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Shape sensitivity for the Laplace–Beltrami operator with singularities

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Abstract

This paper considers shape sensitivity analysis for the Laplace–Beltrami operator formulated on a two-dimensional manifold with a fracture. We characterize the shape gradient of a functional as a bounded measure on the manifold and decompose it into a "distributed gradient" supported on the manifold, plus a singular part that we derive as the limit of a "jump" through the crack and Dirac measures at the crack extremities. The important point is that we introduce a technique that is not dimension dependent, and makes no use of classical arguments such as the maximum principle or continuation uniqueness. The technique makes use of a family of envelopes surrounding the fracture which enable us to relax certain terms and to overcome the lack of regularity resulting from the presence of the fracture. We use the *min–max* differentiation in order to avoid taking the derivative of the state equation and to manage the crack's singularities. Therefore, we write the functional in a *min–max* formulation on a space which takes into account the hidden boundary regularity established by the tangential extractor method.

Keywords: Laplace–Beltrami operator; Flow; Fracture; Oriented distance; Shape derivative; Min–max derivation; Hidden regularity; Tangential extractor; Compactivor property; Kuratowski continuity

1. Introduction and motivation

This work concerns techniques for detecting a *fracture* contained in an elastic structure, usually a thin Shell. This study relies on the theory of intrinsic geometry

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(see [1,7,9]) and results on the Laplace–Beltrami operator established by Desaint–Zolesio (see [12]).

Here, we provide a new shape sensitivity result for a non-smooth case which is needed when dealing with control and shape optimization on non-smooth domains.

One of the aims of this paper is to get the boundary expression for the shape gradient of the cost functional $J((\omega\backslash\sigma)) = \frac{1}{2}\int_{(\omega\backslash\sigma)}(\Phi - \Phi_d)^2$ (given in Theorem 6.1), where ω is a bounded open subset of a C^2 two-dimensional manifold Γ (Γ in $\mathbb{R}^3$), with the relative boundary $\partial_\Gamma\omega$ denoted by $\partial\omega$, σ is a connected *fracture* contained in ω with ends s_1 and s_2. Φ is the solution to the tangential Neumann problem, associated to the Laplace–Beltrami operator, with right-hand side in L^2 formulated on the non-Lipschitizian open set $(\omega\backslash\sigma)$ and Φ_d is a given heat measure.

On the one hand, from a heuristic point of view, looking for the shape sensitivity consists in observing the perturbation effect on the solution defined in a perturbed domain. For this we adopt the so-called velocity method [16] in order to move the domain through the flow mapping associated to a vector field.

On the other hand, since we deal with an oriented compact manifold it is convinient to use the space topology generated by the so-called oriented distance function established by Delfour–Zolésio [7].

As a first step, we will investigate the tangential Neumann problem on a Lipschitizian manifold ω. In this case, we supply an existence result of the material derivative $\dot{\Phi}$ of the state as a unique solution of a variational problem. We begin by establishing a uniform a priori estimate and by the reflexivity property of the Sobolev space and a convergence in norms we get the result.

We recover the distributed shape gradient expression of the associated cost functional governed by the Lipschitizian domain $J(\omega)$ via the adjoint state.

We consider the piecewise smooth case (ω is C^2-piecewise). In order to supply the shape gradient boundary expression of the associated cost functional $J(\omega)$, and to avoid differentiating the state equation we use the *min–max* theory [13] through a hidden boundary regularity of the state provided by the *tangential extractor*, see [17]. This theory requires building a family of vector fields vanishing in a neighborhood of singularities.

We also give a continuity result for the tangential Neumann problem with respect to a family of *envelopes* surrounding the *fracture* σ. That allows us to avoid of the lack of regularity due to the fracture.

The shape gradient $dJ(\omega\backslash\sigma, V)$ turns out to be characterized by a distributed gradient supported on the closure of the *fracture* $\bar{\sigma}$ and the boundary γ, its expression is given as a sum of a distributed term on γ, a jump distributed term in $L^1(\sigma)$ plus Dirac measures at the two extremities s_i.

The second result deals with the shape boundary derivative. Indeed, according to the identity $\Phi'_\Gamma = \dot{\Phi}|_\Gamma - \nabla_\Gamma\Phi V(0)$ (see [16]), it transpires that Φ'_Γ is less regular than the material derivative $\dot{\Phi}$ of the state Φ, that point requires technicalities. On the one hand, we consider the smooth case. We characterize the shape boundary derivative Φ'_Γ of the state as the solution of a non-homogeneous elliptic tangential problem. Thereafter, we extend the previous result to the piecewise smooth case. On the other hand, we relax the gradient tangential, normal component, of the shape boundary derivative in the fractured case.

