \text{for all } \theta \in S^{n-1}. \tag{33}$$

Condition (32) clearly holds for it is a direct computation of the integral

$$\left\| \nabla^j k(x) \right\|_{\mathcal{H}}^2 = \int_0^\infty \left\| \nabla^j k(x) \right\|^2 t \, dt.$$

First we note that for all $x \neq 0$,

$$\left\| k(x) \right\|_{\mathcal{H}}^2 = \int_0^\infty \left| t^{-d/2-1} \left(-\frac{d}{2} + \frac{\pi \|x\|^2}{t} \right) e^{-\frac{\pi \|x\|^2}{t}} \right|^2 t \, dt$$

$$= \int_0^\infty t^{-d-1} \left(-\frac{d}{2} + \frac{\pi \|x\|^2}{t} \right)^2 e^{-\frac{2\pi \|x\|^2}{t}} \, dt$$

$$= \left(2\pi \|x\|^2 \right)^{-d-1} \int_0^\infty \left(\frac{2\pi \|x\|^2}{t} \right)^{d+1} \left(-\frac{d}{2} + \frac{1}{2} \left[\frac{2\pi \|x\|^2}{t} \right] \right)^2 e^{-\frac{2\pi \|x\|^2}{t}} \, dt$$

$$= \left(2\pi \|x\|^2 \right)^{-d} \int_0^\infty \tau^{d-1} \left(-\frac{d}{2} + \frac{1}{2} \tau \right)^2 e^{-\tau} \, d\tau.$$

Let

$$C(0) = \left[(2\pi)^{-d} \int_0^\infty \tau^{d-1} \left[\frac{d^2}{4} + \frac{1}{4}\tau^2 \right] e^{-\tau} \, d\tau \right]^{1/2},$$

which is finite. Then we have

$$\left\| k(x) \right\|_{\mathcal{H}} \leqslant C(0) \|x\|^{-d}.$$

For $j \geqslant 0$, let

$$k_j(x) = t^{-d/2-1} \left(-\frac{d}{2} - j + \frac{\pi \|x\|^2}{t} \right) e^{-\frac{\pi \|x\|^2}{t}}.$$

We observe that ith partial derivative of k is

$$\partial_{x_i} k(x) = \left(-\frac{2\pi}{t} \right) x_i k_1(x),$$

and

$$\left\| \nabla k(x) \right\|^2 = \left\| \left(\partial_{x_1} k(x), \dots, \partial_{x_n} k(x) \right) \right\|^2 = \left[\frac{2\pi}{t} \|x\| k_1(x) \right]^2 = \left(\|x\|^2 \right)^{-1} \left[\frac{2\pi \|x\|^2}{t} k_1(x) \right]^2.$$

But,

$$\left\| \frac{2\pi \|x\|^2}{t} k_1(x) \right\|_{\mathcal{H}}^2 = \left(2\pi \|x\|^2 \right)^{-d} \int_0^\infty \tau^d \left(-\frac{d}{2} - 1 + \frac{1}{2}\tau \right)^2 e^{-\tau} \, d\tau \leqslant C \|x\|^{-2d}.$$

Therefore,

$$\left\| \nabla k(x) \right\|_{\mathcal{H}} \leqslant C_1 \|x\|^{-d-1},$$

for some constant C_1. Now the general case follows, since

$$\partial_{x_i}^j k(x) = \partial_{x_i}^{j-1} \left(-\frac{2\pi}{t} x_i k_1(x) \right).$$

Next, we would like to show here condition (33). Since the kernel $k(x)$ is symmetric, it suffices to show the uniform bound

$$\sup_{\epsilon > 0} \left\| \int_{\{r > \epsilon\}} r^{-d/2} k(\theta/\sqrt{r}) \frac{dr}{r} \right\|_{\mathcal{H}} \leqslant C \quad \text{for all } \theta \in S^{n-1}, \tag{34}$$

where C does not depend on θ. For each $\epsilon > 0$, we have

$$\int_0^\infty \left| \int_{\{r > \epsilon\}} r^{-d/2} k(\theta/\sqrt{r}) \frac{dr}{r} \right|^2 t \, dt = \int_0^\infty \left| \int_{\{r > \epsilon\}} r^{-d/2} \frac{\partial K_t}{\partial t}(\theta/\sqrt{r}) \frac{dr}{r} \right|^2 t \, dt$$

$$= \int_0^\infty \left| \int_{\{r > \epsilon\}} r^{-d/2-1} t^{-d/2-1} \left[-(d/2) + \frac{\pi \|\theta\|^2}{tr} \right] e^{-\pi \frac{\|\theta\|^2}{tr}} \, dr \right|^2 t \, dt$$

$$= \int_0^\infty \left| t^{-d/2-1} \int_\epsilon^\infty \frac{\partial}{\partial r} K_r(\theta/\sqrt{t}) \, dr \right|^2 t \, dt = \int_0^\infty \left| t^{-d/2-1} K_\epsilon(\theta/\sqrt{t}) \right|^2 t \, dt$$

$$= \int_0^\infty \left| t^{-d/2-1} \epsilon^{-d/2} e^{-\pi \frac{\|\theta\|^2}{t\epsilon}} \right|^2 t \, dt = \left(\pi \|\theta\|^2 \right)^{-d} \int_0^\infty r^{d-1} e^{-2r} \, dr = C_d \pi^{-d},$$

where $C_d = \int_0^\infty r^{d-1} e^{-2r} \, dr < \infty$ with $1 \leqslant d \leqslant n$. Thus (34) holds with $C = [C_d \pi^{-d}]^{1/2}$.

To show (31) for general g_k, it suffices to show that the function $h_i(f)$ defined by

$$h_i(f)(x) = \left[\int\limits_0^\infty \left| t^i \frac{\partial^i}{\partial t^i} K_t * f(x) \right|^2 \frac{dt}{t} \right]^{1/2}$$

is bounded on $L^2(\Gamma, \mu)$, for all $i \geqslant 1$. In this case, the Hilbert space $\mathcal{H}$ in consideration is the L^2 space on $(0, \infty)$ with the measure $t^{2i-1} dt$ and the kernel is given by $k(x) = \frac{\partial^i K_t}{\partial t^i}$. $\quad \square$

2.2. Properties of local scales

For the remainder of the paper, when not specified we assume that $\Gamma \subset \mathbb{R}^n$ is a regular d-dimensional Lipschitz graph with μ being the d-dimensional surface or Hausdorff measure restricted to Γ satisfying (8) such that for all integer $k \geqslant 0$,

$$\sup_{x \in \Gamma} \left| \frac{\partial^k}{\partial t^k} \mathbf{S}\Gamma(x, t) \right| \leqslant \frac{C_{k,\Gamma}}{t^k}, \tag{35}$$

for some constant $C_{k,\Gamma}$ that depends only on k and Γ. We will also assume that the Littlewood–Paley function g_k is bounded in $L^2(\Gamma, \mu)$, i.e.

$$\left\| g_k(f) \right\|_{L^2(\Gamma)} = \left[\int\limits_\Gamma \int\limits_0^\infty \left| t^k \frac{\partial^k \psi_t}{\partial t^k} * f(x) \right|^2 \frac{dt}{t} \, d\mu(x) \right]^{1/2} \leqslant C \|f\|_{L^2(\Gamma)}. \tag{36}$$

Next, we would like to discuss some properties and characterizations of local scales on Γ. For $\delta > 0$, let $d_\delta \Gamma$ denote the dilated version of Γ, that is

$$d_\delta \Gamma = \{\delta x : x \in \Gamma\}.$$

Definition 4 *(Dilating consistency property).* We say the set of local scales $T_\Gamma(x)$ satisfies the *dilating consistency property* if

$$T_{d_\delta \Gamma}(x) = \left\{ \delta^s t : t \in T_\Gamma(\delta x) \right\}, \text{ for some } s \in \mathbb{R}.$$

Remark 7. The dilating consistency property of local scales also implies that the number (cardinality) of local scales is invariant under dilation. In other words, we do not introduce or remove local scales as a result of dilating (zoom-in or zoom-out) Γ.

In the following proposition (see proof in Appendix C), we see that the local scales on Γ (defined in Definition 3) satisfy the dilating consistency property.

Proposition 3. *Let μ be the d-dimensional surface (or Hausdorff) measure on a Lipschitz graph Γ. Then for all $\delta > 0$,*

$$T_{d_\delta \Gamma}(x) = \left\{ \delta^2 t : t \in T_\Gamma(\delta x) \right\}. \tag{37}$$

We are now ready to state the characterizations of local scales on Γ. For each $\delta > 0$ and $z \in \Gamma$, denote by

$$\mathcal{T}_\delta(z) := \left\{ \tau \in \mathcal{T}_\Gamma(z) : \left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(z, \tau) \right| > \delta \right\},$$

and

$$\Gamma_{\delta, N} := \left\{ z \in \Gamma : \#\mathcal{T}_\delta(z) > N \right\},$$

for all $N > 0$. Then the same result as in Corollary 1 also holds for Γ.

Corollary 4. *Let Γ be a d-dimensional Lipschitz graph in $\mathbb{R}^n$. Then there exist constants C_1 and C_2 (depending on Γ) such that for all balls $B_R(x)$, $x \in \Gamma$, we have*

$$\mu(\Gamma_{\delta, N} \cap B_R) \leqslant \mu(\Gamma \cap B_R) \cdot C_1 e^{-C_2 \delta^3 N}.$$

See Appendix D for the proof of Corollary 4. For $x \in \Gamma$ and $\beta, \delta > 0$, define

$$\tau_{\beta,\delta}(x) := \left\{ \tau \in \mathcal{T}_\Gamma(x) \colon \left| \mathbf{S}\Gamma(x,\tau) \right| > \beta, \left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(x,\tau) \right| > \delta \right\},$$

and

$$\Gamma_{\beta,\delta,N} := \left\{ x \in \Gamma \colon \#\tau_{\beta,\delta}(x) \geqslant N \right\}. \tag{38}$$

A similar result to Corollary 2 also holds for Γ.

Corollary 5. *Assume Γ is as in Corollary 4. Then there exist constants C_1 and C_2 which depend on Γ such that for all balls $B_R(x)$, $x \in \Gamma$,*

$$\mu(\Gamma_{\beta,\delta,N} \cap B_R) \leqslant \mu(\Gamma \cap B_R) \cdot C_1 e^{-C_2 N \delta^2 \alpha},$$

where $\alpha = \min(\beta, \delta)$.

As in [20], we can also define the nontangential control of $\mathbf{S}\Gamma(x,t)$, as a tool for lifting the visibility of local scales, by

$$\mathbf{S}^*\Gamma(x,t) := \sup_{\|x-y\| < \frac{\sqrt{t}}{\pi}} \left| \mathbf{S}\Gamma(y,t) \right| e^{-\pi \|x-y\|^2 / t}.$$

The nontangential local scales can be defined as before using $\mathbf{S}^*\Gamma(x,t)$.

Definition 5 *(Nontangential local scales).* The nontangential local scales of Γ at x is defined as the set

$$T_\Gamma^*(x) := \left\{ t \in (0, \infty) \colon \mathbf{S}^*\Gamma(x,t) \text{ is a local maximum} \right\}.$$

It can be shown in this case that the nontangential local scales $T_\Gamma^*(x)$ also satisfy the dilating consistency property with $s = 2$, i.e. for all $\delta > 0$,

$$T_{d_\delta \Gamma}^*(x) = \left\{ \delta^2 t \colon t \in T_\Gamma^*(\delta x) \right\}. \tag{39}$$

The proof is the replica of the proof of (37), and therefore we omit it.

Remark 8. From Remark 6, we note that these results can be extended to Γ being CBPLG.

3. Geometric interpretations and examples

Through the following examples and mathematical statements, we argue that the local maxima of $|\mathbf{S}\Gamma(x,\cdot)|$ do provide meaningful scale information of Γ; and when Γ is smooth, $\mathbf{S}\Gamma$ is related to the local curvature, which provides a quantitative way of measuring local curviness of Γ.

In Section 3.1 we first study the scale function of cones such as the one depicted in Fig. 1: this cone is in fact a planar Lipschitz curve. We explicitly compute its scale function, and its local scales in Section 3.1.1, and then generalize these computations to cones in higher dimensional Euclidean spaces in Section 3.1.2, with Proposition 4 summarizing our most general construction of cones. In these cone-like examples we are able to identify zero-dimensional subsets (the vertices of the cones) where the scale function identically vanishes. We also give an example of a 3-dimensional cone in $\mathbb{R}^4$ where the scale function vanishes at all points on the cone, not just the vertices. In Section 3.1.3 we look into another class of non-flat shapes, surfaces in this case, that contain points for which the scale function vanishes identically, but in this case these points form a subset of dimension equal to 1. All the examples in Section 3.1 share the feature that they are curves/surfaces which are non-flat, yet they contain points for which the scale function vanishes identically. Thus, the geometric objects constructed there would be examples of geometric shapes for which our construction of local scales would not be able to unequivocally detect their planarity or lack thereof, at all points.

In Section 3.2 we restrict our attention to smooth planar curves. The purpose of that section is to connect the scale function to differential invariants of the underlying curve. Our most general result in this respect is Theorem 1 in which we make explicit a connection between the scale function of a smooth curve and its curvature. This result implies in particular that for smooth curves, the scale function *exactly measures deviation of the curve from being contained in a line.* Notice that all examples constructed in Section 3.1 are Lipschitz but in fact are non-smooth.

Section 3.3 deals with the case of surfaces; there we follow the same path as in Section 3.2 and first study the case of spheres of a certain radius, and then go on to prove a general result, Theorem 2 that connects the scale function with the principal curvatures of the surface under study.

