JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 13, 265-295 (1966)

Limiting Behavior for Age and Position-Dependent Branching Processes*

HOWARD E. CONNER

Mathematics Research Center, United States Army, The University of Wisconsin, Madison, Wisconsin

Submitted by Richard Bellman

1. INTRODUCTION

In this paper we study a model for the population transition probabilities for a branching process composed of particles moving in a finite interval with absorbing boundary. In general the proposed model does not have the Markov property since we assume the branching properties for each particle depend on its age and position. To establish this dependence we propose the following structure.

The process is generated by one particle initially at x in a bounded interval I with interior I_0. The boundary Γ is an absorbing barrier for which $a(x, t)$ is the probability of absorption at Γ by time t and $a(x) = \lim_{t\to\infty} a(x, t)$ is the probability of ultimate absorption for a particle initially at x. Consequently we assume

(a) $a(x, t)$ is nonnegative and continuous for x in I and $0 < t < \infty$ and satisfies $a(x, t) = 1$ for x in Γ and $0 \leqslant t < \infty$, (b) for each x in I, $a(x, t)$ is nondecreasing as t increases, and (c) $a(x) = \lim_{t\to\infty} a(x, t)$ is continuous in I and satisfies $a(x) \equiv 1$ in Γ and $a(x) \not\equiv 1$ in I_0. (1.1)

The life span of a particle and its motion are so related that $k(x, y; t)\, dydt$ is the conditional probability density function for the position y of a particle with life span t, provided it is initially at x and its motion is restricted to I_0. We assume

$$k(x, y; t) \text{ is nonnegative and continuous for } x, y \text{ in } I, \quad 0 < t < \infty \tag{1.2a}$$

and

$$k(x, y, t) = 0 \text{ for either } x \text{ and/or } y \text{ in } \Gamma, \quad 0 \leqslant t < \infty. \tag{1.2b}$$

* Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin under Contract No.: DA-11-022-ORD-2059.

At any time t a particle with initial position x has either been absorbed at Γ with probability $a(x, t)$, ended its life in I_0 with probability $\int_I \int_0^t k(x, y; s)\, ds dy$ or is moving about in I_0 with probability $b(x, t)$. Consequently we assume

$$b(x, t) \text{ is nonnegative and continuous with } \lim_{t\to\infty} b(x, t) = 0 \quad \text{for} \quad x \text{ in } I, \quad 0 < t < \infty \tag{1.3}$$

and

$$1 = a(x, t) + b(x, t) + \int_I \int_0^t k(x, y; s)\, dsdy \quad \text{for} \quad x \text{ in } I, \quad 0 \leqslant t < \infty ,$$

$$1 = a(x) + \int_I \int_0^\infty k(x, y, s)\, dsdy \quad \text{for} \quad x \text{ in } I. \tag{1.4}$$

A particle ending its life at an age t at the point x is transformed with the conditional probability $h_k(x, t)$ into k particles, $k = 0, 1, 2, \cdots$, with identical independent properties. It is convenient to define

$$h(x, t; z) = \sum_{k=0}^{\infty} h_k(x, t)\, z^k \quad \text{and} \quad \mu(x, t) = \sum_{k=1}^{\infty} k h_k(x, t) \tag{1.5}$$

and to assume

$$\text{each } h_k(x, t) \text{ is nonnegative and continuous and}$$

$$\sum_{k=0}^{\infty} h_k(x, t) = 1 \quad \text{for} \quad x \text{ in } I_0 , \quad 0 < t < \infty \tag{1.6a}$$

and

$$\mu(x, t) \text{ is positive and continuous} \quad \text{for} \quad x \text{ in } I_0 , \quad 0 < t < \infty. \tag{1.6b}$$

In the simplest case the process can be considered as a collection of independent particles each with a random position x_t, at time t, described by an ordinary Brownian motion in I with absorbing boundary Γ and a random life span l described by a density function g. That is, let $p(x, y; t)$ be the fundamental solution to

$$\frac{\partial}{\partial t} p = \frac{\partial^2}{\partial x^2} p, \qquad p(x, 0; t) = p(x, L; t) = 0, \qquad p(x, y; 0) = \delta(y - x)$$

for $0 \leqslant x, y \leqslant L$ and $0 < t$ and let $g(t)$, $0 \leqslant t$, be a nonnegative continuous function. Then

$$k(x, y; t) = p(x, y; t)\, g(t), \qquad b(x, t) = \left(\int_0^L p(x, y; t)\, dy\right) \int_t^\infty g(\tau)\, d\tau$$

and

$$a(x, t) = 1 - b(x, t) - \int_0^t \int_0^L p(x, y; t)\, dydt.$$

A quantity of interest for this model is the random number $N_t(x)$ of particles at time t in the interior I_0 generated by one particle initially at the point x. By considering $N_t(x)$ as a regenerative process with respect to the random age and position of the initial particle when transformed, we have the generating function $E(s^{N_t(x)})$ characterized as a solution to the following functional equation:

$$f(x, t; z) = a(x, t) + zb(x, t) + \int_I \int_0^t h[y, s; f(y, t-s; z)]\, k(x, y; s)\, ds dy \tag{1.7}$$

for x in I, $0 \leqslant t$ and $|z| \leqslant 1$.

The existence of a regular solution to this system is given by the following theorem of H. E. Conner [1].

EXISTENCE THEOREM. *Assume the given functions $a(x, t)$, $b(x, t)$, and $k(x, y; t)$ satisfy (1.1) through (1.4) and the functions $h(x, t; z)$ and $\mu(x, t)$, defined by (1.5), satisfy (1.6). Assume the given functions are further restricted to satisfy*

$$\int_0^\infty \max\{k(x, y; t) \mid x, y \text{ in } I\}\, dt < \infty, \qquad \int_0^\infty \max\{a(x, t) \mid x \text{ in } I\}\, \mathrm{dt} < \infty,$$
$$\int_0^\infty \max\{\mu(y, t)\, k(x, y; t) \mid x, y \text{ in } I\}\, dt < \infty. \tag{1.8}$$

With these assumptions there exists a unique solution $f(x, t; z)$ to (1.7), continuous and bounded in magnitude by 1 for x in I, $0 \leqslant t$ and $|z| \leqslant 1$. Furthermore $f(x, t; z)$ can be represented in $|z| < 1$ by

$$f(x, t; z) = \sum_{n=0}^{\infty} f_n(x, t)\, z^n, \qquad \sum_{n=0}^{\infty} f_n(x, t) = 1 \tag{1.9}$$

where each $f_n(x, t)$ is a nonnegative function continuous and bounded by 1 for x in I and $0 \leqslant t$.

The processes defined by (1.7) are a synthesis of age-dependent and position-dependent processes. A systematic study of the mathematical theory for position-dependent (neutron) branching processes and age-dependent branching processes is developed in, "The Theory of Branching Processes," by T. E. Harris [2]. The book has a comprehensive bibliography of papers in this field. For age-dependent processes we refer to the papers by R. Bellman and T. E. Harris [3], D. G. Kendall [4], N. Levinson [5], and W. A. O'N. Waugh [6]. For position-dependent processes we refer to the papers by B. A. Sevastyanov [7], H. E. Conner [8], J. E. Moyal [9], and S. R. Adke and J. Moyal [10].

2. Summary of Results

For the remaining work, we assume the functions $a(x, t)$, $b(x, t)$, $k(x, y; t)$, and $h(x, t; z)$ satisfy the conditions of the existence theorem stated in the introduction and we let $f(x, t; z)$ be the unique solution to (1.7), satisfying (1.9). A formal differentiation of (1.7) with respect to z suggests the mean population size $(\partial/\partial z) f(x, t; 1)$ is a solution to the following integral equation

$$m(x, t) = b(x, t) + \int_I \int_0^t m(y, t - s)\, \mu(y, s)\, k(x, y; s)\, ds dy$$

$$m(x, 0) = b(x, 0) \qquad x \text{ in } I, \qquad 0 \leqslant t < \infty. \tag{2.1}$$

To proceed we wish to define the expected population size $m(x, t)$ by the expression

$$m(x, t) \equiv \sum_{k=1}^{\infty} k f_k(x, t) \qquad x \text{ in } I, \qquad 0 \leqslant t < \infty \tag{2.2}$$

where the $f_k(x, t)$ are those in the representation (1.9). The validity of this definition and the desired characterization are given by

Theorem 1. (i) *There exists a positive constant* λ_0 *for which* $\sum_{k=1}^{\infty} k f_k(x, t) \leqslant 2e^{\lambda_0 t}$, *$x$ in I and* $0 \leqslant t < \infty$; *so that $m(x, t)$ can be defined by (2.2).*

(ii) *For any* $T > 0$,

$$\lim_{\eta \to 1-} \frac{1 - f(x, t; \eta)}{1 - \eta} = m(x, t)$$

uniformly for x in I and $0 \leqslant t \leqslant T$, *and*

(iii) *$m(x, t)$ is a unique continuous solution to (2.1) for x in I and* $0 \leqslant t < \infty$.

The probability of ultimate extinction

$$f_0(x) = \lim_{t \to \infty} f_0(x, t)$$

can be characterized as the minimum nonnegative continuous solution to $Uf = f$ where U is the Urysohn operator defined by

$$Uf(x) = a(x) + \int_I \left\{ \int_0^{\infty} h[y, s; f(y)]\, k(x, y; s)\, ds \right\} dy$$

for $f(x)$ continuous and bounded in modulus by 1 on I. Assumptions (1.4) and (1.6) show $l(x) \equiv 1$ also satisfies $Uf = f$. A n.a.s.c. for $f_0 \neq l$ is given by the spectral properties of the derivative operator $Ef = (\partial/\partial\phi)\, Uf$ evaluated at $\phi = l$,

$$Ef(x) = \int_I \left\{ \int_0^\infty \mu(y, s)\, k(x, y, s)\, ds \right\} f(y)\, dy. \tag{2.3}$$

This equation defines a linear integral operator transforming the class of functions continuous on I into the subclass of functions vanishing on Γ and the subclass of nonnegative functions into itself. These properties are sufficient to assert the existence of a positive characteristic number P of minimum modulus, the Perron root.

In an ordinary branching process with extinction probability f_0 the population size N_t tends to zero with probability f_0 and increases without bound with probability $1 - f_0$ as $t \to \infty$. When $f_0 < 1$, this increase can be counterbalanced by considering the normalized variable $N_t/m(t)$, where $m(t)$ is the expected population size. However, in the present model $f_0(x)$ is dependent on the absorption at the boundaries in addition to the interior absorption brought about by the branching mechanism. This gives the possible existence of interior points x for which $f_0(x) = 1$ even though $f_0 \neq l$ when $P < 1$. These points are completely determined by the absorption function $a(x)$ provided the process satisfies

For any x such that $a(x) < 1$,

$$0 < \int_{y_1}^{y_2} \left\{ \int_0^\infty \mu(y, t)\, k(x, y; t)\, dt \right\} dy \qquad \text{all} \qquad y_1 < y_2 \text{ in } I. \tag{2.4}$$

For processes satisfying (2.4) we have

$$\text{A n.a.s.c. for} \qquad f_0 \neq l \qquad \text{is} \qquad P < 1 \tag{2.5}$$

where P is the Perron root for (2.3). A direct consequence of (2.4) and the strict monotonicity of $h(y, s; \tau)$ in τ, $0 \leqslant \tau \leqslant 1$, is the following property of $f_0(x)$ for a process with $f_0 \neq l$

$$f_0(x) = 1 \qquad \text{iff } x \text{ is in} \qquad A = \{x \mid a(x) = 1\}. \tag{2.6}$$

Another result is the degeneration on the absorption set A of the integral equations (1.7) and (2.1) into

$$\begin{aligned} f(x, t; z) &= a(x, t) + zb(x, t) \\ m(x, t) &= b(x, t) \qquad \text{for} \qquad x \text{ in } A. \end{aligned} \tag{2.7}$$

In particular the mean population size at time t, $m(x, t)$, has very different behavior for x in A and x not in A and is no longer a suitable normalization function.

