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Lower bounds for real solutions to sparse polynomial systems

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Abstract

We show how to construct sparse polynomial systems that have non-trivial lower bounds on their numbers of real solutions. These are unmixed systems associated to certain polytopes. For the order polytope of a poset P this lower bound is the sign-imbalance of P and it holds if all maximal chains of P have length of the same parity. This theory also gives lower bounds in the real Schubert calculus through the sagbi degeneration of the Grassmannian to a toric variety, and thus recovers a result of Eremenko and Gabrielov.

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0. Introduction

A fundamental problem in real algebraic geometry is to understand the real solutions to a system of real polynomial equations. This is of unquestionable importance

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in applications of mathematics. Even the existence of real solutions is not guaranteed; oftentimes there are few or no real solutions, and all complex solutions must be found to determine if this is the case. We give a method to construct families of polynomial systems that have non-trivial lower bounds on their numbers of real solutions, guaranteeing the existence of real solutions.

Geometric problems with lower bounds on their numbers of real solutions are a recent discovery. Kharlamov and Degtyarev showed that of the 12 (a priori complex) rational cubics passing through 8 real points in the plane, at least 8 are real [3, Proposition 4.7.3]. This was generalized by Welschinger [25], Mikhalkin [14,15], and Itenberg et al. [9], to rational curves passing through real points on toric surfaces. Welschinger discovered an invariant which gives a lower bound, and work of Mikhalkin and of Itenberg, Kharlamov, and Shustin shows that this lower bound is non-zero and in fact quite large. If N_d is the Kontsevich number of such complex rational curves [11] and W_d is Welschinger's invariant, then $\log N_d$ and $\log W_d$ are each asymptotic to $3d \log d$.

At the same time, Eremenko and Gabrielov [4,5] computed the degree of the Wronski map on the real Grassmannian of k-planes in n-space. It is non-trivial when n is odd. This degree is a lower bound on the number of real solutions to certain problems from the Schubert calculus on this Grassmannian. In its formulation as a Wronski determinant, their work implies the existence of many inequivalent k-tuples of polynomials of even degree having a given real polynomial as their Wronskian.

These results highlight the importance of developing a theoretical framework to explain this phenomenon. Our main purpose is to provide such a framework for sparse polynomial equations. We are inspired by the work of Eremenko and Gabrielov. Our lower bound is the topological degree of a linear projection on an oriented double cover of a toric variety. In Section 1, we formulate a polynomial system as the fibers of a map from a toric variety and define the characteristic of such a map to be the degree of the map lifted to a canonical double cover. This has the same equations, but is taken in the sphere covering real projective space. (One method used by Eremenko and Gabrielov was to lift the Wronski map to a double cover of non-orientable Grassmannians.) This characteristic is defined only if the smooth points of the double cover are orientable. We give criteria for this to hold in Section 2. In Section 3, we show how to compute the degree for some maps by degenerating the double cover of the toric variety into a union of oriented coordinate spheres and then determine the degree of the same projection on this union of spheres.

This method does not work for all linear projections of toric varieties. For toric varieties associated to the order polytope of a poset P, there are natural Wronski projections with a computable characteristic when the poset P is ranked $\bmod 2$. That is, the lengths of all maximal chains in P have the same parity. In this case, the degree is the sign-imbalance of P—the difference between the numbers of even and of odd linear extensions [26,20]. This pleasing construction is the subject of Section 4.

Section 5 contains further examples of this theory. Grassmannians admit flat sagbi degenerations to such toric varieties [23, Chapter 11]. For these, the Wronski map coincides with a linear projection we study, and we are able to recover the results of Eremenko and Gabrielov in this way. This is the topic of Section 6.

In Section 7, we give alternative proofs of our lower bound for the order polytope of a poset P, when P is the incomparable union of chains of lengths $a_1, \dots, a_d$. We show that the Wronski polynomial system in this case is equivalent to finding all factorizations $f(z) = f_1(z) \cdots f_d(z)$, where $f(z)$ is a fixed polynomial of degree $a_1 + \cdots + a_d$, and the factors $f_1(z), \dots, f_d(z)$ that we seek have respective degrees $a_1, \dots, a_d$. This reformulation reveals the existence of a new phenomenon for real polynomial systems. Not only do each of these systems possess a lower bound on their number of real solutions, but certain numbers of real solutions cannot occur. That is, there are gaps in the possible numbers of real solutions to these polynomial systems.

1. Systems of sparse polynomials as linear projections

Let $F(t_1, t_2, \dots, t_n)$ be a real polynomial. The exponent vector $m = (m_1, m_2, \dots, m_n)$ of a monomial $t^m := t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$ appearing in F is a point in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. The *Newton polytope* $\Delta \subset \mathbb{R}^n$ of a polynomial F is the convex hull of its exponent vectors. We study real solutions to systems of real polynomial equations

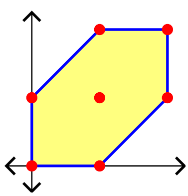
$$F_1(t_1, \dots, t_n) = F_2(t_1, \dots, t_n) = \cdots = F_n(t_1, \dots, t_n) = 0, \tag{1.1}$$

where the polynomials F_i have real coefficients with the same Newton polytope Δ. By Kushnirenko's Theorem [12], there are at most $V(\Delta) := n!\mathrm{vol}(\Delta)$ solutions to (1.1) in the complex torus $(\mathbb{C}^\times)^n$ and this number is attained for generic such systems. We call this number $V(\Delta)$ the *normalized volume* of Δ. We shall always assume that our polynomial systems are generic in that they have $V(\Delta)$ solutions in $(\mathbb{C}^\times)^n$, each necessarily of multiplicity one.

Example 1.2. Suppose that we have a system of two polynomial equations of the form

$$a_i + b_i x + c_i y + d_i xy + e_i x^2 y + f_i xy^2 + g_i x^2 y^2 = 0 \quad \text{for } i = 1, 2.$$

The monomials which appear correspond to the lattice points $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, $(1, 2)$, and $(2, 2)$, whose convex hull is a hexagon.



This hexagon has Euclidean volume 3, and so we expect there to be $3 \cdot 2! = 6$ complex solutions to this set of equations.

The projective toric variety X_Δ associated to the polytope Δ is the variety parametrized by the monomials in Δ. More precisely, let $\mathbb{P}^\Delta$ be the complex projective space with coordinates $\{x_m | m \in \Delta \cap \mathbb{Z}^n\}$ indexed by the points of $\Delta \cap \mathbb{Z}^n$. Then X_Δ is the closure of the image of the map

$$\varphi_\Delta \colon \begin{array}{rcl} (\mathbb{C}^\times)^n & \longrightarrow & \mathbb{P}^\Delta \\ (t_1, t_2, \ldots, t_n) & \longmapsto & [t^m | m \in \Delta \cap \mathbb{Z}^n] \end{array}.$$

This map is injective if and only if the affine span of $\Delta \cap \mathbb{Z}^n$ is equal to $\mathbb{Z}^n$.

Linear forms on $\mathbb{P}^\Delta$ pull back along φ_Δ to polynomials with monomials from $\Delta \cap \mathbb{Z}^n$,

$$\varphi_\Delta^* \left(\sum_{m \in \Delta \cap \mathbb{Z}^n} c_m x_m \right) = \sum_{m \in \Delta \cap \mathbb{Z}^n} c_m t^m .$$

A system (1.1) of real polynomials with Newton polytope Δ corresponds to a system of $n = \dim X_\Delta$ real linear equations on X_Δ, that is, to the intersection of X_Δ with a real linear subspace Λ of codimension n in $\mathbb{P}^\Delta$.

Let $E \subset \Lambda$ be a real hyperplane in Λ disjoint from X_Δ—a linear subspace of $\mathbb{P}^\Delta$ complementary to X_Δ. Let $H (\simeq \mathbb{P}^n) \subset \mathbb{P}^\Delta$ be any real linear subspace of maximal dimension n disjoint from E. Let π_E be the linear projection with center E

$$\begin{aligned} \pi_E \colon \mathbb{P}^\Delta - E &\longrightarrow H, \\ x &\longmapsto \mathrm{Span}(x, E) \cap H. \end{aligned}$$

Then solutions to system (1.1) correspond to points in $X_\Delta \cap \pi_E^{-1}(p)$, where $p := \pi_E(\Lambda) \in H$.

Set $Y_\Delta := X_\Delta \cap \mathbb{RP}^\Delta$, the real points of the toric variety X_Δ, and let f be the restriction of π_E to Y_Δ. We could also consider the closure of the image of $(\mathbb{R}^\times)^n$ under φ_Δ. These objects coincide if and only if the restriction of φ_Δ to $(\mathbb{R}^\times)^n$ is injective, which occurs if and only if the lattice spanned by $\Delta \cap \mathbb{Z}^n$ has odd index in $\mathbb{Z}^n$. We shall always assume that this index is odd.

Then real solutions to system (1.1) are the elements in the fiber $f^{-1}(p)$ of the linear projection f

$$f \colon Y_\Delta \subset \mathbb{RP}^\Delta \overset{\pi_E}{- - \rightarrow} H_\mathbb{R} \simeq \mathbb{RP}^n.$$

If both Y_Δ and $\mathbb{RP}^n$ are oriented, then the absolute value of the topological degree of the map f is a lower bound for the number of points in $f^{-1}(p)$. Our assumption on the genericity of the original system (1.1) implies that p is a regular value of the map f.

In general Y_Δ and $\mathbb{RP}^n$ are not necessarily orientable. Given a normal projective variety $Y \subset \mathbb{RP}^N$ of dimension n, let $Y^+ \subset S^N$ be the subvariety of the sphere given by the same homogeneous equations as Y. Then $Y^+ \to Y$ is a double cover. Likewise, if $f: Y \to \mathbb{RP}^n$ is the restriction of a linear projection $\pi: \mathbb{RP}^N - \to \mathbb{RP}^n$ to Y, then we let $f^+: Y^+ \to S^n$ be the restriction of that projection lifted to the corresponding spheres. We obtain the commutative diagram, where the vertical arrows are 2 to 1 covering maps.

$$
\begin{array}{ccccccc}
f^+: & Y^+ & \subset & S^N & \overset{\pi_E^+}{- - - \to} & S^n \\
 & \downarrow & & \downarrow & & \downarrow \\
f\ : & Y & \subset & \mathbb{RP}^N & \overset{\pi_E}{- - - \to} & \mathbb{RP}^n
\end{array}
$$

Definition 1.3. Suppose that the manifold Y^+_{sm} formed by smooth points of Y^+ is orientable. Fix an orientation of Y^+_{sm} and define the *characteristic* of f, $\mathrm{char}(f)$, to be the absolute value of the topological degree of $f^+: Y^+ \to \mathbb{RP}^n$. This does not depend upon the choice of orientation of Y^+_{sm} if it is connected. If Y^+_{sm} is not connected, then $\mathrm{char}(f)$ could depend upon the choice of orientation of its different components. Since Y is normal, the set of singularities Y^+_{sing} has codimension at least 2. Hence $\mathbb{RP}^n \setminus \pi(Y^+_{\mathrm{sing}})$ is connected and this notion is well-defined.

Suppose that Y is orientable. Consider the orientation on Y^+ that is pulled back from Y along the covering map $S^N \to \mathbb{RP}^N$. If $\mathbb{RP}^n$ is not orientable then $\mathrm{char}(f) = 0$. If $\mathbb{RP}^n$ is orientable then the characteristic $\mathrm{char}(f)$ is equal to the topological degree of f.

We record the obvious, fundamental, and important property of this notion.

Proposition 1.4. *If $p \in \mathbb{RP}^n$ is a regular value of f, then the number of points in a fiber $f^{-1}(p)$ is bounded below by its characteristic* $\mathrm{char}(f)$.

According to Eremenko and Gabrielov [5], this notion is due to Kronecker [13], who defined the characteristic of a regular map $\mathbb{RP}^2 \to \mathbb{RP}^2$ in this manner. Note that if Y^+_{sm} is not connected, then different choices of orientation of the components of Y^+_{sm} may give different values for $\mathrm{char}(f)$. Each value for $\mathrm{char}(f)$ is a lower bound on the number of points in a fiber $f^{-1}(p)$ above a regular value p of f. Optimizing these choices is beyond the scope of this paper.

2. Orientability of real toric varieties

The elementary definition of Y_Δ given in Section 1, as the real points of the variety parametrized by monomials in $\Delta \cap \mathbb{Z}^n$, is inadequate to address the orientability of Y^+_Δ. More useful to us is Cox's construction of X_Δ as a quotient of a torus acting on affine space, as detailed in [1, Theorem 2.1]. Let $\Delta \subset \mathbb{R}^n$ be a polytope with vertices in the integer lattice $\mathbb{Z}^n$ and suppose that it is given by its facet inequalities

$$\begin{aligned}\Delta &= \{x \in \mathbb{R}^n \mid \mathcal{A} \cdot x \geqslant -b\} \\ &= \{x \in \mathbb{R}^n \mid a_i \cdot x \geqslant -b_i,\ i = 1, \ldots, r\},\end{aligned}$$

where $a_i \in \mathbb{Z}^n$ is the primitive inward-pointing normal to the ith facet of Δ.

