Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Ordered cone metric spaces and fixed point results

Ishak Altun [a], Vladimir Rakočević [b,*]

[a] Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450-Yahsihan, Kirikkale, Turkey

[b] Faculty of Mathematics and Sciences, University of Nis, Visegradska 33, 18000 Nis, Serbia

ARTICLE INFO

Article history:
Received 1 April 2009
Received in revised form 23 April 2010
Accepted 22 May 2010

Keywords:
Banach spaces
Cone metric space
Fixed point
Partial order

ABSTRACT

In this paper, we introduce a partial order on a cone metric space and prove a Caristi-type theorem. Furthermore, we prove fixed point theorems for single-valued nondecreasing and weakly increasing mappings, and multi-valued mappings on an ordered cone metric space.

1. Introduction

Non-convex analysis, especially ordered normed spaces, normal cones and Topical functions [1–7], has several applications in optimization theory. In these cases an order is introduced by using vector space cones. Huang and Zang [5] used this approach, and they replaced the real numbers by ordering Banach space and defined a cone metric space. Also, they proved some fixed point theorems of contractive mappings on this new setting.

After the definition of the concept of cone metric space in [5], fixed point theory on these spaces has been developing (see, e.g., [1,8–14,6,15–24,7,25–29]). Generally, this theory on cone metric space is used for contractive-type or contractive-type mappings (see the related references [1–29]). On the other hand, fixed point theory on partially ordered sets has also been developing recently [10,11,30–32].

In this paper, we introduce a partial order on a cone metric space and prove a Caristi-type theorem. Furthermore, we prove fixed point theorems for single-valued nondecreasing and weakly increasing mappings, and multi-valued mappings on an ordered cone metric space.

We recall the definition of cone metric spaces and some of their properties [5]. Let E be a real Banach space and P be a subset of E. By θ we denote the zero element of E and by Int P the interior of P. The subset P is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{\theta\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, and we shall write $x \ll y$ if $y - x \in \text{Int}\, P$.

The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies that $\|x\| \leq M \|y\|$.

The least positive number satisfying the above is called the normal constant of P.

* Corresponding author.
E-mail addresses: ishakaltun@yahoo.com (I. Altun), vrakoc@bankerinter.net (V. Rakočević).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2010.05.038

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been proved in Lemma 1.1 in [25] that every regular cone is normal.

In the following, we always suppose that E is a Banach space, P is a cone in E with Int $P \neq \varnothing$ and $\leq$ is partial ordering with respect to P.

Definition 1 ([5]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d_1) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if $x = y$,
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

Example 1 ([5]). Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2 ([5]). Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is N such that, for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is N such that, for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1 ([5]). *Let (X, d) be a cone metric space, P be a normal cone and let $\{x_n\}$ be a sequence in X. Then*

(i) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$ ($n \to \infty$),*
(ii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ ($n, m \to \infty$).*

Let (X, d) be a cone metric space, $f : X \to X$ and $x_0 \in X$. Then the function f is continuous at x_0 if for any sequence $x_n \to x_0$ we have $fx_n \to fx_0$ [6].

2. Fixed point theorems for nondecreasing mappings

We begin by proving the following lemma. We can find the metric version of it in [33].

Lemma 2. *Let (X, d) be a cone metric space with the Banach space E, P be a cone in E, "$\leq$" be a partial ordering with respect to P and $\phi : X \to E$. Define the relation "$\preceq$" on X as follows:*

$$x \preceq y \Longleftrightarrow d(x, y) \leq \phi(x) - \phi(y).$$

Then "$\preceq$" is a (partial) order on X, named the partial order induced by ϕ.

Proof. For all $x \in X$, $d(x, x) = \theta = \phi(x) - \phi(x)$; that is, "$\preceq$" is reflexive. Again, for $x, y \in X$, let $x \preceq y$ and $y \preceq x$. Then,

$$d(x, y) \leq \phi(x) - \phi(y)$$

and

$$d(y, x) \leq \phi(y) - \phi(x).$$

This shows that $d(x, y) = \theta$; that is, $x = y$. Thus "$\preceq$" is antisymmetric. Now for $x, y, z \in X$, let $x \preceq y$ and $y \preceq z$. Then,

$$d(x, y) \leq \phi(x) - \phi(y) \tag{2.1}$$

and

$$d(y, z) \leq \phi(y) - \phi(z). \tag{2.2}$$

Then, using (2.1) and (2.2) we have

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &\leq \phi(x) - \phi(y) + \phi(y) - \phi(z) \\ &= \phi(x) - \phi(z). \end{aligned}$$

This shows that $x \preceq z$. □

Now we give some examples.