Finally, the last main result we prove the necessary optimality condition of the initial domain and we establish the existence of an optimal domain by using the Kuratowski continuity of the Sobolev spaces.

The techniques used allow us to deal with the situation in which the *fracture* σ needs not to be smooth.

2. Preliminaries

2.1. Velocity method

Let D be a smooth bounded domain of $\mathbb{R}^N$. We consider a regular open subset Ω of D. Its relative boundary will be denoted by Γ ; Γ is an oriented compact manifold. Let X be a given point of $\bar{D}$ and $t \in [0, \delta[$, where δ is a positif number. We define the point $x(t) = T_t(X)$ which moves on the trajectory $x \to x(t)$ with velocity $||\partial_t x(t)||$ equal to $||\partial_t T_t(X)||$,

$$T_t \in C^1([0, \delta[, C^1(D; \mathbb{R}^N)). \tag{1}$$

Let

$$V(t, x) = \frac{\partial T_t}{\partial t} \circ T_t^{-1}(x) \tag{2}$$

it follows that

$$V \in C^0([0, \delta[, C^1(D; \mathbb{R}^N)). \tag{3}$$

Conversely, it is possible to associate transformations T_t to some vector fields V satisfying (2).

Let $\mathscr{V}$ be the set of vector fields satisfying (3), with $\langle V(x,t), n_{\partial D}(x) \rangle = 0$ for $x \in \partial D$ almost everywhere and $V(x, t) = 0$ for all singular point x of ∂D. The transformation T_t is called the flow mapping associated to V.

We refer to [16] for the proof of the subsequent theorem.

Theorem 2.1. *We have the two following assertions*:

(i) *Let V be a vector field of $\mathscr{V}$. Transformations $T_t \in C^1([0, \delta[, C^1(D; \mathbb{R}^N))$ may be associated to V, moreover* (2) *holds.*
(ii) *Let T_t be a transformation satisfying* (1) *then there exists $V \in \mathscr{V}$ verifying* (2).

The transformations T_t is solution of the ordinary differential equation

$$\partial_t x(X, t) = V(x(X, t), t); \quad x(X, 0) = X.$$

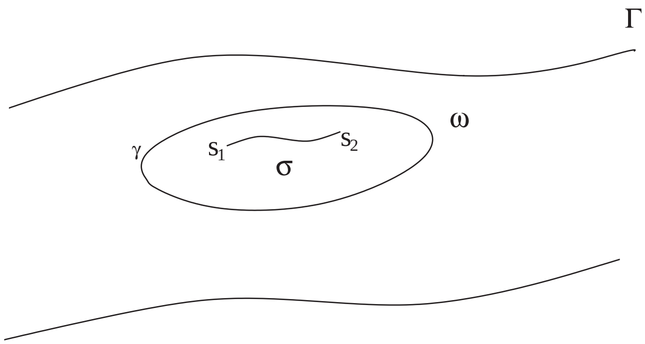

Fig. 1. Fractured manifold.

In the sequel, we point out that, in the general problem, an important issue is to keep the surface Γ fixed in the perturbation process. Such constraint is obviously solved by choosing, in a general setting, the speed vector field $V(t,x)$ tangent to the surface Γ: $V(t,x).n_\Gamma(x) = 0$.

We consider an open subset ω of Γ containing a fracture denoted by σ. The boundary of the open subset is also of class C^2. We design by n the out normal field on the surface Γ and by $\nu(X)$ the normal field on γ outside of ω contained in the tangent space to Γ at X (Fig. 1).

2.2. Intrinsic geometry

Given a bounded open set Ω in R^3 we consider its boundary Γ that we assume to be a C^2 manifold, n being the unitary outgoing normal field. We recall here some basic facts of intrinsic geometry from [7–9].

2.2.1. The oriented distance function

Definition 2.1. The oriented distance function is defined through R^3 as follows:

$$b_\Omega(x) = \begin{cases} d_\Gamma(x) & \text{if } x \in \bar{\Omega}^c, \\ -d_\Gamma(x) & \text{if } x \in \Omega. \end{cases}$$

Among the intrinsic geometrical properties of the oriented distance function we quote the following:

∇b_Ω is an extension of the normal field n on Γ.