The behavior of $m(x, t)$ as $t \to \infty$ is developed from its characterization as a solution to (2.1). The interpretation of (2.1) as a vector-valued renewal equation suggests there is some real ρ such that $\lim_{t\to\infty} e^{-\rho t}\, m(x, t) = m(x, \rho)$ exists and is positive for x not in A. If so, then multiplying (2.1) by $e^{-\rho t}$ and letting $t \to \infty$ formally gives

$$m(x, \rho) = \int_I m(y, \rho) \left\{ \int_0^\infty e^{-\rho s}\, \mu(y, s)\, k(x, y; s)\, ds \right\} dy.$$

This shows the limit $m(x, \rho)$, when it exists, is a characteristic function and 1 is a characteristic number for the indicated kernel. To show the existence of the number ρ, we examine the spectral properties of the family of kernels defined by

$$E(x, y; z) = \int_0^\infty e^{-zt}\, \mu(x, t)\, k(x, y; t)\, dt, \qquad x, y \text{ in } I, \tag{2.8}$$

with parameter z, $\operatorname{Re}[z] \geqslant 0$.

The first result is

Lemma 2.1. *Assume the extinction probability $f_0(x)$, is not identically 1, the absorption set A, (2.6), is at most denumerable and*

$$\mu(x, t)\, k(x, x; t) = 0 \qquad \textit{all} \qquad t \geqslant 0 \qquad \textit{iff} \qquad x \textit{ is in } A.$$

Then there exists a single positive number ρ such that 1 is a characteristic number for the kernel $E(x, y; \sigma)$, $\sigma \geqslant 0$, iff $\sigma = \rho$.

The processes for which Lemma 2.1 is valid are henceforth called self-generating processes. The behavior of $m(x, t)$ for a self-generating process is given by

Theorem 2. *Let ρ be as determined in Lemma 2.1. Suppose $b(x, t)$ and $\mu(y, t)\, k(x, y; t)$ satisfy the restriction*

$$\int_0^\infty [e^{-\rho t} \max\{\mu(y, t)\, k(x, y, t) \mid x, y \text{ in } I\}]^2\, dt < \infty,$$
$$\int_0^\infty [e^{-\rho t} \max\{b(x, t) \mid x \text{ in } I)\}^2\, dt < \infty, \tag{2.9}$$

then there exists

$$\lim_{t\to\infty} e^{-\rho t}\, m(x, t) = m(x) \tag{2.10}$$

uniformly for x in I. The limit $m(x)$ is determined up to a multiplicative constant as a continuous solution to

$$m(x) = \int_I \left\{ \int_0^\infty e^{-\rho t}\, \mu(y, t)\, k(x, y; t)\, dt \right\} m(y)\, dy. \tag{2.11}$$

In addition

$$m(x) = 0 \quad \text{iff} \quad x \quad \text{is in} \quad A = \{x \mid a(x) = 1\}. \tag{2.12}$$

Let $G(x, t; u)$ be the family of distribution functions defined by

$$G(x, t; u) = \sum_{k \leqslant u} f_k(x, t), \qquad 0 \leqslant u, \tag{2.13}$$

with parameters x in I and $t \geqslant 0$. We proceed to establish for each x in I-A the weak convergence of $G(x, t; e^{-\rho t} u)$ to a proper distribution $G(x; u)$ as $t \to \infty$ and to determine the dependence on x.

The Laplace-Stieltjes transform of $G(x, t; e^{-\rho t} u)$ is given by

$$\Psi(x, t; z) = \sum_{k=0}^{\infty} f_k(x, t) \exp[-ke^{-\rho t}z], \qquad x \text{ in } I, \qquad t \geqslant 0, \qquad \text{Re}[z] \geqslant 0. \tag{2.14}$$

Consequently it satisfies the integral equation

$$\Psi(x, t; z) = a(x, t) + \exp[-e^{-\rho t}z]\, b(x, t) + \int_I \left\{ \int_0^t h[y, s; \Psi(y, t-s; e^{-\rho s}z)]\, k(x, y; s)\, ds \right\} dy. \tag{2.15}$$

The correspondence relation between distribution functions on a half-line and their L.-S. transforms allows us to concentrate our attention on the transforms $\psi(x, t; z)$ as $t \to \infty$. Letting $t \to \infty$ in (2.15) gives an auxiliary equation to be satisfied by the limit transform $\psi(x; z)$,

$$\psi(x; z) = a(x) + \int_I \left\{ \int_0^\infty h[y, s; \psi(y, e^{-\rho s}z]\, k(x, y; s)\, ds \right\} dy \tag{2.16}$$

for x in I and $\text{Re}[z] \geqslant 0$. This equation has the improper solution $\psi \equiv 1$ and we are faced with an existence and uniqueness problem for proper solutions to (2.16). The convergence properties of uniformly bounded sequences of analytic functions and the law of permancy of functional equations suggests we first consider (2.16) restricted to $z = \sigma \geqslant 0$.

THEOREM 3. *Assume the conditions for Theorems 1 and 2 are satisfied. Let $\alpha(\sigma)$ be a twice-continuously differentiable function defined for $0 \leqslant \sigma < \infty$ and satisfying*

$$\alpha(0) = 0, \qquad \alpha''(\sigma) \geqslant 0 \qquad \textit{and} \qquad \int_0^1 \frac{\alpha(\sigma)\, d\sigma}{\sigma} < \infty \tag{2.17}$$

and

$$\left| \mu(x, t) - \frac{1 - h[x, t; e^{-\sigma}]}{\sigma} \right| \leqslant \alpha(\sigma)\, \beta(x, t),$$

where $\beta(x, t)/\mu(x, t)$ *is uniformly bounded for* x *in* I *and* $0 \leqslant t$. *(This implies we can replace* $\beta(x, t)$ *by* $\mu(x, t)$.*) Then there exists a continuous solution* $\psi(x, \sigma)$ *to (2.16) for* x *in* I *and* $0 \leqslant \sigma$ *and satisfying*

(i) $0 < \psi(x, \sigma) \leqslant 1$ *for* x *in* I *and* $0 \leqslant \sigma$, *and*

(ii) $\psi(x, 0) = 1$ *for* x *in* I *and* $\psi(x, \sigma) = 1$ *for* x *in* A *and* $0 \leqslant \sigma$. *In addition* $\psi(x, \sigma)$ *is uniquely determined within the class of continuous solutions to (2.16), satisfying (i) and (ii), by the additional property: for each* x *in* I

(iii) $$\lim_{\sigma \to 0+} \frac{1 - \psi(x, \sigma)}{\sigma} = m(x),$$

and uniformly for x *in closed subsets of* I-A.

The nature of the condition (2.17) and its relation to a second moment requirement is thoroughly discussed in a paper of N. Levinson [5].

The behavior of $\psi(x, t; z)$, (2.14), as $t \to \infty$ is now developed. The functional relations which $\psi(x, t; z)$ and the iterative solution $\psi(x, \sigma)$, Theorem 3, satisfy are used to develop a functional inequality for

$$\Delta(x, t, \sigma) = \frac{\psi(x, t; \sigma) - \psi(x; \sigma)}{m(x)}. \tag{2.18}$$

This inequality is exploited to show $\Delta(x, t; \sigma) \to 0$ as $t \to 0$ and to obtain

THEOREM 4. *Assume the conditions for* Theorem 3 *are satisfied.*

Let $\psi(x, \sigma)$ *be the solution to* (2.16) *found in* Theorem 3 *and let* $\psi(x, t; \sigma)$ *be defined by* (2.14). *Then for each* x *in* I *and* $0 \leqslant \sigma$

$$\lim_{t \to \infty} \psi(x, t; \sigma) = \psi(x, \sigma)$$

and uniformly for x *in closed subsets of* I-A *and* $0 \leqslant \sigma \leqslant \sigma_0 < \infty$.

Using standard techniques in the theory of Laplace-Stieltjes transforms of distribution functions, we extend the last result to

THEOREM 5. *Assume the conditions for Theorem 3 are satisfied. Then for each* x *in* I *the distributions* $G(x, t; e^{-\rho t} u)$ *converge weakly as* $t \to \infty$ *to a distribution* $U(x, u)$, $0 \leqslant u < \infty$, *with the following properties. Its transform*

$$\Phi(x, z) = \int_{0-}^{\infty} e^{-uz}\, dU(x, u)$$

satisfies

(i) $0 < | \Phi(x, z) | \leqslant 1$ *x in I*, $\operatorname{Re}(z) \geqslant 0$.

(ii) $\Phi(x, 0) = 1$ *x in I and for x in the absorption set* $A = \{a(x) = 1\}$ $\Phi(x, z) = 1$, $\operatorname{Re}(z) \geqslant 0$,

$$\text{(iii)} \quad \lim_{\sigma \to 0+} \frac{1 - \Phi(x, \sigma)}{\sigma} = m(x) \tag{2.19}$$

for x in I and uniformly for x in closed subsets of I-A, and

$$\Phi(x, z) = a(x) = \int_I \left\{ \int_0^\infty h[y, s; \Phi(y; e^{-\rho s} z)]\, k(x, y; s)\, ds \right\} dy \tag{2.20}$$

for x in I and $\operatorname{Re}(z) \geqslant 0$.

3. Proof for Theorem 1

To begin we list some properties for $h(x, t; z)$ which follow directly from its definition (1.5), assumption (1.6), and an application of the mean value theorem:

(i) $0 < h(x, t; \tau_1) < h(x, t; \tau_2)$,

(ii) $0 \leqslant | h(x, t; z) | \leqslant h(x, t; | z |)$,

(iii) $0 \leqslant \left| \frac{\partial}{\partial z} h(x, t; z) \right| \leqslant \mu(x, t)$

$$\text{(iv)} \quad | h(x, t; z) - h(x, t; w) | \leqslant \mu(x, t) \, | z - w | \tag{3.1}$$

for x in I, $0 \leqslant t$, $| z |$ and $| w | \leqslant 1$, and $0 < \tau_1 < \tau_2 \leqslant 1$.

A useful estimate for which we have repeated use is the following

Lemma 3.1. *Suppose* $r(x, t)$ *is a continuous function for x in I and* $0 \leqslant t < \infty$ *satisfying*

$$r(x, t) \leqslant | A(x, t) | + \int_I \int_0^t r(y, t - s) \, | B(x, y; s) | \, ds dy,$$

$$x \text{ in } I, \qquad 0 \leqslant t < \infty$$

where

(i) $| A(x, t) | \leqslant K$ *for x in I and* $0 \leqslant t$ *and*

(ii) $\int_I \int^\infty \max \{| B(x, y; s) | \, | \, x \text{ in } I\} \, ds dy < \infty$, *then there is a positive constant* λ_0, *independent of* K, *for which*

$$r(x, t) \leqslant 2Ke^{\lambda_0 t} \quad \text{for} \quad x \text{ in } I \quad \text{and} \quad 0 \leqslant t.$$

To prove this, let

$$R(T, \lambda) = \max \{e^{-\lambda t} r(x, t) \mid x \text{ in } I, 0 \leqslant t \leqslant T\},$$

defined and finite for T and λ positive. Then

$$R(T, \lambda) \leqslant K + \left[\int_I \int_0^\infty e^{-\lambda s} \max \{| B(x, y; s) | \mid x \text{ in } I\} \, ds dy\right] R(T, \lambda).$$

Choosing λ_0 so that the convergent integral in brackets is $< \frac{1}{2}$ shows $R(T, \lambda_0) \leqslant 2K$. This implies $r(x, t) \leqslant 2Ke^{\lambda_0 t}$.