Example 2. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = \{a, b, c\}$ and $d : X \times X \to E$ such that $d(x, x) = (0, 0)$ for all $x \in X$, $d(a, b) = d(b, a) = (1, 2)$, $d(a, c) = d(c, a) = (1, 3)$ and $d(b, c) = d(c, b) = (2, 3)$. Then it is obvious that (X, d) is a cone metric space. Now let $\phi : X \to E$, $\phi(a) = (3, 4)$, $\phi(b) = (2, 2)$ and $\phi(c) = (2, 1)$. Now, since $d(a, b) = (1, 2) \le (1, 2) = \phi(a) - \phi(b)$, then $a \preceq b$. Again, since $d(a, c) = (1, 3) \le (1, 3) = \phi(a) - \phi(c)$, then $a \preceq c$. Since $d(b, c) = (2, 3) \not\le (0, 1) = \phi(b) - \phi(c)$ and $d(c, b) = (2, 3) \not\le (0, -1) = \phi(c) - \phi(b)$, $b \not\preceq c$ and $c \not\preceq b$. Therefore, by using Lemma 2, "$\preceq$" is a partially order induced by ϕ.

Example 3. Let E, P, X and d be as in Example 1. Let $\phi : X \to E$, $\phi(x) = (-x, -\alpha x)$ for all $x \in X$. Then we have the usual order on X.

Our main result for single-valued nondecreasing mappings is as follows.

Theorem 1. *Let (X, d) be a complete cone metric space with the Banach space E, P be a regular cone in E, "$\le$" be a partial ordering with respect to P and $\phi : X \to E$ be a bounded below function and "$\preceq$" be the partial order induced by ϕ. If $f : X \to X$ is a continuous nondecreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then f has a fixed point in X.*

Proof. Consider a point $x_0 \in X$ satisfying $x_0 \preceq fx_0$. Now we define a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ for $n = 1, 2, \ldots$. Then, since f is nondecreasing we have $x_0 \preceq x_1 \preceq x_2 \preceq \cdots$; that is, the sequence $\{x_n\}$ is nondecreasing. By the definition of "$\preceq$" we have $\cdots \le \phi(x_2) \le \phi(x_1) \le \phi(x_0)$; that is, the sequence $\{\phi(x_n)\}$ is a nonincreasing sequence in E. Since P is regular and ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent, and hence it is Cauchy. That is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m > n > n_0$, we have $\|\phi(x_m) - \phi(x_n)\| < \varepsilon$. On the other hand, since $x_n \preceq x_m$, we have $d(x_n, x_m) \le \phi(x_n) - \phi(x_m)$. Therefore, since P is regular and so normal, there exists $M > 0$ such that

$$\begin{aligned}\|d(x_n, x_m)\| &\le M\,\|\phi(x_n) - \phi(x_m)\| \\ &< M\varepsilon.\end{aligned}$$

This implies that $d(x_n, x_m) \to \theta$ $(n, m \to \infty)$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $x_n \to z$ $(n \to \infty)$. Consequently, by the continuity of f, we have $fz = z$. □

If we assume that $\phi(X)$ is compact in E, then we can remove the boundedness of ϕ and regularity of P in Theorem 1, and we can have the following theorem.

Theorem 2. *Let (X, d) be a complete cone metric space with the Banach space E, P be a normal cone in E, "$\le$" the partial ordering with respect to P, $\phi : X \to E$ a function such that $\phi(X)$ is compact and "$\preceq$" the partial order induced by ϕ. If $f : X \to X$ is a continuous nondecreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then f has a fixed point in X.*

Example 4. Let E, P, X, d and ϕ be as in Example 2. Let $f : X \to X$, $f(a) = b$, $f(b) = b$ and $f(c) = c$; then it is obvious that all conditions of Theorem 1 or Theorem 2 are satisfied. Therefore f has a fixed point. But since f is not contractive, the result of [5] is not applicable to this example.

Now we prove a Caristi-type theorem on cone metric spaces.