3.1. Local scales of cones and cylinders

3.1.1. Cones in $\mathbb{R}^2$

Consider the Lipschitz curve from Fig. 1:

$$\Gamma = \big\{(x, |x|), \, x \in \mathbb{R}\big\} \subset \mathbb{R}^2. \tag{40}$$

Let μ be the length measure on Γ, then we have for any $x \in \mathbb{R}$ and $t > 0$,

$$\mathbf{S}\Gamma\big((x, |x|), t\big) = -\frac{e^{-\frac{2\pi x^2}{t}} |x|(\sqrt{2}\sqrt{t} - 2\pi |x|)}{2t}. \tag{41}$$

Indeed, assume that $x \geqslant 0$. Then

$$
\begin{aligned}
2^{-1/2} t^{1/2} \mathbf{K}\Gamma\big((x, x), t\big) &= \int_{-\infty}^{\infty} e^{-\frac{\pi}{t}((s-x)^2 + (|s|-x)^2)} \, ds \\
&= \int_{-\infty}^{0} e^{-\frac{\pi}{t}((s-x)^2 + (s+x)^2)} \, ds + \int_{0}^{\infty} e^{-2\pi(s-x)^2/t} \, ds \\
&= e^{-2\pi x^2/t} \int_{-\infty}^{0} e^{-2\pi s^2/t} \, ds + \int_{-x}^{\infty} e^{-2\pi u^2/t} \, du \\
&= e^{-2\pi x^2/t} \int_{-\infty}^{0} e^{-2\pi s^2/t} \, ds + \int_{0}^{\infty} e^{-2\pi u^2/t} \, du + \int_{0}^{x} e^{-2\pi u^2/t} \, du.
\end{aligned}
$$

Since $\int_{-\infty}^{0} e^{-2\pi s^2/t} \, ds = \int_{0}^{\infty} e^{-2\pi s^2/t} \, ds = t^{1/2} 2^{-3/2}$, we find that

$$
\begin{aligned}
\mathbf{K}\Gamma\big((x, x), t\big) &= \frac{1}{2}\left(e^{-2\pi x^2/t} + 1 + 2^{3/2} t^{-1/2} \int_{0}^{x} e^{-2\pi s^2/t} \, ds \right) \\
&= \frac{1}{2}\left(e^{-2\pi x^2/t} + 1 + \int_{0}^{x(2/t)^{1/2}} e^{-\pi u^2} \, du \right).
\end{aligned}
$$

Now, (41) is found by a direct application of Remark 5. By direct differentiation one finds that for all $x \geqslant 0$, and $t > 0$

$$\frac{\partial \mathbf{S}\Gamma((x, x), t)}{\partial t} = e^{-2\pi x^2/t} \frac{x(\sqrt{2}t^{3/2} - 4\sqrt{2}\pi\sqrt{t}x^2 - 4\pi tx + 8\pi^2 x^3)}{4t^3}.$$

Solving, for $t > 0$ s.t. the RHS of the above expression vanishes yields that the local scales at (x, x) are given by

$$T_\Gamma\big((x, x)\big) = \big\{x^2 \cdot C_\Gamma\big\},$$

where

$$C_\Gamma = \frac{8}{3}\pi\left(1 + \pi + \sqrt{1 + 2\pi + 4\pi^2} \cos\left(\frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{3\pi(8 + \pi(13 + 16\pi))}}{-4 + \pi(15 + 8\pi(3 + 4\pi))}\right)\right)\right) \simeq 91.9502.$$

However, for $x = 0$ one sees from (41) that $\mathbf{S}\Gamma((0, 0), t) = 0$ for all $t > 0$, as claimed in Example 1. This means that the scale function fails to detect that Γ is not affine at $(0, 0)$ at any scale. This is consistent with the fact that $\{0\}$ is the only local scale of Γ at $x = 0$.

For a general cone $\Gamma = \{(x, \alpha|x|), \, x \in \mathbb{R}\} \subset \mathbb{R}^2$, one can also show that

$$T_\Gamma\big((x, x)\big) = \big\{x^2 \cdot C_\alpha\big\},$$

where $C_{\alpha_1} < C_{\alpha_2}$, whenever $\alpha_1 < \alpha_2$.

3.1.2. Cones in $\mathbb{R}^{d+1}$

Consider, for some constant c, the cone $\Gamma = \{(s, c\|s\|): s \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$. Let μ be the d-dimensional Hausdorff measure (i.e. $d\mu = ds$). We claim that $\mathbf{S}\Gamma(0, t) = 0$ for all $t > 0$.

Indeed, let $x = (s, c\|s\|) \in \Gamma$, $s \in \mathbb{R}^d$, we have

$$K_t(x) = t^{-d/2}e^{-\frac{\pi(\|s\|^2 + c^2\|s\|^2)}{t}} = (1+c^2)^{-d/2}(t(1+c^2)^{-1})^{-d/2}e^{-\frac{\pi\|s\|^2}{t(1+c^2)^{-1}}}.$$

Let $t' = t(1+c^2)^{-1}$, then

$$\psi_t(x) = t\frac{\partial}{\partial t}K_t(x) = (1+c^2)^{-d/2}t'\frac{\partial}{\partial t'}k_{t'}(s),$$

where

$$k_{t'}(s) = t'^{-d/2}e^{-\pi\|s\|^2/t'}, \quad s \in \mathbb{R}^d, \tag{42}$$

which is the heat kernel on $\mathbb{R}^d$. This implies

$$\mathbf{S}\Gamma(0, t) = \int_{\mathbb{R}^d} \psi_t(x)\,ds = (1+c^2)^{-d/2}\int_{\mathbb{R}^d} t'\frac{\partial}{\partial t'}k_{t'}(s)\,ds = 0,$$

for all $t' > 0$, and hence for all $t > 0$.

Remark 9. It is possible to generalize the construction to a certain degree. We do this by first noting that for all $\theta \in S^{d-1}$ and $t > 0$,

$$\int_0^\infty t\frac{\partial}{\partial t}k_t(r\theta)r^{d-1}\,dr = 0. \tag{43}$$

Then, the more general result holds.

Proposition 4. Let $A : S^{d-1} \subset \mathbb{R}^d \to \mathbb{R}$ be continuous. Define $\widehat{A} : \mathbb{R}^d \to \mathbb{R}$ by

$$\widehat{A}(s) = \begin{cases} \|s\|\,A(\frac{s}{\|s\|}) & s \in \mathbb{R}^d\setminus\{0\}, \\ 0 & s = 0, \end{cases}$$

and let $\Gamma = \{(s, \widehat{A}(s)) \in \mathbb{R}^{d+1}, s \in \mathbb{R}^d\}$. Then $\mathbf{S}\Gamma(0, t) = 0$ for all $t > 0$. Here the reference measure is the d-dimensional Hausdorff measure.

Proof. We have

$$\int_{\mathbb{R}^d} \psi_t\big((s, \widehat{A}(s))\big)\,ds = \int_{S^{d-1}}\int_0^\infty \psi_t\big((r\theta, \widehat{A}(r\theta))\big)r^{d-1}\,dr\,d\theta.$$

Then for each $\theta \in S^{d-1}$, we have by (43)

$$\int_0^\infty \psi_t\big((r\theta, \widehat{A}(r\theta))\big)r^{d-1}\,dr = (1 + A(\theta)^2)^{-d/2}\int_0^\infty t'\frac{\partial}{\partial t'}k_{t'}(r)r^{d-1}\,dr = 0.$$

Here $t' = t(1 + A(\theta)^2)^{-1}$. Thus, $\mathbf{S}\Gamma(0, t) = 0$ for all $t > 0$, and of course Γ is far from the graph of any linear function at 0. $\square$

Remark 10. We borrow the following convention from [13]. A positive Borel measure μ is d-uniform if

$$\mu(B_r(x)) = \omega_d r^d, \quad \text{for all } x \in \text{supp}(\mu), \text{ and } r > 0,$$

where ω_d is the Lesbegue measure of the Euclidean unit ball in $\mathbb{R}^d$. Kevin Vixie pointed out to the authors that the following 3-dimensional cone

$$\Gamma = \{x \in \mathbb{R}^4: x_4^2 = x_1^2 + x_2^2 + x_3^2\}$$

in $\mathbb{R}^4$ is 3-uniform with respect to the 3-dimensional Hausdorff measure $\mathcal{H}^3$ [13]. In other words, for all $x \in \Gamma$ and $r > 0$,

$$\mathcal{H}^3\big(\Gamma \cap B(x, r)\big) = \omega_3 r^3, \tag{44}$$

which implies that $\mathbf{S}\Gamma(x, t) = 0$ for all $x \in \Gamma$ and $t > 0$.

When Γ is a d-dimensional surface embedded in $\mathbb{R}^{d+1}$ with $d < 3$, using Theorem 6.12 from [13] on p. 59, one can show that the scale function vanishes for all points on Γ if and only if the measure μ is flat.

3.1.3. Cylinders

Another family of examples stems from constructions of surfaces from a given curve. Let $\Gamma \subset \mathbb{R}^2$ be a Lipschitz curve with $(0, 0) \in \Gamma$. Then, consider the cylindric surface with section Γ given by

$$\Gamma' = \Gamma \times \mathbb{R} \subset \mathbb{R}^3.$$

Let $(x(s), y(s), 0)$, $s \in J$ be an arc length parametrization of Γ, then one can parametrize Γ' by $(x(s), y(s), z)$ with $s \in J$ and $z \in \mathbb{R}$. Furthermore, the area element on Γ' is given by $ds\,dz$. Then, since

$$\iint_{J\ \mathbb{R}} e^{-\pi(x^2(s)+y^2(s)+z^2)/t}\,ds\,dz = \int_J e^{-\pi(x^2(s)+y^2(s))/t}\,ds \int_{\mathbb{R}} e^{-\pi z^2/t}$$

$$= t^{1/2} \int_J e^{-\pi(x^2(s)+y^2(s))/t}\,ds,$$

one finds that the convolution of the Gaussian kernel K_t (with $n = 2$) with the surface measure $\mu_{\Gamma'}$ of Γ' at point $(0, 0, 0)$ is

$$\mathbf{K}\Gamma'\big((0, 0, 0), t\big) = K_t * \mu_{\Gamma'}(0, 0, 0) = t^{-1/2} \int_J e^{-\pi(x^2(s)+y^2(s))/t}\,ds$$

$$= K_t * \mu_\Gamma = \mathbf{K}\Gamma\big((0, 0, 0), t\big)$$

where the K_t in the right hand side has normalization factor corresponding to $n = 1$. By applying Remark 5 we find $\mathbf{S}\Gamma'((0, 0, 0), t) = \mathbf{S}\Gamma((0, 0, 0), t)$.

Proposition 5. *Let $\Gamma \subset \mathbb{R}^2 \subset \mathbb{R}^3$ be a Lipschitz curve. Then, for any (x, y, z) such that $(x, y, 0) \in \Gamma$, one has that*

$$\mathbf{S}(\Gamma \times \mathbb{R})\big((x, y, z), t\big) = \mathbf{S}\Gamma\big((x, y, 0), t\big)$$

for all $t > 0$.

This is a geometric version of the property of scales for functions: if one adds a linear function h to f then $\mathbf{S}(f + h) = \mathbf{S}f$.

Remark 11. Using this result it is possible to construct an example of a Lipschitz surface Γ' in $\mathbb{R}^3$ s.t. the set

$$\Gamma'_0 := \big\{x \in \Gamma' \colon \mathbf{S}\Gamma'(x, t) = 0 \text{ for all } t > 0\big\}$$

has codimension 1. Indeed, let $\Gamma' = \Gamma \times \mathbb{R}$ where Γ is the curve described in (40). Then, the scale function is identically zero at all points on the line where the two planes meet, see Fig. 2.

3.2. The scale function and curvature of smooth curves

One would expect the scale function of Γ to be related to differential invariants of Γ for $t > 0$ small. This is made precise by the results that follow.

We start by considering the case of a circle in the plane; we first compute its scale function in terms of Bessel functions, and then, by means of a Taylor expansion, we show how the radius of the circle (and hence its curvature) is connected to the scale function up to first order in t.

Proposition 6. *Fix $R > 0$ and let $\Gamma \subset \mathbb{R}^2$ be a circle of radius R. Then, for all $t > 0$ and $x \in \Gamma$,*

$$\mathbf{S}\Gamma(x, t) = e^{-\frac{2\pi R^2}{t}} \pi R t^{-3/2}\left(4\pi R^2 \mathrm{I}_0\left(\frac{2\pi R^2}{t}\right) - t \mathrm{I}_0\left(\frac{2\pi R^2}{t}\right) - 4\pi R^2 \mathrm{I}_1\left(\frac{2\pi R^2}{t}\right)\right), \tag{45}$$

where I_0 and I_1 are the modified Bessel functions of order 0 and 1, respectively.
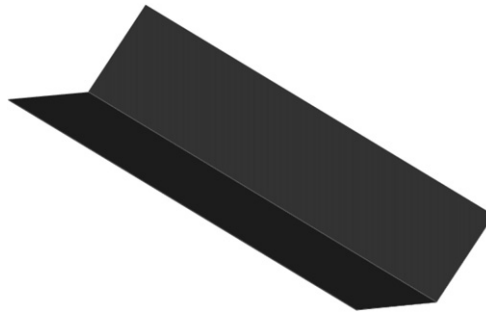
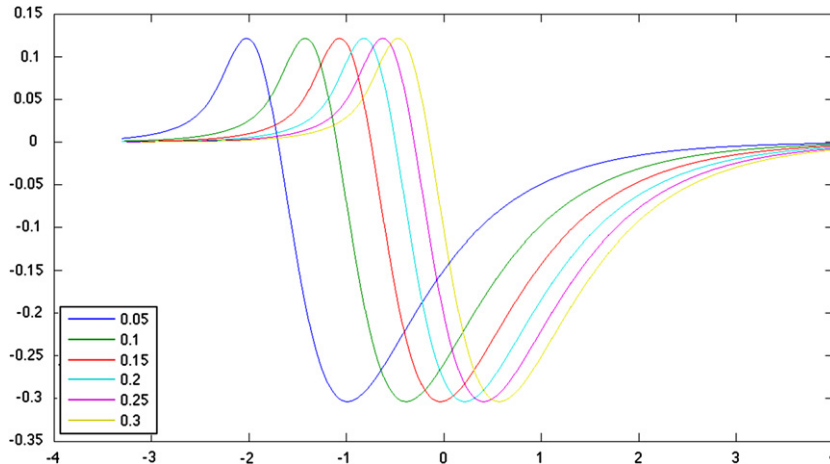**Fig. 2.** A Lipschitz surface with a subset of codimension one where the scale function vanishes identically.

Fig. 3. Scale function for a circle of radius R for different values of R (see legend on the bottom-left). The horizontal axis is given in logarithmic scale. Here, μ is the surface measure.

Proof. Fix the parametrization of Γ given by $\Gamma = \{C(\theta), \, \theta \in [0, 2\pi]\}$ where $C(\theta) = (R(\cos(\theta) - 1), R\sin(\theta))$ for $\theta \in [0, 2\pi]$. Hence, since $\|C(\theta)\|^2 = 2R^2(1 - \cos(\theta))$ and $\|C'(\theta)\| = R$, it follows that for $x = (0, 0)$ we have

$$\mathbf{K}\Gamma\big((0,0), t\big) = Rt^{-1/2}e^{-\frac{2\pi R^2}{t}} \int_0^{2\pi} e^{\frac{2\pi R^2}{t}\cos(\theta)} \, d\theta$$

$$= 2\pi R \, t^{-1/2} e^{-\frac{2\pi R^2}{t}} \left[\frac{1}{\pi} \int_0^{\pi} e^{\frac{2\pi R^2}{t}\cos(\theta)} \, d\theta \right]. \tag{46}$$

Recall the modified Bessel function: $I_n(u) = \frac{1}{\pi} \int_0^{\pi} e^{u\cos\theta} \cos(n\theta) \, d\theta$. This implies

$$\mathbf{K}\Gamma\big((0,0), t\big) = 2\pi R t^{-1/2} e^{-\frac{2\pi R^2}{t}} I_0\left(\frac{2\pi R^2}{t}\right).$$

Now invoking Remark 5 and using the fact that $\frac{\partial I_0(\rho)}{\partial \rho} = I_1(\rho)$ for all $\rho > 0$ we find the claim. $\square$

Fig. 3 shows a plot of the scale function of a circle of radius R for different values of R and for $t \in [0, 1]$, where μ is the surface measure. Note that the value of $\mathbf{S}\Gamma(x, t)$ at local maxima t is invariant under dilation, which agrees with (67).