Example 2.1. If Δ is the hexagon of Example 1.2, then

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}^T \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}^T.$$

Let $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$. For each $m \in \Delta \cap \mathbb{Z}^n$, set

$$z(m) := \prod_{i=1}^{r} z_i^{a_i \cdot m + b_i}$$

and consider the map ψ_Δ defined by

$$\psi_\Delta(z) = [z(m) \mid m \in \Delta \cap \mathbb{Z}^n] \in \mathbb{P}^\Delta.$$

This map is undefined on the zero locus B_Δ of the monomial ideal

$$\langle z(m) \mid m \in \Delta \cap \mathbb{Z}^n \rangle.$$

Note that z_i appears in $z(m)$ if and only if m does not lie on the ith facet. Define the vertex monomial $z^{(v)}$ to be the product of all z_i such that v misses the ith facet. Then B_Δ is the zero locus of the monomial ideal

$$\langle z^{(v)} \mid v \text{ a vertex of } \Delta \rangle.$$

The monomial $z^b = z_1^{b_1} \cdots z_r^{b_r}$ divides each component $z(m)$ of $\psi_\Delta(z)$. Removing these common factors from $\psi_\Delta(z) \in \mathbb{P}^\Delta$ shows that ψ_Δ factors through φ_Δ, at least for z in the torus $(\mathbb{C}^\times)^r$. For $z \in (\mathbb{C}^\times)^r$, we have $\psi_\Delta(z) = \varphi_\Delta \circ \phi_\Delta(z)$, where

$$\phi_\Delta \colon (z_1, z_2, \ldots, z_r) \longmapsto (\ldots, z_1^{a_{1i}} z_2^{a_{2i}} \cdots z_r^{a_{ri}}, \ldots).$$

Since Δ has full dimension, this map is surjective and so the image of ψ_Δ is dense in the projective toric variety X_Δ. Since φ_Δ is injective, two points of $(\mathbb{C}^\times)^r$ have the same image under ψ_Δ if and only if they are equal modulo the kernel G_Δ of ϕ_Δ

$$G_\Delta := \left\{ \mu \in (\mathbb{C}^\times)^r \,|\, 1 = \prod_{i=1}^{r} \mu_i^{a_{ij}} \text{ for each } j = 1, \dots, n \right\}.$$

The map ψ_Δ almost identifies X_Δ as the quotient of $\mathbb{C}^r - B_\Delta$ by G_Δ. The difficulty is that G_Δ-orbits on $\mathbb{C}^r - B_\Delta$ are not necessarily closed and so the geometric quotient $(\mathbb{C}^r - B_\Delta)/G_\Delta$ may not be Hausdorff. If Δ is a simple polytope (each vertex lies on exactly n facets), then this does not occur and X_Δ is the geometric quotient. In general, X_Δ is the closest variety to the non-Hausdorff quotient. More precisely, it is the quotient in the category of schemes, the *categorical quotient*, written $(\mathbb{C}^r - B_\Delta)//G_\Delta$.

Proposition 2.2 (*Cox [1, Theorem 2.1]*). *Suppose that $\Delta \cap \mathbb{Z}^n$ affinely spans $\mathbb{Z}^n$. Then the abstract toric variety X_Σ defined by the normal fan Σ of Δ is the categorical quotient $(\mathbb{C}^r - B_\Delta)//G_\Delta$, and the map ψ_Δ induces an isomorphism of toric varieties $X_\Sigma \to X_\Delta$. This categorical quotient is a geometric quotient if and only if Δ is simple.*

If we restrict the map ψ_Δ to $\mathbb{R}^r - B_\Delta$, then its image lies in the real toric variety Y_Δ, but this image is not in general equal to Y_Δ.

Proposition 2.3. *The image of $\mathbb{R}^r - B_\Delta$ under the map ψ_Δ is equal to Y_Δ if and only if the index of the lattice $\Lambda_{\mathcal{A}}$ spanned by the columns of $\mathcal{A}$ in its saturation $\Lambda_{\mathcal{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is odd.*

The lattice $\Lambda_{\mathcal{A}}$ for the hexagon of Example 1.2 is saturated as $\mathcal{A}$ has a 2×2 minor with absolute value 1.

Proof of Proposition 2.3. It suffices to show that the image of $(\mathbb{R}^\times)^r$ under the map ϕ_Δ is equal to the real points $(\mathbb{R}^\times)^n$ of $(\mathbb{C}^\times)^n$ if and only if the index of $\Lambda_{\mathcal{A}}$ in its saturation is odd.

Invertible integer row and column operations reduce $\mathcal{A}$ to its Smith normal form

$$\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

These operations do not change the index of the lattice $\Lambda_{\mathcal{A}}$ in its saturation. It follows that the index is equal to the product $a_1 \cdots a_n$. The image of $(\mathbb{R}^\times)^r$ under the map ϕ_Δ is equal to $(\mathbb{R}^\times)^n$ if and only if the map from $(\mathbb{R}^\times)^r$ to $(\mathbb{R}^\times)^n$ defined by $(z_1, \ldots, z_r) \mapsto (z_1^{a_1}, \ldots, z_n^{a_n})$ is surjective, which happens if and only if the product $a_1 \cdots a_n$ is odd. $\square$

We address the orientability of Y_Δ^+. First, set $\ell := \#\Delta \cap \mathbb{Z}^n$ and consider the map $g\colon \mathbb{C}^r \to \mathbb{C}^\ell$ which lifts the map $\psi_\Delta\colon \mathbb{C}^r \to \mathbb{P}^{\ell-1} = \mathbb{P}^\Delta$

$$g\colon z \longmapsto (z(m) | m \in \Delta \cap \mathbb{Z}^n).$$

If we let γ be the map from G_Δ to $\mathbb{C}^\times$ defined by

$$\gamma(\mu_1, \mu_2, \ldots, \mu_r) = \mu_1^{b_1} \mu_2^{b_2} \cdots \mu_r^{b_r} =: \mu^b,$$

then points $w, z \in (\mathbb{C}^\times)^r$ have the same image in $\mathbb{C}^\ell$ if and only if $wz^{-1} \in \ker(\gamma)$. More generally, the fibers of the map $\mathbb{C}^r - B_\Delta \to \mathbb{C}^\ell$ are unions of orbits of $\ker(\gamma)$.

Let $\mathbb{R}_>$ be the positive real numbers. The real cone over Y_Δ is $Z_\Delta := (\mathbb{R}^\ell - \{0\}) \cap g(\mathbb{C}^r)$. Then the double cover Y_Δ^+ of Y_Δ is the quotient $Z_\Delta / \mathbb{R}_>$. If we assume that the column space of $\mathcal{A}$ has odd index in its saturation, then there are two cases to consider.

(i) $Z_\Delta = g(\mathbb{R}^r)$, or
(ii) Z_Δ is the disjoint union of $g(\mathbb{R}^r)$ and $-1 \cdot g(\mathbb{R}^r)$.

These cases are distinguished by the image of the map γ, when restricted to the real points $G_\Delta(\mathbb{R})$ of G_Δ. This also serves to describe Y_Δ^+. Set $K := \gamma^{-1}(\mathbb{R}_>)$.

Proposition 2.4. *With the above definitions, we have*

(i) *If $K \subsetneq G_\Delta(\mathbb{R})$ so that $\gamma(G_\Delta(\mathbb{R})) = \mathbb{R}^\times$, then $Z_\Delta = g(\mathbb{R}^r - B_\Delta)$, and so Y_Δ^+ is the image of $\mathbb{R}^r - B_\Delta$ under the composition*

$$\mathbb{R}^r - B_\Delta \xrightarrow{g} Z_\Delta \longrightarrow Z_\Delta / \mathbb{R}_> = Y_\Delta^+. \tag{2.5}$$

(ii) *If $K = G_\Delta(\mathbb{R})$, so that $\gamma(G_\Delta(\mathbb{R})) = \mathbb{R}_>$, then $Z_\Delta \neq g(\mathbb{R}^r - B_\Delta)$ but we have*

$$Z_\Delta = g(\mathbb{R}^r - B_\Delta) \coprod -g(\mathbb{R}^r - B_\Delta).$$

Furthermore, Y_Δ^+ has two components, each isomorphic to Y_Δ, and these components are interchanged by the antipodal map on the sphere $S^\Delta = S^{\ell-1}$, and one component is the image of $\mathbb{R}^r - B_\Delta$ under map (2.5).

We state our main result on the orientability of Y_Δ^+.

Theorem 2.6. *Suppose that the lattice affinely spanned by* $\Delta \cap \mathbb{Z}^n$ *has odd index in* $\mathbb{Z}^n$ *and that* $\Lambda_{\mathcal{A}}$ *has odd index in its saturation. If there is a vector* v *in the integer column span of* $[\mathcal{A} : b]$, *all of whose components are odd, then the standard orientation of* $\mathbb{R}^r$ *induces an orientation on the smooth part of* Y_Δ^+ *via the map* $\psi_\Delta : \mathbb{R}^r - B_\Delta \to Y_\Delta^+$.

Remark 2.7. If there is a vector v in the integer column span of $\mathcal{A}$, all of whose components are odd, then the orientation of $\mathbb{R}^r$ induces an orientation on the smooth part of Y_Δ. The proof of this statement is analogous to the proof of Theorem 2.6.

Remark 2.8. In general, we may not know if either Y_Δ or Y_Δ^+ are orientable. The positive part $Y_\Delta^>$ of Y_Δ is the intersection of Y_Δ with the positive orthant of $\mathbb{P}^\Delta$ is always orientable, as it is isomorphic to Δ, as a manifold with corners [6, §4].

When the hypotheses of Theorem 2.6 are satisfied, we assume that the smooth points of Y_Δ^+ have the orientation induced by ψ_Δ, and we say that Y_Δ^+ is *Cox-oriented*. If Δ is the hexagon of Example 1.2, then Y_Δ^+ is Cox-oriented as it is smooth and the vector with all components 1 is the sum of the three columns of the 6×3-matrix $[\mathcal{A} : b]$.

Proof of Theorem 2.6. Recall that the subgroup $K \subset G_\Delta(\mathbb{R})$ is

$$K := \gamma^{-1}(\mathbb{R}_>) = \{\mu \in G_\Delta(\mathbb{R}) | \mu^b > 0\}.$$

We claim that if $\mu \in K$, then $\det(\mu) = \mu_1\mu_2 \cdots \mu_r > 0$, so that K preserves the standard orientation on $\mathbb{R}^r$. Indeed, let $c = (c_1, \ldots, c_r)$ be an integer vector with each component c_i odd such that $c - kb \in \Lambda_{\mathcal{A}}$ for some $k \in \mathbb{Z}$. Then $\mu^c = (\mu^b)^k > 0$, and so we have $\det \mu > 0$, as each component of c is odd (for then $\mu^c / \det(\mu)$ is a square).

Thus if $U \subset \mathbb{R}^r$ is an open subset with $K \cdot U = U$ such that every orbit of K is closed in U, then the smooth part of the quotient U/K has an orientation induced by the standard orientation of $\mathbb{R}^r$.

For each face F of the polytope Δ, let $\tau_F \subset \mathbb{R}^n$ be the cone generated by the primitive inward-pointing normal vectors to the facets containing F—these generators are the rows of $\mathcal{A}$ corresponding to the facets containing F. Set $U_F \subset \mathbb{C}^r$ to be the complement of the variety defined by the monomial ideal $\langle z(m) | m \in F \rangle$. This is the set of points $(z_1, \ldots, z_r) \in \mathbb{C}^r$ such that $z_i \neq 0$ if the ith facet of Δ does not contain F. We have $G_\Delta \cdot U_F = U_F$.

If the cone τ_F is simplicial, then every G_Δ-orbit of U_F is closed. The arguments that show this in the proof of Theorem 2.1 of [1] show that the same is true of the K-orbits of $U_F(\mathbb{R})$. Furthermore, U_F/G_Δ and also $U_F(\mathbb{R})/K$ is smooth if the generators of τ_F in addition generate a saturated sublattice of $\mathbb{Z}^n$.

If F is a facet, then τ_F is just a ray generated by a primitive vector, and is thus simplicial. If we let F run over the facets of Δ, the quotients $U_F(\mathbb{R})/K$ are glued together along the common subtorus, which is $U_\Delta(\mathbb{R})$. As each piece and the torus is oriented by the canonical orientation of $\mathbb{R}^r$ under the quotient by K, this union W is a smooth and oriented subset of Y_Δ^+. Moreover, the difference $\overline{W} - W$ has codimension 2 (this is part of the proof that toric varieties are smooth in codimension 1).