Let (X, d) be a cone metric space, $C \subset X$ and $\phi : C \to E$ a function; then ϕ is called a lower semicontinuous on C whenever $x_n \to x$ implies that $\phi(x) \le \liminf_{n\to\infty} \phi(x_n)$ [9].

Theorem 3. *Let (X, d) be a complete cone metric space with the Banach space E, P be a regular cone in E, "$\le$" the partial ordering with respect to P and $\phi : X \to E$ a lower semicontinuous and bounded below function. Now, if $f : X \to X$ satisfies*

$$d(x, fx) \le \phi(x) - \phi(fx) \tag{2.3}$$

for all $x \in X$, then f has a fixed point in X.

Proof. We define a partial order on X in the following way:

$$x \preceq y \iff d(x, y) \le \phi(x) - \phi(y).$$

We wish to show that X has a maximal element. Let $\{x_\alpha\}_{\alpha\in I}$ be a nondecreasing chain; then $\{\phi(x_\alpha)\}_{\alpha\in I}$ is a nonincreasing net in E. Let $\{\alpha_n\}$ be an increasing sequence of element from I such that $\lim_{n\to\infty} \phi(x_{\alpha_n}) = r$ (This is possible, since ϕ is bounded from below and P is a regular cone). Using the definition of "$\preceq$", one can show that $\{x_{\alpha_n}\}$ is Cauchy and therefore converges to $z \in X$. By the lower semicontinuity of ϕ, we have $\phi(z) \le r$. Now, for $x_{\alpha_n} \preceq x_{\alpha_m}$, we have

$$d(x_{\alpha_n}, x_{\alpha_m}) \le \phi(x_{\alpha_n}) - \phi(x_{\alpha_m}),$$

and letting $m \to \infty$, we have

$$d(x_{\alpha_n}, z) \le \phi(x_{\alpha_n}) - \phi(z).$$

This shows that $x_{\alpha_n} \preceq z$ for all $n \ge 1$, which means that z is an upper bound for $\{x_{\alpha_n}\}_{n\ge1}$. In order to see that z is also an upper bound for $\{x_\alpha\}_{\alpha\in I}$, let $\beta \in I$ be such that $x_{\alpha_n} \preceq x_\beta$ for all $n \ge 1$. Then we have $\phi(x_\beta) \le \phi(x_{\alpha_n})$ for all $n \ge 1$, which implies that $\phi(x_\beta) = r$. Since $d(x_\beta, x_{\alpha_n}) \le \phi(x_\beta) - \phi(x_{\alpha_n})$, we get $\lim_{n\to\infty} x_{\alpha_n} = x_\beta$, which implies that $x_\beta = z$. Therefore, for any $\alpha \in I$, there exists $n \ge 1$ such that $x_\alpha \preceq x_{\alpha_n}$, which implies that $x_\alpha \preceq z$; that is, z is an upper bound of $\{x_\alpha\}_{\alpha\in I}$.

Thus, by Zorn's lemma, X has a maximal element v. Finally, we prove that v is the desired point. In fact, we have from (2.3)

$$d(v, fv) \leq \phi(v) - \phi(fv);$$

that is, $v \preceq fv$, and again by the maximality of v we have $f(v) \preceq v$. Thus $f(v) = v$. □

In the following we provide multi-valued versions of the preceding theorem. The results are related to those in [34].

Let X be a topological space and $\preceq$ be a partial order on X. Let 2^X denote the family of all nonempty subsets of X.

Definition 3 ([34]). Let A, B be two nonempty subsets of X; the relations between A and B are defined as follows:

(r_1) If, for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
(r_2) If, for every $b \in B$, there exists $a \in A$ such that $a \preceq b$, then $A \prec_2 B$.
(r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 1 ([34]). $\prec_1$ and $\prec_2$ are different relations between A and B. For example, let $X = \mathbb{R}$, $A = \left[\frac{1}{2}, 1\right]$, $B = [0, 1]$, $\preceq$ be the usual order on X; then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = \left[0, \frac{1}{2}\right]$, then $A \prec_2 B$ while $A \not\prec_1 B$.

Remark 2 ([34]). $\prec_1$, $\prec_2$ and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\preceq$ be the usual order on X; then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders on 2^X.

Definition 4 ([34]). A multi-valued operator $T : X \to 2^X$ is called order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X$, $u_n \to u_0$, $v_n \to v_0$ and $v_n \in Tu_n$ imply that $v_0 \in Tu_0$.