By invoking standard expansions of $I_0(u)$ and $I_1(u)$ for large u one finds:

Corollary 6. *Fix $R > 0$ and let $\Gamma \subset \mathbb{R}^2$ be a circle of radius R. Then, for $0 < t \ll R^2$ one has*

$$\mathbf{S}\Gamma(x, t) = \frac{t}{16\pi R^2} + O\big(t^2\big), \quad \text{for all } x \in \Gamma.$$

Proof. One has [46] that for large $u > 0$, $I_0(u) = \frac{e^u}{\sqrt{2\pi u}}(1 + \frac{1}{8u} + \frac{9}{128u^2} + O(\frac{1}{u^3}))$ and $I_1(u) = \frac{e^u}{\sqrt{2\pi u}}(1 - \frac{3}{8u} - \frac{15}{128u^2} + O(\frac{1}{u^3}))$. Direct substitution of these expansions with $u = \frac{2\pi R^2}{t}$ into (45) yields the claim. $\quad\square$

Hence, the interpretation is indeed, that as R increases,[1] the scale function tends to zero. This result can be generalized to smooth curves.

3.2.1. The general case
In a fashion similar to Corollary 6 we now prove the following theorem for any smooth planar curve:

Theorem 1 *(Scale function of a smooth curve). Let Γ be any smooth simple planar curve which is bounded and let $x \in \Gamma$. Then, for all $t > 0$ sufficiently small,*

$$\mathbf{S}\Gamma(z, t) = \frac{1}{16\pi} \cdot \big(\kappa(z)\big)^2 t + O\big(t^2\big).$$

Here $\kappa(z)$ stands for the curvature of Γ at point z. Here, the reference measure is the length measure on the curve.

Remark 12.

- The theorem above makes apparent that the scale function exactly reflects the change in deviation from flatness.
- Note that as opposed to (21), the theorem above suggests that a better bound than (21) is possible when Γ is smooth and not just merely Lipschitz.

We need this standard lemma whose proof we omit.

Lemma 2. *For each $n \in \mathbb{N}$ there exist a polynomial p_n such that*

$$\int_\rho^\infty e^{-\pi x^2/t} x^n \, dx \leqslant \rho \, p_n(t/\rho) e^{-\pi\rho^2/t},$$

for all $\rho > 0$ and $t > 0$.

Proof of Theorem 1. Without loss of generality one can assume that z coincides with the origin, that the normal to the curve at z is the vertical direction (y), and that the tangent to the curve is the horizontal coordinate (x). Let κ denote the curvature of Γ at z and let $D > 0$ be large enough s.t. $\Gamma \subset B(z, D)$.

Below for $p \geqslant 0$ by $O(w^p)$ we denote continuous functions of w that vanish faster than any power of w order $0 < \ell < p$ as $w \to 0$.

Fix t small. We will compute

$$S(t) := \mathbf{S}\Gamma(0, t) := \int_\Gamma \psi_t(q) \, d\mu(q)$$

for the length measure μ on Γ. For a given $R > 0$ write

$$S(t) = \int_{\Gamma \cap B(0,R)} \psi_t\big(\|q\|\big) \, d\mu(q) + \int_{\Gamma \setminus B(0,R)} \psi_t\big(\|q\|\big) \, d\mu(q).$$

Notice that $\psi_t(r) = t^{-1/2} k(r^2/t)(\frac{\pi r^2}{t} - \frac{1}{2})$ where for $\eta \geqslant 0$ we write $k(\eta) = e^{-\pi\eta}$. Thus,

$$\left| \int_{\Gamma \setminus B(0,R)} \psi_t\big(\|q\|\big) \, d\mu(q) \right| \leqslant t^{-1/2} \int_{\Gamma \setminus B(0,R)} e^{-\pi\|q\|^2/t} \left(\frac{\pi\|q\|^2}{t} + \frac{1}{2} \right) d\mu(q)$$

$$\leqslant t^{-1/2} e^{-\pi R^2/t} \left(\frac{\pi D^2}{t} + \frac{1}{2} \right) \mu(\Gamma).$$

For $\gamma \in (0, 1/2)$ and $R = t^\gamma$, the RHS vanishes faster than t^2 as $t \to 0$.

---

[1] And therefore the curvature, which is equal to $1/R$, decreases.

When $R > 0$ is small enough [4] one can write $\Gamma \cap B(0, R) = \{P(x) := (x, y(x)), x \in I_R\}$ for some open interval $I_R \subset (-R, R)$ where for each $x \in I_R$,

$$y(x) = \frac{\kappa}{2}x^2 + O(R^3).$$

In particular, one can write

$$\int_{\Gamma \cap B(0,R)} \psi_t(\|q\|)\mu(dq) = \int_{I_R} \psi_t(\|P(x)\|)\|P'(x)\| \, dx.$$

Notice that for $x \in I_R$, one has $\|P(x)\|^2 = x^2 + \frac{1}{4}\kappa^2 x^4 + O(R^5)$ and $\|P'(x)\| = 1 + \frac{1}{2}\kappa^2 x^2 + O(R^3)$. This can be seen as expanding

$$g(x) = \|P'(x)\| = \sqrt{1 + \kappa^2 x^2 + O(R^4)} = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + O(R^3).$$

Furthermore, if one writes $I_R = (x^-(R), x^+(R))$ then one can see that $x^+(R) = R - O(R^3)$ and $x^-(R) = -R + O(R^3)$ are the solutions to the equation $\|P(x)\|^2 = R^2$ for R small.

Write the Taylor expansion of order one of k around a given point η with increment $\nu - \eta$:

$$k(\nu) = k(\eta) + k'(\eta)(\nu - \eta) + O(|\nu - \eta|^2).$$

In particular, letting $\eta = x^2/t$ and $\nu = \|P(x)\|^2/t$ one finds that for all $x \in I_R$

$$k(\|P(x)\|^2/t) = k(x^2/t) + k'(x^2/t)\frac{\kappa^2}{4}\frac{x^4}{t} + O(R^3),$$

and therefore

$$\psi_t(\|P(x)\|)\|P'(x)\| = Q_t(x) + O(R^3),$$

where

$$Q_t(x) := -\frac{e^{-\frac{\pi x^2}{t}}(x^2\kappa^2 + 2)(\pi^2\kappa^2(x^2\kappa^2 + 4)x^6 - 2\pi t(3x^2\kappa^2 + 8)x^2 + 8t^2)}{32t^{5/2}}.$$

Notice that $Q_t(x)$ is well defined for all $x \in \mathbb{R}$. In particular, one can compute explicitly

$$\int_{-\infty}^{\infty} Q_t(x) \, dx = \frac{\kappa^2}{16\pi}t + O(t^2).$$

Applying Lemma 2 we bound the error

$$\left| \int_{-\infty}^{\infty} Q_t(x) \, dx - \int_{x^-(R)}^{x^+(R)} Q_t(x) \, dx \right| \leqslant \int_{-\infty}^{x^-(R)} |Q_t(x)| \, dx + \int_{x^+(R)}^{\infty} |Q_t(x)| \, dx$$

$$\leqslant t^{-5/2}e^{-\pi(R^2 + O(R^4))/t}(R + O(R^3))U(t/(R + O(R^3))),$$

for some polynomial U. Hence, for $R = t^\gamma$ with $\gamma \in (0, 1/2)$, the RHS vanishes faster than t^2. Now,

$$\left| S(t) - \frac{\kappa^2}{16\pi}t \right| \leqslant \left| S(t) - \int_{I_R} \psi_t(\|P(x)\|)\|P'(x)\| \, dx \right|$$

$$+ \left| \int_{I_R} \psi_t(\|P(x)\|)\|P'(x)\| \, dx - \int_{I_R} Q_t(x) \, dx \right|$$

$$+ \left| \int_{I_R} Q_t(x) \, dx - \int_{\mathbb{R}} Q_t(x) \, dx \right| + \left| \int_{\mathbb{R}} Q_t(x) \, dx - \frac{\kappa^2}{16\pi}t \right|$$

$$\leqslant O(t^2) + O(R^4) + O(t^2) + O(t^2).$$

Finally notice that with $R = t^\gamma$ and $\gamma \in (3/4, 1/2)$, $O(R^4) = O(t^2)$ and we are done. $\quad\square$

Remark 13. A smooth planar curve is flat if and only if it has zero curvature at all points, and hence Theorem 1 provides a bit more evidence for the claim that the scale function measures "curviness" at scale $t > 0$. In other words, what this proposition says is that the claim that the scale function measures local deviation from a line/plane is true in the category of smooth curves.

3.3. The scale function and curvature of surfaces

Interestingly, the scale function for a sphere is easy to compute explicitly.

Proposition 7. *For each $R > 0$ let Γ denote a sphere of radius R in $\mathbb{R}^3$. Then, for all $t > 0$ and all $x \in \Gamma$,*

$$\mathbf{S}\Gamma(x,t) = -\frac{4\pi R^2}{t} e^{-\frac{4\pi R^2}{t}}.$$

Proof. Write the parametrization

$$\Gamma = \big\{ P(\theta, \varphi), \ \theta \in [0, \pi], \ \varphi \in [0, 2\pi] \big\},$$

where $P(\theta, \varphi) = R(\sin\theta\cos\varphi, \sin\theta\sin\varphi, 1 + \cos\theta)$, for which $P(0, \varphi) = (0, 0, 0)$, all φ. Note that the area element is $dA = R^2 \sin\theta\, d\theta\, d\varphi$ and that $\|P(\theta, \varphi)\|^2 = 2R^2(1 + \cos\theta)$. Thus

$$
\begin{aligned}
\mathbf{K}\Gamma\big((0,0,0),t\big) &:= R^2 t^{-1} \int\limits_0^\pi \int\limits_0^{2\pi} e^{-\pi \|P(\theta,\varphi)\|^2/t} \sin\theta\, d\theta\, d\varphi \\
&= R^2 t^{-1} \int\limits_0^\pi \int\limits_0^{2\pi} e^{-2\pi R^2(1+\cos\theta)/t} \sin\theta\, d\theta\, d\varphi \\
&= 2\pi R^2 t^{-1} e^{-2\pi R^2/t} \int\limits_0^\pi e^{-2\pi R\cos\theta/t} \sin\theta\, d\theta \\
&= 1 - e^{-\frac{4\pi R^2}{t}}.
\end{aligned}
$$

Now, we obtain the claim by observing that, by Remark 5,

$$\mathbf{S}\Gamma\big((0,0,0),t\big) = t\frac{\partial \mathbf{K}\Gamma\big((0,0,0),t\big)}{\partial t}. \qquad \square$$

The interpretation is again that as R goes to ∞, the scale function for a fixed t vanishes, thus reflecting the fact that the sphere is becoming flatter. Another way of stating this is that in order to see the curvature of the sphere of radius R, t needs to be large enough. Note that since the measure μ is the surface measure, the minimum value of the scale function remains constant at $-e^{-1}$, independent of R which coincides with (67) (this follows also from the explicit computation of the local scales below). Plots of the scale function for a few values of R are shown in Fig. 4.

This is a case in which one can also explicitly compute the local scales. We omit the easy proof of the following result.

Corollary 7. *For a sphere of radius R in $\mathbb{R}^3$ the set of local scales is singleton with element $4\pi R^2$.*

3.3.1. The general case

We obtain a theorem analogous to Theorem 1 but now in the context of surfaces in $\mathbb{R}^3$.

Theorem 2. *Let $\Gamma \subset \mathbb{R}^3$ be a smooth open surface. Then, for all $q \in \Gamma$ and $t > 0$ sufficiently small,*

$$\mathbf{S}\Gamma(q,t) = \frac{1}{16\pi}(\kappa_1 - \kappa_2)^2 t + O\big(t^{3/2}\big),$$

where κ_1 and κ_2 are the principal curvatures of Γ at the q. Here, the reference measure is the area measure on the surface.

Remark 14. Theorem 2 above can be interpreted as expressing the fact that for small t, at *umbilic* points of Γ, *up to first order* the scale function does not see the curviness of Γ. A further point is that in analogy with Theorem 1 one would expect that up to first order the *mean curvature* of Γ at q, given by $H(q) = \frac{\kappa_1 + \kappa_2}{2}$, would be the dominating term in $\mathbf{S}\Gamma(q,t)$. This is not the case, and in fact one can show that irrespective of the kernel used (as long as it is isotropic) the squared
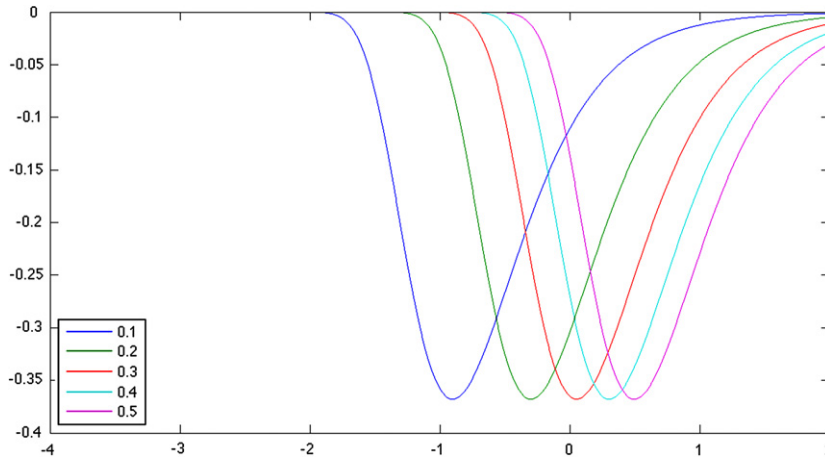
Fig. 4. Scale function for a sphere of radius R, for different values of R, which corresponds to a dilation (see legend on the bottom-left). The horizontal axis is given in logarithmic scale. Here μ is the surface measure.

difference of the principal curvatures will be the dominating term. In general, it is not possible to independently recover the two principal curvatures by observing the scale function as a function of t. This is not surprising since the scale function depends on only one parameter (t). Note that in the case of a circle, one can recover its radius (and hence curvature) by $\lim_{t\to 0} \frac{\partial \mathbf{S}\Gamma}{\partial t}$. However, in the case of a sphere (in $\mathbb{R}^3$), we see that $\lim_{t\to 0} \frac{\partial^k \mathbf{S}\Gamma}{\partial t^k} = 0$ for all $k \geqslant 0$. Even so, from Corollary 7 one can recover the radius of a sphere by looking at its local scales. See Section 4 for a discussion on non-isotropic kernels.