Since $f(x, t; z)$ satisfies (1.7) and (1.4) is assumed, we can write

$$\frac{1 - f(x, t; \eta)}{1 - \eta} = b(x, t) + \int_I \int_0^t \frac{1 - h[y, s; f(y, t - s; \eta)]}{1 - \eta} k(x, y; s) \, ds dy \quad (3.2)$$

for x in I, $0 \leqslant t$ and $0 \leqslant \eta < 1$. The definitions for $h(x, t; z)$ and $\mu(x, t)$, (1.5), and the inequality $(1 - x^k)/k(1 - x) < 1$, valid for $0 \leqslant x < 1$ and $k = 1, 2, 3, \cdots$, imply

$$\mu(x, t) - \frac{1 - h(x, t; \lambda)}{1 - \lambda} = \sum_{k=1}^{\infty} kh_k(x, t) \left[1 - \frac{1 - \lambda^k}{k(1 - \lambda)}\right] > 0 \quad (3.3)$$

for x in I, $0 \leqslant t$ and $0 \leqslant \lambda < 1$. Substituting (3.3) with $\lambda = f(y, t - s; \eta)$ into (3.2) gives

$$\frac{1 - f(x, t; \eta)}{1 - \eta} \leqslant b(x, t) + \int_I \int_0^t \frac{1 - f(y, t - s; \eta)}{1 - \eta} \mu(y, s) \, k(x, y; s) \, ds dy \quad (3.4)$$

for x in I, $0 \leqslant t$ and $0 \leqslant \eta < 1$. The application of Lemma 3.1, with $K = 1$, to (3.4) gives the existence of some $\lambda_0 > 0$ for which

$$0 < \frac{1 - f(x, t; \eta)}{1 - \eta} \leqslant 2e^{\lambda_0 t}, \quad (3.5)$$

for x in I, $0 \leqslant t$ and $0 \leqslant \eta < 1$. Since

$$\lim_{x \to 1-} \frac{1 - x^k}{k(1 - x)} = 1, \qquad k = 1, 2, 3, \cdots,$$

letting $\eta \to 1-$ in the inequality

$$\sum_{k=1}^{N} kf_k(x, t) \frac{1-\eta^k}{k(1-\eta)} \leqslant \frac{1-f(x, t; \eta)}{1-\eta}$$

and using (3.5) shows for $0 \leqslant t$ the partial sums $\sum_{k=1}^{N} kf_k(x, t)$ are bounded uniformly in x by $2e^{\lambda_0 t}$. Therefore $m(x, t)$, as defined by (2.2), exists and satisfies for some $\lambda_0 > 0$

$$m(x, t) \leqslant 2e^{\lambda_0 t} \qquad x \text{ in } I, \qquad 0 \leqslant t < \infty \tag{3.6}$$

In the same way we show for positive N the validity of

$$m(x, t) - \sum_{k=N}^{\infty} kf_k(x, t) \leqslant \liminf_{\eta \to 1-} \frac{1-f(x, t; \eta)}{1-\eta}$$
$$\leqslant \limsup_{\eta \to 1-} \frac{1-f(x, t; \eta)}{1-\eta} \leqslant m(x, t),$$

for x in I and $0 \leqslant t$. Letting $N \to \infty$ and using (3.6) establishes

$$\lim_{\eta \to 1-} \frac{1-f(x, t; \eta)}{1-\eta} = m(x, t) \tag{3.7}$$

for x in I and $0 \leqslant t < \infty$.

The determination of uniform convergence requires more information about the function $(1 - f(x, t; \eta))/(1 - \eta)$. To establish this we use the triangle inequality

$$\left| \frac{1-f(x_1, t_1; \eta)}{1-\eta} - \frac{1-f(x_2, t_2; \eta)}{1-\eta} \right|$$
$$\leqslant \left| \frac{f(x_2, t_2; \eta) - f(x_1, t_2; \eta)}{1-\eta} \right| + \left| \frac{f(x_1, t_2; \eta) - f(x_1, t_1; \eta)}{1-\eta} \right| \tag{3.8}$$

to replace differences on oblique lines by differences on sections.

We also require the following inequality

$$\frac{1 - h[y, s; f(y, t-s; \eta)]}{1-\eta} \leqslant 2\mu(y, s)\, e^{\lambda_0 (t-s)}, \tag{3.9}$$

which is a direct result of the estimates (3.1) and (3.5).

Setting

$$\Delta f(x_1, x_2, t_2; \eta) = \frac{f(x_1, t_2; \eta) - f(x_2, t_2; \eta)}{1-\eta},$$
$$\Delta b(x_1, x_2, t) = b(x_1, t) - b(x_2, t)$$

and

$$\Delta k(x_1, x_2, y; s) = k(x_1, y; s) - k(x_2, y, s)$$

and using (3.4) and (3.9), we have the inequality

$$| \Delta f(x_1, x_2, t_2; \eta) | \leqslant | \Delta b(x_1, x_2, t_2) | + 2 \int_I \int_0^{t_2} e^{\lambda_0 (t_2 - s)} \mu(y, s) | \Delta k(x_1, x_2, y; s) | dsdy. \quad (3.10)$$

For each $T > 0$ and $0 \leqslant t_1, t_2 \leqslant T$, the continuity properties of $b(x, t)$ and $\mu(y, t) k(x, y; t)$ for x and y in I and $0 \leqslant t \leqslant T$ imply the existence of a positive $\delta_1(\epsilon, T)$, $0 < \epsilon, T$, for which the r.h.s. of (3.10) is uniformly small; so that

$$\left| \frac{f(x_2, t_2, \eta) - f(x_1, t_2, \eta)}{1 - \eta} \right| \leqslant \frac{\epsilon}{2} \quad \text{for} \quad 0 \leqslant t_2 \leqslant T,$$

$$| x_2 - x_1 | < \delta_1(\epsilon, T). \quad (3.11)$$

Setting

$$\Delta f(x_1, t_1, t_2; \eta) = \frac{f(x_1, t_2; \eta) - f(x_1, t_1; \eta)}{1 - \eta},$$

$$\Delta b(x_1, t_1, t_2) = b(x_1, t_1) - b(x_1, t_2)$$

and again using (3.4) and (3.9), we have the inequality

$$| \Delta f(x_1, t_1, t_2; \eta) | \leqslant | \Delta b(x_1, t_1, t_2) | + \int_I \int_{t_1}^{t_2} e^{\lambda_0 (t_2 - s)} \mu(y, s) k(x_1, y; s) dsdy + \int_I \int_0^{t_1} | \Delta f(y, t_1 - s; t_2 - s; \eta) | \mu(y, s) k(x_1, y, s) dsdy.$$

Taking the maximum with respect to x_1 over I of the r.h.s. of the previous inequality and applying Lemma 3.1 gives the existence of a positive λ such that

$$| \Delta f(x_2, t_1, t_2; \eta) | \leqslant 2K(T, t_1, t_2) e^{\lambda T}$$

where

$$K(T, t_1, t_2) = \max \left\{ \Delta b(x_1, t_1, t_2) + \int_I \int_{t_1}^{t_2} e^{\lambda_0 (t_2 - s)} \mu(y, 0) k(x_1, y, s) dsdy \mid x_1 \text{ in } I \right\}.$$

For each $T > 0$ and $0 \leqslant t_1 \leqslant t_2 \leqslant T$, the previously stated continuity properties of $b(x, t)$ and $\mu(y, t) k(x, y, t)$ are sufficient to assert $K(T, t_1, t_2)$

is continuous in $0 \leqslant t_1 \leqslant t_2 \leqslant T$ and $K(T, t_1, t_1) = 0$. Consequently there exists $\delta_2(\epsilon, T)$, $0 < \epsilon$, T, for which $K(T, t_1, t_2) < (e^{-\lambda T}/4)\, \epsilon$ for $t_2 - t_1 < \delta_2$; so that,

$$\left|\frac{f(x_1, t_2; \eta) - f(x_1, t_1; \eta)}{1 - \eta}\right| < \frac{\epsilon}{2} \qquad \text{for} \qquad x_1 \text{ in } I,$$

$$0 \leqslant t_1 \leqslant t_2 \leqslant T, \qquad t_2 - t_1 < \delta_2(\epsilon, T). \tag{3.12}$$

The estimates (3.11) and (3.12) combine with (3.8) to show for each $T > 0$ $(1 - f(x, t, \eta))/(1 - \eta)$ is continuous for x in I and $0 \leqslant t \leqslant T$ uniformly in $0 \leqslant \eta < 1$.

This is sufficient to assert

$$m(x, t) = \lim_{\eta \to 1-} \frac{1 - f(x, t; \eta)}{1 - \eta}$$

is continuous for x in I and $0 \leqslant t \leqslant T$. Therefore by the Moore-Osgood theorem on iterated limits, we have for each $T > 0$,

$$\lim_{\eta \to 1-} \frac{1 - f(x, t; \eta)}{1 - \eta} = m(x, t) \tag{3.13}$$

uniformly for x in I and $0 \leqslant t \leqslant T$.

Let

$$\Gamma(x, t; \eta) = \mu(x, t) - \frac{1 - h(x, t; \eta)}{1 - \eta}$$

for x in I, $0 \leqslant t$ and $0 \leqslant \eta < 1$.

Then we have, for $0 \leqslant \eta_1 \leqslant \eta_2 < 1$,

$$\Gamma(x, t, \eta_1) - \Gamma(x, t; \eta_2) = \sum k h_k(x, t) \left[\frac{1 - \eta_2{}^k}{k(1 - \eta_2)} - \frac{1 - \eta_1{}^k}{k(1 - \eta_1)}\right].$$

This implies $\Gamma(x, t; \eta)$ is a decreasing function on $0 \leqslant \eta < 1$ since $(1 - x^k)/k(1 - x)$ is increasing for $0 \leqslant x < 1$. An argument similar to that used to develop (3.7) shows $\lim_{\eta \to 1-} \Gamma(x, t; \eta) = 0$ for x in I and $0 \leqslant t$. Moreover the convergence is uniform since $\Gamma(x, t; \eta)$ is decreasing in η; so that we can assert the existence of a positive $\delta(\epsilon, T)$, $0 < \epsilon$, T, for which

$$0 < \mu(x, t) - \frac{1 - h(x, t; \eta)}{1 - \eta} < \epsilon \tag{3.14}$$

when x in I, $0 \leqslant t \leqslant T$ and $1 - \lambda < \delta(\epsilon_1 T)$. This and the uniform convergence as $\eta \to 1-$ of $f(x, t; \eta)$ to 1 for x in I and $0 \leqslant t \leqslant T$ imply for each $T > 0$

$$\lim_{\eta \to 1-} \frac{1 - h[x, s; f(x, t-s; \eta)]}{1 - f(x, t-s; \eta)} = \mu(x, t) \qquad \text{uniformly for} \qquad x \text{ in } I,$$

and

$$0 \leqslant s \leqslant t \leqslant T. \tag{3.15}$$

Rewritting (3.2) in the form

$$\frac{1 - f(x, t; \eta)}{1 - \eta} = b(x, t) + \int_I \int_0^t \frac{1 - h[y, s; f(y, t-s, \eta)]}{1 - f(y, t-s, \eta)} \times \frac{1 - f(y, t-s; \eta)}{1 - \eta} k(x, y; s)\, ds dy$$

and letting $\eta \to 1-$, we use the uniform convergence established in (3.13) and (3.15) to show $m(x, t)$ is a solution to (2.1) and Lemma 3.1 to show it is unique in the class of continuous solutions for x in I and $0 \leqslant t$.