Thus the image of $\mathbb{R}^r - B_\Delta$ in Y_Δ^+ is smooth and oriented in codimension 1. This either is dense in Y_Δ^+ or in one of the two isomorphic components of Y_Δ^+. This completes our proof of the theorem. □

3. Computation of the characteristic

Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope with a regular triangulation Δ_ω defined by a lifting function $\omega : \Delta \cap \mathbb{Z}^n \to \mathbb{Z}_{\geqslant 0}$, and $Y_\Delta \subset \mathbb{P}^\Delta$ the real toric variety parametrized by the monomials in Δ. Assume that ω is convex, which means that all integer points of Δ are vertices of Δ_ω. Call simplices with odd normalized volume *odd* and simplices with even normalized volume *even*.

Definition 3.1. A triangulation Δ_ω is *balanced* if its vertex-edge graph is $(n+1)$-colorable. This means that there exists a map κ from the integer points of $\Delta \cap \mathbb{Z}^n$ to the vertices of the standard simplex which is a bijection on each simplex in the triangulation Δ_ω. We call this map κ a *folding* of Δ_ω.

A triangulation is balanced if and only if its dual graph is bipartite. For the direct implication, note that an orientation of the standard simplex induces orientations of the simplices in the triangulation Δ_w via the map κ. This induced orientation changes when passing to an adjacent simplex. The other implication is [10, Corollary 11].

Definition 3.2. For a balanced triangulation Δ_ω, assign $+$ or $-$ to each of the simplices so that every two adjacent simplices have opposite signs. Disregard even simplices and define the *signature* $\sigma(\Delta_\omega)$ of Δ_ω to be the absolute value of the difference of the numbers of odd simplices with $+$ and odd simplices with $-$.

For each $m \in \Delta \cap \mathbb{Z}^n$, fix a non-zero real number α_m whose sign depends only upon $\kappa(m)$. Call this vector $(\alpha_m | m \in \Delta \cap \mathbb{Z}^n)$ a *weight function* for Δ. As the vertices of the standard simplex are the standard basis vectors in $\mathbb{P}^n$ and the vertices of the triangulation are the basis vectors of $\mathbb{P}^\Delta$, the folding κ defines a linear projection, called the *Wronski projection* $\pi_\alpha \colon \mathbb{P}^\Delta \to \mathbb{P}^n$ sending each basis vector e_m of $\mathbb{P}^\Delta$ to $\alpha_m e'_{\kappa(m)}$, where e'_i is a basis vector of $\mathbb{P}^n$. If α is constant, then we omit it from our system of notation as it has no effect.

A linear form Λ on $\mathbb{P}^n$ pulls back along π_α to a polynomial F of the form

$$F = \sum_m c_{\kappa(m)} \alpha_m x^m ,$$

where $c_0, c_1, \ldots, c_n \in \mathbb{R}$. We call such a polynomial a Wronski polynomial for the triangulation Δ_w and the weight function α. A *Wronski polynomial system* for the triangulation Δ_w and weight function α is a system of n such polynomials, all with

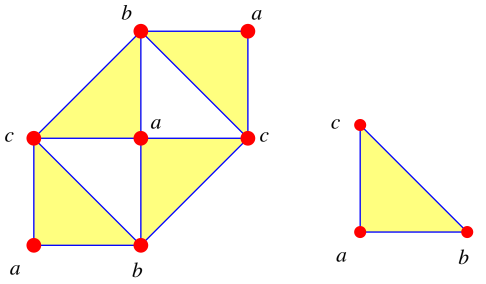


Fig. 1.

weight function α. Solutions to such a Wronski system correspond to a fiber of the Wronski map π_α.

Example 3.3. The lattice hexagon of Example 1.2 admits a regular unimodular balanced triangulation induced by the lifting function taking value 0 at its center $(1, 1)$, 3 at the vertices $(0, 0)$ and $(2, 2)$, and 1 at the remaining 4 vertices. This triangulation defines a 3-coloring of the vertices indicated by the labels a, b, c in Fig. 1. We illustrate the bipartite dual graph by shading the positive simplices. For the constant weight function, this defines a Wronski projection from $\mathbb{P}^6$ to $\mathbb{P}^2$ by

$$[x_{00}, x_{10}, x_{01}, x_{11}, x_{21}, x_{12}, x_{22}] \longmapsto [x_{00} + x_{11} + x_{22}, x_{10} + x_{12}, x_{01} + x_{21}].$$

The corresponding Wronski polynomials have the form

$$a(1 + xy + x^2y^2) + b(x + xy^2) + c(y + x^2y) = 0,$$

where a, b, and c are arbitrary real numbers.

The lifting function $\omega = (\omega_m : m \in \Delta \cap \mathbb{Z}^n)$ defines a partial term order on the coordinate ring $\mathbb{R}[x_m | m \in \Delta \cap \mathbb{Z}^n]$ of $\mathbb{P}^\Delta$ by $x^b > x^c$ if $b \cdot \omega < c \cdot \omega$, where $b \cdot \omega$ and $c \cdot \omega$ denote the standard scalar product. It also defines an action of $\mathbb{R}^\times$ on $\mathbb{P}^\Delta$ by

$$s.x_m = s^{-\omega_m} \cdot x_m. \tag{3.4}$$

The corresponding action on $\mathbb{R}[x_m | m \in \Delta \cap \mathbb{Z}^n]$ is the dual action

$$s.g(x) = g(s^{-1}.x)$$

Thus a monomial x^b is transformed into $s^{b \cdot \omega} x^b$. The monomials in the initial form $\mathrm{in}_\omega\, g$ of g are multiplied by the same power of s in $g(x)$, which is less than the power of s for the other monomials. Dividing $s.g$ by this lowest power $s^{b\cdot\omega}$ of s we see that

$$\lim_{s\to 0} s^{b\cdot\omega} s.g(x) = \mathrm{in}_\omega\, g(x).$$

Consider this for Y_Δ. The ideal $I(s.Y_\Delta)$ of $s.Y_\Delta$ is

$$I(s.Y_\Delta) = \{s.g(x) | g \in I(Y_\Delta)\}.$$

Let $\mathrm{in}(Y_\Delta)$ be the variety defined by the initial ideal $\mathrm{in}_\omega I(Y_\Delta)$. These arguments show that it is the scheme-theoretic limit of the family $s.Y_\Delta$,

$$\lim_{s\to 0} s.Y_\Delta = \mathrm{in}(Y_\Delta).$$

If we define $s.\alpha$ by $(s.\alpha)_m = s^{m\cdot\omega}\alpha_m$, then the Wronski map π_α on $s.Y_\Delta$ is equivalent to the Wronski map $\pi_{s.\alpha}$ on Y_Δ.

The family $\{s.Y_\Delta | s \in (0, 1]\} \cup \mathrm{in}(Y_\Delta)$ is a *toric degeneration* of Y_Δ. This action lifts to the sphere, giving the family $\{s.Y_\Delta^+ | s \in (0, 1]\} \cup \mathrm{in}(Y_\Delta^+)$ in which $s.Y_\Delta^+$ and $\mathrm{in}(Y_\Delta^+)$ are the subvarieties of the sphere S^Δ given by the same homogeneous equations as $s.Y_\Delta$ and $\mathrm{in}(Y_\Delta)$.

By Kushnirenko's Theorem [12] the number of complex solutions of the Wronski system is equal to the normalized volume $V(\Delta)$, which is an upper bound for the number of real solutions. When the triangulation Δ_w is unimodular, Sturmfels [22] used these toric degenerations to show that this upper bound is attained. We use a toric degeneration to compute $\mathrm{char}(f)$ which is by Proposition 1.4 a lower bound for the number of real solutions of a Wronski polynomial system.

Theorem 3.5. *Suppose that the toric degeneration of Y_Δ does not meet the center of the Wronski projection π_α and Y_Δ^+ is Cox-oriented. Then* $\mathrm{char}(f)$ *is equal to the signature $\sigma(\Delta_\omega)$ of the triangulation Δ_ω. Moreover, if $s_0 \in \mathbb{R}_>$ is minimal such that $s_0.Y_\Delta$ meets the center of projection, then* $\mathrm{char}\left(\pi_\alpha|_{s.Y_\Delta}\right) = \sigma(\Delta_\omega)$, *for any* $0 < s < s_0$.

Example 3.6. We observed that if Δ is the hexagon Example 1.2, then Y_Δ^+ is Cox-oriented. In Example 3.3, we saw that Δ has a regular unimodular balanced triangulation (illustrated in Fig. 2), which has a signature of 2. The Wronski polynomials for the family $s.Y_\Delta$ have the form

$$a(s^2 + xy + s^2x^2y^2) + bs(x + xy^2) + cs(y + x^2y) = 0, \tag{3.7}$$

where a, b, and c are arbitrary real numbers. The coefficients (in s, x, y) of a, b, and c vanish where $s.Y_\Delta$ meets the center of projection. There are no real values of x, y where these coefficients vanish for $s \neq 0$. Thus no variety $s.Y_\Delta$ in the family induced by the weight function meets the center of projection.

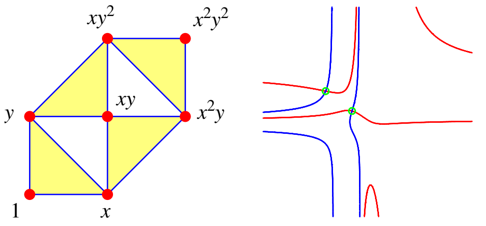


Fig. 2. Hexagonal system.

By Theorem 3.5, for any given $s \neq 0$, two general polynomial equations of form (3.7) will have at least 2 common real solutions. Fig. 2 shows the two curves given by equations of form (3.7) when $s = 1$ with coefficients (a, b, c) equal to $(3, 5, 1)$ and to $(1, -2, -3)$, which meet in the two points indicated. We computed 1 million random instances of this polynomial system with $s = 1$. Each one had exactly 2 real solutions.

These computations, like all computations reported here, were done purely symbolically. The computation procedure involved generating random polynomial systems, and then computing a univariate eliminant for each system. This eliminant has the property that its number of real solutions equals the number of real solutions to the original system. This part of the computation was done with the computer algebra system Singular [7]. For all computations, except those reported in the last paragraph, the number of real roots for the eliminant were determined using the authors' implementation of Sturm sequences in Singular. That implementation is inefficient for polynomials of degree 30, so the last computations in this paper used Maple's `realroot` routine to compute the number of real solutions.

We also computed 500,000 instances of system (3.7) for $s \in (0, 1)$. Of these, 429,916 had 2 real solutions, 70,084 had 6 real solutions, and none had 4 solutions. More precisely, $1000 \cdot s$ was an integer chosen uniformly in $[1, 999]$ and the coefficients a, b, c were chosen uniformly in $[-60, 60]$.

Given fixed weights α_m for $m \in \Delta \cap \mathbb{Z}^2$, a Wronski polynomial with these weights is

$$a(\alpha_1 + \alpha_{xy}xy + \alpha_{x^2y^2}x^2y^2) + b(\alpha_x x + \alpha_{xy^2}xy^2) + c(\alpha_y y + \alpha_{x^2y}x^2y) = 0.$$

We computed instances of such Wronski systems with 2, 4, or 6 real solutions.

Proof of Theorem 3.5. We can assume that Δ_ω has at least one odd simplex, for otherwise the lower bound is trivial. Write π for the Wronski projection π_α. It lifts to $\pi^+ : S^\Delta \to S^n$ given by the same equations as π. Let f_s^+ be the restriction of π^+ to $s.Y_{\Delta^+}$ for $s \in (0, s_0)$ and f_0^+ the restriction of π^+ to $\mathrm{in}(Y_\Delta^+)$. Since the toric degeneration of Y_Δ does not meet the center of projection π, the characteristic $\mathrm{char}(f)$ is equal to the characteristic of $f_s^+ : s.Y_\Delta^+ \to S^n$ for any $s \in (0, s_0)$.

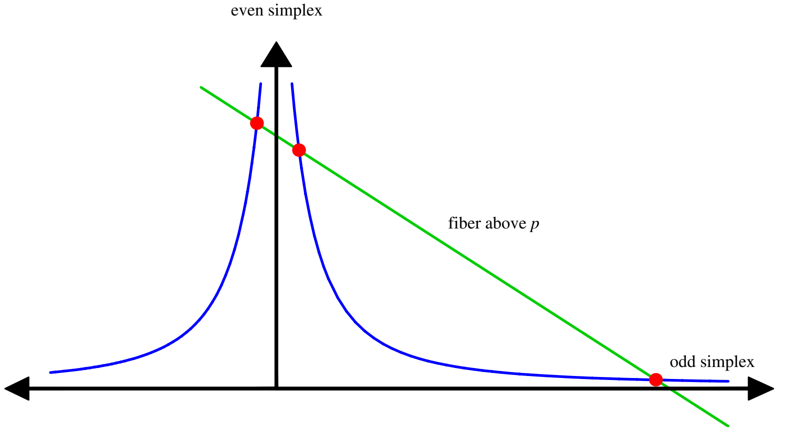


Fig. 3. Preimages near coordinate spheres.