Proof of Theorem 2. The proof is similar to that of Theorem 1 and here we indicate the main differences only. Firstly, one can always assume that q coincides with the origin of $\mathbb{R}^2$ and one can rotate the axes so that the normal to Γ at q coincides with the z direction, and the principal curvature directions coincide with the x and y axes. Then, inside $B(0,R)$ one can write

$$\Gamma = \big\{ P(\rho,\phi) = \big(\rho\cos\phi, \rho\sin\phi, z(\rho,\phi)\big),\ \rho \in \big[0,\rho(\phi)\big), \phi \in [0,2\pi]\big\}$$

where $\rho(\phi) = R + O(R^3)$, and

$$z(\rho,\phi) = \frac{\rho^2}{2}\big(\kappa_1\cos^2\phi + \kappa_2\sin^2\phi\big) + O\big(R^3\big).$$

Furthermore, $\|P(\rho,\phi)\|^2 = \rho^2 + \frac{\rho^4}{4}(\kappa_1\cos^2\phi + \kappa_2\sin^2\phi)^2 + O(R^5)$ and the area element is

$$dA = \left(\rho + \frac{\rho^3}{2}\big(\kappa_1^2\cos^2\phi + \kappa_2^2\sin^2\phi\big) + O\big(\rho^5\big)\right)d\rho\,d\phi.$$

As in the proof of Theorem 1 we write

$$\int\limits_{\Gamma\cap B(0,R)} \psi_t\big(\|q\|^2\big)\,dA(q) = \int\limits_0^{2\pi}\int\limits_0^{\rho(\phi)} Q_t(\rho,\phi)\,d\rho\,d\phi + O\big(R^4\big),$$

where

$$Q_t(\rho,\phi) := e^{-\frac{\pi\rho^2}{t}}\left(1 - \frac{\pi\rho^4(\kappa_1\cos^2(\phi) + \sin^2(\phi)\kappa_2)^2}{4t}\right)\cdot\left(\frac{1}{2}\big(\cos^2(\phi)\kappa_1^2 + \sin^2(\phi)\kappa_2^2\big)\rho^3 + \rho\right).$$

By direct integration one finds that

$$\int\limits_0^{2\pi}\int\limits_0^{+\infty} Q_t(\rho,\phi)\,d\rho\,d\phi = \frac{(\kappa_1 - \kappa_2)^2}{16\pi}t + O\big(t^2\big).$$

The rest of the proof follows similar lines to those used in the proof of Theorem 1. $\quad\square$
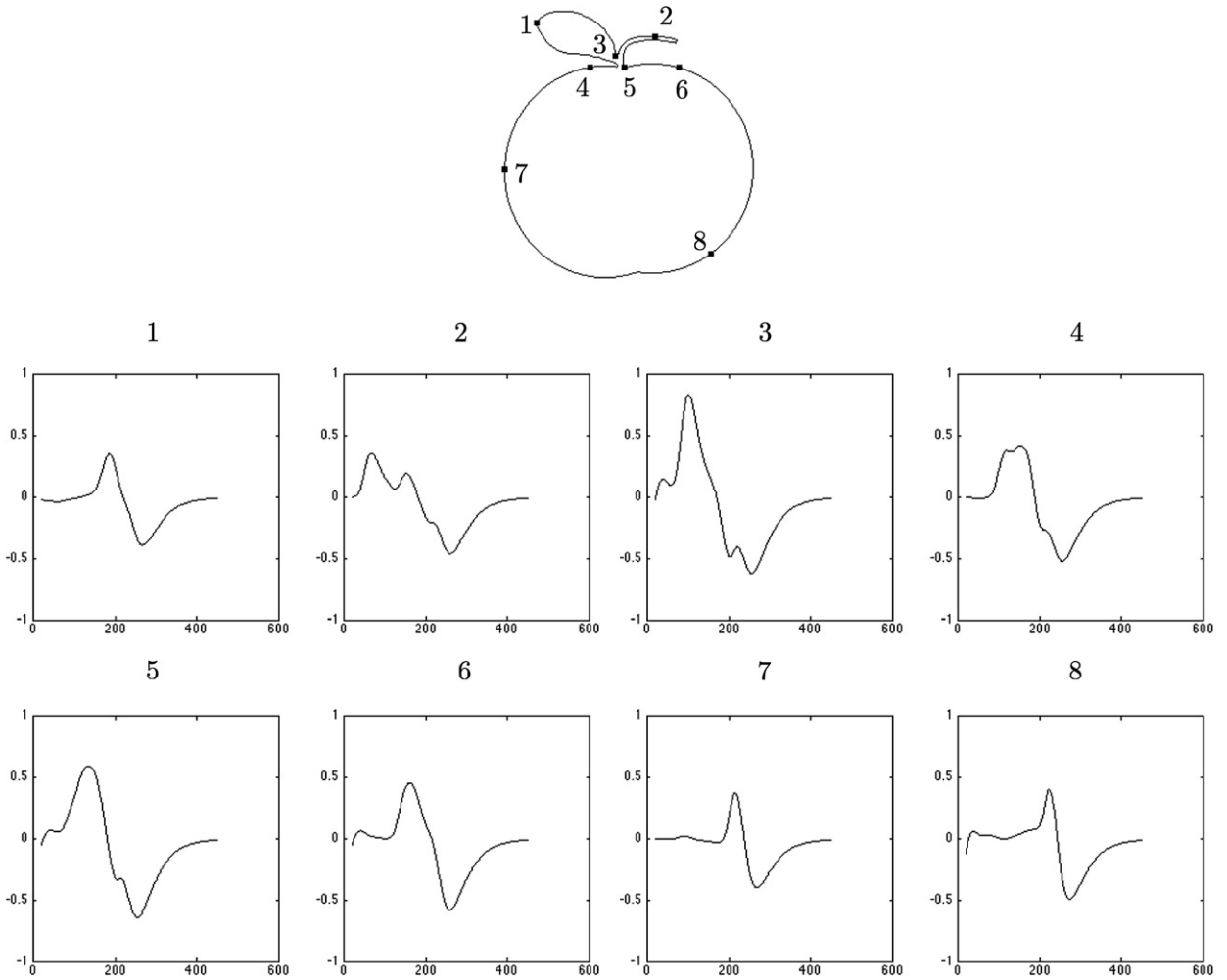
Fig. 5. In this figure, we show the scale function $\mathbf{S}\Gamma(x, \tau)$ for different points x's on the boundary of the apple silhouette from the MPEG7 Shape 1-B database. The horizontal axis given in each plot is in logarithmic scale τ. Points $x = 7$ and 8 have the same scale functions, which from their perspective look like the scale functions of a circle. Note that the two largest local maxima are the same. The scale functions of points $x = 2$, 3, and 4 are shifted version of one another. The first local scale corresponds to the deviations of the apple from its leaf.

3.4. A computational example with planar curves

We present a numerical computation of a planar curve, which is the boundary of a binary image I from the MPEG7 Shape 1-B database. We estimate the scale function as follows:

- We apply the Matlab function `bwboundaries` to I to obtain a polygonal curve representing the boundary of the object.
- Using this polygonal curve we estimate the length element μ at each point on the curve by centered differences. Notice that this step effectively constructs a (polygonal) parametrization of the boundary.
- We then perform numerical integration to obtain an estimate of the scale function for each value $t = a^\tau$ for $a = 1.05$ and a range of values $\tau = 20, \ldots, 450$. Images are discretized with $\triangle x = \triangle y = 1$.

We approximate the measures discretely by their empirical counterparts $\sum_i \delta_{x_i} \alpha_i$, where x_i are points on the curve, and α_i are non-negative weights attached to each of those points. In the case that we wish to approximate the length measure, we use the polygonal approximation to properly estimate the α_i. We also consider another case, which we call singular measure, where we use $\alpha_i = 1$ for all i. The latter is the preferred option in practical applications. Note that when Γ is the boundary of some set Ω, then the length measure μ is the norm of the distributional derivative $\|Du\|$, where $u = \chi_\Omega$. Results of carrying out these computations appear in Figs. 5 and 6.

In Fig. 6, μ is the length measure. Notice how points x and x' on the curve Γ belonging to similar areas have similar scale functions $\mathbf{S}\Gamma(x, \cdot)$ and $\mathbf{S}\Gamma(x', \cdot)$. For example, points 7 and 8 have very similar scale functions. Notice as well how
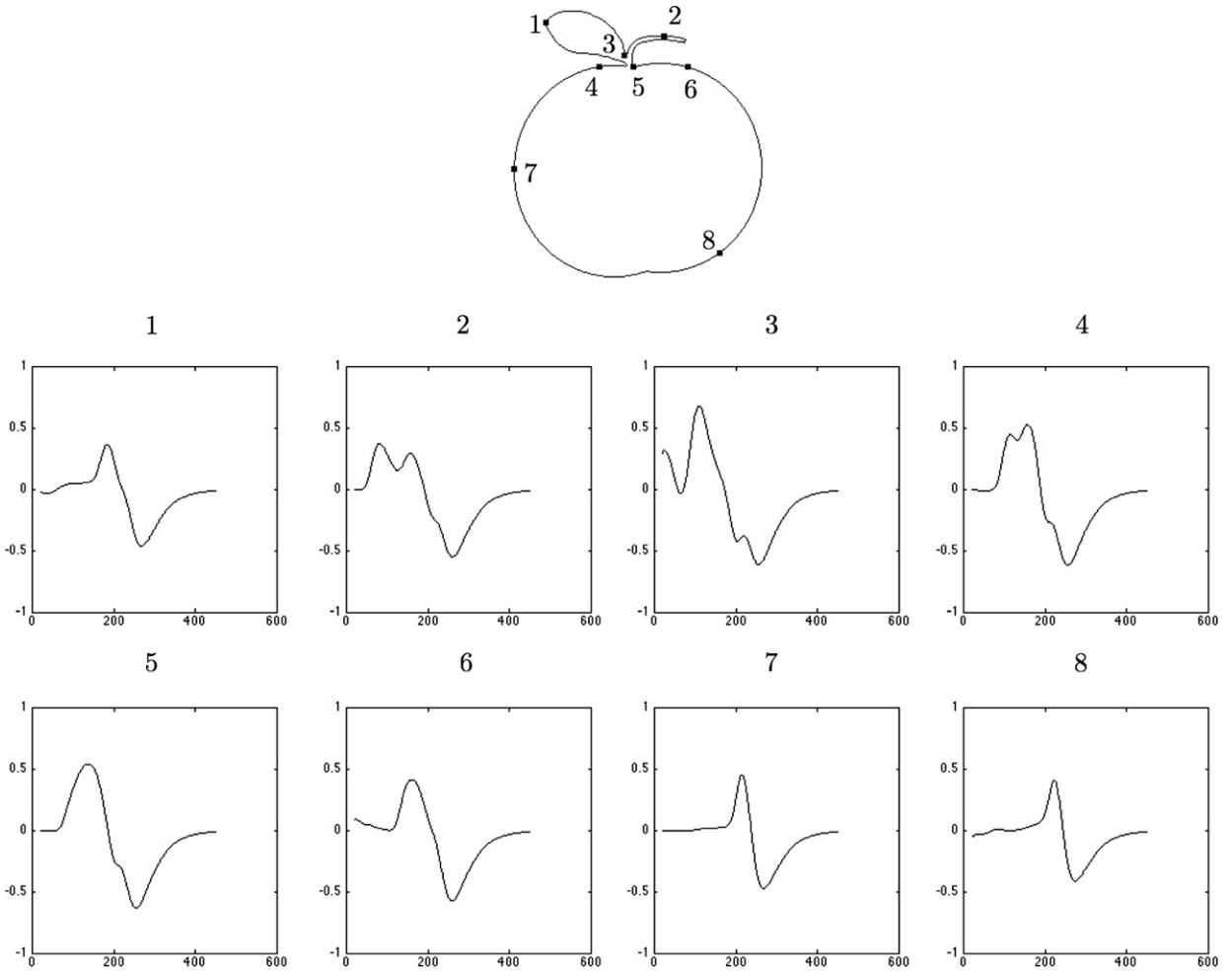
Fig. 6. As in Fig. 5, we show the scale function $\mathbf{S}\Gamma(x, \tau)$ for different points x's on the boundary of the apple silhouette from the MPEG7 Shape 1-B database. But now μ is a singular measure on the boundary Γ of the apple.

points that are near a feature, such as point 4, have a scale function that is significantly different from the scale function of a point such as 7, which lies in the middle of a rather smooth chord. By the same token, points 1, 2, 3, 4, and 5 all have scale functions that exhibit features very different from each other and also very different from those of points 7 and 8. These observations, can be used to devise shape matching methods in which one would match a point x in curve (shape) X to a point y in curve (shape) Y whenever their corresponding scale functions are similar, i.e. $\|\mathbf{S}X(x, \cdot) - \mathbf{S}Y(y, \cdot)\|$ is small for some suitable norm $\|\cdot\|$.

Fig. 6 is the same as in Fig. 5 but now μ is a singular measure on the curve. Here, we do not find significant differences between the scale functions using the length and the singular measure. In practice, the singular measure is easier to compute.

In Fig. 7, we show the first (logarithmic) β-visible local scale τ_1 where $\beta = 0.25$. In (a), μ is the length measure computed as in Fig. 5. In (b), the measure μ is the singular measure on the curve. Note that most points on the apple have similar local scale τ_1, and points near corners have smaller local scale τ_1. This aligns with the cone example (Section 3.1.1), which shows that the local scale of a point x (near a corner) with distance $d(x)$ from the corner is of order $d(x)^2$.

3.4.1. Local scales and oscillations

Let $\Lambda \subset \mathbb{R}^+$ be a finite set of frequencies, and for each $\lambda \in \Lambda$, let $A_\lambda : \mathbb{R} \to \mathbb{R}$ be given by $A_\lambda(s) = \sin(\lambda \pi s)$, $s \in \mathbb{R}$. Let $\Gamma_\lambda = \{P_\lambda(s) := (s, A_\lambda(s)): s \in \mathbb{R}\}$. Then for any fixed $r \in \mathbb{R}$ we have

$$\mathbf{S}\Gamma_\lambda\big((r, A_\lambda(r)), t\big) = \int\limits_{-\infty}^{\infty} \psi_t\big(P_\lambda(s) - P_\lambda(r)\big)\sqrt{1 + A_\lambda'(s)^2}\, ds.$$

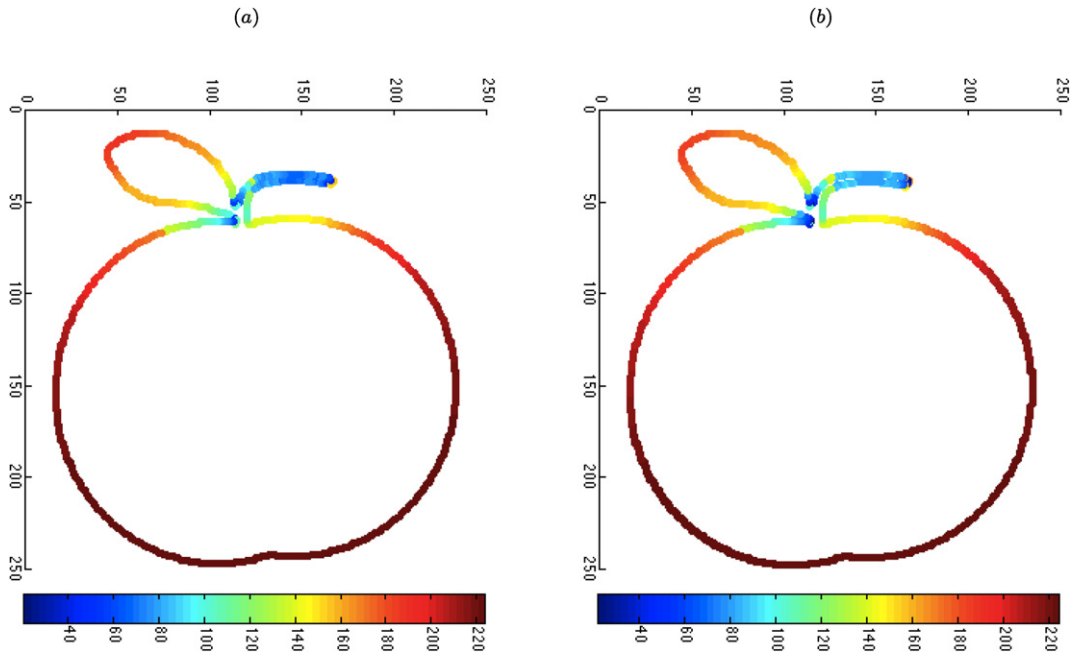(a)   (b)

Fig. 7. In this figure, we show the first (logarithmic) β-visible local scale $\tau_1 = \log_a t_1$ where $\beta = 0.25$. In (a), μ is the length measure computed as in Fig. 5. In (b), the measure μ is the singular measure on the curve corresponding to the outline of the apple.
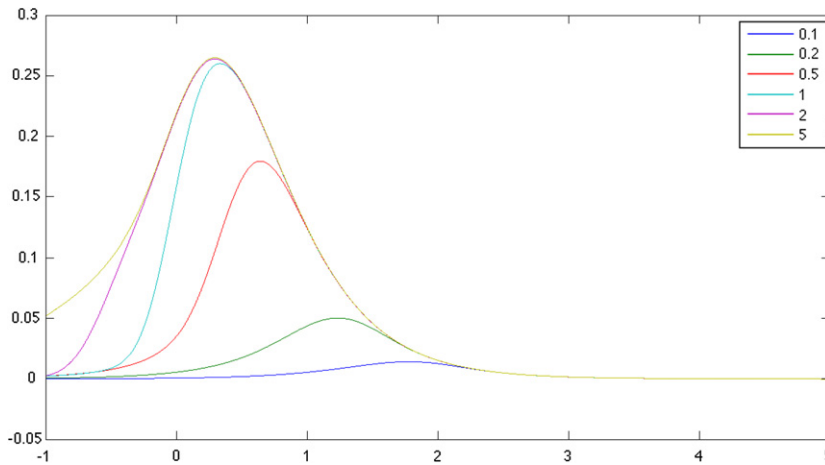

Fig. 8. Scale function for the curves $\Gamma_\lambda = \{(s, \sin(\lambda\pi s)): s \in \mathbb{R}\}$ for $\lambda \in \{0.1, 0.2, 0.5, 1, 2, 5\}$ (see legend on the top-right). The horizontal axis is given in logarithmic scale. These curves were obtained by numerical integration.