The multi-valued version of the preceding theorem is as follows.

Theorem 4. *Let (X, d) be a complete cone metric space with the Banach space E, P be a regular cone in E, "$\leq$" the partial ordering with respect to P, $\phi : X \to E$ a bounded below function and "$\preceq$" the partial order induced by ϕ. $F : X \to 2^X$ is an order closed operator with $\{x_0\} \prec_1 Fx_0$ for some $x_0 \in X$. If $\forall x, y \in X, x \preceq y \Longrightarrow Fx \prec_1 Fy$ (that is, F is nondecreasing with respect to $\prec_1$), then F has a fixed point in X.*

Proof. Since Fx is nonempty for all $x \in X$, there exists $x_1 \in Fx_0$ such that $x_0 \preceq x_1$. Now, since $Fx_0 \prec_1 Fx_1$, there exists $x_2 \in Fx_1$ such that $x_1 \preceq x_2$. Continuing this process, we will get a nondecreasing sequence $\{x_n\}$, which satisfies $x_{n+1} \in Fx_n$. By the definition of "$\preceq$", we have $\cdots \leq \phi(x_2) \leq \phi(x_1) \leq \phi(x_0)$; that is, the sequence $\{\phi(x_n)\}$ is a nonincreasing sequence in E. Since P is regular and ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy. By the same argumentation as in the proof of Theorem 1, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $x_n \to z$ $(n \to \infty)$. Consequently, we have $z \in Fz$ since F is order closed and $x_{n+1} \in Fx_n$. □

Similarly, we can prove the following theorem.

Theorem 5. *Let (X, d) be a complete cone metric space with the Banach space E, P be a regular cone in E, "$\leq$" the partial ordering with respect to P, $\phi : X \to E$ a bounded above function and "$\preceq$" the partial order induced by ϕ. $F : X \to 2^X$ is an order closed operator with $Fx_0 \prec_2 \{x_0\}$ for some $x_0 \in X$. If $\forall x, y \in X, x \preceq y \Longrightarrow Fx \prec_2 Fy$ (that is, F is nondecreasing with respect to $\prec_2$), then F has a fixed point in X.*

3. Fixed point theorems for weakly increasing mappings

Definition 5 ([35,36]). Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.

Note that two weakly increasing mappings need not be nondecreasing. We can find the following examples in [30].

Example 5. Let $X = \mathbb{R}_+$ be endowed with the usual ordering. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } 1 < x < \infty \end{cases} \quad \text{and} \quad gx = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } 1 < x < \infty; \end{cases}$$

then it is obvious that $fx \leq gfx$ and $gx \leq fgx$ for all $x \in X$. Thus f and g are weakly increasing mappings. Note that both f and g are not nondecreasing.

Example 6. Let $X = [1, \infty) \times [1, \infty)$ be endowed with coordinate-wise ordering; that is, $(x, y) \preceq (z, w) \Leftrightarrow x \leq z$ and $y \leq w$. Let $f, g : X \to X$ be defined by $f(x, y) = (2x, 3y)$ and $g(x, y) = (x^2, y^2)$; then $f(x, y) = (2x, 3y) \preceq gf(x, y) = g(2x, 3y) = (4x^2, 9y^2)$ and $g(x, y) = (x^2, y^2) \preceq fg(x, y) = f(x^2, y^2) = (2x^2, 3y^2)$. Thus f and g are weakly increasing mappings.

Example 7. Let $X = \mathbb{R}^2$ be endowed with lexicographical ordering; that is, $(x, y) \preceq (z, w)$ if and only if $x < z$ or ($x = z$ and $y \leq w$). Let $f, g : X \to X$ be defined by

$$f(x, y) = (\max\{x, y\}, \min\{x, y\})$$

and

$$g(x,y)=\left(\max\{x,y\},\frac{x+y}{2}\right);$$

then

$$\begin{aligned} f(x,y) &= (\max\{x,y\},\min\{x,y\}) \\ &\preceq gf(x,y) \\ &= g\,(\max\{x,y\},\min\{x,y\}) \\ &= \left(\max\{\max\{x,y\},\min\{x,y\}\},\frac{\max\{x,y\}+\min\{x,y\}}{2}\right) \\ &= \left(\max\{x,y\},\frac{x+y}{2}\right) \end{aligned}$$

and

$$\begin{aligned} g(x,y) &= \left(\max\{x,y\},\frac{x+y}{2}\right) \\ &\preceq fg(x,y) \\ &= f\left(\max\{x,y\},\frac{x+y}{2}\right) \\ &= \left(\max\left\{\max\{x,y\},\frac{x+y}{2}\right\},\min\left\{\max\{x,y\},\frac{x+y}{2}\right\}\right) \\ &= \left(\max\{x,y\},\frac{x+y}{2}\right). \end{aligned}$$