4. Proof for Theorem 2

The functions $B(x, z)$ and $E(x, y, z)$ defined by

$$B(x, z) = \int_0^\infty e^{-zt}\, b(x, t)\, dt \tag{4.1}$$

and (2.8) exist, are continuous for x, y in I and $\operatorname{Re}[z] > 0$ and are analytic in z, $\operatorname{Re}[z] > 0$, for fixed x, y in I. This is a direct result of the assumption (2.9). By Theorem 1, $m(x, t)$ is a solution to (2.1), and satisfies $m(x, t) \leqslant 2e^{\lambda_0 t}$ for some positive λ_0, (3.6). Therefore

$$M(x, z, \delta) = \int_0^\infty e^{-zt} e^{-\delta t}\, m(x, t)\, dt, \qquad \delta > 0, \tag{4.2}$$

exists and is continuous in (x, z) and analytic in z for fixed x in I and $\operatorname{Re}[z] > (+\lambda_0 - \delta)$. Since $m(x, t)$ is nonnegative, the abscissa of convergence $\alpha(x, \delta)$ is a singularity for $M(x, z, \delta)$ as an analytic function of z with fixed x and δ.

Multiplying (2.1) by $e^{-(z+\delta)t}$ and integrating with respect to t shows

$$M(x, z, \delta) = B(x, z+\delta) = \int_I E(x, y, z+\delta)\, M(y, z, \delta)\, dy. \tag{4.3}$$

The assumptions of Lemma 2.1 and condition (1.8) are sufficient to conclude for each $\sigma \geqslant 0$ the kernel $E(x, y; \sigma)$ has a positive char. number $P(\sigma)$ with the following properties

(i) $P(\sigma)$ has algebraic multiplicity 1 and $P(\sigma) < | \lambda_i(\sigma) |$ for any char. number $\lambda_i(\sigma) \neq P(\sigma)$ associated with $K(x, y; \sigma)$,

(ii) the char. function $\phi(x, \sigma)$ associated with $P(\sigma)$, *uniquely determined up to a multiplicative constant, is nonnegative and can be zero only for those x in the exceptional set A given in the assumptions.* (4.4)

Since $E(x, y; \sigma)$ is continuous in (x, y), the statements in (4.4) also apply to the transposed kernel $E(y, x, \sigma)$. These properties, extensions of the Perron-Frobenius theory for nonnegative matrices to compact integral operators with nonegative kernels, are developed in the monograph by M. G. Krein and M. A. Rutman [11, Chapter 6].

The dependence of $P(\sigma)$ on σ is given by

LEMMA 4.1. *When conditions for Lemma 2.1 are satisfied, $P(\sigma)$ is a continuous, strictly increasing function of σ on $0 \leqslant \sigma < \infty$.*

PROOF. For each z, $\operatorname{Re}[z] \geqslant 0$, let $d(\lambda; z)$ be the Fredholm determinant for the kernel $E(x, y; z)$. The characteristic numbers of $E(x, y; z)$ are the zeros in λ of $d(\lambda; z)$. Condition (1.8) is sufficient for us to assert $d(\lambda, z)$ is

(i) continuous in (λ, z) on $| \lambda | < \infty$ and $\operatorname{Re}[z] \geqslant 0$,

(ii) entire in λ for fixed z on $\operatorname{Re}(z) \geqslant 0$ and

(iii) analytic in z on $\operatorname{Re}(z) > 0$ for fixed finite λ. (4.5)

Since the char. number $P(\sigma)$ can be identified as the zero of $d(\lambda; \sigma)$ of least modulus, Roche's theorem can be applied to show $P(\sigma)$ is continuous for $0 \leqslant \sigma < \infty$.

Let $P(\sigma_i)$ and $\phi(x, \sigma_i)$, $(\psi(x, \sigma_i))$, be the positive characteristic pair described in (4.4) for $E(x, y; \sigma_i)$, $(E(y, x; \sigma_i))$, $i = 1, 2$, when $0 \leqslant \sigma_1 < \sigma_2$. The strictly decreasing behavior of $\max \{E(x, y; \sigma \mid x, y \text{ in } I\}$ for $0 \leqslant \sigma < \infty$ and the fact that $\phi(x, \sigma)$ $(\psi(x, \sigma))$ can be zero only on the exceptional set A imply

$$\int_I P(\sigma_2) \left\{ \int_I K(x, y; \sigma_1) \phi(y, \sigma_2)\, dy \right\} \psi(x, \sigma_1)\, dx > \int_I \phi(x, \sigma_2)\, \psi(x, \sigma_1)\, dx > 0.$$

However the left-side of the above inequality is evaluated to be

$$\frac{P(\sigma_2)}{P(\sigma_1)} \int_I \phi(y, \sigma_2)\, \psi(y, \sigma_1)\, dy.$$

Therefore $P(\sigma_2)/P(\sigma_1) > 1$ or $P(\sigma_1) < P(\sigma_2)$ for $0 \leqslant \sigma_1 < \sigma_2$, completing the proof for Lemma 4.1.

Since $E(\sigma) = \max\{E(x, y; \sigma) \mid 0 \leqslant x, y \leqslant L\} \to 0$ as $\sigma \to \infty$, the inequality $1 \leqslant P(\sigma) E(\sigma)$ shows that $P(\sigma) \to \infty$ as $\sigma \to \infty$. The assumption $f_0 \neq 1$ is equivalent to $P(0) < 1$, (2.5). Since $P(\sigma)$ is continuous and strictly increasing, this shows the existence of a single positive number ρ for which $P(\rho) = 1$ and proves Lemma 2.1.

In addition since $P(\sigma)$ is the zero of minimum modulus for the Fredholm determinant $d(\lambda; \sigma)$ for the kernel $E(x, y; \sigma)$, we can state

$$d(1, \sigma) \neq 0 \quad \text{for} \quad \rho < \sigma < \infty \quad \text{and} \quad d(1, \rho) = 0. \tag{4.6}$$

This result is now extended to all the kernels $E(x, y; z)$, Re $[z] \geqslant 0$.

LEMMA 4.2. *Assume the conditions of Lemma 2.1 are satisfied. Let ρ be the unique real number for which $P(\rho) = 1$. Then we have:*

(i) *If $\lambda(z)$ is a char. number for $E(x, y; z)$ and* Re $[z] > \rho$ *then* $1 < |\lambda(z)|$.

(ii) *When $\eta \neq 0$, 1 is not a char. number for $E(x, y; \rho + in)$.*

PROOF. Suppose for some z, Re $[z] = \sigma > \rho$, $g(x, z)$ is a continuous solution to

$$g(x, z) = \lambda(z) \int_I E(x, y; z)\, g(y, z)\, dy.$$

Since $|E(x, y; z)| \leqslant E(x, y; \sigma)$,

$$|g(x, z)| \leqslant |\lambda(z)| \int_I E(x, y; \sigma)\, |g(y, z)|\, dy, \qquad x \text{ in } I.$$

Multiplying this inequality by the unique solution $\psi(x, \sigma)$ associated with $P(\sigma)$ for the transposed kernel $E(y, x; \sigma)$, (4.4), and integrating over I gives

$$0 < \int_I |g(x, z)|\, \psi(x, \sigma)\, dx \leqslant \frac{|\lambda(z)|}{P(\sigma)} \int_I |g(x, z)|\, \psi(x, \sigma)\, dx.$$

Therefore $|\lambda(z)| \geqslant P(\sigma) > P(\rho) = 1$, proving part (i).

Part (ii) is proved by contradiction. Suppose for some $\eta \neq 0$, the continuous function $q(x, \eta)$, $0 \leqslant x \leqslant L$, satisfies

$$g(x, \eta) = \int_I E(x, y; \rho + i\eta)\, g(y; \eta)\, dy, \qquad x \text{ in } I. \tag{4.7}$$

Taking absolute values and using $| E(x, y; \rho + i\eta) | \leqslant E(x, y; \rho)$ shows

$$| g(x; \eta) | \leqslant \left| \int_I E(x, y; \rho + i\eta) g(y; \eta) \, dy \right|$$

$$\leqslant \int_I | E(x, y; \rho + i\eta) | \, | g(y; \eta) | \, dy$$

$$\leqslant \int_I E(x, y; \rho) \, | g(y; \eta) | \, dy, \qquad x \text{ in } I. \tag{4.8}$$

Multiplying the last inequality by the unique solution $\psi(x, \rho)$ associated with $P(\rho) = 1$ for the transposed kernel and integrating gives, after a little manipulation,

$$\int_I \psi(x, \rho) \left\{ \int_I E(x, y; \rho) \, | g(y, \eta) | \, dy - | g(x; \eta) | \right\} dx = 0.$$

Since $\psi(x, \rho)$ is positive outside the countable exceptional set A, and the term in brackets is nonnegative, it follows that

$$| g(x; \eta) | = \int_I K(x, y; \rho) \, | g(y, \eta) | \, dy, \qquad x \text{ in } I. \qquad (4.9$$

The resulting equalities in (4.8) show

$$| g(x; \eta) | = \int_I | K(x, y; \rho + i\eta) | \, | g(y; \eta) | \, dy, \qquad x \text{ in } I. \tag{4.10}$$

Since $P(\rho) = 1$ is simple, (4.9) implies $| g(x; \eta) |$ is a constant multiple of the char. function $\phi(x, \rho)$ associated with the char. number 1 for the kernel $E(x, y; \rho)$. Substituting this into (4.9) and (4.10) and using the positivity properties of $\phi(x; \rho)$ shows

$$| E(x, y; \rho + i\eta) | = E(x, y; \rho).$$

In particular for all x in I.

$$\left| \int_0^\infty e^{-i\eta t} e^{-\rho t} \mu(x, t) \, k(x, x; t) \, dt \right| = \int_0^\infty e^{-\rho t} \mu(x, t) \, k(x, x; t) \, dt.$$

This is only valid if for all x in I $\mu(x, t) \, k(x, x; t)$ is identically zero for $t \geqslant 0$, contradicting the assumption (2.9). Therefore (4.7) cannot have continuous solutions and 1 is not a char. number for the kernel $E(x, y; \rho + i\eta)$, $\eta \neq 0$, completing the proof.

We now develop a representation for $d(\lambda; z)$ in a bicircular neighborhood of $\lambda = P(\rho) = 1$ and $z = \rho$.

LEMMA 4.3. *Assume the conditions of Lemma 2.1 are satisfied. Then there exists a* $\delta_1 > 0$ *and two functions* $d_1(\lambda, z)$ *and* $w(z)$ *such that*

$$d(\lambda; z) = (\lambda - w(z))\, d_1(\lambda; z)$$

$$d_1(\lambda, z) \neq 0, \qquad w(\rho) = 1 \qquad |\lambda - 1| < \delta_1, \qquad |z - \rho| < \delta_1, \tag{4.11}$$

where $d_1(\lambda, z)$ *and* $w(z)$ *are analytic separately in* λ *and* z *and* $(\partial/\partial z)\, w(\rho) > 0$.