Fig. 8 shows plots of $\mathbf{S}\Gamma_\lambda((0,0), t)$ for different values of λ. We haven't been able to obtain an explicit formula for the scale function. Nevertheless, this computational example provides evidence that strongly suggests that oscillatory behavior on geometric objects may as well be captured by using the constructions of local scales that we describe in this paper.

4. Discussion

Next, we would like to make some remarks about other possible notions of local scales and future interests:

1. The kernel ψ_t dictates the type of local scales we see. In our case, $\psi_t = t\frac{\partial K_t}{\partial t}$ is symmetric and has zero mean and zero first moments. In particular, by considering $\psi_t = t^k \frac{\partial^k K_t}{\partial t^k}$, we see that if f is a surface which can be represented by a polynomial of degree $< 2k$, then $\psi_t * f = 0$. In other words, the resulting scale function would be insensitive to surfaces that arise from linear combinations of such polynomials.
2. Let Γ be a connected and bounded d-dimensional subset in $\mathbb{R}^n$, and suppose that it can be parametrized as $\Gamma = \{(f_1(r), \ldots, f_n(r)): r \in [0, 1]^d\}$, where f_i is continuous for each i. For each i, define

$$u_i(r,t) = K_t * f_i(r) = \int\limits_{\mathbb{R}^d} K_t(r-s) f_i(s) \, ds,$$

where $K_t(r) = t^{-d/2} e^{-\pi \|r\|^2 / t}$. For each $t > 0$, let

$$\Gamma(t) = \left\{ \left(u_1(r,t), \ldots, u_n(r,t) \right), \; r \in [0,1]^d \right\}.$$

We can think of $\Gamma(t)$ as the diffused version of Γ at scale t. Note that each $u_i(r,t)$ depends on the parametrization of f_i, which could be problematic for surfaces (d-dimensional surfaces with $d \geqslant 2$). However, given a parametrization, this method provides a tool for obtaining a diffused d-dimensional $\Gamma(t)$. Define $\mathbf{S}\Gamma(r,t)$ by

$$\mathbf{S}\Gamma(r,t) = \left\| \left(t \frac{\partial K_t}{\partial t} * f_1(r), \ldots, t \frac{\partial K_t}{\partial t} * f_n(r) \right) \right\|.$$

For each $r \in \mathbb{R}^d$, as before, the local scales of Γ can be defined as the local maxima of $\mathbf{S}\Gamma(r,t)$. This approach is considered by L.-M. Reissell in [34] and by P.L. Rosin [35] to represent curves in a multiscale fashion using wavelets. Note that $u_i(r,t)$ can be viewed as a heat diffusion with the initial condition given by $f_i(r)$. One can also use nonlinear diffusions for $u_i(r,t)$.

3. Note that in principle one could define a more sophisticated notion of local scale based on non-isotropic kernels. Assume that one has a notion of scale of objects in $\mathbb{R}^2$ that depends on a choice of $v \in \mathbb{R}^2_+$, denoted by $\mathbf{S}_v \Gamma(x,t)$ for $x \in \Gamma$. The condition that the scale function is well behaved under rigid isometries is now that

$$\mathbf{S}_{Tv}(T\Gamma)(Tx,t) = \mathbf{S}_v \Gamma(x,t),$$

where $T \in E(n)$. One example of such a notion is that coming from using a non-isotropic heat kernel.

4. We note that the definition of $\mathbf{K}\Gamma(y,t)$ can be defined for $y \notin \Gamma$ (in particular near Γ). Hence, the eigenvalues and eigenvectors of the Hessian matrix $H(\mathbf{K}\Gamma)(x,t)$, for $x \in \Gamma$, can be used to obtain the local features of Γ at x at multiple scales.

5. As a future interest, we would like to use $\mathbf{K}\Gamma$ and $\mathbf{S}\Gamma$ for variational problems involving constraints on the regularity of Γ.

Acknowledgments

Appendix A. Proof of Lemma 1

The proof of Lemma 1 is a collection of known results from [37,38,32] for the case of scalar valued functions. For the purpose of completeness, we (re)present and adapt these results to the case of vector-valued functions.

We follow Chapter VII, Section 3.2 in [38] with the following set up for vector-valued functions. Let $\mathcal{H}$ be a Hilbert space. Denote by $\mathcal{S}(\mathbb{R}^n)$ the space of scalar-valued test functions defined on $\mathbb{R}^n$, and by $\mathcal{S}(\mathbb{R}^n, \mathcal{H})$ the space of $\mathcal{H}$-valued distributions defined on $\mathbb{R}^n$. Fix a positive integer $M > 0$, the set of normalized bump function consists of smooth functions ϕ, supported on the unit ball, that satisfy

$$\left| \partial_x^\alpha \phi(x) \right| \leqslant 1, \quad 0 \leqslant |\alpha| \leqslant M. \tag{47}$$

Denote by $B(x_0, R)$ the ball of center x_0 and radius R. Define

$$\phi^{R,x_0}(x) = \phi\left(\frac{x - x_0}{R} \right).$$

The functions ϕ^{R,x_0} are called the normalized bump functions for the ball $B(x_0, R)$.

Let $K(x,y)$ be an $\mathcal{H}$-valued kernel, defined for $x, y \in \mathbb{R}^n$ with $x \neq y$, that satisfies the following conditions: For some γ, $0 < \gamma < 1$, we have

$$\left\| K(x,y) \right\|_{\mathcal{H}} \leqslant C \|x - y\|^{-n},$$

$$\left\| K(x,y) - K(x',y) \right\|_{\mathcal{H}} \leqslant C \frac{\|x - x'\|^\gamma}{\|x - y\|^{n+\gamma}}, \quad \text{if } \|x - x'\| \leqslant \|x - y\|/2, \quad \text{and}$$

$$\left\| K(x,y) - K(x,y') \right\|_{\mathcal{H}} \leqslant C \frac{\|y - y'\|^\gamma}{\|x - y\|^{n+\gamma}}, \quad \text{if } \|y - y'\| \leqslant \|x - y\|/2. \tag{48}$$

Let $T : \mathcal{S} \to \mathcal{S}'$ be a continuous mapping and assume that associated to T is the kernel K satisfying condition (48). That is, if $f \in \mathcal{S}$ has compact support, then for a point x outside of the support of f, $Tf(x)$ agrees with

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x, y) f(y) \, dy. \tag{49}$$

We say T is restrictedly bounded if the distribution $T(\phi^{R,x_0})$ belongs to $L^2(\mathbb{R}^n, \mathcal{H})$ and

$$\left\| T\left(\phi^{R,x_0}\right) \right\|_{L^2(\mathbb{R}^n, \mathcal{H})} \leqslant C R^{n/2} \tag{50}$$

holds with some constant C independent of R, x_0 and ϕ. Denote by T^* the adjoint operator defined as

$$\langle Tf, g \rangle = \langle f, T^*g \rangle, \quad \text{whenever } f, g \in \mathcal{S}.$$

Similarly, we say T^* is restrictedly bounded if

$$\left\| T^*\left(\phi^{R,x_0}\right) \right\|_{L^2(\mathbb{R}^n, \mathcal{H})} \leqslant C R^{n/2} \tag{51}$$

holds with some constant C independent of R, x_0 and ϕ. Note that the kernel associated to T^* is the adjoint kernel $K^*(x, y) = \overline{K}(y, x)$.

We will need the following theorem, which is the vector-valued version of Theorem 3 in Chapter VII, Section 3.2 in [38]. The proof, which we omit, is an exact replica of Stein's proof given in [38].

Theorem 3. *Suppose T is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ associated to a kernel K satisfying (48) and (49). Then T extends to a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n, \mathcal{H})$ if and only if both T and T^* are restrictedly bounded in the sense of (50) and (51).*

For $0 < \epsilon < N$, define

$$I_{\epsilon,N}(x) = \int\limits_{\epsilon < \|x-y\| < N} K(x, y) \, dy, \quad \text{and}$$

$$I^*_{\epsilon,N}(x) = \int\limits_{\epsilon < \|x-y\| < N} \overline{K}(y, x) \, dy.$$

The following theorem is the vector-valued version of Theorem 4 in Chapter VII, Section 3.4 in [38]. Again, the proof, which we omit, is an exact replica of Stein's proof given in [38].

Theorem 4. *Suppose K satisfies (48). Then there exists a bounded linear operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathcal{H})$ so that (49) holds if and only if*

$$\int\limits_{\|x_0-x\| < N} \left\| I_{\epsilon,N}(x) \right\|^2_{\mathcal{H}} dx \leqslant C N^n, \quad \text{for all } \epsilon, N, \text{ and } x_0, \tag{52}$$

with a same condition for $I^*_{\epsilon,N}(x)$.

Before proving Lemma 1, we wish to establish the following preliminary results, restricting to the 1-dimensional case. Let $k : \mathbb{R} \setminus \{0\} \to \mathcal{H}$ such that for some constant C_0,

$$\left\| k(x) \right\|_{\mathcal{H}} \leqslant C_0 |x|^{-1}, \qquad \left\| k'(x) \right\|_{\mathcal{H}} \leqslant C_0 |x|^{-2}. \tag{53}$$

The kernel K that we consider is given by $K(x, y) = k(x - y)$.

Remark 15. Let k be a kernel satisfying (53), and define $K(x, y) = k(x - y)$, for $x \neq y$. Then K satisfies condition (48) with $\gamma = 1$. Indeed, suppose $|x - x'| \leqslant |x - y|/2$. Then we have

$$\left\| K(x, y) - K\left(x', y\right) \right\|_{\mathcal{H}} = \left\| k(x - y) - k\left(x' - y\right) \right\|_{\mathcal{H}} = \left\| k'(z) \right\|_{\mathcal{H}} |x - x'|,$$

where z is between $x - y$ and $x' - y$. Since $|x - x'| \leqslant |x - y|/2$, we have $|z| \geqslant |x - y|/2$. Now, using (53), we have

$$\left\| k(x - y) - k\left(x' - y\right) \right\|_{\mathcal{H}} \leqslant C_0 \frac{|x - x'|}{|z|^2} \leqslant 4 C_0 \frac{|x - x'|}{|x - y|^2}.$$

Similarly, we have

$$\left\| K(x, y) - K(x, y') \right\|_{\mathcal{H}}' \leqslant 4C_0 \frac{|y - y'|}{|x - y|^2}, \quad \text{whenever } \left| y - y' \right| \leqslant |x - y|/2.$$

Remark 16. Let k be a kernel satisfying (53). Then

$$\left\| k(x) \right\|_{\mathcal{H}} \leqslant C_0 |x|^{-1}, \quad \text{and for all } |y| > 0, \quad \int\limits_{|x| \geqslant 2|y|} \left\| k(x - y) - k(x) \right\|_{\mathcal{H}} dx \leqslant 2C_0. \tag{54}$$

Indeed, we have $k(x - y) - k(x) = -k'(z)y$, for some $z = z(x, y)$ between x and $x - y$.

$$\int\limits_{|x| \geqslant 2|y|} \left\| k(x - y) - k(x) \right\|_{\mathcal{H}} dx = \int\limits_{|x| \geqslant 2|y|} \left\| k'(z)y \right\|_{\mathcal{H}} dx \leqslant |y| \int\limits_{|x| \geqslant 2|y|} C_0 |z|^{-2} dx.$$

But we have either $|x| \leqslant |z|$ or $|x - y| \leqslant |z|$. In the first case,

$$|y| \int\limits_{|x| \geqslant 2|y|} C_0 |z|^{-2} dx \leqslant |y| \int\limits_{|x| \geqslant 2|y|} C_0 |x|^{-2} dx = C_0.$$

In the latter,

$$|y| \int\limits_{|x| \geqslant 2|y|} C_0 |z|^{-2} dx \leqslant |y| \int\limits_{|x| \geqslant 2|y|} C_0 |x - y|^{-2} dx \leqslant |y| \int\limits_{|x| \geqslant |y|} C_0 |x|^{-2} dx = 2C_0.$$

Proposition 8. *Let k be a* 1*-dimensional kernel satisfying* (53) *and*

$$\sup_{0 < \epsilon < N} \left\| \int\limits_{\epsilon < |x| < N} k(x) \, dx \right\|_{\mathcal{H}} \leqslant C_0, \quad \text{for all } \epsilon, N > 0. \tag{55}$$

Then the operator T defined by

$$Tf(x) = \int\limits_{\mathbb{R}} k(x - y) f(y) \, dy \tag{56}$$

is bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mathcal{H})$.