It is proved in Chapter 8 of [23] that

$$\mathrm{Rad}(\mathrm{in}_\omega I(Y_\Delta)) = \bigcap_\tau \langle x_m \mid m \notin \tau \rangle,$$

where the intersection is taken over all simplices τ of Δ_ω. Thus $\mathrm{in}_\omega(Y_\Delta^+)$ is the union of coordinate n-planes $\mathbb{RP}^\tau$, one for each simplex τ in Δ_ω and $\mathrm{in}_\omega(Y_\Delta^+)$ is a similar union of coordinate n-spheres S^τ. Thus a point $p \in S^n$ with non-zero coordinates has one preimage a_τ under f_0^+ on each sphere S^τ. The preimages of p under f_s^+ on Y_s^+ for small s are clustered around these $\{a_\tau | \tau \in \Delta_\omega\}$. This is illustrated in Fig. 3. The preimages are the dots, the linear subspace $(f^+)^{-1}(p)$ is the line, and the toric variety Y_Δ^+ is the curve.

When s is small, consider the contribution to the characteristic of f_s to the solutions near a_τ. In a neighborhood of the point a_τ the projection f_s^+ is homotopic to the coordinate projection π_τ to S^τ and therefore we compute this local contribution using π_τ.

This is easiest when τ is an even simplex, as in that case the restriction $\pi_\tau|_{Y_\Delta^+}$ is not surjective and therefore this contribution to the characteristic of f_s^+ is zero. To see this, it is best to consider this projection in $\mathbb{RP}^\Delta$. The composition

$$(\mathbb{R}^\times)^n \xrightarrow{\varphi_\Delta} \mathbb{RP}^\Delta \xrightarrow{\pi_\tau} \mathbb{RP}^\tau \simeq \mathbb{RP}^n$$

is the parametrization φ_τ of $\mathbb{RP}^\tau$ by the monomials corresponding to integer points of τ. Since the affine span of the lattice points in Δ has odd index in $\mathbb{Z}^n$, the map φ_Δ is an isomorphism between $(\mathbb{R}^\times)^n$ and the dense torus in Y_Δ. Thus the restriction $\pi_\tau|_{Y_\Delta}$ is surjective if and only if φ_τ maps $(\mathbb{R}^\times)^n$ onto the dense torus in $\mathbb{RP}^n$. But this is not the case, as the integer points in τ span a sublattice of $\mathbb{Z}^n$ with even index. For an odd simplex τ, the map $\varphi_\tau : (\mathbb{R}^\times)^n \to (\mathbb{R}^\times)^n$ is an isomorphism and therefore the degree of π_τ is 1.

Pick a point $p = (p_0, \ldots, p_n)$ in S^n such that $\operatorname{sign} p_i = -\operatorname{sign} \kappa(m)$ whenever $\kappa(m) = i$ where κ is the folding of Δ_ω. Then for each odd simplex there exists a unique preimage of p under π_τ and all of its components are positive. For an even simplex, there is an even number of preimages with one of them having all components positive.

Orient each of the coordinate spheres S^τ pulling back the orientation of S^n along π. Each of these orientations induces an orientation of the positive part of Y_Δ^+. It remains to compare these induced orientations.

Consider two adjacent simplices in Δ_ω. Let the vertices of the common facet be indexed by the variables $x_1, \ldots, x_n$, and the remaining two vertices by x_0 and x_{n+1}. Then $x_0^{a_0} x_{n+1}^{a_{n+1}} = x_1^{a_1} \cdots x_n^{a_n}$ for some integers $a_0, \ldots, a_{n+1}$ with a_0 and a_{n+1} positive. Projections to the coordinate spheres of each simplex give local coordinate charts for Y_Δ^+, namely $x_0, x_1, \ldots, x_n$ and $x_{n+1}, x_1, \ldots, x_n$. The Jacobian matric for this change of coordinates has the form

$$\begin{bmatrix} \dfrac{\partial x_{n+1}}{\partial x_0} & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{bmatrix}, \quad \text{where } \frac{\partial x_{n+1}}{\partial x_0} = -\frac{a_0}{a_{n+1}} \frac{x_{n+1}}{x_0}.$$

Since the Jacobian determinant is negative, these two charts belong to different orienting atlases, and we need to count the corresponding preimages with opposite signs. Therefore, $\operatorname{char}(f)$ is equal to the signature $\sigma(\Delta_\omega)$. $\square$

Remark 3.8. Notice that this computation does not depend on the choice of orientation of different connected components of the smooth part of Y_Δ^+. This implies that $\operatorname{char}(f) = 0$ whenever the smooth part of Y_Δ^+ is not connected. In particular, $\operatorname{char}(f) = 0$ if Y_Δ^+ is isomorphic to two copies of Y_Δ. We have noted before that if Y_Δ is orientable but $\mathbb{RP}^n$ is not then $\operatorname{char}(f) = 0$ if the orientation on Y_Δ^+ is pulled back from Y_Δ. We have proved that this last assumption is redundant: if Y_Δ is orientable but $\mathbb{RP}^n$ is not then $\operatorname{char}(f) = 0$.

Lemma 3.9. *If Δ_ω contains only odd simplices and the sign of α_m depends only upon $\kappa(m)$, then there exists a regular value in $z \in S^n$ all of whose preimages in Y_Δ^+ under f have all components positive.*

Proof. Since $f(s^{-1}.x) = s^{-1}.z$ whenever $f_s(x) = z$, the statement follows from above. $\square$

While in general we may not know if either Y_Δ or Y_Δ^+ are orientable, the topological degree $\operatorname{char}(f_>)$ of $f_> := f|_{Y_\Delta^>} : Y_\Delta^> \to f(Y_\Delta^>)$ is always well defined, as $Y_\Delta^>$ is orientible.

Corollary 3.10. *Suppose that the toric degeneration of* Y_Δ *does not meet the center of projection* π. *Then* $\mathrm{char}(f_>)$ *is equal to* $\sigma(\Delta_\omega)$.

4. Toric varieties from posets

Let P be a finite partially ordered set (poset) with n elements. We recall some definitions from the paper of Stanley [18].

Definition 4.1. *The order polytope* $O(P)$ of a finite poset P is the set of points y in the unit cube $[0, 1]^P$ such that $y_a \leqslant y_b$ whenever $a \leqslant b$ in P.

The vertices of the order polytope are the characteristic functions of (upper) order ideals of P. Let $\mathcal{J}(P)$ be the set of such order ideals of P. The *canonical triangulation* of the order polytope $O(P)$ is defined by the linear extensions (order-preserving bijections) of the poset P. Suppose that P has n elements and let $\lambda\colon P \to [n]$ be a linear extension of P. For each $k = 1, \ldots, n$, let a_k be the element of P such that $\lambda(a_k) = k$. Then λ defines an n-dimensional simplex $\tau_\lambda \subset O(P)$ consisting of all y satisfying

$$0 \leqslant y_{a_1} \leqslant \cdots \leqslant y_{a_n} \leqslant 1.$$

The τ_λ are the simplices in a unimodular triangulation of $O(P)$. It is balanced as the association of an order ideal to its number of elements is a proper coloring of its vertex-edge graph. We will show in Lemma 4.6 that this triangulation is regular.

Fixing one linear extension of P identifies each linear extension of P with a permutation of P, where the fixed extension is identified with the identity permutation. The sign of a linear extension is the sign of the corresponding permutation.

Definition 4.2. The *sign imbalance* $\sigma(P)$ of a poset P is the absolute value of the difference between the numbers of the positive and negative linear extensions of P. If $\sigma(P) = 0$ we say that P is sign-balanced. Stanley studied this notion of sign-balanced posets [20].

For an order ideal J, let $t^J := \prod_{a\in J} t_a$ be the monomial in $\mathbb{R}[t_a | a \in P]$ whose exponent vector is the vertex of $O(P)$ corresponding to the order ideal J. Let $|J|$ be the number of elements in the order ideal J. Fix a system of weights $\{\alpha_J \in \mathbb{R}^\times | J \in \mathcal{J}(P)\}$. This gives the Wronski projection π_α, Wronski polynomials, and Wronski polynomial systems as in Section 3. Wronski polynomials for π_α have the form

$$\sum_{J\in\mathcal{J}(P)} c_{|J|}\alpha_J t^J,$$

where $c_0, \ldots, c_{|P|} \in \mathbb{R}$.

Theorem 4.3. *Suppose that a finite poset* P *is ranked* mod 2. *For any choice* α *of weights, a Wronski polynomial system for the canonical triangulation of the order polytope of* P *with weight* α *has at least* $\sigma(P)$ *real solutions.*

The set $\mathcal{J}(P)$ of order ideals, ordered by inclusion, forms a ranked distributive lattice $\mathcal{J}(P)$. Equations for the toric variety $Y_{O(P)}$ parametrized by the monomials in the order polytope are nicely described by this lattice. The lattice operations are $J \vee K = J \cap K$ and $J \wedge K = J \cup K$. Its ideal I is the Hibi ideal of this lattice [8]

$$I = \langle x_J x_K - x_{J\wedge K} x_{J\vee K} \,|\, J, K \in \mathcal{J}(P) \text{ are incomparable}\rangle. \tag{4.4}$$

The geometry of toric varieties associated to distributive lattices is discussed in [24]. Maximal chains of $\mathcal{J}(P)$ are the linear extensions of P. If two maximal chains differ by one element, they have opposite signs. Then $\sigma(P)$ is the absolute value of the difference between the number of the positive maximal chains and the number of negative maximal chains. We also call $\sigma(P)$ the sign-imbalance of the lattice $\mathcal{J}(P)$.

Example 4.5. The toric variety $\mathbb{P}^2 \times \mathbb{P}^2$ is defined by the order polytope of the poset with the Hasse diagram

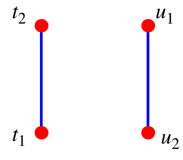
$P =$ 

Fig. 4 shows its lattice $\mathcal{J}(P)$ of order ideals and the six maximal chains in $\mathcal{J}(P)$. The corresponding signs of these maximal chains are $+, -, +, +, -, +$, and so the sign imbalance $\sigma(P)$ is 2. By Theorem 4.3 a generic system of 4 real equations of the form

$$c_0 + c_1(t_2 + u_2) + c_2(t_1t_2 + t_2u_2 + u_1u_2) + c_3(t_1t_2u_2 + t_2u_1u_2) + c_4t_1t_2u_1u_2 = 0,$$

has at least 2 real solutions. (For simplicity, the weights are constant, $\alpha_J = 1$.)

The following four lemmas reduce Theorem 4.3 to Theorem 3.5.

Lemma 4.6. *The canonical triangulation of the order polytope $O(P)$ is regular.*

Proof. Define a lifting function $w(J)$ for each order ideal $J \in \mathcal{J}(P)$ by

$$\omega(J) := -|J|^2.$$

For each simplex τ in the canonical triangulation we give a linear function Λ on $\mathbb{R}^P$ such that $\Lambda(m)+\omega(m)\geqslant 0$ for all vertices of $O(P)$, with equality if and only if $m \in \tau$.

For a linear extension λ, the vertices of the simplex τ_λ are

$$m_0 := (0, \ldots, 0), \quad \text{and} \quad m_k := \sum_{i=1}^{k} e_{\lambda^{-1}(n-i+1)} \quad \text{for } k = 1, \ldots, n,$$

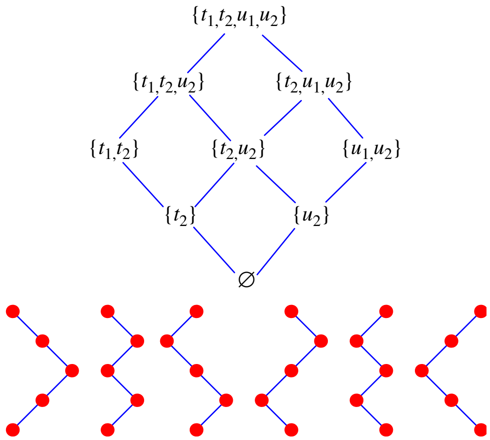


Fig. 4. The distributive lattice $\mathcal{J}(P)$ and its maximal chains.

where $\{e_a | a \in P\}$ is the standard basis for $\mathbb{R}^P$. The value of the lifting function at the vertex m_k is $-k^2$. Define the linear function Λ on $\mathbb{R}^P$ by $\Lambda(e_{\lambda^{-1}(n-k+1)}) = 2k-1$. Then Λ is the unique linear function on $\mathbb{R}^n$ such that $\Lambda(m_k) + \omega(m_k) = 0$. Indeed,

$$\Lambda(m_k) = \sum_{i=1}^{k} \Lambda(e_{\lambda^{-1}(n-i+1)}) = \sum_{i=1}^{k} 2i - 1 = k^2 = -\omega(m_k).$$

This also shows that if a vertex m of $O(P)$ corresponds to an order ideal with k elements, then $\Lambda(m) \geqslant k^2$ with equality only when $m = m_k$. Thus a vertex m of $O(P)$ does not lie in τ_λ exactly when $\Lambda(m) + \omega(m) > 0$. $\square$

Lemma 4.7. *The canonical triangulation of the order polytope $O(P)$ is balanced. Its signature is $\sigma(P)$, the sign imbalance of P.*

Proof. The folding map $m \mapsto |m|$ shows that the canonical triangulation is balanced. Linear extensions corresponding to adjacent simplices differ by a transposition and thus have opposite signs. The second statement is immediate. $\square$

Lemma 4.8. *For any choice of weights $\{\alpha_J | J \in \mathcal{J}(P)\}$, the toric degeneration of $Y_{O(P)}$ does not meet the center of the Wronski projection π_α.*

Proof. We show that on $Y_{O(P)}$ the equations defining the center of the projection π_α

$$\sum_{|J|=k} \alpha_{|J|} x_J = 0, \quad k = 0, \ldots, n,$$

generate the irrelevant ideal $\langle x_J | J \in \mathcal{J}(P)\rangle$.