PROOF. The simplicity of $P(\sigma)$ is equivalent to $\lambda = P(\sigma)$ being a simple zero for $d(\lambda; \sigma)$. The Weierstrass preparation theorem can be applied to obtain the representation (4.11), Saks and Zygmund [12]. The vanishing of $d(\lambda; z)$ on this bicircular set is equivalent to $\lambda = w(z)$. This and the continuity properties of $P(\sigma)$ imply $P(\sigma) = w(\sigma)$ locally at $z = \rho$. Since $P(\sigma)$ is strictly increasing we have $(\partial/\partial z)\, w(\rho) > 0$, completing the proof.

Lemmas 4.2 and 4.3 determine a subset of parameter values

$$\Omega = \{|z - \rho| < \delta\} \cup \operatorname{Re}[z] > \rho\} \cup \{z = \rho + in, \eta \neq 0\} \tag{4.12}$$

for which the corresponding kernels $E(x, y; z)$ do not have 1 as a char. number. Therefore for each $(z + \rho)$ in Ω the equation

$$f(x) = B(x, z + \rho) + \int_I E(x, y; z + \rho) f(y)\, dy$$

has a unique continuous solution $R(x; z)$ given by

$$R(x; z) = B(x; z + \rho) + \frac{1}{d(1; z + \rho)} \int_I D(x, y; 1, z + \rho)\, B(y; z + \rho)\, dy \tag{4.13}$$

where $d(\lambda, z + \rho)$ and $D(x, y; \lambda, z + \rho)$ are the Fredholm determinant and kernel for $E(x, y; z + \rho)$. The condition (2.9) is sufficient to assert $D(x, y; \lambda, z)$ is

(i) continuous for x, y in I, $|\lambda| < \infty$ and $\operatorname{Re}[z] \geqslant 0$,

(ii) entire in λ with fixed x, y and z for x, y in I and $\operatorname{Re}[z] \geqslant 0$,

(iii) analytic in z, $\operatorname{Re}[z] > 0$, with fixed x, y and λ for x, y in I $|\lambda| < \infty$. (4.14)

Properties (4.14) for $D(x, y; \lambda, z)$ and (4.5) for $d(\lambda; z)$ are sufficient for $R(x; z)$, defined by (4.13), to be continuous for x in I and z in Ω and analytic for z in Ω, (4.12), with fixed x in I.

The function $M(x; z, \rho)$, defined by (4.2) with $\delta = \rho$, satisfies (4.3) and consequently (4.13) for Re $[z]$ sufficiently large. Therefore the unicity of solutions to (4.3) shows $R(x, z) = M(x; z, \rho)$, x in I, and Re $[z]$ sufficiently large. For each x in I, $R(x, z)$ is an analytic extension of $M(x, z; \rho)$ to the set Ω. As previously remarked, in a comment following (4.2), for each x in I the absicissa of convergence for the transform $M(x; z, \rho)$ is also a point of singularity for it as a function of z. Therefore $M(x; z, \rho)$ exists for x in I and $(z - \rho)$ in Ω and, being equal to $R(x, z)$ on this set, satisfies (4.13) for x in I and $(z - \rho)$ in Ω.

LEMMA 4.4. *When the conditions of Lemma 2.1 are satisfied, the function $R(x; z)$, defined by (4.13), has for each x in I a simple pole at $z = 0$ with residue $R(x)$. $R(x)$ is continuous, nonzero for x not in the exceptional set A, (2.8), and satisfies*

$$R(x) = \int_I E(x, y; \rho)\, R(y)\, dy, \qquad x \text{ in } I.$$

PROOF. By using (4.11) and restricting to $0 < |z| < \delta$, (4.13) can be written

$$zR(x, z) = \left[\frac{1 - w(z + \rho)}{z}\right]^{-1} [d_1(1; z + \rho)]^{-1} \times \int_I D(x, y; 1, z + \rho)\, B(y; z + \rho)\, dy + zB(x; z + \rho). \tag{4.15}$$

Letting $|z| \to 0$ gives the pointwise existence of

$$R(x) = \lim_{z \to 0} zR(x, z), \qquad x \text{ in } I \tag{4.16}$$

and

$$R(x) = -[w'(\rho)\, d_1(1; \rho)]^{-1} \int_I D(x, y; 1, \rho)\, B(y; \rho)\, dy, \qquad x \text{ in } I. \tag{4.17}$$

Some properties of the functions appearing in the r.h.s. of (4.15) and (4.17) are listed in (4.1) and (4.11). These are sufficient to assert $R(x)$ is continuous for x in I and $zR(x, z)$ is continuous for x in I, uniformly for z sufficiently near $z = 0$. Applying the Moore-Osgood theorem on iterated limits, the limit in (4.16) is uniform for x in I. Therefore letting $z \to 0$ in

$$zR(x; z) = zB(x; z + \rho) + \int_I E(x, y; z + \rho)\, zR(y; z)\, dy,$$

the uniform convergence in (4.16) shows

$$R(x) = \int_I E(x, y; \rho)\, R(y)\, dy, \qquad x \text{ in } I. \tag{4.18}$$

Since $P(\rho) = 1$ is simple and $R(x)$ is continuous on I, (4.18) implies $R(x)$ is some constant multiple of the char. function $\phi(x, \rho)$. Consequently $R(x)$ is nonzero for x not in the exceptional subset A. This completes the proof for Lemma 4.4.

The boundary behavior on $z = i\eta$, $-\infty < \eta < \infty$, of $M(x; z, \rho)$ can now be described using the previous work. In particular

$$\psi(x; z) = R(x, z) - z^{-1}R(x) - B(x; z + \rho) + \frac{R(x)}{z + \rho}$$

is analytic on the line Re $[z] = 0$. Also, Condition (2.9) implies the existence of a positive $N(\delta)$, $0 < \delta < 1$, such that, setting $L =$ length of I,

$$| E | (\rho + i\eta) = \max\{| E(x, y; \rho + i\eta) | \mid x, y \text{ in } I\} \leqslant \frac{\delta}{L}$$

when $| \eta | \geqslant N(\delta)$. Therefore when restricted to $| \eta | \geqslant N(\delta)$, $R(x, \rho + in)$ can be represented by an absolutely uniformly convergent Neumann expansion. In particular we have for $| \eta | \geqslant N(\delta)$

$$\begin{aligned} | \psi | (\rho + i\eta) &= \max\{| \psi(x, \rho + i\eta) | \mid x \text{ in } I\} \\ &\leqslant c_1 \frac{1}{| \eta | \, | \rho + i\eta |} + c_2 | E | (\rho + i\eta) \cdot | B | (\rho + i\eta) \qquad (4.19) \end{aligned}$$

where

$$c_1 = \rho \max\{| R(x) | \mid x \text{ in } I\}, \qquad c_2 = L \sum_{n=1}^{\infty} \delta^{n-1}$$

and

$$| B | (\rho + i\eta) = \max\{| B(y, \rho + i\eta) | \mid y \text{ in } I\}.$$

Therefore using first Schwartz's inequality and then the Hausdorff-Young inequality for Laplace transforms with $p = q = \frac{1}{2}$, (4.19) implies the existence of positive constants c_3, c_4 such that

$$\int_{|\eta| \geqslant N(\delta)} | \psi | (\rho + i\eta) \, d\eta \leqslant c_3 \int_0^{\infty} e^{-2\rho t} | k |^2 (t) \, dt + c_4 \int_0^{\infty} e^{-2\rho t} | b | (t) \, dt + c_1 \int_{|\eta| \geqslant N(\delta)} \frac{d\eta}{| \eta | \, | \rho + i\eta |} \qquad (4.20)$$

where

$$| k | (t) = \max\{k(x, y; t) \mid x, y \text{ in } I\} \quad \text{and} \quad | b | (t) = \max\{b(x, t) \mid x \text{ in } I\}.$$

Since $|\psi|(i\eta)$ is continuous in η, (4.20) and assumption (2.9) are sufficient for the absolute integrability of $\psi(x, i\eta)$ on $-\infty < \eta < \infty$, uniformly for x in I.

Therefore when the Laplace inversion is made, the integrability of $\psi(x, i\eta)$ uniformly in x gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta t}\,\psi(x, i\eta)\,d\eta = e^{-\rho t}\,m(x, t) - R(x) - e^{-\rho t}[b(x, t) + R(x)]$$

for x in I and $0 < t$. Also the r.h.s. goes to zero uniformly in x as $t \to \infty$. This last statement is verified by using the integrability properties of $\psi(x, i\eta)$ to reduce the problem to an application of the Rieman-Lebesgue theorem, uniformly in x. Since $e^{-\rho t}[b(x, t) + R(x)] \to \infty$ uniformly in x as $t \to \infty$, $e^{-\rho t}\,m(x, t) \to R(x)$ uniformly in x as $t \to \infty$. Moreover, $R(x)$ is positive except for x in the exceptional subset A, where it is zero. Summarizing, when the conditions of Lemma 2.1 and Theorem 2 are satisfied

$$\lim_{t\to\infty} e^{-\rho t}\,m(x, t) = R(x) \qquad \text{uniformly for} \qquad x \text{ in } I$$

where ρ is a positive constant, determined uniquely by $P(\rho) = 1$, and $R(x)$ is a nonnegative continuous solution to (4.18). $R(x)$ is uniquely determined by (4.17) and is zero iff x is in A. This completes the proof for Theorem 2, with $m(x) = R(x)$.

Proof for Theorem 3

The sequence $\{\psi_n(x, \sigma)\}$ is defined by the formula

$$\psi_0(x, \sigma) = e^{-m(x)\sigma}$$

$$\psi_{n+1}(x, \sigma) = a(x) + \int_I \left\{\int_0^{\infty} h[y, s; \psi_n(y, e^{-\rho s}\sigma)]\,k(x, y; s)\,ds\right\} dy \qquad (5.1)$$

for x in I and $0 \leqslant \sigma$.

A direct induction using the properties of $h(x, t; z)$, (3.1), and (1.4) shows for each n $\psi_n(x, \sigma)$ is continuous on its domain and satisfies

(i) $0 \leqslant \psi_n(x, \sigma) \leqslant 1$,

(ii) $\psi(x, 0) = 1$ for x in I and $\psi(x, \sigma) = 1$ for x in A and $0 < \sigma$ (5.2)

where $A = \{x \mid a(x) = 1\}$.

Using (5.1) and (1.4) to represent $1 - \psi_n(x, \sigma)$, a direct induction shows

$$1 - \psi_n(x, \sigma) \leqslant \int_0^{\infty} [1 - \psi_{n-1}(y, e^{-\rho s}\sigma)]\,\mu(y, s)\,k(x, y; s)\,ds\,dy \leqslant \sigma m(x) \qquad (5.3)$$

for x in I, $0 \leqslant \sigma$ and $n = 1, 2, 3, \cdots$.

Again using (5.1) and (1.4) to represent $1 - \psi_1(x, \sigma)$ and (2.11) to represent $m(x)$, we have

$$m(x) - \frac{1 - \psi_1(x, \sigma)}{\sigma} = \int_I \int_0^\infty \left\{ \mu(y, s) - \frac{1 - h[y, s; \exp(-m(y)\, e^{-\rho s}\sigma)]}{e^{-\rho s}\sigma m(y)} \right\} \times m(y)\, e^{-\rho s}\, k(x, y; s)\, ds dy$$

for x in I and $0 < \sigma$.