Proof. An obvious proof is to use Remark 15 and invoke Theorem 4. However, we will follow Chapter II, Section 3.3 in [37], and show directly that

$$\sup_{\xi} \left\| \hat{k}(\xi) \right\|_{\mathcal{H}} \leqslant CC_0, \tag{57}$$

for constant $C > 0$. Therefore, for all $f \in L^2(\mathbb{R})$,

$$\|Tf\|_{L^2(\mathbb{R}, \mathcal{H})} = \|\widehat{Tf}\|_{L^2(\mathbb{R}, \mathcal{H})} \leqslant \left[\sup_{\xi} \left\| \hat{k}(\xi) \right\|_{\mathcal{H}} \right] \|\hat{f}\|_{L^2(\mathbb{R})} \leqslant CC_0 \|f\|_{L^2(\mathbb{R})}.$$

By Remark 16, we may assume k also satisfies (54). For each $\epsilon > 0$, define the truncated kernel k_ϵ as

$$k_\epsilon(x) = \begin{cases} k(x) & \text{if } |x| > \epsilon, \\ 0 & \text{if } |x| \leqslant \epsilon. \end{cases}$$

We will show that for all $\epsilon > 0$,

$$\sup_{\xi} \left\| \hat{k}_\epsilon(\xi) \right\|_{\mathcal{H}} \leqslant CC_0,$$

where C is independent on ϵ. Thus, we obtain (57) by taking the taking the supremum over ϵ.

First, consider $\epsilon = 1$. We see that k_1 also satisfies (54)–(55) with C_0 replaced by some new constant CC_0. Next, write

$$\hat{k}_1(\xi) = \lim_{R \to \infty} \int_{|x| \leqslant R} e^{2\pi i x \xi} k_1(x) \, dx$$

$$= \int_{|x| \leqslant 1/|\xi|} e^{2\pi i x \xi} k_1(x) \, dx + \lim_{R \to \infty} \int_{1/|\xi| \leqslant |x| \leqslant R} e^{2\pi i x \xi} k_1(x) \, dx$$

$$= I_1 + I_2.$$

We have by (55)

$$\|I_1\|_{\mathcal{H}} \leqslant \left\| \int_{|x| \leqslant 1/|\xi|} [e^{2\pi i x \xi} - 1] k_1(x) \, dx \right\|_{\mathcal{H}} + \left\| \int_{|x| \leqslant 1/|\xi|} k_1(x) \, dx \right\|_{\mathcal{H}}$$

$$\leqslant \int_{|x| \leqslant 1/|\xi|} \left\| [e^{2\pi i x \xi} - 1] k_1(x) \right\|_{\mathcal{H}} dx + C_0$$

$$\leqslant C \int_{|x| \leqslant 1/|\xi|} \left\| x \xi k_1(x) \right\|_{\mathcal{H}} dx + C_0 = C|\xi| \int_{|x| \leqslant 1/|\xi|} \left\| x k_1(x) \right\|_{\mathcal{H}} dx + C_0$$

$$\leqslant CC_0 |\xi| \int_{|x| \leqslant 1/|\xi|} dx + C_0 = CC_0.$$

To estimate I_2, choose $z = z(\xi)$ so that $e^{2\pi i z \xi} = -1$ (that is $z = \frac{1}{2} \frac{\xi}{|\xi|^2}$ and $|z| = \frac{1}{2|y|}$). This implies

$$\int_{\mathbb{R}} k_1(x) e^{2\pi x \xi} \, dx = \frac{1}{2} \int_{\mathbb{R}} [k_1(x) - k_1(x - z)] e^{2\pi x \xi} \, dx,$$

which shows

$$I_2 = \frac{1}{2} \lim_{R \to \infty} \int_{1/|\xi| \leqslant |x| \leqslant R} [k_1(x) - k_1(x - z)] e^{2\pi x \xi} \, dx - \frac{1}{2} \int_{1/|\xi| \leqslant |x+z|, |x| \leqslant 1/|\xi|} k_1(x) e^{2\pi i x \xi} \, dx$$

$$= J_1 + J_2.$$

The last integral is taken over the set of x contained in $\frac{1}{2|y|} \leqslant |x| \leqslant \frac{1}{|y|}$. Thus,

$$\|J_2\|_{\mathcal{H}} = \left\| \frac{1}{2} \int_{1/|\xi| \leqslant |x+z|, |x| \leqslant 1/|\xi|} k_1(x) e^{2\pi i x \xi} \, dx \right\|_{\mathcal{H}} \leqslant \frac{1}{2} \int_{\frac{1}{2|y|} \leqslant |x| \leqslant \frac{1}{|y|}} \left\| k_1(x) \right\|_{\mathcal{H}} dx$$

$$\leqslant C_0 \frac{1}{2} \int_{\frac{1}{2|y|} \leqslant |x| \leqslant \frac{1}{|y|}} |x|^{-1} \, dx = C_0 \int_{\frac{1}{2|y|} \leqslant x \leqslant \frac{1}{|y|}} x^{-1} \, dx$$

$$= C_0 \big[\log(1/|y|) - \log(1/(2|y|)) \big] = C_0 \log(2).$$

As for J_1, we have

$$\|J_1\|_{\mathcal{H}} \leqslant \frac{1}{2} \int_{1/|\xi| \leqslant |x|} \left\| k_1(x) - k_1(x - z) \right\|_{\mathcal{H}} dx = \frac{1}{2} \int_{2|z| \leqslant |x|} \left\| k_1(x) - k_1(x - z) \right\|_{\mathcal{H}} dx$$

$$\leqslant CC_0.$$

Thus, $\|I_2\|_{\mathcal{H}} \leqslant (2 + C) C_0 = CC_0$. Combining the bounds for I_1 and I_2, we have

$$\left\| \hat{k}_1(\xi) \right\|_{\mathcal{H}} \leqslant CC_0,$$

These constants do not depend on ξ. Now, for any $\epsilon > 0$, we note that the kernel $k'(x) = \epsilon k(\epsilon x)$ also satisfy the same conditions as k with the same bounds. Thus Fourier transform of the truncated kernel $k'_1(x) = k'(x) \chi_{(1,\infty)}(x)$ is bounded in $\mathcal{H}$ by CC_0. However, we note that $\epsilon^{-1} k'_1(\epsilon^{-1} x) = k_\epsilon(x)$, and

$$\hat{k}_\epsilon(\xi) = \hat{k}'_1(\epsilon \xi).$$

Therefore,

$$\|\hat{k}_\epsilon(\xi)\|_{\mathcal{H}} = \|\hat{k}'_1(\epsilon\xi)\|_{\mathcal{H}} \leqslant CC_0,$$

where C does not depend on ξ and ϵ. □

Proposition 9. *Let $k : \mathbb{R} \setminus \{0\} \to \mathcal{H}$ satisfing* (55) *and for all positive integer j,*

$$\left\| \frac{d^j}{dx^j} k(x) \right\|_{\mathcal{H}} \leqslant C(j)|x|^{-1-j}. \tag{58}$$

Let A be a Lipschitz function defined on $\mathbb{R}$. Then the kernel

$$K(x, y) = k(x - y)\left[\frac{A(x) - A(y)}{x - y} \right]$$

defines a bounded operator T_A from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mathcal{H})$ with $\|T_A\| \leqslant C_0 C \|A'\|_\infty$.

Proof. We have the kernel $K(x, y) = k(x - y)[\frac{A(x)-A(y)}{x-y}]$ satisfies (48) with $\gamma = 1$ with the constant $C_0 C \|A'\|_\infty$.

Using Theorem 3, we will show that T_A satisfies (50) with the constant $C_0 C \|A'\|$. Indeed, let $s : \mathbb{R} \setminus \{0\} \to \mathcal{H}$ be defined such that its Fourier transform satisfies

$$\hat{s}(\xi) = \frac{1}{2\pi i \xi} \int\limits_{-\xi}^{\xi} \hat{k}(\gamma) \, d\gamma.$$

Then we have $s'(x) = \frac{k(x)}{x}$. Now, since $\hat{s}(0) = 0$ and $\|s'(x)\|_{\mathcal{H}} \leqslant C_0 C |x|^{-2}$, we have $\|s(x)\|_{\mathcal{H}} \leqslant C_0 C |x|^{-1}$. This shows that the kernel s satisfies (53), and hence (by an abuse of notation) the kernel $s(x, y) = s(x - y)$ satisfies (48) with $\gamma = 1$.

Note that by (57), we have $\sup_{\xi \in \mathbb{R}} \|\hat{k}(\xi)\|_{\mathcal{H}} \leqslant C_0 C$, and therefore,

$$\|\hat{s}(\xi)\|_{\mathcal{H}} \leqslant \frac{C_0 C}{\pi}. \tag{59}$$

Let S be the operator associated to the distributional kernel s, then (59) implies S is a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mathcal{H})$.

Denote

$$[S, A]g(x) := \int\limits_{\mathbb{R}} s(x - y)\big(A(x) - A(y)\big)g(y) \, dy. \tag{60}$$

Let ϕ^{R,x_0} be some normalized bump function on $B(x_0, R) = [x_0 - R, x_0 + R]$. By an integration by parts, we have

$$\begin{aligned}
T_A \phi^{R,x_0}(x) &= \int\limits_{\mathbb{R}} s'(x - y)\big(A(x) - A(y)\big)\phi^{R,x_0}(y) \, dy \\
&= \int\limits_{\mathbb{R}} s(x - y) A'(y) \phi^{R,x_0}(y) \, dy \\
&\quad - \int\limits_{\mathbb{R}} s(x - y)\big(A(x) - A(y)\big)\frac{d}{dx}\big(\phi^{R,x_0}\big)(y) \, dy \\
&= S\big(A'\phi^{R,x_0}\big)(x) - [S, A]\left(\frac{d}{dx}\big(\phi^{R,x_0}\big)\right)(x).
\end{aligned}$$

In the first term using the fact that S is bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mathcal{H})$, we have

$$\big\| S\big(A'\big)\phi^{R,x_0} \big\|_{L^2(\mathbb{R}, \mathcal{H})} \leqslant C_0 C \big\| A' \big\|_\infty \big\| \phi^{R,x_0} \big\|_{L^2} \leqslant C C_0 \big\| A' \big\| R^{1/2}. \tag{61}$$

As for the second term, we have for all $x \neq y$,

$$\big\| s(x - y)\big(A(x) - A(y)\big) \big\|_{\mathcal{H}} \leqslant C C_0 \big\| A' \big\|_\infty.$$

This implies

$$\left\| [S, A]\left(\frac{d}{dx}(\phi^{R,x_0})\right)(x) \right\|_{\mathcal{H}} = \left\| \int_{\mathbb{R}} s(x - y)(A(x) - A(y))\left(\frac{d}{dx}(\phi^{R,x_0})\right)(y)\,dy \right\|_{\mathcal{H}}$$

$$\leqslant C_0 C \|A'\|_\infty \int_{x_0-R}^{x_0-R} \left| \frac{d}{dx}(\phi^{R,x_0}) \right| dy$$

$$\leqslant C_0 C \|A'\|_\infty \int_{x_0-R}^{x_0-R} R^{-1}\,dy = 2C_0 C \|A'\|_\infty.$$

This implies

$$\left\| [S, A]\left(\frac{d}{dx}(\phi^{R,x_0})\right) \right\|_{L^2(\mathbb{R},\mathcal{H})} \leqslant C_0 C \|A'\| R^{1/2}.$$

Thus,

$$\|T_A \phi^{R,x_0}\|_{L^2(\mathbb{R},\mathcal{H})} \leqslant \|S(A')\phi^{R,x_0}\|_{L^2(\mathbb{R},\mathcal{H})} + \left\| [S, A]\left(\frac{d}{dx}(\phi^{R,x_0})\right) \right\|_{L^2(\mathbb{R},\mathcal{H})}$$

$$\leqslant C_0 C \|A'\|_\infty R^{1/2},$$

for some constant C. This shows T_A satisfies (50). The same holds for T_A^*. Therefore, by Theorem 3, T_A is a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mathcal{H})$ with the operator norm $\|T_A\|$ bounded by $C_0 C \|A'\|_\infty$. $\quad\square$

By an inductive step, one obtains the following general result.

Proposition 10. *Let k be as in Proposition 9, and A be a Lipschitz function defined on $\mathbb{R}$. Then the kernel $K(x, y) = k(x - y)[\frac{A(x)-A(y)}{x-y}]^m$ defines a bounded operator T_{A^m} from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mathcal{H})$ with $\|T_{A^m}\| \leqslant C_0 C^m \|A'\|_\infty^m$.*

Proposition 11. *Let k be as in Proposition 9. The kernel $K(x, y) = k(x - y)e^{i\frac{A(x)-A(y)}{x-y}}$ defines a bounded linear operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \mathcal{H})$ with the operator norm bounded above by $C_0 C(1 + \|A'\|_\infty)^5$.*

The proof of this proposition, which we omit, is an exact replica of proof in the case when k is the Cauchy kernel using the rising sun lemma [33]. See for instance [32, pp. 100–106], where $|\cdot|$ is replaced by $\|\cdot\|_{\mathcal{H}}$ where appropriate.

Proof of Lemma 1. Let T be the operator with the corresponding kernel $K(r, s) = k(r - s, A(r) - A(s))$, that is

$$Tf(r) = \int_{\mathbb{R}^d} K(r, s)f(s)\,ds = \int_{\mathbb{R}^d} k(r - s, A(r) - A(s))f(s)\,ds.$$

Here, we assume that μ is the Hausdorff measure (the result for surface measure μ is an easy consequence of the Hausdorff measure). Using the method of rotation (see [32]), we write the above integral in polar coordinates centered at $r \in \mathbb{R}^d$ and letting $s = r + \gamma\theta$, for $\theta \in S^{d-1}$. We then have

$$Tf(r) := Tf(x) = \frac{1}{2} \int_{S^{d-1}} \int_{-\infty}^{\infty} K(r, r + \gamma\theta)f(r + \gamma\theta)|\gamma|^{d-1}\,d\gamma\,d\theta. \tag{62}$$

We will show that the operator T_θ defined by

$$T_\theta f(r) = \int_{-\infty}^{\infty} K(r, r + \gamma\theta)f(r + \gamma\theta)|\gamma|^{d-1}\,d\gamma$$

$$= \int_{-\infty}^{\infty} k(\gamma\theta, A(r) - A(r + \gamma\theta))f(r + \gamma\theta)|\gamma|^{d-1}\,d\gamma$$

is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mathcal{H})$. Since $\theta \in \mathbb{R}^d$ is fixed (by a rotation if necessary) we may assume $\theta = (0, \ldots, 0, 1)$. Write $r = (r', u)$, where $r' \in \mathbb{R}^{d-1}$ and $u \in \mathbb{R}$. Then, writing $r' = (r', 0)$,

$$\|T_\theta f\|^2_{L^2(\mathbb{R}^d, \mathcal{H})} = \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \|T_\theta f(r' + u\theta)\|^2_{\mathcal{H}} \, dr' \, du.$$

Using Fubini's theorem, the integral over du is calculated as

$$\int_{\mathbb{R}} \|T_\theta f(r' + u\theta)\|^2_{\mathcal{H}} \, du = \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} K(r' + u\theta, r' + (u+\gamma)\theta) f(r' + (u+\gamma)\theta) |\gamma|^{d-1} \, d\gamma \right\|^2_{\mathcal{H}} du$$

$$= \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} K(r' + u\theta, r' + \gamma\theta) f(r' + \gamma\theta) |\gamma - u|^{d-1} \, d\gamma \right\|^2_{\mathcal{H}} du.$$

Let

$$T_{r',\theta} f(u) = \int_{\mathbb{R}} K(r' + u\theta, r' + \gamma\theta) |\gamma - u|^{d-1} f(r' + \gamma\theta) \, d\gamma$$

$$= \int_{\mathbb{R}} k\big((u - \gamma)\theta, A(r' + u\theta) - A(r' + \gamma\theta)\big) |\gamma - u|^{d-1} f(r' + \gamma\theta) \, d\gamma.$$