Let I be the ideal of the equations for the center of projection and the equations defining X_Δ. If there is only one order ideal J with $|J| = k$, then $x_J \in I$, in particular, $x_J \in I$ when $|J| = 0$. Suppose that we have $x_J \in I$ for all J with $|J| < k$. Given two order ideals J and K of size k, we have $x_J x_K = x_{J\wedge K} x_{J\vee K} \in I$ as $|J \vee K| < k$. Together with the equation defining the center of projection, this implies that if $|J| = k$, then $x_J \in I$. By induction on k, $I = \langle x_J | J \in \mathcal{J}(P)\rangle$.

This argument also shows that $s.Y_{O(P)}$ does not meet the center of projection. □

Lemma 4.9. *If a finite poset P is ranked* mod 2, *then $Y^+_{O(P)}$ is Cox-oriented.*

Proof. Let the order polytope be defined by facet inequalities

$$O(P) = \left\{ y \in \mathbb{R}^n | \mathcal{A}y \geqslant -b \right\},$$

where $\mathcal{A}$ is an integer $r \times n$ matrix. By Theorem 2.6 it is enough to check that the vector consisting of all ones is in the mod 2 integer column span of the matrix $[\mathcal{A} : b]$ and the lattice $\Lambda_{\mathcal{A}}$ spanned by the columns of $\mathcal{A}$ is saturated. The integral points of $O(P)$ affinely span $\mathbb{Z}^P$ as $O(P)$ has a unimodular triangulation.

Each facet of the order polytope $O(P)$ is defined by one of the following conditions:

$$\begin{array}{ll} y_a = 0 & \text{for a minimal } a \in P, \\ y_b = 1 & \text{for a maximal } b \in P, \\ y_a = y_b & \text{for } a \text{ covering } b \text{ in } P. \end{array}$$

Fix a maximal chain $a_1 < \cdots < a_k$ in P. The corresponding facets of $O(P)$ are

$$y_{a_1} = 0, \quad y_{a_2} - y_{a_1} = 0, \ldots, \quad y_{a_k} - y_{a_{k-1}} = 0, \quad y_{a_k} = 1,$$

and the corresponding rows of the matrix $[\mathcal{A} : b]$ are

$$\left[\begin{array}{rrrcrrrcr|r} 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 \end{array}\right].$$

The columns of $\mathcal{A}$ are indexed by the elements of P. Consider the linear combination of the columns of $\mathcal{A}$ where the coefficient of a column corresponding to an element a of P is $1 - \mathrm{rk}(a)$, where $\mathrm{rk}(a)$ is its mod 2 rank. This will be a vector with all components odd if P has mod 2 rank 0. If P has mod 2 rank 1, then adding the vector b to this combination gives a vector with all components odd. (Here, a minimal element has rank 0.)

Consider the submatrix of $\mathcal{A}$ consisting of its rows corresponding to minimal elements of P, together with one row for each non-minimal element a of P corresponding to some cover $a' \lessdot a$. This submatrix has determinant ± 1, which implies that column space of the matrix $\mathcal{A}$ is saturated. $\square$

We noted earlier that $\mathrm{char}(f) = 0$ whenever Y_Δ is orientable but $\mathbb{RP}^n$ is not. By Remark 2.7 Y_Δ is orientable if there exists a vector all of whose components are odd in the integer column span of $\mathcal{A}$. For posets, this translates to: $Y_{O(P)}$ is orientable if all the maximal chains of P are odd. We obtain:

Corollary 4.10. *If all the maximal chains of a finite poset P are odd but the number of elements in P is even then the poset P is sign-balanced.*

This is Corollary 2.2 of [20], where it is given a purely combinatorial proof.

5. Further examples

This theory applies to other toric varieties besides those associated to the order polytopes of finite posets. The hexagon of Example 3.6 is one instance. We present three additional instances based on particular triangulations of polytopes, and one infinite family that is based on the chain polytopes of [18].

5.1. Three examples of polytopes

Example 5.1. Let Δ be the convex hull of the points $(0, 0)$, $(0, 3)$, and $(3, 0)$, a triangle. Then T_Δ is a Veronese embedding of $\mathbb{RP}^2$ and Y_Δ^+ is the 2-sphere, and so it is orientable. The triangle has a regular unimodular balanced triangulation with signature 3 illustrated in Fig. 5 below. This is induced by a weight function whose values are 0 at the center, 3 at each of the three vertices, and 1 at the remaining six points. A Wronski polynomial with constant weight 1 on members of the family $s.Y_\Delta$ has the form

$$a(s^3 + xy + s^3x^3 + s^3y^3) + bs(x + x^2y + y^2) + cs(y + xy^2 + x^2) = 0. \tag{5.2}$$

The center of projection does not meet $s.Y_\Delta$, for any $s \neq 0$. Thus any two polynomial equations of form (5.2) will have at least 3 common real solutions. Fig. 5 also shows two curves given by equations of form (5.2) with $s = 1$ and coefficients $(4, -11, 4)$ and $(-13, -1, 24)$. These meet in 9 points, giving 9 solutions to the system. We computed 10 million instances of the Wronski system on Y_Δ, where the coefficients a, b, c were integers chosen uniformly from the interval $[-1000, 1000]$. Of these, 9, 976, 701 (99.8%) had 3 real solutions and 23, 299 had 9 real solutions. Computing 500, 000 instances of the Wronski system (5.2) with $s \in (0, 1)$, we found 414, 592 with 3 real solutions, 85, 408 with 9 real solutions, and did not find any with either 5 or 7 real solutions.

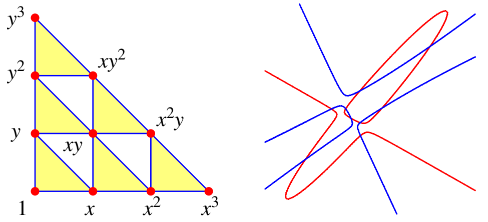


Fig. 5. Cubic system.

Example 5.3. Let Δ be the unit cube. Then $Y_\Delta = (S^1)^3$ and so it is oriented. Consider the regular unimodular balanced triangulation of the unit cube $[0, 1]^3$ illustrated on the left in Fig. 6. It has signature $4 - 2 = 2$ and is given by a weight function taking values 3 at $(0, 0, 0)$ and $(1, 1, 1)$, 0 at $(1, 0, 0)$ and $(0, 1, 1)$, and 1 at the remaining vertices. The corresponding Wronski polynomials on $s.Y_\Delta$ have the form

$$a(s^3 + yz) + b(x + s^3xyz) + cs(y + xz) + ds(z + xy).$$

The family $s.Y_\Delta$ meets the center of projection only when $s^3 = \pm 1$. These points for $s = 1$ are

$$(x, y, z) \in \{(1, 1, -1), (1, -1, 1), (1, i, i), (1, -i, -i)\}.$$

Thus Y_Δ meets the center of projection in 2 real and 2 complex points. Theorem 3.5 implies that for $s \in (0, 1)$, there will be at least 2 real solutions, and we have computed such systems with 2, 4, and 6 real solutions.

Example 5.4. On the right of Fig. 6 is the regular triangulation of the cube given by the lifting function that takes values 0 at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$, and 1 at the remaining four vertices. It is balanced with 4 unimodular simplices of the same color and one with normalized volume 2 of the opposite color, and thus has signature 4. The corresponding Wronski polynomials on $s.Y_\Delta$ have the form

$$a(1 + sxyz) + b(sx + yz) + c(sy + xz) + d(sz + xy).$$

The variety $s.Y_\Delta$ meets the center of projection only when $s^4 = 1$. When $s = 1$, there are four points of intersection

$$\{(x, y, z) | x, y, z \in \{\pm 1\}, xyz = -1\}.$$

Since the sign imbalance is 4, Theorem 3.5 implies that for $s \in (0, 1)$, three polynomials of this form will have at least 4 real solutions. Computing 500,000 instances, we found 453,811 with 4 solutions and 46,189 with 6 solutions.

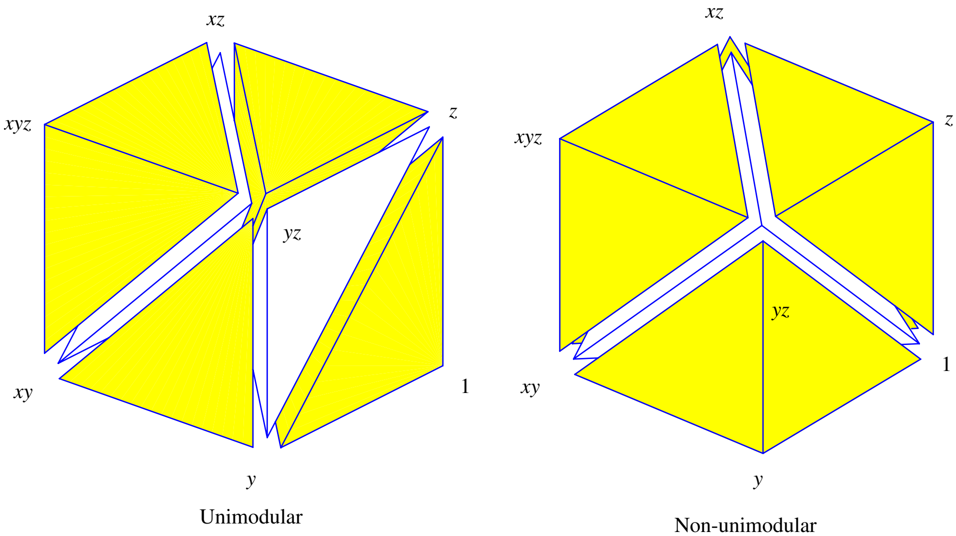


Fig. 6. Triangulations of cubes.

5.2. *Systems from chain polytopes*

Let P be a poset with n elements. Stanley [18] defined the *chain polytope* $C(P)$ to be the set of points y in the unit cube $[0,1]^P$ such that

$$y_a + y_b + \cdots + y_c \leqslant 1 \quad \text{whenever } a < b < \cdots < c \text{ is a chain in } P.$$

This polytope is intimately related to the order polytope $O(P)$ of Section 4. It has no interior lattice points but its vertices are the characteristic functions of the antichains of P, and the bijection between (upper) order ideals J and antichains A given by

$$\begin{aligned} J &\longmapsto \text{minimal elements in } J \\ A &\longmapsto \langle A\rangle := \{b \in P \mid a \leqslant b \text{ for some } a \in A\} \end{aligned}$$

extends to a bijection φ between the polytopes. Let $y \in O(P)$ be a point in $[0,1]^P$ with $y_a \leqslant y_b$ whenever $a \leqslant b$ in P. For $a \in P$, define

$$\varphi(y)_a = \min\{y_a - y_b \mid a \text{ covers } b \text{ in } P\}. \tag{5.5}$$

This is piecewise linear on the simplices of the canonical triangulation of $O(P)$ and it extends the bijection given above. This induces the *canonical triangulation* of the chain polytope, which is unimodular, balanced, and has the same signature as the canonical triangulation of the order polytope. It is regular, by Lemma 5.9.

Let $\mathcal{A}(P)$ denote the set of antichains of P. Let $\mathrm{rk}(A)$ be the number of elements in the (upper) order ideal generated by the antichain A. For an antichain A, let $t^A := \prod_{a\in A} t_a$ be the monomial in $\mathbb{R}[t_p | p \in P]$ whose exponent vector is the vertex of $C(P)$ corresponding to the antichain A. Fix a system of weights $\{\alpha_A \in \mathbb{R}^\times | A \in \mathcal{A}(P)\}$. Given a coefficient vector $c = (c_0, c_1, \ldots, c_n) \in (\mathbb{R}^\times)^{n+1}$, set

$$F_c(t) := \sum_{A\in\mathcal{A}(P)} c_{\mathrm{rk}(A)} \alpha_A t^A. \tag{5.6}$$

A system of n such polynomials for a fixed choice α of weights is a Wronski polynomial system for the canonical triangulation of the chain polytope of P with weight α.