Thus f and g are weakly increasing mappings. Note that, since $(1,4)\preceq(2,3)$ but $f(1,4)=(4,1)\npreceq(3,2)=f(2,3)$, then f is not nondecreasing. Similarly g is not nondecreasing.

Let us remark that in the next theorem we remove the condition "there exists an $x_0\in X$ with $x_0\preceq f(x_0)$" of Theorem 1.

Theorem 6. *Let (X,d) be a complete cone metric space with the Banach space E, P be a regular cone in E, "$\leq$" the partial ordering with respect to P, $\phi:X\to E$ a bounded below function and "$\preceq$" the partial order induced by ϕ. If $f,g:X\to X$ are two continuous weakly increasing functions, then f and g have a common fixed point in X.*

Proof. Let x_0 be an arbitrary point of X and let us define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1}=fx_{2n}\quad\text{and}\quad x_{2n+2}=gx_{2n+1}\quad\text{for } n\in\{0,1,\ldots\}.$$

Note that, since f and g are weakly increasing, we have

$$\begin{aligned} x_1 &= fx_0\preceq gfx_0=gx_1=x_2, \\ x_2 &= gx_1\preceq fgx_1=fx_2=x_3, \end{aligned}$$

and continuing this process, we have

$$x_1\preceq x_2\preceq\cdots\preceq x_n\preceq x_{n+1}\preceq\cdots.$$

That is, the sequence $\{x_n\}$ is nondecreasing. By the definition of "$\preceq$", we have $\cdots\phi(x_2)\leq\phi(x_1)\leq\phi(x_0)$; that is, the sequence $\{\phi(x_n)\}$ is a nonincreasing sequence in E. Since P is regular and ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent, and hence it is Cauchy. By the same argumentation as in the proof of Theorem 1, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $z\in X$ such that $x_n\to z$ $(n\to\infty)$. Therefore, $x_{2n+1}\to z$ and $x_{2n+2}\to z$. Consequently, by the continuity of f and g, we have $fz=gz=z$. □

Now we introduce the following definition.

Definition 6. Let $(X,\preceq)$ be a partially ordered set. Two mappings $F,G:X\to 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x\in X$ we have $Fx\prec_1 Gy$ for all $y\in Fx$ and $Gx\prec_1 Fy$ for all $y\in Gx$. Similarly, two maps $F,G:X\to 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x\in X$ we have $Gy\prec_2 Fx$ for all $y\in Fx$ and $Fy\prec_2 Gx$ for all $y\in Gx$.

Now we give some examples.

Example 8. Let $X=[1,\infty)$ and $\leq$ be the usual order on X. Consider two mappings $F,G:X\to 2^X$ defined by $Fx=[1,x^2]$ and $Gx=[1,2x]$ for all $x\in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_2$ but not $\prec_1$. Indeed, since

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \quad \text{for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \quad \text{for all } y \in Gx,$$

F and G are weakly increasing with respect to $\prec_2$ but $F2 = [1, 4] \not\prec_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to $\prec_1$.

Example 9. Let $X = [0, 1]$ and $\leq$ be the usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = \{0, 1\}$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_1$ but not $\prec_2$. Indeed, since

$$Fx = \{0, 1\} \prec_1 [y, 1] = Gy \quad \text{for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 \{0, 1\} = Fy \quad \text{for all } y \in Gx,$$

F and G are weakly increasing with respect to $\prec_1$ but $G1 = \{1\} \not\prec_2 \{0, 1\} = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to $\prec_2$.