Claim 1.

$$\int_{\mathbb{R}} \|T_\theta f(r' + u\theta)\|^2_{\mathcal{H}} \, du = \int_{\mathbb{R}} \|T_{r',\theta}(u)\|^2_{\mathcal{H}} \, du \leqslant C \int_{\mathbb{R}} |f(r' + \gamma\theta)|^2 \, d\gamma.$$

Thus, by integrating over dr', we obtain

$$\|T_\theta f\|^2_{L^2(\mathbb{R}^d, \mathcal{H})} = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \|T_\theta f(r' + u\theta)\|^2_{\mathcal{H}} \, du \, dr' \leqslant C \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |f(r' + \gamma\theta)|^2 \, d\gamma \, dr' = C \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Therefore,

$$\|Tf\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{2} \int_{S^{d-1}} \|T_\theta f\|^2_{L^2(\mathbb{R}^d)} \, d\theta \leqslant \frac{C\omega_d}{2} \|f\|^2_{L^2(\mathbb{R}^d)},$$

and hence Lemma 1 holds. $\square$

Proof of Claim 1. Since r' and θ are fixed, denote

$$\tilde{A}(u) = A(r' + u\theta),$$
$$\tilde{f}(u) = f(r' + u\theta),$$
$$\tilde{k}(u, z') = k(u\theta, z'), \quad z' \in \mathbb{R}^{n-d},$$
$$\tilde{K}(u, v) = \tilde{k}(u - v, \tilde{A}(u) - \tilde{A}(v)), \quad \text{and}$$
$$\tilde{T}\tilde{f}(u) = T_{r',\theta} f(u).$$

Then we have

$$\tilde{T}\tilde{f}(u) = \int_{\mathbb{R}} \tilde{K}(u, v) \tilde{f}(v) |u - v|^{d-1} \, dv = \int_{\mathbb{R}} \tilde{k}(u - v, \tilde{A}(u) - \tilde{A}(v)) \tilde{f}(v) |u - v|^{d-1} \, dv.$$

Let $z = (u, z') \in \mathbb{R}^{1+(n-d)}$. Next, define

$$\tilde{s}(u, z') = \tilde{k}(u, z') |u|^{d-1} \quad \text{and} \quad \bar{s}(u, \alpha) = \tilde{s}(u, z'),$$

where $\alpha = \frac{z'}{u} \in \mathbb{R}^{n-d}$.

Let $M = \|A'\|_\infty$. Next, we replace $\bar{s}(u, \alpha)$ outside of the ball $\|\alpha\| \leqslant M$ by extending $\bar{s}(u, \alpha)$ to be an infinitely differentiable periodic function of period $4M$.

We note that $\bar{s}(u, \cdot)$ is C^∞ in both u and α. For each $\alpha = \frac{z'}{u}$ with $\|\alpha\| \leqslant M$, let

$$x = (u\theta, z') \in \mathbb{R}^n.$$

We then have

$$\begin{aligned}
\big\|\bar{s}(u, \alpha)\big\|_{\mathcal{H}} &= \big\||u|^{d-1}\tilde{k}(u, z')\big\|_{\mathcal{H}} = |u|^{d-1}\big\|k(x)\big\|_{\mathcal{H}} \leqslant C_0|u|^{d-1}\|x\|^{-d} \\
&= C_0|u|^{d-1}\big(|u|^2 + \|z'\|^2\big)^{-d/2} \leqslant C_0|u|^{-1}.
\end{aligned}$$

Inductively, one can also show that for any fixed $\|\alpha\| \leqslant M$,

$$\left\|\frac{\partial^j}{\partial u^j}\bar{s}(u, \alpha)\right\|_{\mathcal{H}} \leqslant C_0|u|^{-1-j}.$$

Note also,

$$\bar{s}(u, \alpha) = k(x)|u|^{d-1} = k\left(u\|y\|\frac{y}{\|y\|}\right)|u|^{d-1},$$

where $y = (\theta, \alpha) \in \mathbb{R}^n$. Let $\theta_n = \frac{y}{\|y\|} \in S^{n-1}$. Then by (29), we have

$$\begin{aligned}
\sup_{0 < \epsilon < N}\left\|\int_{\epsilon < |u| < N}\bar{s}(u, \alpha)\,du\right\|_{\mathcal{H}} &= \sup_{0 < \epsilon < N}\left\|\int_{\epsilon < |u| < N}k\big(\|y\|u\theta_n\big)|u|^{d-1}\,du\right\|_{\mathcal{H}} \\
&= \sup_{0 < \epsilon < N}\left\|\|y\|^{-d}\int_{\epsilon\|y\| < |u| < N\|y\|}k(u\theta_n)|u|^{d-1}\,du\right\|_{\mathcal{H}} \leqslant \|y\|^{-d}C_0 \leqslant C_0,
\end{aligned}$$

since $\|y\| \geqslant 1$. Thus, for each fixed α, $\bar{s}(u, \alpha)$ is a 1-dimensional C^∞ kernel satisfying (55) and (58).

Since $\bar{s}(u, \alpha)$ is C^∞ in α, we write $\bar{s}(u, \alpha)$ as

$$\bar{s}(u, \alpha) = \sum_{j \in \mathbb{Z}^{n-d}}s_{[j]}(u)e^{i\delta\alpha \cdot j},$$

where $\delta = \frac{\pi}{2M}$.

Claim 2. *For each $j \in \mathbb{Z}^{n-d}$, $s_{[j]}$ is a 1-dimensional kernel satisfying* (55) *and* (58) *with the constant $C_0(j)$ decays rapidly, that is for any $N > 0$, there exists $C(N)$ such that $C_0(j) = C_0C(N)\|j\|^{-N}$.*

This implies

$$\tilde{s}\big(u - v, \tilde{A}(u) - \tilde{A}(v)\big) = \bar{s}\left(u - v, \frac{\tilde{A}(u) - \tilde{A}(v)}{u - v}\right) = \sum_{j \in \mathbb{Z}^{n-d}}s_{[j]}(u)e^{i\delta j \cdot \frac{\tilde{A}(u) - \tilde{A}(v)}{u - v}}.$$

Let $\tilde{T}_{[j]}$ be the singular integral with the distributional kernel $s_{[j]}(u)e^{i\delta j \cdot \frac{\tilde{A}(u) - \tilde{A}(v)}{u - v}}$. Then from Proposition 11, we have

$$\|\tilde{T}_{[j]}\| \leqslant C_0(j)\big(1 + \delta\|j\|\big\|A'\big\|_\infty\big)^5 = \frac{C_0C(N)}{\|j\|^N}\big(1 + \delta\|j\|\big\|A'\big\|_\infty\big)^5.$$

This implies

$$\|\tilde{T}\| \leqslant \sum_{j \in \mathbb{Z}^{n-d}}\frac{C_0C(N)}{\|j\|^N}\big(1 + \delta\|j\|\big\|A'\big\|_\infty\big)^5 < \infty,$$

if N is large enough. In other words, there exists $C > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}}\big\|T_{r',\theta}(u)\big\|_{\mathcal{H}}^2\,du = \int_{\mathbb{R}}\big\|\tilde{T}\tilde{f}(u)\big\|_{\mathcal{H}}^2\,du &\leqslant C\int_{\mathbb{R}}\big|\tilde{f}\big(u, \tilde{A}(u)\big)\big|^2\,du \\
&= C\int_{\mathbb{R}}\big|f\big(r' + u\theta, A(r' + u\theta)\big)\big|^2\,du.
\end{aligned}$$

This proves Claim 1. $\quad\square$

Proof of Claim 2. Let

$$\alpha = (\alpha_1, \ldots, \alpha_{n-d}) = \frac{1}{u}(z'_1, \ldots, z'_{n-d}) = \frac{z'}{u}, \quad \text{and} \quad x = (u\theta, z') \in \mathbb{R}^n,$$

with $\|\alpha\| \leqslant M$. We have for any $1 \leqslant m \leqslant n - d$,

$$\frac{\partial}{\partial \alpha_m} \bar{s}(u, \alpha) = \frac{\partial}{\partial z'_m} \tilde{s}(u, z') \frac{dz'_m}{d\alpha_m} = u \frac{\partial}{\partial z'_m} \tilde{s}(u, z').$$

But since k satisfies (28), we have

$$\left\| \frac{\partial}{\partial z'_m} \tilde{s}(u, z') \right\|_{\mathcal{H}} = |u|^{d-1} \left\| \frac{\partial}{\partial z'_m} k(u\theta, z') \right\|_{\mathcal{H}} \leqslant C_0 |u|^{d-1} \big(|u|^2 + \|z'\|^2 \big)^{(-d-1)/2} \leqslant C_0 |u|^{-2}.$$

This implies,

$$\left\| \frac{\partial}{\partial \alpha_m} \bar{s}(u, \alpha) \right\|_{\mathcal{H}} = |u| \left\| \frac{\partial}{\partial z'_m} \tilde{s}(u, z') \right\|_{\mathcal{H}} \leqslant C_0 |u|^{-1}.$$

Inductively, one can show that

$$\left\| \frac{\partial^\ell}{\partial u^\ell} \frac{\partial}{\partial \alpha_i} \bar{s}(u, \alpha) \right\|_{\mathcal{H}} \leqslant C_0 |u|^{-\ell-1}. \tag{63}$$

We have

$$\frac{\partial}{\partial \alpha_m} \bar{s}(u, \alpha) = \sum_{j \in \mathbb{Z}^{n-d}} s_{[j]}(u) (i\delta j_m) e^{i\delta\alpha \cdot j},$$

and

$$\left\| s_{[j]}(u)(i\delta j_m) \right\|_{\mathcal{H}} = \left\| \frac{1}{(4M)^m} \int\limits_{[-2M,2M]^m} \frac{\partial}{\partial \alpha_m} \bar{s}(u, \alpha) e^{-i\delta\alpha \cdot j} \, d\alpha \right\|_{\mathcal{H}} \leqslant \left\| \frac{\partial}{\partial \alpha_m} \bar{s}(u, \alpha) \right\|_{\mathcal{H}} \leqslant C_0 |u|^{-1},$$

which implies

$$\left\| s_{[j]}(u) \right\|_{\mathcal{H}} \leqslant \frac{C_0}{|\delta j_m|} |u|^{-1}.$$

Inductively, we also have

$$\left\| s_{[j]}^{(\ell)}(u) \right\|_{\mathcal{H}} \leqslant \frac{C_0}{|\delta j_m|} |u|^{-\ell-1}.$$

Note also,

$$\sup_{0<\epsilon<N} \left\| \int\limits_{\epsilon<|u|<N} s_{[j]}(u)(i\delta j_m) \, du \right\|_{\mathcal{H}} = \sup_{0<\epsilon<N} \left\| \int\limits_{\epsilon<|u|<N} \frac{1}{(4M)^{n-d}} \int\limits_{[-2M,2M]^{n-d}} \frac{\partial}{\partial \alpha_m} \bar{s}(u, \alpha) e^{i\delta j \cdot \alpha} \, d\alpha \, du \right\|_{\mathcal{H}}$$

$$= \sup_{0<\epsilon<N} \left\| \frac{1}{(4M)^{n-d}} \int\limits_{[-2M,2M]^{n-d}} \frac{\partial}{\partial \alpha_m} \left[\int\limits_{\epsilon<|u|<N} \bar{s}(u, \alpha) \, du \right] e^{i\delta j \cdot \alpha} \, d\alpha \right\|_{\mathcal{H}}.$$

Let $G(\alpha) = \int_{\epsilon<|u|<N} \bar{s}(u, \alpha) \, du$, which is C^∞ in α, and $\|G(\alpha)\|_{\mathcal{H}} \leqslant C_0$. Then

$$\sup_{0<\epsilon<N} \left\| \int\limits_{\epsilon<|u|<N} s_{[j]}(u)(i\delta j_m) \, du \right\|_{\mathcal{H}} = \sup_{0<\epsilon<N} \left\| \frac{1}{(4M)^{n-d}} \int\limits_{[-2M,2M]^{n-d}} \frac{\partial}{\partial \alpha_m} G(\alpha) e^{i\delta j \cdot \alpha} \, d\alpha \right\|_{\mathcal{H}}$$

$$= \sup_{0<\epsilon<N} \left\| \frac{\partial}{\partial \alpha_m} G \right\|_{\mathcal{H}} = C_0 C_1 < \infty,$$

which simply using the fact that $G(\alpha)$ is C^∞ in α. This implies,

$$\sup_{0<\epsilon<N} \left\| \int\limits_{\epsilon<|u|<N} s_{[j]}(u) \, du \right\|_{\mathcal{H}} \leqslant \frac{C_0 C_1}{|\delta j_m|}.$$

Inductively, by applying ∇_α^N to $\bar{s}(u, \alpha)$ for any $N > 0$, we see that there exists a positive C_N such that

$$\left\| s_{[j]}^{(\ell)}(u) \right\|_{\mathcal{H}} \leqslant \frac{C_0 C_N}{\|j\|^N} |u|^{-\ell - 1}, \quad \text{and} \tag{64}$$

$$\sup_{0 < \epsilon < N} \left\| \int_{\epsilon < |u| < N} s_{[j]}(u) \, du \right\|_{\mathcal{H}} \leqslant \frac{C_0 C_N}{\|j\|^N}. \tag{65}$$

Thus, $s_{[j]}(u)$ is a 1-dimensional kernel satisfying (55) and (58) with the constant $C_0(j) = \frac{C_0 C_N}{\|j\|^N}$ for any N. This proves Claim 2. $\square$

Appendix B. Proof of Proposition 2

Let $x = (r, A(r))$ and $y = (s, A(s))$ in Γ. By a translation, we may assume $x = (r, A(r)) = 0$. We have

$$\int_{\mathbb{R}^d} \psi_{t,k}(x - y) \, ds = \int_{\mathbb{R}^d} \psi_{t,k}(y) \, ds = \int_{\mathbb{R}^d} t^{-d/2} \left[\pi \frac{\|s\|^2 + \|A(s)\|^2}{t} \right]^k e^{-\pi \left(\frac{\|s\|^2 + \|A(s)\|^2}{t} \right)} ds$$

$$= \int_{\mathbb{R}^d} \left[\pi \left(\|s\|^2 + \left\| \frac{A(\sqrt{t}s)}{\sqrt{t}} \right\|^2 \right) \right]^k e^{-\pi \left(\|s\|^2 + \| \frac{A(\sqrt{t}s)}{\sqrt{t}} \|^2 \right)} ds$$

$$= \int_{S^{d-1}} \int_0^\infty \left[\pi \left(\gamma^2 + \left\| \frac{A(\sqrt{t}\gamma\theta)}{\sqrt{t}} \right\|^2 \right) \right]^k e^{-\pi \left(\gamma^2 + \| \frac{A(\sqrt{t}\gamma\theta)}{\sqrt{t}} \|^2 \right)} \gamma^{d-1} \, d\gamma \, d\theta$$

$$\leqslant \int_{S^{d-1}} \int_0^\infty \left[\pi \left(\gamma^2 + \left\| \frac{A(\sqrt{t}\gamma\theta)}{\sqrt{t}} \right\|^2 \right) \right]^{k + \frac{d-1}{2}} e^{-\pi \left(\gamma^2 + \| \frac{A(\sqrt{t}\gamma\theta)}{\sqrt{t}} \|^2 \right)} \, d\gamma \, d\theta.$$