Using (2.17) to estimate the integrand, we have

$$\left| m(x) - \frac{1 - \psi_1(x, 0)}{\sigma} \right| \leqslant \int_I \int_0^\infty \alpha(e^{-\rho s}\, m(y)\, \sigma)\, m(y)\, e^{-\rho s}\, \mu(y, s)\, k(x, y; s)\, ds dy.$$

Since $\alpha(\sigma)$ is nondecreasing and

$$0 \leqslant e^{-\rho s}\, m(y) \leqslant M = \max \{m(x) \mid x \text{ in } I\},$$

we have

$$\left| m(x) - \frac{1 - \psi_1(x, \sigma)}{\sigma} \right| \leqslant \alpha(M\sigma)\, m(x), \qquad x \text{ in } I \qquad \text{and} \qquad 0 < \sigma.$$

This and the inequality

$$\left| m(x) - \frac{1 - e^{-m(x)\sigma}}{\sigma} \right| \leqslant \sigma M$$

show

$$| \psi_1(x, \sigma) - \psi_0(x, \sigma) | \leqslant \sigma m(x)\, \alpha_1(\sigma), \qquad x \text{ in } I \qquad \text{and} \qquad 0 < \sigma. \tag{5.4}$$

The function $\alpha_1(\sigma) \equiv \alpha(M\sigma) + M\sigma$, $\sigma > 0$, satisfies the conditions (2.17). We have, using (5.1),

$$\psi_2(x, \sigma) - \psi_1(x, \sigma) = \int_I \int_0^\infty \{h[y, s; \psi_1(y, e^{-\rho s}\sigma)] - h[y, s; \psi_0(y, e^{-\rho s}\sigma]\} \times k(x, y; s)\, ds dy.$$

This, (3.1) (iv) and (5.4), show

$$| \psi_2(x, \sigma) - \psi_1(x, \sigma) | \leqslant \sigma \int_I \int_0^\infty \alpha_1(e^{-\rho s}\sigma)\, m(y)\, e^{-\rho s}\, \mu(y, s)\, k(x, y; s)\, ds dy \tag{5.5}$$

for x in I and $0 < \sigma$. When x is in the exceptional set A, defined in (2.6), both sides degenerate to zero.

For x not in A $m(x)$ is positive and there exists a single positive $S(x)$ such that

$$\int_I \int_0^{S(x)} m(y)\, e^{-\rho s}\, \mu(y, s)\, k(x, y; s)\, dy ds = \tfrac{1}{2}\, m(x).$$

Substituting this into (5.5) and using the convexity of $\alpha_1(\sigma)$ gives

$$| \psi_2(x, \sigma) - \psi_1(x, \sigma) | \leqslant \sigma \Big\{ \alpha_1(\sigma) \int_I \int_0^{S(x)} m(y)\, e^{-\rho s} \mu(y, s)\, k(x, y; s)\, dyds$$

$$+ \alpha_1(\sigma\, e^{-\rho S(x)}) \int_I \int_{S(x)}^{\infty} m(y)\, e^{-\rho s} \mu(y; s)\, k(x, y; s)\, dyds \Big\}$$

$$\leqslant \sigma m(x) \left[\tfrac{1}{2} \alpha_1(\sigma) + \tfrac{1}{2} \alpha_1(\sigma e^{-S(x)})\right]$$

$$\leqslant \sigma m(x)\, \alpha_1 \Big\{ \sigma \left[\frac{1 + e^{-\rho S(x)}}{2}\right] \Big\},$$

$$x \text{ in } I\text{-}A \qquad \text{and} \qquad 0 < \sigma.$$

For $\epsilon > 0$ let $I(\epsilon)$ be that closed subset of I-A where $m(x) \geqslant \epsilon$. Then there exists some positive $S(\epsilon)$ for which $S(\epsilon) \leqslant S(x)$ and

$$| \psi_2(x, \sigma) - \psi_1(x, \sigma) | \leqslant \sigma \left[m(x)\, \alpha_1 \left(\sigma \frac{1 + e^{-\rho S(\epsilon)}}{2} \right) \right] \qquad \text{for} \qquad x \text{ in } I(\epsilon).$$

We continue by induction. Assuming

$$| \psi_{n-1}(x, \sigma) - \psi_{n-2}(x, \sigma) | \leqslant \sigma m(x)\, \alpha_1[\sigma l^{n-2}(\epsilon, \rho)],$$

where

$$l^n(\epsilon, \rho) = \left[\frac{1 + e^{-\rho s(\epsilon)}}{2}\right]^n,$$

the same arguments give

$$| \psi_n(x, \sigma) - \psi_{n-1}(x, \sigma) | \leqslant \int_I \int_0^{\infty} | \psi_{n-1}(y, e^{-\rho s} \sigma) - \psi_{n-2}(y, e^{-\rho s} \sigma) | \cdot \mu(y, s)\, k(x, y; s)\, dsdy$$

$$\leqslant \sigma \int_I \int_0^{\infty} \alpha_1[\sigma\, e^{-\rho s}\, l^{n-2}(\epsilon, \rho)]\, m(y)\, e^{-\rho s} \mu(y, s) \times k(x, y; s)\, dsdy$$

$$\leqslant \sigma m(x)\, \alpha_1[\sigma l^{n-1}(\epsilon, \rho)], \qquad x \text{ in } I\text{-}A \qquad \text{and} \qquad 0 < \sigma.$$

This and the previous inequality show

$$\sum_{j=n+1}^{n+p-1} | \psi_{j+1}(x, \sigma) - \psi_j(x, \sigma) | \leqslant \sigma m(x) \sum_{j=n+1}^{\infty} \alpha_1[l^j(\epsilon, \rho)],$$

$$x \text{ in } I(\epsilon) \qquad \text{and} \qquad 0 < \sigma. \tag{5.6}$$

Since α_1 is nondecreasing,

$$\sum_{j=n+1}^{\infty} \alpha_1[\sigma l^j(\epsilon, \rho)] \leqslant \int_n^{\infty} \alpha_1[\sigma l^t(\epsilon, \rho)]\, dt, \qquad x \text{ in } I(\epsilon) \qquad \text{and} \qquad 0 < \sigma. \tag{5.7}$$

Choosing $\delta(\epsilon)$ so that $e^{-\delta(\epsilon)} = l(\epsilon, \rho)$ and setting $u = \sigma e^{-\delta(\epsilon)t}$, (5.6) and (5.7) imply for fixed $I(\epsilon)$ and arbitrary positive p

$$|\psi_{n+p}(x, \sigma) - \psi_n(x, \sigma)| \leqslant \frac{\sigma M}{\delta(\epsilon)} \int_0^{\sigma e^{-\delta(\epsilon)n}} \frac{\alpha_1(u)}{u}\, du, \qquad x \text{ in } I(\epsilon) \quad \text{and} \quad 0 < \sigma, \tag{5.8}$$

where $M = \max\{m(x) \mid x \text{ in } I\}$. By assumption (2.17), the r.h.s, tends to zero as $n \to \infty$.

Estimates (5.2) and (5.8) show $\psi_n(x, \sigma)$ converges boundedly for each x in I and $0 \leqslant \sigma$ and for each $I(\epsilon)$ and positive σ_0 converges uniformly for x in $I(\epsilon)$ and $0 \leqslant \sigma \leqslant \sigma_0$ to a limit $\psi(x, \sigma)$. Consequently $\psi(x, \sigma)$ is continuous for x in I-A and $0 < \sigma$ and satisfies properties (i) and (ii) of the theorem. Letting $n \to \infty$ in (5.3) shows

$$1 - \psi(x, \sigma) \leqslant \sigma m(x), \qquad x \text{ in } I \quad \text{and} \quad 0 \leqslant \sigma. \tag{5.9}$$

This shows $\psi(x, \sigma)$ is continuous for x in A and $0 \leqslant \sigma$; and so, $\psi(x, \sigma)$ is continuous for x in I and $0 \leqslant \sigma$. Letting $n \to \infty$ in (5.1) and applying the dominated convergence theorem shows $\psi(x, \sigma)$ is a solution to (2.16).

To verify property (iii) we choose $\delta_1(\epsilon)$, $0 < \epsilon$, so that

$$\left| m(x) - \frac{1 - e^{-\sigma m(x)}}{\sigma} \right| < \frac{\epsilon}{2}$$

$$\left| m(x) - \frac{1 - \psi(x, \sigma)}{\sigma} \right| < \frac{\epsilon}{2} + |\psi(x, \sigma) - \psi_0(x, \sigma)|,$$

$$x \text{ in } I \quad \text{and} \quad 0 < \sigma \leqslant \sigma_1 .$$

Setting $n = 0$ and letting $p \to \infty$ in (5.8) gives the existence of a $\sigma_2(\epsilon)$, $0 < \epsilon$, such that for any $I(\epsilon_1)$, $0 < \epsilon_1$,

$$|\psi(x, \sigma) - \psi_0(x, \sigma)| \leqslant \frac{\sigma M}{\delta(\epsilon_1)} \int_0^{\sigma} \frac{\alpha_1(u)}{u}\, du < \frac{\epsilon}{2}$$

for x in $I(\epsilon_1)$ and $0 < \sigma \leqslant \sigma_2(\epsilon)$. This and the previous inequality show

$$\lim_{\sigma \to 0+} \frac{1 - \psi(x, \sigma)}{\sigma} = m(x) \qquad x \text{ in } I \tag{5.10}$$

and uniformly for x in closed subsets of I-A.

The verification of uniqueness is the final step. Suppose $\psi_1(x, \sigma)$ and $\psi_2(x, \sigma)$ are two solutions to (2.16) with the listed properties. The difference

$$\Delta(x, \sigma) = \left| \frac{\psi_1(x, \sigma) - \psi_2(x, \sigma)}{\sigma m(x)} \right|$$

satisfies

$$\Delta(x, \sigma) \leqslant \frac{1}{m(x)} \int_I \int_0^\infty \Delta(y, e^{-\rho s} \sigma)\, m(y)\, e^{-\rho s} \mu(y, s)\, k(x, y; s)\, ds dy \tag{5.11}$$

for x in I-A and $0 < \sigma$.