Theorem 5.7. *Suppose that a finite poset P is ranked* mod 2. *For any choice of weights, a Wronski polynomial system for the canonical triangulation of the chain polytope of P with weight α will have at least $\sigma(P)$ real solutions.*

In Section 7, we consider such systems when P is a incomparable union of chains.

Example 5.8. Despite the similarities between our results for the chain and order polytopes, the polytopes $O(P)$ and $C(P)$ are not isomorphic, in general. For example, if B_n is the boolean poset $\{0, 1\}^n$, then B_n has 2^n elements and $n!$ maximal chains. It also has unique maximal and minimal elements and exactly $4 \cdot 3^{n-2}$ covers. Thus, for $n \geqslant 2$,

$$O(B_n) \text{ has } 2{+}4{\cdot}3^{n-2} \text{ facets, while } C(B_n) \text{ has } 2^n{+}n! \text{ facets.}$$

In particular, $O(B_4)$ has 38 facets while $C(B_4)$ has 40. When $n = 10$, these numbers of facets are 26, 246 and 3, 629, 824, respectively.

The following four lemmas reduce Theorem 5.7 to Theorem 3.5.

Lemma 5.9. *The canonical triangulation of the chain polytope is regular.*

We defer the proof until the end of this section. There, we will show that the lifting function $\omega(A) := 3^{\mathrm{rk}(A)}$ induces the canonical triangulation of the chain polytope.

Lemma 5.10. *The canonical triangulation of the chain polytope $C(P)$ is balanced. Its signature is $\sigma(P)$, the sign imbalance of P.*

Proof. The map φ between simplices in the canonical triangulations of the chain and order polytopes shows that the two triangulations are combinatorially equivalent. The statement then follows from Lemma 4.7. $\square$

The lifting function ω which induces the canonical triangulation of $C(P)$ give a $\mathbb{R}^\times$-action on the projective space $\mathbb{RP}^{C(P)}$ in which $Y_{C(P)}$ lives. Let $s.Y_{C(P)}$ be the associated toric deformation of $Y_{C(P)}$.

Lemma 5.11. *For any choice of weights $\{\alpha_A | A \in \mathcal{A}(P)\}$, the toric degeneration of $Y_{C(P)}$ induced by the lifting function $\omega(A) := 3^{\mathrm{rk}(A)}$ does not meet the center of the projection π_α defining the Wronski polynomial system.*

Proof. The center of the projection π_α is defined by

$$\sum_{\mathrm{rk}(A)=k} \alpha_A t^A = 0 \quad \text{for } k = 0, 1, \ldots, n = |P|.$$

As in the proof of Lemma 4.8, it suffices to show that if A and B are two antichains with the same rank $k > 0$, then there is a relation in the ideal of $Y_{C(P)}$ of the form

$$x_A x_B - x_C x_D, \tag{5.12}$$

where C, D are antichains with $\mathrm{rk}(C) < k$.

Let A and B be two antichains, each of rank k. Let D be the antichain of minimal elements in $\langle A\rangle \cup \langle B\rangle$. Then $D \subset A \cup B$. Set $C := [(A \cup B) - D] \cup (A \cap B)$. This is a subset (possibly proper) of the minimal elements of $\langle A\rangle \cap \langle B\rangle$. Since $A \neq B$, we have $|\langle A\rangle \cap \langle B\rangle| < k$, and so C is an antichain with rank less than k. Finally, the multiset equality $A \cup B = C \cup D$ implies relation (5.12), which completes the proof. $\square$

Remark 5.13. The sets C and D may be constructed from any two incomparable antichains A and B of P. From the construction, we have $\mathrm{rk}(A)+\mathrm{rk}(B) \geqslant \mathrm{rk}(C)+\mathrm{rk}(D)$, with equality only when C is the set of minimal elements of $\langle A\rangle \cap \langle B\rangle$.

Despite the similarity of the equations for the order polytope (4.4) to those in the proof above (5.12) for the chain polytope, the equations for the chain polytope do not come from a distributive lattice.

Lemma 5.14. *If a finite poset P is ranked* mod 2, *then $Y^+_{C(P)}$ is Cox-oriented.*

Proof. The chain polytope contains the standard basis of $\mathbb{R}^P$ as some of its vertices and it contains the origin. Thus its integral points affinely span $\mathbb{Z}^P$. The facet inequalities $\mathcal{A}x \geqslant b$ for $C(P)$ come in two forms

$$\begin{aligned} f(a) &\geqslant 0 \quad \text{for } a \in P, \text{ and} \\ f(a) + f(b) + \cdots + f(c) &\leqslant 1 \quad \text{for } a < b < \cdots < c \text{ a maximal chain in } P. \end{aligned}$$

The first collection of inequalities ensures that the columns of the matrix $\mathcal{A}$ span an n-dimensional saturated sublattice of $\mathbb{Z}^{n+c}$, where c is the number of maximal chains. If P has mod 2 rank 0, then the sum of the columns of $\mathcal{A}$ is a vector in Z^{n+c} with every component odd. If P has mod 2 rank 1, then we add b to the sum of columns of A gives a vector in Z^{n+c} with every component odd.

The lemma follows by Theorem 2.6. $\square$

Proof of Lemma 5.9. For an antichain A of P generating an order ideal with k elements, set $\omega(A) := 3^k - 1$. We show that ω induces the canonical triangulation.

Suppose that P has n elements and fix a linear extension $\pi: P \to [n]$. Let Δ_π be the simplex in the canonical triangulation corresponding to this linear extension. For each $k = 0, \ldots, n$, let I_k be the order ideal $\pi^{-1}\{n+1-k, \ldots, n-1, n\}$ and let A_k be the antichain of minimal elements of I_k. Then the vertices of Δ_π are $m_k = \sum_{x \in A_k} e_x$, for $k = 0, \ldots, n$, where $\{e_x | x \in P\}$ is the standard basis for $\mathbb{R}^P$.

Let Λ be the unique linear function satisfying $\Lambda(m_k) + \omega(m_k) = 0$. We will show that if m is a vertex of the chain polytope but not of Δ_π, then $\Lambda(m) + \omega(m) < 0$, which will prove the proposition. This requires a more precise description of Λ. Write $p \lessdot q$ if q covers p in P. For $x, y \in P$, define the function $\beta_{x,y} \in \{0, 1\}$ recursively as follows:

(1) $\beta_{x,y} = 0$ if $x \not\leqslant y$,
(2) $\beta_{y,y} = 1$ and
(3) $\beta_{x,y} = \sum\{\beta_{z,y} | x \lessdot z$ and $z \in A_{\pi(x)+1}\}$.

These functions $\beta_{x,y}$ have the following elementary and obvious properties:

Lemma 5.15. *Let $\pi: P \to [n]$ be a linear extension and define α as above. Then*

(1) *For any $y \in P$ and antichain A, $\sum_{x \in A} \beta_{x,y} \leqslant 1$.*
(2) *If $\pi(y) \geqslant j$, then $1 = \sum_{x \in A_j} \beta_{x,y}$.*

Set $f(k) := 3^{k-1} - 3^k$ and note that if

$$\Lambda(e_x) := \sum_y \beta_{x,y} f(n - \pi(y) + 1),$$

then $\Lambda(m_k) = 1 - 3^k$. Indeed,

$$\begin{aligned}
\Lambda(m_k) &= \sum_{x \in A_k} \Lambda(e_x) = \sum_{x \in A_k} \sum_y \beta_{x,y} f(n - \pi(y) + 1) \\
&= \sum_{y \in I_k} f(n - \pi(y) + 1) \sum_{x \in A_k} \beta_{x,y} \\
&= \sum_{y \in I_k} f(n - \pi(y) + 1) = \sum_{i=0}^{k} f(k) = 1 - 3^k.
\end{aligned}$$

Suppose now that m is a vertex of $C(P)$, but $m \notin \Delta_\pi$. Let A be the antichain corresponding to m and let z be the least element of A under the linear extension π. Set $k := n + 1 - \pi(z)$. Then $A \subset I_k$ but A does not generate I_k (for otherwise $m = m_k \in \Delta_\pi$), and so $\omega(m) \leqslant 3^{k-1} - 1$, as the order ideal generated by A has at most

$k-1$ elements. However, e_z occurs in m, so $\Lambda(m) \leqslant f(n+1-\pi(z)) = f(k) = 3^{k-1}-3^k$. But then $\Lambda(m)+\omega(m) < 0$, as claimed. $\square$

6. Lower bounds from sagbi degenerations

The Grassmannian has a flat deformation to the toric variety of an order polytope induced by a canonical subalgebra or sagbi basis of its coordinate ring. We use this sagbi deformation to compute the characteristic of the real Wronski map and recover the result of Eremenko and Gabrielov [5] which motivated our work. More generally, we compute the characteristic of Wronski projections for many projective varieties whose coordinate rings are algebras with a straightening law on a distributive lattice.

We review some definitions from [5]. Let $f_1(z), \ldots, f_p(z)$ be real polynomials in one variable z, each of degree at most $m+p-1$. Their Wronski determinant is

$$W(f_1(z), \ldots, f_p(z)) := \begin{vmatrix} f_1(z) & \cdots & f_p(z) \\ f_1'(z) & \cdots & f_p'(z) \\ \vdots & & \vdots \\ f_1^{(p-1)}(z) & \cdots & f_p^{(p-1)}(z) \end{vmatrix}.$$

This Wronskian has degree at most mp, and, up to a scalar factor, it depends only upon the linear span of the polynomials $f_1(z), f_2(z), \ldots, f_p(z)$. If we identify a polynomial of degree $m+p-1$ as a linear form on $\mathbb{R}^{m+p}$, then p linearly independent polynomials cut out a m-plane. Thus the Wronski determinant induces the *Wronski map*

$$W : G(m, p) \longrightarrow \mathbb{P}^{mp},$$

where $G(m, p)$ is the Grassmannian of m-planes in $\mathbb{R}^{m+p}$, and $\mathbb{P}^{mp}$ is the space of polynomials of degree mp, modulo scalars.

Consider this in more detail. Represent a polynomial $f(z)$ by the column vector f of its coefficients. Set

$$\gamma := [1, z, z^2, \ldots, z^{m+p-1}]$$

and define $K(z)$ to be the matrix with rows $\gamma(z), \gamma'(z), \ldots, \gamma^{(m-1)}(z)$. Then

$$\begin{bmatrix} f_1(z) & \cdots & f_p(z) \\ \vdots & \ddots & \vdots \\ f_1^{(m-1)}(z) & \cdots & f_p^{(m-1)}(z) \end{bmatrix} = K(z) \cdot [f_1, \ldots, f_p].$$

Expanding the determinant using the Cauchy–Binet formula gives

$$W(f_1(z), \ldots, f_p(z)) = \sum_J k_J(z) x_J(f_1, \ldots, f_p),$$

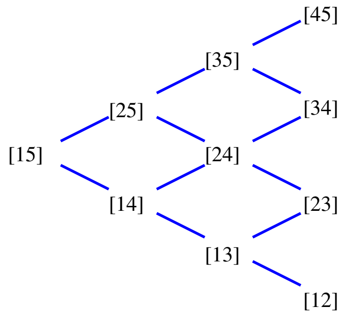


Fig. 7. The distributive lattice $\mathcal{C}_{2,3}$.

where the summation is over all sequences $J : 1 \leqslant j_1 < \cdots < j_p \leqslant m+p$, $x_J(f_1, \ldots, f_p)$ is the determinant of the $p \times p$ submatrix of $[f_1, \ldots, f_p]$ formed by the rows in J, and $(-1)^{mp-|J|}k_J(z)$ is the determinant of the complementary rows of $K(z)$. These functions $x_J(f_1, \ldots, f_p)$ are the *Plücker coordinates* of the m-plane cut out by $f_1, \ldots, f_p$. They define a projective embedding of $G(m, p)$ into Plücker space, $\mathbb{P}^N$, where $N = \binom{m+p}{m} - 1$.

If $|J| := \sum j_i - i$, then $k_J(z) = z^{mp-|J|}k_J(1)$. Moreover, $(-1)^{mp-|J|}k_J(1) > 0$ for all J [16, (Eq. (5.5))].[3] If we write $\alpha_J := k_J(1)$, then, in the Plücker coordinates for $G(m, p)$ and the basis of coefficients for polynomials in $\mathbb{P}^{mp}$, the Wronski map is

$$W(x_J | J \in \mathcal{C}_{m,p}) = \sum_{j=0}^{mp} z^{mp-j} \sum_{|J|=j} \alpha_J x_J,$$

where $\mathcal{C}_{m,p}$ is the set of indices of Plücker coordinates. We recognize this as the restriction of a linear projection π_α on $\mathbb{P}^{mp}$ to the Grassmannian $G(m, p)$.