In the last inequality, we use

$$\gamma^{2(d-1)} \leqslant \gamma^{2(d-1)} \left(1 + \left\| \frac{A(\sqrt{t}\gamma\theta)}{\sqrt{t}} \right\|^2 \right).$$

Let $p = k + (d - 1)/2$. The function $g(\gamma) = (\pi\gamma^2)^p e^{-\pi\gamma^2}$, for $\gamma \geqslant 0$, achieves its maximum when $\gamma = \sqrt{p/\pi}$, and

$$p^p e^{-p} = \sup_{\gamma \geqslant 0} \left\{ (\pi\gamma^2)^p e^{-\pi\gamma^2} \right\}.$$

We have that g increases on $[0, \sqrt{p/\pi}]$ and decreases on $[\sqrt{p/\pi}, \infty]$, and $\int_0^\infty g(\gamma) \, d\gamma = C_p < \infty$, where C_p depends only on p. Let

$$G(\gamma, \theta) = \left[\pi \left(\gamma^2 + \left\| \frac{A(\sqrt{t}\gamma\theta)}{\sqrt{t}} \right\|^2 \right) \right]^p e^{-\pi \left(\gamma^2 + \| \frac{A(\sqrt{t}\gamma\theta)}{\sqrt{t}} \|^2 \right)},$$

we have

$$G(\gamma, \theta) \leqslant \begin{cases} p^p e^{-p} & \text{if } 0 \leqslant \gamma \leqslant \sqrt{p/\pi}, \\ g(\gamma) & \text{if } \gamma > \sqrt{p/\pi}. \end{cases}$$

This implies,

$$\int_{S^{d-1}} \int_0^\infty G(\gamma, \theta) \, d\gamma \, d\theta = \int_{S^{d-1}} \left[\int_{\gamma < \sqrt{\frac{p}{\pi}}} G(\gamma, \theta) \, d\gamma + \int_{\gamma \geqslant \sqrt{\frac{p}{\pi}}} G(\gamma, \theta) \, d\gamma \right] d\theta$$

$$\leqslant \int_{S^{d-1}} \left[\int_{\gamma < \sqrt{\frac{p}{\pi}}} p^p e^{-p} \, d\gamma + \int_{\gamma \geqslant \sqrt{\frac{p}{\pi}}} g(\gamma) \, d\gamma \right] d\theta$$

$$\leqslant \theta_d \left[p^p e^{-p} \sqrt{p/\pi} + \int_0^\infty g(\gamma) \, d\gamma \right] = \theta_d \left[p^p e^{-p} \sqrt{p/\pi} + C_p \right] < \infty.$$

Here, $\omega_d = \int_{S^{d-1}} 1 \, d\theta$. Let $C_{k,d} = \theta_d [p^p e^{-p} \sqrt{p/\pi} + C_p]$, then

$$\int_{\mathbb{R}^d} \psi_{t,k}(x - y) \, ds \leqslant C_{k,d}.$$

Appendix C. Proof of Proposition 3

Let $\Gamma = \{(r, A(r)); r \in \mathbb{R}^d\}$. For $\delta > 0$, we have $d_\delta \Gamma = \{(\delta r, \delta A(r)): r \in \mathbb{R}^d\}$. We first prove the case where μ is the d-dimensional Hausdorff measure restricted to Γ. Recall that for $x = (r, A(r))$, we have

$$\mathbf{K}\Gamma(x, t) = \int_\Gamma K_t(x - y) \, d\mu(y) = \int_{\mathbb{R}^d} t^{-d/2} e^{-\pi [\|r - s\|^2 + \|A(r) - A(s)\|^2]/t} \, ds,$$

where $K_t(y) = t^{-d/2} e^{-\pi \frac{\|y\|^2}{t}}$.

By Remark 3, we may assume $x = 0$. We have

$$\begin{aligned}
\mathbf{K}(d_\delta \Gamma)(0, t) &= \int_{d_\delta \Gamma} K_t(y) \, d\mu(y) = \int_{\mathbb{R}^d} t^{-d/2} e^{-\pi \delta^2 [\|s\|^2 + \|A(s)\|^2]/t} \, ds \\
&= \int_{\mathbb{R}^d} t^{-d/2} e^{-\pi [\|s\|^2 + \|A(s)\|^2]/(t\delta^{-2})} \, ds \\
&= \delta^{-d} \int_{\mathbb{R}^d} (t\delta^{-2})^{-d/2} e^{-\pi [\|s\|^2 + \|A(s)\|^2]/(t\delta^{-2})} \, ds = \delta^{-d} \mathbf{K}\Gamma(0, t\delta^{-2}).
\end{aligned}$$

Let

$$t_\Gamma^+(0) = \arg\max_{t>0} \left\{ \left| t \frac{\partial}{\partial t} \mathbf{K}\Gamma(0, t) \right| \right\}.$$

We will show that $t_{d_\delta \Gamma}^+(0) = \delta^2 t_\Gamma^+(0)$ (the same techniques apply to local maxima). Indeed, let $p = t\delta^{-2}$, we have

$$t \frac{\partial}{\partial t} \mathbf{K}(d_\delta \Gamma)(0, t) = t \frac{\partial}{\partial t} \left[\delta^{-d} \mathbf{K}\Gamma(0, p) \right] = \delta^{-d} t \left[\frac{\partial}{\partial p} \mathbf{K}\Gamma(0, p) \right] \frac{dp}{dt} = \delta^{-d} p \frac{\partial}{\partial p} \mathbf{K}\Gamma(0, p).$$

This implies

$$t_{d_\delta \Gamma}^+(0) = \arg\max_{t>0} \left\{ \left| t \frac{\partial}{\partial t} \mathbf{K}(d_\delta \Gamma)(0, t) \right| \right\} = \arg\max_{t>0} \left\{ \left| p \frac{\partial}{\partial p} \mathbf{K}\Gamma(0, p) \right| \right\}.$$

This shows that the maximum occurs at $p^+ = t_\Gamma^+(0)$, but $p^+ = \delta^{-2} t_{d_\delta \Gamma}^+(0)$. Therefore,

$$t_{d_\delta \Gamma}^+(0) = \delta^2 t_\Gamma^+(0).$$

Note that in this case

$$\mathbf{S}(d_\delta \Gamma)(0, \delta^2 t) = \delta^{-d} \mathbf{S}\Gamma(0, t) \quad \text{for all } t \in T_\Gamma(0). \tag{66}$$

Now suppose μ is the d-dimensional surface measure on Γ. Let

$$\Gamma = \{z(r) = (r, A(r)): r \in \mathbb{R}^d\}.$$

Then from the definition of $\|z'(r)\|$ in (19), we have

$$\| (\delta z)'(r) \| = \delta^d \| z'(r) \|.$$

Thus,

$$\mathbf{K}(d_\delta \Gamma)(0, t) = \mathbf{K}\Gamma(0, t\delta^{-2}).$$

Following the same techniques as above, we also have

$$t_{d_\delta \Gamma}^+(0) = \delta^2 t_\Gamma^+(0).$$

Note that in this case,

$$\mathbf{S}(d_\delta \Gamma)(0, \delta^2 t) = \mathbf{S}\Gamma(0, t) \quad \text{for all } t \in T_\Gamma(0). \tag{67}$$

Appendix D. Proof of Corollary 4

In providing the characterizations of local scales on Γ, we rely on the John–Nirenberg theorem for functions in BMO(Γ). We note that the function space BMO(X) is well defined for X being the space of homogeneous type introduced by Coifman–Weiss [9]. As noted in [9], the John–Nirenberg theorem [17] can be extended to functions in BMO(X). See [36] for a proof of this extension. We recall the definition of spaces of homogeneous type.

Definition 6. Let X be a topological space endowed with a Borel measure μ and a quasi-metric d satisfying: (a) $d(x, y) = d(y, x)$ for all $x, y \in X$, (b) $d(x, y) > 0$ if and only if $x \neq y$, (c) there exists a constant K such that $d(x, y) \leqslant K(d(x, z) + d(y, z))$ for all $x, y, z \in X$. For each $x \in X$ and $r > 0$, define

$$B_r(x) = \{y \in X\colon d(x, y) < r\}.$$

Suppose the measure μ satisfies: (a) $\mu(B_r(x)) > 0$ for all $r > 0$, and (b) there exists a constant c such that $\mu(B_r(x)) \leqslant c\mu(B_{r/2}(x))$ for all $r > 0$ and $x \in X$. Then (X, d, μ) is called a *space of homogeneous type*.

The function space BMO(X) can be defined as usual.

Definition 7. Let $f \in L^1_{loc}(X)$, where X is a space of homogeneous type. We say $f \in \text{BMO}(X)$ whenever

$$\|f\|_{\text{BMO}} = \sup_B \frac{1}{\mu(B)} \int_B |f(x) - f_B| \, d\mu(x) < \infty,$$

where $f_B = \frac{1}{\mu(B)} \int_B f(x) \, d\mu(x)$. The supremum is taken over all balls $B \subset X$. Equipped with the norm $\|\cdot\|_{\text{BMO}}$, BMO(X) is a Banach space of functions modulo constants.

The classical John–Nirenberg theorem for functions in BMO($\mathbb{R}^n$) can be extended to BMO(X).

Theorem 5 *(John–Nirenberg theorem [9,36]). Suppose $f \in \text{BMO}(X)$, with X being the space of homogeneous type. Then*

1. *For any $1 < p < \infty$, $f \in L^p_{loc}(X)$, and*

$$\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \leqslant C_p \|f\|^p_{\text{BMO}}.$$

   Define

$$\|f\|_{\text{BMO}_p} = \sup_B \left[\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right]^{1/p}.$$

   Then $\|\cdot\|_{\text{BMO}}$ and $\|\cdot\|_{\text{BMO}_p}$ are equivalent norms on BMO.
2. *There exist positive constants c_1 and c_2 so that, for each $\alpha > 0$ and every ball B,*

$$\mu\big(\{x \in B\colon |f(x) - f_B| > \alpha\}\big) \leqslant \mu(B) \cdot c_1 e^{-c_2 \alpha / \|f\|_{\text{BMO}}}. \tag{68}$$

Remark 17. Suppose Γ is regular (Lipschitz $\Rightarrow$ regular) with μ being the d-dimensional Hausdorff or surface measure satisfying (8). Endow Γ with the Euclidean metric. Then Γ being regular implies that (Γ, d, μ) is the space of homogeneous type, and hence (68) holds for $f \in \text{BMO}(\Gamma)$.

Proof of Corollary 4. The proof can be carried out in the exact same manner as in [20]. For completeness, we show the steps here. By a change of variable, $\tau = \log_a(t)$, let $\mathbf{S}\Gamma(x, \tau) := \mathbf{S}\Gamma(x, a^\tau)$. Then by Proposition 1, we have

$$\frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(x, \tau) = \Phi_t * \mu(x),$$

where $\Phi_t = (\ln(a))^2 [t^2 \frac{\partial^2}{\partial t^2} \psi_t + t \frac{\partial}{\partial t} \psi_t]$. Define the square function (of $\frac{\partial^2}{\partial t^2} \mathbf{S}\Gamma$)

$$\mathcal{S}^2 \Gamma(x) = \int_0^\infty |\Phi_t * \mu(x)|^2 \frac{dt}{t} = \ln(a) \int_{-\infty}^\infty \left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(x, \tau) \right|^2 d\tau.$$

Then by Corollary 3, we have that for all balls $B \subset \mathbb{R}^n$,

$$\int_{\Gamma \cap B} \mathcal{S}^2 \Gamma(x) \, d\mu(x) \leqslant C_\Gamma \|\chi_B\|_{L^2(\Gamma)}^2,$$

which shows that $\mathcal{S}^2 \Gamma \in \mathrm{BMO}(\Gamma)$ with the BMO-norm bounded by C_Γ. Let C be a constant such that

$$\sup_{\tau \in \mathbb{R}} \left\| \frac{\partial^3}{\partial \tau^3} \mathbf{S}\Gamma(\cdot, \tau) \right\|_{L^\infty(\Gamma)} \leqslant C.$$

For each $x \in \Gamma_{\delta, N}$ and $\tau_i \in \mathcal{T}_\delta(x)$. Let $\epsilon = \delta/(2C)$ and $I_i = (\tau_i - \epsilon, \tau_i + \epsilon)$, then we have

$$\left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(x, \tau) \right| \geqslant \frac{\delta}{2}, \quad \text{for all } \tau \in I_i,$$

and $I_i \cap \mathcal{T}_\Gamma(x) = \{\tau_i\}$ and the $\{I_i\}$ are disjoint. We have

$$\mathcal{S}^2 \Gamma(x) = \ln(a) \int_{-\infty}^{\infty} \left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(x, \tau) \right|^2 d\tau \geqslant \sum_{\tau_i \in \mathcal{T}_\delta(x)} \int_{I_i} \left| \frac{\partial^2}{\partial \tau^2} \mathbf{S}\Gamma(x, \tau) \right|^2 d\tau$$

$$\geqslant (\delta/2)^2 |I_i| \left(\#\mathcal{T}_\delta(x) \right) > CN\delta^3,$$

for some new constant C. This implies

$$\Gamma_{\delta, N} \subset \left\{ x \in \Gamma \colon \mathcal{S}^2 \Gamma(x) > CN\delta^3 \right\} \subset \left\{ x \in \Gamma \colon \left| \mathcal{S}^2 \Gamma(x) - \mathcal{S}_\Gamma \right| > CN\delta^3 - \mathcal{S}_\Gamma \right\},$$

where $\mathcal{S}_\Gamma = \frac{1}{\mu(\Gamma)} \int_\Gamma \mathcal{S}^2 \Gamma(x) \, d\mu(x)$.

If $CN\delta^3 > \mathcal{S}_\Gamma$, then by the John–Nirenberg inequality (68), there exist positive constants C_1' and C_2' independent of Γ such that

$$\mu\left(\left\{ x \in \Gamma \cap B_R \colon \left| \mathcal{S}^2 \Gamma(x) - \mathcal{S}_\Gamma \right| > CN\delta^3 - \mathcal{S}_\Gamma \right\} \right) \leqslant |\Gamma \cap B_R| C_1' e^{-\frac{C_2'(CN\delta^3 - \mathcal{S}_\Gamma)}{\|\mathcal{S}^2 \Gamma\|_{\mathrm{BMO}}}}. \tag{69}$$

On the other hand, if $CN\delta^3 \leqslant \mathcal{S}_\Gamma$, then (69) still holds with $C_1' \geqslant 1$. Let

$$C_1 = C_1' e^{\frac{C_2' \mathcal{S}_\Gamma}{\|\mathcal{S}^2 \Gamma\|_{\mathrm{BMO}}}} \quad \text{and} \quad C_2 = \frac{C_2' C}{\|\mathcal{S}^2 \Gamma\|_{\mathrm{BMO}}},$$

then

$$\mu(\Gamma_{\delta, N}) \leqslant \mu(\Gamma \cap B_R) \cdot C_1 e^{-C_2 \delta^3 N}. \qquad \square$$

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