Theorem 7. *Let (X, d) be a complete cone metric space with the Banach space E, P be a regular cone in E, "$\leq$" the partial ordering with respect to P, $\phi : X \to E$ a bounded below function and "$\preceq$" the partial order induced by ϕ. If $F, G : X \to 2^X$ are two order closed and weakly increasing mappings with respect to $\prec_1$, then F and G have a common fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point. Since $Fx_0 \neq \emptyset$, we can choose $x_1 \in Fx_0$. Now, since F and G are weakly increasing with respect to $\prec_1$, we have $x_1 \in Fx_0 \prec_1 Gx_1$. Thus there exists some $x_2 \in Gx_1$ such that $x_1 \preceq x_2$. Again, since F and G are weakly increasing with respect to $\prec_1$, we have $x_2 \in Gx_1 \prec_1 Fx_2$. Thus there exists some $x_3 \in Fx_2$ such that $x_2 \preceq x_3$. Continuing this process, we will get a nondecreasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Fx_{2n}, x_{2n+2} \in Gx_{2n+1}$, $n = 0, 1, 2, \ldots$. By the definition of "$\preceq$", we have

$$\cdots \leq \phi(x_3) \leq \phi(x_2) \leq \phi(x_1);$$

that is, the sequence $\{\phi(x_n)\}$ is a nonincreasing sequence in E. Since P is regular and ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy. By the same argumentation as in the proof of Theorem 1, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $x_n \to z$ $(n \to \infty)$. Therefore, $x_{2n+1} \to z$ and $x_{2n+2} \to z$. Consequently, since F and G are order closed, $\{x_n\}_{n=1}^{\infty}$ monotone and $x_{2n+1} \in Fx_{2n}, x_{2n+2} \in Gx_{2n+1}$, we deduce that $z \in Fz$ and $z \in Gz$; i.e., z is a common fixed point of F and G. □

Theorem 8. *Let (X, d) be a complete cone metric space with the Banach space E, P be a regular cone in E, "$\leq$" the partial ordering with respect to P, $\phi : X \to E$ a bounded above function and "$\preceq$" the partial order induced by ϕ. If $F, G : X \to 2^X$ are two order closed and weakly increasing mappings with respect to $\prec_2$, then F and G have a common fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary point. Since $Fx_0 \neq \emptyset$, we can choose $x_1 \in Fx_0$. Now since F and G are weakly increasing with respect to $\prec_2$, we have $Gx_1 \prec_2 Fx_0$. Thus there exists some $x_2 \in Gx_1$ such that $x_2 \preceq x_1$. Again, since F and G are weakly increasing with respect to $\prec_2$, we have $Fx_2 \prec_2 Gx_1$. Thus there exists some $x_3 \in Fx_2$ such that $x_3 \preceq x_2$. Continue this process, we will get a nonincreasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Fx_{2n}, x_{2n+2} \in Gx_{2n+1}$, $n = 0, 1, 2, \ldots$. By the definition of "$\preceq$", we have

$$\varphi(x_1) \leq \varphi(x_2) \leq \varphi(x_3) \leq \cdots.$$

Since P is regular and ϕ is bounded from above, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy. By the same argumentation as in the proof of Theorem 1, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $x_n \to z$ $(n \to \infty)$. Therefore, $x_{2n+1} \to z$ and $x_{2n+2} \to z$. Consequently, since F and G are order closed, $\{x_n\}_{n=1}^{\infty}$ monotone and $x_{2n+1} \in Fx_{2n}, x_{2n+2} \in Gx_{2n+1}$, we deduce that $z \in Fz$ and $z \in Gz$; i.e. z is a common fixed point of F and G. □

Acknowledgements

The authors thank the referees for their valuable comments and suggestions. The second author was supported by the Ministry of Science, Technology and Development, Republic of Serbia.

References

[1] M. Abbas, G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008) 416–420.
[2] C. Di Bari, P. Vetro, φ-pairs and common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo 57 (2008) 279–285.
[3] C. Di Bari, P. Vetro, Weakly φ-pairs and common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo 58 (2009) 125–132.
[4] R.H. Haghi, Sh. Rezapour, Fixed points of multifunctions on regular cone metric spaces, Expo. Math. 28 (2010) 71–77.
[5] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476.
[6] D. Ilić, V. Rakočević, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341 (2008) 876–882.
[7] Sh. Rezapour, R.H. Haghi, Fixed point of multifunctions on cone metric spaces, Numer. Funct. Anal. Optim. 30 (7–8) (2009) 825–832.