Let

$$\Delta(\tau, \epsilon) = \sup \{\Delta(x, \sigma) \mid x \text{ in } I(\epsilon), 0 < \sigma \leqslant \tau\}.$$

It is finite, by (5.9), and is nondecreasing in τ. This and (5.11) show

$$\Delta(\tau, \epsilon) \leqslant \sup \left\{ \frac{1}{m(x)} \int_I \int_0^\infty \Delta(e^{-\rho s} \tau, \epsilon)\, m(y)\, e^{-\rho s} \mu(y, s)\, k(x, y; s)\, ds dy \mid x \text{ in } I(\epsilon) \right\}.$$

The function enclosed in brackets is continuous for x in $I(\epsilon)$ and therefore assumes its maximum value at some point $\bar{x}$ in $I(\epsilon)$. Applying a previous argument, we have

$$\begin{aligned} \Delta(\tau, \epsilon) &\leqslant \frac{1}{m(\bar{x})} \int_I \int_0^{S(\bar{x})} \Delta(\tau, \epsilon)\, m(y)\, e^{-\rho s} \mu(y, s)\, k(\bar{x}, y; s)\, ds dy \\ &\quad + \frac{1}{m(\bar{x})} \int_I \int_{S(\bar{x})}^\infty \Delta(e^{-\rho S(\bar{x})} \tau, \epsilon)\, m(y)\, e^{-\rho s} \mu(y, s)\, k(\bar{x}, y; s)\, ds dy \\ &\leqslant \tfrac{1}{2} \Delta(\tau, \epsilon) + \tfrac{1}{2} \Delta(e^{-\rho S(\bar{x})} \tau, \epsilon), \qquad 0 < \tau. \end{aligned}$$

Therefore for each $I(\epsilon)$, $0 < \epsilon$, there is some positive $S(\epsilon)$ such that

$$\Delta(\tau, \epsilon) \leqslant \Delta(e^{-\rho S(\epsilon)} \tau, \epsilon), \qquad 0 < \tau. \tag{5.12}$$

Since

$$\Delta(x, \sigma) \leqslant \frac{1}{m(x)} \left| m(x) - \frac{1 - \psi_1(x, \sigma)}{\sigma} \right| + \frac{1}{m(x)} \left| m(x) - \frac{1 - \psi_2(x, \sigma)}{\sigma} \right|,$$

we have

$$\begin{aligned} \Delta(\tau, \epsilon) &\leqslant \frac{1}{\epsilon} \left[\sup \left\{ \left| m(x) - \frac{1 - \psi_1(x, \sigma)}{\sigma} \right| \mid x \text{ in } I(\epsilon), 0 < \sigma \leqslant \tau \right\} \right] \\ &\quad + \frac{1}{\epsilon} \left[\sup \left\{ \left| m(x) - \frac{1 - \psi_2(x, \sigma)}{\sigma} \right| \mid x \text{ in } I(\epsilon), 0 < \sigma \leqslant \tau \right\} \right]. \end{aligned}$$

Since $\psi_1(x, \sigma)$ and $\psi_2(x, \sigma)$ each satisfy (5.10), this implies for each $I(\epsilon)$ $\Delta(0+, \epsilon) = 0$. The iteration of (5.12) shows $\Delta(\tau) \leqslant \Delta(e^{-n\rho S(\epsilon)} \tau, \epsilon)$ for $n = 1, 2, \cdots$. Therefore for each $I(\epsilon)$, $\Delta(\tau, \epsilon) \leqslant \Delta(0+, \epsilon) = 0$. This and property (ii) shows $\psi_1(x, \sigma) = \psi_2(x, \sigma)$ for x in I and $0 \leqslant \sigma$.

6. Proof for Theorem 4

Let $\Delta(x, t, \sigma)$ be defined by (2.18) for x in I-A, $0 < \sigma$ and $0 \leqslant t$. The continuity of $\psi(x, t; \sigma)$, $\psi(x, \sigma)$ and $m(x)$ imply $\Delta(x, t, \sigma)$ is continuous on its domain. Define

$$\Pi(x, t) = \frac{e^{-\rho t}\, m(x, t)}{m(x)}, \qquad x \text{ in } I - A \qquad \text{and} \qquad 0 \leqslant t. \tag{6.1}$$

By Theorem 3, $\Pi(x, t)$ is continuous on its domain and $\lim_{t\to\infty} \Pi(x, t) = 1$ uniformly for x in I-A. In particular $\Pi(x, t) \leqslant c$ for some $c > 0$. The definition (2.14) for $\psi(x, t; \sigma)$ and (2.2) for $m(x, t)$ show

$$\Pi(x, t) - \frac{1 - \psi(x, t; \sigma)}{\sigma m(x)}$$

$$= e^{-\rho t}\, m(x)^{-1} \left\{ \sum_{k=1}^{\infty} k f_k(x, t) \left[1 - \frac{1 - \exp(-ke^{-\rho t}\sigma)}{ke^{-\rho t}\sigma} \right] \right\} \geqslant 0 \tag{6.2}$$

for x in I-A, $0 < \sigma$ and $0 \leqslant t$ since $m(x)$, $f_k(x, t)$ and $1 - (1 - e^{-t})/t$ are positive on this set. This and inequality (5.9) show

$$\Delta(x, t; \sigma) \leqslant \left| \frac{1 - \psi(x, t; \sigma)}{\sigma m(x)} \right| + \left| \frac{1 - \psi(x, \sigma)}{\sigma m(x)} \right| \leqslant \Pi(x, t) + 1 \leqslant c + 1; \tag{6.3}$$

so that $\Delta(x, t; \sigma)$ is bounded on its domain.

The functional equations (2.15) and (2.16), inequality (3.1) (i) for $h(x, t, \tau)$, inequality (5.9) for $1 - \psi(x, \sigma)$, and $a(x) - a(x, t) \leqslant b(x, t)$ are used to show

$$\begin{aligned} \Delta(x, t; \sigma) \leqslant{} & \frac{1}{m(x)} \int_I \int_0^t \Delta(y, t - s; e^{-\rho s}\sigma)\, m(y)\, e^{-\rho s} \mu(y, s)\, k(x, y; s)\, ds dy \\ & + \frac{1}{m(x)} \int_I \int_t^\infty m(y)\, e^{-\rho s} \mu(y, s)\, k(x, y; s)\, ds dy \\ & + \frac{1}{m(x)} \left[\frac{1 - \exp(-e^{-\rho t}\sigma)}{\sigma} \right] b(x, t). \end{aligned} \tag{6.4}$$

Let $\bar{\Delta}(x, t; \tau) = \sup\{\Delta(x, t; \sigma) \mid 0 < \sigma \leqslant \tau\}$. Then it is continuous for x in I-A, $0 < \tau$ and $0 \leqslant t$ and it is nondecreasing in τ for fixed x, t.

We again let $I(\epsilon) = \{x \mid m(x) \geqslant \epsilon\}$, $0 < \epsilon$. We apply a previous argument, see development for (5.12), to show the existence of some $\bar{x}$ in $I(\epsilon)$ such that for $t > S(\bar{x})$ the first term $J(x, t, \sigma)$ in (6.4) satisfies

$$J(x, t, \sigma) \leqslant \frac{1}{m(\bar{x})} \int_I \int_0^{S(\bar{x})} \bar{\Delta}(y, t-s, \tau)\, m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy$$
$$+ \frac{1}{m(\bar{x})} \int_I \int_{S(\bar{x})}^{t} \bar{\Delta}(y, t-s, e^{-\rho S(\bar{x})} \tau)\, m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy \tag{6.5}$$

for x in $I(\epsilon)$, $0 < \sigma \leqslant \tau$ and $S(\bar{x}) < t$.

The function $\Delta(t, \tau, \epsilon) = \sup \{\bar{\Delta}(x, t, \tau) \mid x \text{ in } I(\epsilon)\}$ is continuous for $0 < \tau, t$ and is nondecreasing in τ for fixed t. Denote the terms on the r.h.s. of (6.5) by J_1 and J_2 respectively. The properties for $\Delta(t, \tau, \epsilon)$ imply

$$\text{(i)} \quad J_1(\bar{x}, t, \tau) \leqslant \frac{1}{m(\bar{x})} \int_I \int_0^{S(\bar{x})} \Delta(t-s, \tau, \epsilon)\, m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy$$

$$\text{(ii)} \quad J_2(\bar{x}, t, \tau) \leqslant \frac{1}{m(\bar{x})} \int_I \int_{S(\bar{x})}^{t} \Delta(t-s, \tau e^{-\rho S(\bar{x})}, \epsilon)\, m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy \tag{6.6}$$

for $\bar{x}$ in $I(\epsilon)$, $0 < \tau$ and $S(\bar{x}) < t$.

The function $\bar{\Delta}(\tau, \epsilon) = \lim_{t \to \infty} \sup \Delta(t, \tau, \epsilon)$ is finite by the uniform boundedness of $\Delta(x, t, \sigma)$, (6.3). From (6.6) (i) we obtain

$$\limsup_{t \to \infty} J_1(\bar{x}, t, \tau) \leqslant \frac{\bar{\Delta}(\tau, \epsilon)}{m(\bar{x})} \int_I \int_0^{S(\bar{x})} m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy = \tfrac{1}{2} \bar{\Delta}(\tau, \epsilon) \tag{6.7}$$

for $\bar{x}$ in $I(\epsilon)$, and $0 < \tau$. Using (6.3) to extend (6.6) (ii).

$$J_2(\bar{x}, t, \tau) \leqslant \frac{1}{m(\bar{x})} \int_I \int_{S(\bar{x})}^{t/2} \Delta(t-s, e^{-\rho S(\bar{x})} \tau)\, m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy$$
$$+ \frac{c+1}{m(\bar{x})} \int_I \int_{t/2}^{t} m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy$$

for $\bar{x}$ in $I(\epsilon)$, $0 < \tau$ and $S(\bar{x}) < t/2$. This shows

$$\limsup_{t \to \infty} J_2(\bar{x}, \tau, t) \leqslant \frac{\bar{\Delta}(e^{-\rho S(\bar{x})} \tau, \epsilon)}{m(\bar{x})} \int_I \int_{S(\bar{x})}^{\infty} m(y)\, e^{-\rho s}\, \mu(y, s)\, k(\bar{x}, y; s)\, ds dy$$
$$= \tfrac{1}{2} \bar{\Delta}(e^{-\rho S(\bar{x})} \tau). \tag{6.8}$$

for $\bar{x}$ in $I(\epsilon)$ and $0 < \tau$. The estimates (6.4) through (6.8) show for every $I(\epsilon)$ there exists $S(\epsilon)$ such that

$$\limsup_{t \to \infty} \Delta(x, t, \sigma) \leqslant \bar{\Delta}(\tau, \epsilon) \leqslant \bar{\Delta}(e^{-\rho S(\epsilon)} \tau, \epsilon) \tag{6.9}$$

for x in $I(\epsilon)$ and $0 < \sigma \leqslant \tau$.

We proceed to show $\bar{\Delta}(0+, \epsilon) = 0$. An application of (2.10) and (5.10) to the inequality

$$\Delta(x, t, \tau) \leqslant \left| 1 - \frac{1 - \psi(x, \sigma)}{\sigma m(x)} \right| + | \Pi(x, t) - 1 | + \left| \Pi(x, t) - \frac{1 - \psi(x, t, \sigma)}{\sigma m(x)} \right|$$

reduces our work to considering

$$\delta(x, t, \sigma) = \Pi(x, t) - \frac{1 - \psi(x, t, \sigma)}{\sigma m(x)} .$$

If we use (2.1) to represent $m(x, t)$ and (2.15) to represent $\psi(x, t, \sigma)$, we can show $\delta(x, t, \sigma)$ satisfies

$$\begin{aligned} \delta(x, t, \sigma) = {} & \frac{1}{m(x)} \int_I \int_0^t \left\{ \Pi(y, t-s) - \frac{1 - \psi(y, t-s, e^{-\rho s} \sigma)}{e^{-\rho s} \sigma m(y)} \right\} m(y)\, e^{-\rho s} \\ & \times k(x, y; s)\, ds dy \\ & + \frac{1}{m(x)} \int_I \int_0^t \left\{ \frac{1 - \psi(y, t-s, e^{-\rho s} \sigma)}{e^{-\rho s} \sigma m(y)} \mu(y, s) \right. \\ & \left. - \frac{1 - h[y, s, \psi(y, t-s, e^{-\rho s} \sigma]}{e^{-\rho s} \sigma m(y)} \right\} m(y)\, e^{-\rho s} k(x, y; s)\, ds dy \\ & + e^{-\rho t} \frac{b(x, t)}{m(x)} \left[1 - \frac{1 - \exp(-e^{-\rho t} \sigma)}{e^{-\rho t} \sigma} \right] \end{aligned} \tag{6.10}$$

for x in I-A, $0 < \sigma$ and $0 < t$.