This is a Wronski projection, in the sense of Section 3. Indeed, indices $\mathcal{C}_{m,p}$ of Plücker coordinates are partially ordered by componentwise comparison

$$J = [j_1, \ldots, j_p] \leqslant K = [k_1, \ldots, k_p] \iff j_i \leqslant k_i \quad \text{for } i = 1, \ldots, p.$$

This poset $\mathcal{C}_{m,p}$ is the lattice of order ideals of the poset $[m] \times [p]$ of two chains of lengths m and p and the rank of J is $|J|$. Fig. 7 shows $\mathcal{C}_{3,2}$.

Following [5] we define the characteristic of the Wronski map W. This map sends the subset X of $G(m, p)$ where $x_{1,2,\ldots,p} \neq 0$ to the subset Y of $\mathbb{P}^{mp}$ of monic polynomials of degree mp, and the complement of X to the complement of Y. Both X and Y are orientable as they are identified with $\mathbb{R}^{mp}$. Let $\mathrm{char}(W)$ be the absolute value of the topological degree of $W|_X$.

An equivalent definition of $\mathrm{char}(W)$ similar to Definition 1.3 also appears in [5]. Lift the Grassmannian and the projection to the double covers S^N and S^{mp} of $\mathbb{RP}^N$ and

[3] There is a misprint in the cited paper at this point, $F_j(0)$ should be $F_i(1)$.

$\mathbb{RP}^{mp}$. The pullback of $G(m,m)$ is the upper Grassmannian $G^+(m,m)$ of all oriented m-planes in $\mathbb{R}^{m+p}$. Let $W^+\colon G^+(m,m) \to S^{mn}$ be the pullback of W. Since $G^+(m,m)$ is orientable [5], $\mathrm{char}(W)$ is well defined as in Definition 1.3.

Theorem 6.1 (*Eremenko and Gabrielov [5, Theorem 2]*). *The characteristic of the real Wronski map W is equal to the sign-imbalance of $\mathcal{C}_{m,p}$.*

Remark 6.2. White [26] computed this sign-imbalance, showing that $\sigma(\mathcal{C}_{m,p}) = 0$ unless $m+p$ is odd, and then it equals

$$\frac{1!2!\cdots(p-1)!(m-1)!(m-2)!\cdots(m-p+1)!\left(\frac{mp}{2}\right)!}{(m-p+2)!(m-p+4)!\cdots(m+p-2)!\left(\frac{m-p+1}{2}\right)!\left(\frac{m-p+3}{2}\right)!\cdots\left(\frac{m+p-1}{2}\right)!}.$$

Proof of Theorem 6.1. The Plücker ideal I of the Grassmannian $G(m,p)$ has a quadratic Gröbner basis whose elements are indexed by incomparable pairs J, K in $\mathcal{C}_{m,p}$ and have the form

$$x_J x_K - x_{J\wedge K} x_{J\vee K} + \text{other terms}, \tag{6.3}$$

where the other terms have the form $a x_L x_M$ with $L \leqslant J, K \leqslant M$ [21, Chapter 3]. The term order here is degree reverse lexicographic on $\mathbb{C}[x_J \mid J \in \mathcal{C}_{m,p}]$ where the variables are first linearly ordered by the ordinary lexicographic order on their indices.

Any lifting function $\omega\colon \mathcal{C}(m,p) \to \mathbb{Z}$ defines a $\mathbb{R}^\times$-action on Plücker space by $s.x_J = s^{-\omega(J)} x_J$. Restricting this action to the Grassmannian gives a family $s.G(m,p)$ whose scheme theoretic limit as $s \to 0$ is cut out by the initial ideal $\mathrm{in}_\omega I$. There is a lifting function ω so that $\mathrm{in}_\omega I$ is the toric ideal

$$x_J x_K - x_{J\wedge K} x_{J\vee K} \quad \text{for } J, K \text{ incomparable}$$

of the distributive lattice $\mathcal{C}_{m,p}$ [23, Theorem 11.4]. We call the corresponding family $s.G(m,p)$ the *sagbi* deformation of the Grassmannian, which deforms it into the toric variety $Y(m,p)$ of the distributive lattice $\mathcal{C}_{m,p}$. As in the proof of Lemma 4.8, the form of Eqs. (6.3) for the Grassmannian imply that the sagbi degeneration does not meet the center of the projection π_α.

For $s \in (0,1]$, let W_s be the restriction of the Wronski π_α to $s.G(m,p)$. Then the characteristic $\mathrm{char}(W)$ coincides with $\mathrm{char}(W_s)$ for $s \in (0,1]$. By Lemma 3.9, there exists a regular value z of W all of whose preimages in $Y(m,p)$ have all components positive. By the implicit function theorem, every preimage of z in $s.G(m,p)$ for s sufficiently small has the same property. Thus $\mathrm{char}(W)$ equals to the degree of the restriction of W to $s.G_>(m,p)$, the intersection of the Grassmannian with the positive orthant.

Let $Y_>(m,p)$ be the positive part of the toric variety $Y(m,p)$. Then $Y_>(m,p)$ has coordinates and orientation defined by the projection to the coordinate plane corresponding to some (fixed) maximal chain in $\mathcal{C}_{m,m+p}$. By the implicit function theorem, the same is true for $s.G_>(m,p)$ when s is sufficiently small. For each preimage of z in $Y_>(m,p)$ there is a nearby preimage of z in $s.G_>(m,p)$. Hence the projections of these preimages to the coordinate plane are nearby, and the signs of $\det W_s$ and $\det W_0$ coincide. This proves that $\mathrm{char}(W) = \mathrm{char}\left(W_0|_{Y_>(m,p)}\right)$. Finally, by Corollary 3.10, $\mathrm{char}(W_0|_{Y_>(m,p)})$ is equal to the sign-imbalance of $\mathcal{C}_{m,p}$, which completes the proof. □

We only used that the characteristic of the Wronski map was defined and that $G(m,p)$ has equations of form (6.3), for some distributive lattice D. The projective coordinate ring of a variety Y with such equations is an algebra with straightening law on the distributive lattice D [2], and this has the geometric consequence that Y admits a flat degeneration to the toric variety Y_D of the distributive lattice. There are many examples of such varieties, besides the Grassmannian. These include Schubert varieties of the Grassmannian, the classical flag variety, and the Drinfel'd compactification of the space of curves on the Grassmannian [17], as well as products of such spaces.

Such a variety Y has projective coordinates $\{x_J \mid J \in D\}$. Given a set α of weights, a Wronski map for the lattice D is a linear projection π_α of the form

$$\pi_\alpha\colon (x_J \mid J \in D) \longmapsto \left(\sum_{|J|=j} \alpha_J x_J \mid = 0, \ldots, \mathrm{rank}(D) \right).$$

We say that π_α has constant sign if the sign of α_J depends only upon $|J|$. Let $\hat{0}$ be the unique minimal element in D. For $\mathcal{C}_{m,p}$, this is $(1,2,\ldots,p)$.

Theorem 6.4. *Let Y be a projective variety whose coordinate ring is an algebra with straightening law on a distributive lattice D and let π_α be a Wronski projection for this lattice with constant sign. If either Y^+ or $Y \cap \{x \mid x_{\hat{0}} \neq 0\}$ are oriented, then the characteristic of the Wronski projection on Y is equal to the sign-imbalance of D.*

This result for Schubert varieties, the Drinfel'd compactification, and products of such varieties was communicated to us by Eremenko and Gabrielov, to whom it should be accredited.

7. Incomparable chains and factoring polynomials

We give a different proof of Theorems 4.3 and 5.7, when the poset P is a disjoint union of chains of lengths $a_1, a_2, \ldots, a_k$, and the weights α are constant. Our method will be to show that solutions to a general Wronski polynomial system for the chain polytope of P are certain factorizations of a particular univariate polynomial. This reformulation shows that there are certain numbers of real solutions to these systems that are forbidden to occur, which is a new phenomenon, which we seem to have also observed in Example 5.1. It also proves the sharpness of the lower bound of Theorems 4.3 and 5.7 for these posets, and shows that the conclusion holds even when the hypotheses of those theorems do not, as $Y^+_{C(P)}$ is not orientable if the a_i do not all have the same parity. This analysis extends to posets which are the incomparable unions of other posets.

Let P be the incomparable union of chains of lengths $a_1, a_2, \ldots, a_k$. For each $i = 1, \ldots, k$, set $x_{i,0} := 1$ and let

$$x_{i,1} > x_{i,2} > \cdots > x_{i,a_k}$$

be indeterminates which we identify with the elements in the ith chain, ordered as indicated. Observe that the upper order ideal generated by $x_{i,j}$ has j elements. Antichains of P correspond to monomials

$$x_{1,i_1}x_{2,i_2}\ldots x_{k,a_k}$$

and the order ideal generated by this antichain has $i_1 + i_2 + \cdots + i_k$ elements.

A Wronski polynomial with constant weight 1 for the canonical triangulation of the chain polytope $C(P)$ has the form

$$\begin{aligned} F &= \sum_{i_1,\ldots,i_k} c_{i_1+\cdots+i_k} x_{1,i_1}x_{2,i_2}\ldots x_{k,a_k} \\ &= \sum_{j=0}^{a_1+\cdots+a_k} c_t \left(\sum_{\substack{i_1,\ldots,i_k \\ i_1+\cdots+i_k=j}} x_{1,i_1}x_{2,i_2}\ldots x_{k,a_k} \right). \end{aligned} \tag{7.1}$$

A general system of such Wronski polynomials is equivalent to one of the form

$$\sum_{\substack{i_1,\ldots,i_k \\ i_1+\cdots+i_k=j}} x_{1,i_1}x_{2,i_2}\ldots x_{k,i_k} = b_j \quad \text{for } j = 1, 2, \ldots, a_1 + \cdots + a_k. \tag{7.2}$$

Suppose that we have a solution to (7.2). For each $i = 1, \ldots, k$, define the univariate polynomial

$$f_i(z) := 1 + \sum_{j=1}^{a_i} x_{i,j} z^j.$$

Then we clearly have

$$f_1(z) f_2(z) \cdots f_k(z) = 1 + \sum_{j=1}^{a_1+\cdots+a_k} b_j z^j = f(z). \tag{7.3}$$

Similarly, any such factorization of $f(z)$ where $\deg(f_i) = a_i$ gives a solution to (7.2), and hence to our original system. We have proven the following theorem:

Theorem 7.4. *The solutions to a general Wronski system with constant weights for the chain polytope of the incomparable union of chains of lengths* $a_1, \ldots, a_k$ *are the factorizations of a univariate polynomial* f *of degree* $a_1 + \cdots + a_k$ *into polynomials* $f_1, \ldots, f_k$, *where* f_i *has degree* a_i.

Remark 7.5. For each variable $x_{i,j}$ above, set

$$\varphi(x_{i,j}) := \prod_{j \leqslant l \leqslant a_i} x_{i,l}.$$

If we apply φ to a Wronski polynomial F (7.1) of the chain polytope of P, we obtain a Wronski polynomial for the canonical triangulation of the order polytope of P. In this way, Wronski systems for the order polytope and chain polytope of P are equivalent, and thus the results of this section also hold for the order polytope of P.

We investigate the consequences of Theorem 7.4. A factorization

$$f_1(z) f_2(z) \cdots f_k(z) = f(z), \tag{7.6}$$

where f_i is a complex polynomial of degree a_i for $i = 1, \ldots, k$ and $f(z)$ has degree $a_1+\cdots+a_k$ and distinct roots, is a distribution of the roots of f between the polynomials $f_1, \ldots, f_k$, with f_i receiving a_i roots. Thus the number of such factorizations is the multinomial coefficient

$$\binom{a_1 + \cdots + a_k}{a_1, a_2, \ldots, a_k} = \frac{(a_1 + \cdots + a_k)!}{a_1! a_2! \cdots a_k!}, \tag{7.7}$$

which is also the number of linear extensions of P. Indeed, the positions taken by the elements from each chain in a linear extension of P give a distribution of $a_1+\cdots+a_k$ positions among k chains with the ith chain receiving a_i positions. We already knew that the number of such linear extensions is the number of complex solutions to a Wronski polynomial system for the chain polytope of P.

Suppose now that $f(z)$ is a real polynomial with r real roots and c pairs of complex conjugate roots, all distinct. In every factorization of $f(z)$ into real polynomials, each conjugate pair of roots must be distributed to the same polynomial. This imposes stringent restrictions on the numbers of such real factorizations.