[8] M. Abbas, B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 22 (2009) 511–515.
[9] T. Abdeljawad, E. Karapinar, Quasicone metric spaces and generalizations of Caristi Kirk's theorem, Fixed Point Theory Appl. (2009) Article ID 574387, 9 pages.
[10] I. Altun, B. Damjanović, D. Djorić, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. 23 (2010) 310–316.
[11] I. Altun, G. Durmaz, Some fixed point theorems on ordered cone metric spaces, Rend. Circ. Mat. Palermo 58 (2009) 319–325.
[12] M. Arshad, A. Azam, I. Beg, Common fixed points of two maps in cone metric spaces, Rend. Circ. Mat. Palermo 57 (2008) 433–441.
[13] M. Arshad, A. Azam, P. Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory Appl. (2009) Article ID 493965, 11 pages.
[14] A. Azam, M. Arshad, Common fixed points of generalized contractive maps in cone metric spaces, Bull. Iranian Math. Soc. 35 (2) (2009) 255–264.
[15] D. Ilić, V. Rakočević, Quasi-contraction on cone metric spaces, Appl. Math. Lett. 22 (2009) 728–731.
[16] S. Janković, Z. Kadelburg, S. Radenović, B.E. Rhoades, Assad–Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces, Fixed Point Theory Appl. (2009) Article ID 761086, 16 pages.
[17] Z. Kadelburg, M. Pavlović, S. Radenović, Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces, Comput. Math. Appl. 59 (2010) 3148–3159.
[18] Z. Kadelburg, S. Radenović, V. Rakočević, Remarks on "quasi-contraction on a cone metric space", Appl. Math. Lett. 22 (2009) 1674–1679.
[19] Z. Kadelburg, S. Radenović, B. Rosić, Strict contractive conditions and common fixed point theorems in cone metric spaces, Fixed Point Theory Appl. (2009) Article ID 173838, 14 Pages.
[20] D. Kilm, D. Wardowski, Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces, Nonlinear Anal. 71 (2009) 5170–5175.
[21] S. Radenović, Common fixed points under contractive conditions in cone metric spaces, Comput. Math. Appl. 58 (2009) 1273–1278.
[22] S. Radenović, B.E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, Comput. Math. Appl. 57 (2009) 1701–1707.
[23] Sh. Rezapour, M. Derafshpour, N. Shahzad, Best proximity points of cyclic φ-contractions on reflexive Banach spaces, Fixed Point Theory Appl. (2010) Article ID 946178, 7 Pages.
[24] Sh. Rezapour, R.H. Haghi, N. Shahzad, Some notes on fixed points of quasicontraction maps, Appl. Math. Lett. 23 (2010) 498–502.
[25] Sh. Rezapour, R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings ", J. Math. Anal. Appl. 345 (2008) 719–724.
[26] Sh. Rezapour, H. Khandani, S.M. Vaezpour, Efficacy of cones on topological vector spaces and application to common fixed points of multifunctions, Rend. Circ. Mat. Palermo 59 (2010) 185–197.
[27] D. Wardowski, Endpoints and fixed points of set-valued contractions in cone metric spaces, Nonlinear Anal. 71 (2009) 512–516.
[28] D. Wei-Shih, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal. 72 (2010) 2259–2261.
[29] K. Wlodarczyk, R. Plebaniak, C. Obczynski, Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces, Nonlinear Anal. 72 (2010) 794–805.
[30] I. Altun, H. Simsek, Some fixed point theorems on ordered metric spaces and application, Fixed Point Theory Appl. (2010) Article ID 621469, 17 pages.
[31] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. 72 (2010) 1188–1197.
[32] Z. Zhao, X. Chen, Fixed points of decreasing operators in ordered Banach spaces and applications to nonlinear second order elliptic equations, Comput. Math. Appl. 58 (2009) 1223–1229.
[33] A. Brøndsted, On a lemma of Bishop and Phelps, Pacific J. Math. 55 (1974) 335–341.
[34] Y. Feng, S. Liu, Fixed point theorems for multi-valued increasing operators in partially ordered spaces, Soochow J. Math. 30 (4) (2004) 461–469.
[35] B.C. Dhage, Condensing mappings and applications to existence theorems for common solution of differential equations, Bull. Korean Math. Soc. 36 (3) (1999) 565–578.
[36] B.C. Dhage, D. O'Regan, R.P. Agarwal, Common fixed point theorems for a pair of countably condensing mappings in ordered Banach spaces, J. Appl. Math. Stoch. Anal. 16 (3) (2003) 243–248.