For each $I(\epsilon)$ the function

$$\delta(u, \tau; \epsilon) = \sup \{ \delta(x, t, \sigma) \mid x \text{ in } I(\epsilon), 0 < \sigma \leqslant \tau, 0 \leqslant t \leqslant u \}$$

is finite and nondecreasing in τ and t. Designate the first and second terms in (6.10) by J_1 and J_2 and let $\bar{x}$ and $S(\bar{x})$ be so chosen as in the development for (5.12) and (6.5). Then for each $I(\epsilon)$, we can show by considering separately $t < S(\bar{x})$ and $t \geqslant S(\bar{x})$

$$\begin{aligned} J_1(x, \sigma, t) & \leqslant \frac{1}{m(\bar{x})} \delta(u, \tau, \epsilon) \int_I \int_0^{S(\bar{x})} m(y)\, e^{-\rho s} \mu(y, s)\, k(\bar{x}, y; s)\, ds dy \\ & \quad + \frac{1}{m(\bar{x})} \delta(u, e^{-\rho S(\bar{x})}) \int_I \int_{S(\bar{x})}^{\infty} m(y)\, e^{-\rho s} \mu(y, s)\, k(\bar{x}, y; s)\, ds dy \\ & \leqslant \tfrac{1}{2} \delta(u, \tau, \epsilon) + \delta(u, e^{-\rho S(\bar{x})} \tau, \epsilon) \end{aligned} \tag{6.11}$$

for x in $I(\epsilon)$, $0 < t \leqslant u$.

Denote the first factor in the integrand of the second term J_2 by J_{21}. It can be written

$$J_{21}(y, t, s, \eta) = \frac{1 - \psi(y, t-s; \eta)}{\eta m(y)} \left\{ \mu(y, s) - \frac{1 - h[y, s; \psi(y, t-s, \eta)]}{1 - \psi(y, t-s; \eta} \right\}.$$

The first factor can be estimated using (6.2) and the uniform boundedness of $\Pi(x, t)$. The second factor can be estimated using the function $\alpha(\sigma)$ given in (2.17), under the provisions of the following lemma.

Lemma 6.1. *When $\frac{1}{2} < \eta < 1$ then*

$$0 \leqslant \mu(x, t) - \frac{1 - h(x, t; \eta)}{1 - \eta} \leqslant \alpha[2(1-\eta)]\, \mu(x, t) \quad \textit{for} \quad x \textit{ in } I \quad \textit{and} \quad 0 \leqslant t.$$

Consequently there exists positive τ_1, c_1, and c_2 such that

$$J_{21}(y, t, s, \eta) \leqslant c_1 \alpha[c_2 \tau]$$

for y in I-A, $0 \leqslant s < t$ and $0 < \tau \leqslant \tau_1$; and so,

$$J_2(x, \sigma, t) \leqslant c_1 \alpha[c_2 \tau] \tag{6.12}$$

for x in I-A, $0 < \sigma \leqslant \tau_1$ and $0 \leqslant t$.

Since $1 - (1 - e^{-\eta})/\eta \leqslant \min(1, \eta)$ for positive η there is a positive τ_2 such that the third term J_3 in 6.10 satisfies

$$J_3(x, t; \sigma) \leqslant c_3(t)\, \tau, \qquad c_3(\epsilon) = \max \left\{ \frac{1}{m(x)} \mid x \text{ in } I(\epsilon) \right\} \tag{6.13}$$

for x in $I(\epsilon)$, $0 \leqslant \sigma \leqslant \tau \leqslant \tau_2$ and $0 \leqslant t$.

Taking sup. of (6.10) over x in $I(\epsilon)$, $0 < \sigma \leqslant \tau$ and $0 \leqslant t \leqslant u$, the estimates (6.11)-(6.13) show for each $I(\epsilon)$ there exists a positive $S(\epsilon)$ such that

$$\delta(u, \tau, \epsilon) \leqslant \delta(u, e^{-\rho S(\epsilon)}\tau, \epsilon) + 2c_1 \alpha[c_2 \tau] + 2c_3(\epsilon)\, \tau$$

for x in $I(\epsilon)$, $0 \leqslant u$ and $0 < \tau \leqslant \min(\tau_1, \tau_2)$. If this is iterated we obtain for each N ($N = 1, 2, 3, \cdots$),

$$\delta(u, \tau, \epsilon) \leqslant \delta(u, e^{-N\rho S(\epsilon)}, \epsilon) + 2c_1 \sum_{k=0}^{N} \alpha[c_2 e^{-k\rho S(\epsilon)}\, \tau] + 2c_3(\epsilon) \sum_{k=0}^{N} e^{-k\rho S(\epsilon)}\, \tau. \tag{6.14}$$

Theorem 1 shows

$$\lim_{\sigma \to \infty +} \frac{1 - \psi(x, t, \sigma)}{\sigma m(x)} = \Pi(x, t)$$

uniformly for x in $I(\epsilon)$ and $0 \leqslant t \leqslant u$. Therefore for each positive u, $\delta(0+, u) = 0$; and so, letting $N \to \infty$ in (6.14) gives

$$\delta(x, \sigma, t) \leqslant \delta(u, \tau, \epsilon) \leqslant 2c_1 \sum_{k=0}^{\infty} \alpha[c_2 e^{-k\rho S(\epsilon)} \tau] + 2c_3(\epsilon) \tau[1 - e^{-\rho S(\epsilon)}]^{-1}$$

for x in $I(\epsilon)$, $0 < \sigma \leqslant \tau \leqslant \tau_0$ and $0 < t \leqslant u$. As previously shown, (5.7), the assumed properties for $\alpha(\sigma)$ are sufficient to show

$$\sum_{k=0}^{\infty} \alpha[c_2 e^{-k\rho S(\epsilon)} \tau] \leqslant \frac{1}{\rho S(\epsilon)} \int_0^{c_2\tau} \frac{\alpha(u)}{u} du$$

which tends to zero with τ. This shows $\bar{\Delta}(0+, \epsilon) = 0$.

An iteration of (6.9) gives for each N, $N = 1, 2, 3, \cdots$, and $I(\epsilon)$

$$\limsup_{t\to\infty} \Delta(x, \sigma, t) \leqslant \bar{\Delta}(e^{-N\rho S(\epsilon)} \tau, \epsilon).$$

Letting $N \to \infty$ shows for each $I(\epsilon)$ and positive τ

$$\limsup_{t\to\infty} \Delta(x, t, \sigma) \leqslant \bar{\Delta}(0+, \epsilon) = 0$$

for x in $I(\epsilon)$ and $0 < \sigma \leqslant \tau$. Since $\psi(x, 0, t) = \psi(x, 0) = 1$ for x in I and $\psi(x, \sigma, t) = \psi(x, \sigma) = 1$ for x in A and $0 \leqslant \sigma$, we have for each x in I and $0 \leqslant \sigma$

$$\lim_{t\to\infty} \psi(x, \sigma, t) = \psi(x, \sigma)$$

and uniformly for x in closed subsets of I-A and $0 \leqslant \sigma \leqslant \sigma_0 < \infty$.

We give a proof for Lemma 6.1. As shown by the work preceding (3.14), the function $\mu(x, t) - (1 - h(x, t; \eta))/(1 - \eta)$ is nonnegative for $0 \leqslant \eta < 1$. Setting $\sigma = -\log \eta$, we have by (2.17)

$$0 \leqslant \mu(x, t) - \frac{1 - h(x, t; \eta)}{1 - \eta} \leqslant \mu(x, t) - \frac{1 - h[x, t; e^{-\sigma}]}{\sigma} - \alpha[-\log \eta] \mu(x, t)$$

Since $-\log \eta \leqslant 2[1 - \eta]$ for $\frac{1}{2} < \eta \leqslant 1$ and since $\alpha(\sigma)$ is a nondecreasing function of σ, $\alpha[-\log \eta] \leqslant \alpha[2(1 - \eta)]$ for $\frac{1}{2} \leqslant \eta \leqslant 1$. This and the previous inequality complete the proof.

7. Proof for Theorem 5

Let $U(x, t; u) = G(x, t; e^{-\rho t})$ where $G(x, t; u)$ is defined by (2.13). The L.-S. transform of $U(x, t; u)$ is $\psi(x, t; z)$, (2.14), and satisfies (2.15). If for some x in I $U(x, t; u)$ does not converge weakly to a limit distribution there exists, by application of the selection principle, two weakly convergent subsequences with distinct limit distributions and limit transforms $\psi^{(1)}(x, z)$ and $\psi^{(2)}(x, z)$. Theorem 4 shows $\psi^{(1)}(x, \sigma) = \psi^{(2)}(x, \sigma)$ for $0 < \sigma$, implying $\psi^{(1)} = \psi^{(2)}$ since each is analytic for Re $(z) > 0$. This contradiction implies the existence of a family of distribution functions $U(x; u)$, $0 \leqslant u < \infty$, with parameter x in I such that $U(x, t; u)$ converge weakly to $U(x, u)$ for each x in I. The continuity theorem for L.-S. transforms shows this is equivalent to

$$\lim_{t\to\infty} \psi(x, t; z) = \int_{0-}^{\infty} e^{-zu}\, dU(x, u) \equiv \Phi(x, z).$$

Theorem 5 shows $\Phi(x, \sigma) = \psi(x, \sigma)$ so that $\Phi(x, \sigma)$ satisfies the functional equation (2.20) and has the property (2.19). Since $|\psi(x, z)| \leqslant 1$, the r.h.s. of (2.20) is continuous in x and z and analytic in z with Re $(z) > 0$ for fixed x in I. Therefore by the law of permanancy of functional equations $\psi(x, z)$ satisfies (2.20) for Re $(z) > 0$ and x in I.

References

1. H. E. Conner. Extinction probabilities for Age and position-dependent branching processes. *J. Soc. Ind. Appl. Math.*, in press.
2. T. E. Harris. "The Theory of Branching Processes," Die Grundlehren der Mathematischen Wissenschaften, Band 119. Springer, Berlin, 1963.
3. R. Bellman and T. E. Harris. On age-dependent branching processes. *Ann. Math.* 55 (1952), 280.
4. M. G. Krein and M. A. Rutman. Linear operators leaving invariant a Cone in a Banach space. *Usp. Mat. Nauk* 3 (1948), 1 (Russian), Am. Math. Soc. Translation, No. 26, 1950.
5. N. Levinson. Limiting theorems for age-dependent branching processes. *Illinois J. Math.* 4 (1960), 100.
6. W. A. O'N. Waugh. An age-dependent birth and death process. *Biometrika* 42 (1955), 291.
7. B. A. Sevastyanov. Branching processes for particles diffusing in a bounded region with absorbing boundaries. *Theory of Probability and Applications* 3 (1958), 121, (English Translation *SIAM*).
8. H. E. Conner. A limit theorem for a position-dependent branching process. *J. Math. Anal. Appl.* 3 (1961), 560.
9. J. E. Moyal. The general theory of stochastic population processes. *Acta Math.* 108 (1092), 1.
10. S. R. Adke and J. E. Moyal. A birth, death, and diffusion process. *J. Math. Anal. Appl.* 5 (1963), 209.
11. N. Levinson. Limiting theorems for Galton-Watson branching processes. *Illinois J. Math.* 3 (1959), 554.
12. S. Saks and A. Zygmund. "Analytic Functions." Warzawa, Uakl. Polskiego Tow. Matematycznego, 1952 (Translated by E. J. Scott).
13. D. G. Kendell. On the role of variable generation time in the development of a stochastic birth process. *Biometrika* 35 (1948), 316.