If every root of $f(z)$ is real, so that $c = 0$, then the number of real factorizations (7.6) is the multinomial coefficient (7.7). Also, there are no such factorizations if $f(z)$ has fewer than $|\{j | a_j \text{ is odd}\}|$ real roots. In particular, the minimum number of real factorizations is 0 if more than one a_j is odd. Recall that if $B \neq b_1 + b_2 + \cdots + b_k$, then we have

$$\binom{B}{b_1, b_2, \ldots, b_k} = 0.$$

Theorem 7.8. *Suppose that $f(z)$ is a real polynomial of degree $a_1 + \cdots + a_k$ with distinct roots. Let n be the number of real factorizations* (7.6) *of f where f_i has degree a_i. Then n depends only on the number of real roots of $f(z)$ and satisfies*

$$\binom{\lfloor \frac{a_1+\cdots+a_k}{2} \rfloor}{\lfloor \frac{a_1}{2} \rfloor, \ldots, \lfloor \frac{a_k}{2} \rfloor} \leqslant n \leqslant \binom{a_1 + a_2 + \cdots + a_k}{a_1, a_2, \ldots, a_k}.$$

The minimum is attained when $f(z)$ has at most one real root, and the maximum occurs when $f(z)$ has all roots real. Moreover, at most

$$1 + \left\lfloor \frac{a_1}{2} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_k}{2} \right\rfloor$$

distinct values of n can occur.

For example, if $k = 3$ and $(a_1, a_2, a_3) = (4, 4, 5)$, then $f(z)$ has degree 13. The number n of real factorizations of $f(z)$ into polynomials of degrees 4, 4, and 5 as a function of the number of real roots r of $f(z)$ is given in the table below

r	1	3	5	7	9	11	13
n	90	210	666	2226	7434	25,410	90,090

Proof of Theorem 7.8. A factorization of a polynomial $f(z)$ with r distinct real roots and c distinct pairs of conjugate roots into real polynomials of degrees $a_1, \ldots, a_k$ is a distribution of the roots of f among the factors where the ith factor receives a_i roots, and the conjugate pairs are distributed to the same factor.

The upper bound was described previously, so we consider the lower bound. The binomial coefficient lower bound vanishes when more than one a_i is odd, and we already observed that there are no real factorizations of f in this case. If every a_i is even and $f(z)$ has no real roots, then the root distribution is enumerated by this binomial coefficient. Lastly, if a_i is the only odd number among $a_1, a_2, \ldots, a_k$, and f has exactly one real root, that root must be given to the factor f_i. If we replace a_i by $a_i - 1$ and this problem of distributing roots reduces to the previous case.

The last statement follows as n depends only on the number of real roots of $f(z)$ and $n = 0$ unless $f(z)$ has at least $|\{j | a_j \text{ is odd}\}|$ real roots. $\square$

The number of real factorizations (7.6) is given by a generating function. We thank Ira Gessel who explained this to us.

Proposition 7.9. *The coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in $(x_1 + \cdots + x_k)^r (x_1^2 + \cdots + x_k^2)^c$ is the number of factorizations*

$$f_1(z) \cdot f_2(z) \cdots f_k(z) = f(z),$$

where $f(z)$ is real and has degree $r+2c=a_1+\cdots+a_k$ with r distinct real roots and c distinct pairs of complex conjugate roots, and $f_i(z)$ is real and has degree a_i for $i=1,\ldots,k$.

Proof. This is a standard use of generating functions, as described in Chapter 1 of [19]. We have r red balls and c cyan balls to distribute among k boxes such that if r_i is the number of red balls in box i and c_i is the number of cyan balls in box i, then $r_i+2c_i=a_i$. $\square$

We relate the lower bound of Theorem 7.8 to Theorem 5.7.

Proposition 7.10. *Let P be the incomparable union of chains of lengths $a_1,a_2,\ldots,a_k$. The sign-imbalance of P is*

$$\sigma(a_1,a_2,\ldots,a_k):=\binom{\lfloor\frac{a_i+\cdots+a_k}{2}\rfloor}{\lfloor\frac{a_1}{2}\rfloor,\ldots,\lfloor\frac{a_k}{2}\rfloor}.$$

This equals zero unless at most one a_i is odd.

Proof. If we precompose a linear extension with the inverse of the extension where every element of the ith chain precedes every element of the $(i+1)$st chain, then we have identified the set of all linear extensions of P with the set of minimal coset representatives S^a of the subgroup $S_{a_1}\times S_{a_2}\times\cdots\times S_{a_k}$ of the symmetric group $S_{a_1+\cdots+a_m}$, which we call $(a_1,\ldots,a_k)$-shuffles. The generating function for the distribution of lengths of these shuffles is the q-multinomial coefficient (the case $k=2$ is [19, Proposition 1.3.7])

$$\sum_{w\in S^a} q^{\ell(w)}=\binom{a_1+a_2+\cdots+a_k}{a_1,a_2,\ldots,a_k}_q,$$

where if $k>2$, then

$$\binom{a_1+a_2+\cdots+a_k}{a_1,a_2,\ldots,a_k}_q=\binom{a_1+a_2+\cdots+a_{k-1}}{a_1,\ldots,a_{k-1}}_q \cdot\binom{a_1+a_2+\cdots+a_k}{a_1+\cdots+a_{k-1},a_k}_q \tag{7.11}$$

and $\binom{a+b}{a,b}_q$ is the q-binomial coefficient

$$\binom{a+b}{a,b}_q=\frac{(1-q^{a+b})(1-q^{a+b-1})\cdots(1-q^2)(1-q)}{(1-q^a)\cdots(1-q^2)(1-q)\cdot(1-q^b)\cdots(1-q^2)(1-q)}. \tag{7.12}$$

We evaluate the q-multinomial coefficient at $q = -1$ to compute the sign-imbalance of P. If k is odd, then $1 - q^k = 2$ when $q = -1$. For even exponents, we have

$$1 - q^{2a} = (1 - q^2)(1 + q^2 + q^4 + \cdots + q^{2a-2}).$$

Now consider (7.12) when $q = -1$. If both a and b are odd, then (7.12) has one more factor with an even exponent in its numerator then in its denominator, and so it vanishes when $q = -1$. Otherwise (7.12) has the same number of factors with even exponents in its numerator as in its denominator, and so we cancel all factors of $(1 - q^2)$. If we substitute $q = -1$, then each factor $(1 - q^c)$ with odd exponent c becomes 2, and these cancel as there is the same number of such factors in the numerator and denominator. Since $(1 + q^2 + q^4 + \cdots + q^{2l-2}) = l$ when $q = -1$, we see that

$$\binom{a+b}{a, b}_{q=-1} = \binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}.$$

Applying (7.11) to this formula completes the proof. $\square$

Remark 7.13. By Theorem 7.8 and Proposition 7.10, the sign-imbalance of P is the sharp lower bound for the Wronski polynomial systems of the chain polytopes chain polytope of P. Thus Theorem 5.7 is sharp. Moreover, if the a_i do not all have the same parity, then the hypotheses of Theorem 5.7 do not hold, and in fact the toric variety Y_P^+ is not orientable. Despite this, the conclusion of Theorem 5.7 does hold.

The ideas in Proposition 7.10 can be used to compute the sign-imbalance of a product of posets. If P is the incomparable union of posets $P_1, P_2, \ldots, P_k$ with $|P_i| = a_i$, then the linear extensions of P are $(a_1, \ldots, a_k)$-shuffles of linear extensions of each component P_i. If we let $\eta(P)$ be the number of linear extensions of a poset P, then we have the following corollary:

Corollary 7.14. *Let P be as described. Then we have*

$$\eta(P) = \binom{a_1 + a_2 + \cdots + a_k}{a_1, a_2, \ldots, a_k} \cdot \prod_{i=1}^{k} \eta(P_i)$$

$$\sigma(P) = \binom{\lfloor \frac{a_1 + \cdots + a_x}{2} \rfloor}{\lfloor \frac{a_1}{2} \rfloor, \ldots, \lfloor \frac{a_k}{2} \rfloor} \cdot \prod_{i=1}^{k} \sigma(P_i).$$

Example 7.15. The Grassmannian $G = G(2, 2)$ has a sagbi degeneration to the toric variety Y associated to the distributive lattice of order ideals on a product $C_2 \times C_2$ of two chains of length 2. Let Z be the toric variety associated to the chain polytope of this poset. Since $C_2 \times C_2$ is sign-balanced, the lower bound here is 0.

If we take the product of G with the projective plane, we obtain a variety to which Theorem 6.4 applies. It has a sagbi degeneration into $Y \times \mathbb{RP}^2$, which is the toric variety of the distributive lattice of order ideals on the disjoint union of a chain C_2 of length 2 with $C_2 \times C_2$. Similarly, the toric variety associated to the chain polytope of this poset is $Z \times \mathbb{RP}^2$. By Corollary 7.14, the Wronski polynomial systems on these varieties will have $30 = 2 \cdot \binom{2+4}{2,4}$ complex solutions with at least 2 real.

The table below records the percentage that a given number of real roots was observed in Wronski polynomial systems on these varieties. The entries of 0 indicate values that were not observed.

# Real	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$G \times \mathbb{RP}^2$	0	11	55	24	5.3	1.6	1.2	1.4	0	0	0	0	0	0	0	0.31
$Y \times \mathbb{RP}^2$	0	2.2	25	14	23	1.2	0.09	22	0	0	0	0.001	0.001	0.07	0.01	12
$Z \times \mathbb{RP}^2$	0	0.07	6	33	4.6	1.3	2.9	39	0	0	0.003	0.01	1.5	0.4	0.37	10

We do not yet understand the apparent gaps in these data.

References

[1] D. Cox, The homogeneous coordinate ring of a toric variety, J. Alg. Geom. 4 (1995) 17–50.

[2] C. De Concini, D. Eisenbud, C. Procesi, Hodge algebras, Astérisque, vol. 91, Société Mathématique de France, Paris, 1982, With a French summary.

[3] A.I. Degtyarev, V.M. Kharlamov, Topological properties of real algebraic varieties: Rokhlin's way, Uspekhi Mat. Nauk 55 (4)(334) (2000) 129–212.

[4] A. Eremenko, A. Gabrielov, The Wronski map and Grassmannians of real codimension 2 subspaces, Comput. Methods Funct. Theory 1 (1) (2001) 1–25.

[5] A. Eremenko, A. Gabrielov, Degrees of real Wronski maps, Discrete Comput. Geom. 28 (3) (2002) 331–347.

[6] W. Fulton, Introduction to toric varieties, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[7] G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 2.0, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2001, http://www.singular.uni-kl.de

[8] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws, Commutative Algebra and Combinatorics, Advanced Studies in Pure Mathematics, vol. 11, North-Holland, Amsterdam, 1987, pp. 93–109.

[9] I. Itenberg, V. Kharlamov, E. Shustin, Welschinger invariant and enumeration of real plane rational curves, Internat. Math. Res. Notes 49 (2003) 2639–2653.

[10] M. Joswig, Projectivities in simplicial complexes and colorings of simple polytopes, Math. Z. 240 (2) (2002) 243–259.

[11] M. Kontsevich, Yu. Manin, Gromov–Witten classes, quantum comomology, and enumerative geometry, Comm. Math. Phys. 164 (1994) 525–562.

[12] A.G. Kushnirenko, A Newton polyhedron and the number of solutions of a system of k equations in k unknowns, Usp. Math. Nauk. 30 (1975) 266–267.

[13] L. Kronecker, Leopold Kronecker's Werke, Chelsea, NY, 1968.

[14] G. Mikhalkin, Counting curves via lattice paths in polygons, C.R. Math. Acad. Sci. Paris 336 (8) (2003) 629–634.

[15] G. Mikhalkin, Enumerative tropical algebraic geometry, J. Amer. Math. Soc. 18 (2005) 313–377.

[16] F. Sottile, Enumerative real algebraic geometry, Algorithmic and Quantitative Real Algebraic Geometry, Piscataway, NJ, 2001, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 60, American Mathematical Society, Providence, RI, 2003, pp. 139–179.

[17] F. Sottile, B. Sturmfels, A sagbi basis for the quantum Grassmannian, J. Pure Appl. Algebra 158 (2–3) (2001) 347–366.
[18] R. Stanley, Two poset polytopes, Discrete Comput. Geom. 1 (1986) 9–23.
[19] R. Stanley, Enumerative combinatorics, vol. 1, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.
[20] R. Stanley, Some remarks on sign-balanced and maj-balanced posets, Adv. Appl. Math. 34 (2005) 880–902.
[21] B. Sturmfels, Algorithms in invariant theory, Texts and Monographs in Symbolic Computation, Springer, Vienna, 1993.
[22] B. Sturmfels, On the number of real roots of a sparse polynomial system, Hamiltonian and Gradient Flows, Algorithms and Control, Fields Inst. Commun., vol. 3, American Mathematical Society, Providence, 1994, pp. 137–143.
[23] B. Sturmfels, Gröbner bases and convex polytopes, American Mathematical Society, Providence, RI, 1996.
[24] D.G. Wagner, Singularities of toric varieties associated with finite distributive lattices, J. Algebraic Combin. 5 (2) (1996) 149–165.
[25] J.-Y. Welschinger, Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry, C.R. Math. Acad. Sci. Paris 336 (4) (2003) 341–344.
[26] D.E. White, Sign-balanced posets, J. Combin. Theory Ser. A 95 (1) (2001) 1–38.