Δb_Ω is the mean curvature H of the surface Γ (i.e. $H = \Delta b_\Omega|_\Gamma$).

2.2.2. The projection mapping

Let U be a tubular neighborhood of Γ given, for h small enough, by

$$U(\Gamma) = \{x \in D;\ \ |b_\Omega(x)| < h\};$$

we can associate to the oriented distance function b_Ω a projection mapping on the compact manifold Γ.

Definition 2.2. The projection mapping p is defined in [7] by

$$p : U \to \Gamma; p_\Gamma(x) = x - b_\Omega(x).\nabla b_\Omega(x)$$

2.2.3. Laplace–Beltrami operator

Definition 2.3. The Laplace–Beltrami operator is denoted by Δ_Γ and specified, in [3], for such a regular function φ by

$$\Delta_\Gamma \varphi = div_\Gamma\, \nabla_\Gamma \varphi$$

with

$\nabla_\Gamma \varphi = (\nabla\phi - \langle \nabla\phi, \nabla b_\Omega \rangle \nabla b_\Omega)|_\Gamma, \phi$ being any extension of φ to a neighborhood of Γ

and

$div_\Gamma\, e = (div\, E - \langle DE.\nabla b_\Omega, \nabla b_\Omega \rangle)|_\Gamma, E$ being any extension of e to a neighborhood of Γ.

2.3. The Neumann tangential problem

Let F be an element given in $H^{\frac{1}{2}+\delta}(D)$ such that $F \mid \omega = f$ and $F \mid \omega_t = f_t$.

We consider the tangential Neumann problem formulated in the fractured subset $\omega \backslash \sigma$:

$$\mathcal{N}T \begin{cases} -\Delta_\Gamma \Phi = f & \text{in } \omega\backslash\sigma, \\ \dfrac{\partial \Phi}{\partial v} = 0 & \text{on } \partial(\omega\backslash\sigma). \end{cases}$$

Lemma 2.1. *We notice that*

(1) *The previous problem has a unique solution in the following Hilbert space*

$$H^1_*(\omega\backslash\sigma) = \{v \in H^1(\omega\backslash\sigma);\ \ \langle v, 1 \rangle = 0\}$$

where $\langle v, 1 \rangle = 0$ *means that* $\int_{(\omega\backslash\sigma)} v = 0$.

(2) *The optimal regularity of the solution of the problem* $\mathcal{N}T$ *is* $H^{\frac{3}{2}-\mu}$ *with* $\mu > 0$, *and to* $H^2(\omega')$ *for every open subset* ω' *contained in* $(\omega \backslash \sigma)$ *with empty intersection with* $\bar{\sigma}$, *we refer to* [15].

The principal aim of this paper is to exhibit the shape gradient of the cost function J and to characterize the shape boundary derivative of the state. But because of the existence of the fracture the open subset $\omega \backslash \sigma$ is not Lipschitzian, this lack of regularity involves many technical problems.

As a first step, we will investigate the tangential Neumann problem successively in a Lipschitzian and a piecewise smooth domain. Therefore, we will approach the fractured manifold $\omega \backslash \sigma$ by a family of piecewise smooth domains.

3. Study over manifolds in several cases

Throught out this section we deal with the shape gradient expressions of the considered cost functional. In the first section we investigate the tangential Neumann problem on a Lipshitizian manifold ω. In this case, we supply an existence result of the material derivative $\dot{\Phi}$ of the state as an unique solution of a variational problem. We begin by establishing an uniform a priori estimate and by the reflexivity property of the Sobolev space and a convergence in norms we get the result.

We then recover the shape gradient distributed expression of the associated functional cost governed by the Lipshitizian domain $J(\omega)$ via the adjoint state.

Thereafter, in the second section we consider the piecewise smooth case (ω is C^2-piecewise). In order to supply the shape gradient boundary expression of the associated cost functional $J(\omega)$ we have to avoid differentiating the state equation that's why we use the *min–max* theory through a hidden boundary regularity of the state provided by the *tangential extractor*. This theory requires a building of a family of vector field vanishing in a neighborhood of singularities. thus, in this case it arises the shape gradient boundary expression is splitting in a continuous term and a pointwise one mapped on the singularities.

In the third section we deal with the fractured case. In fact the lack of regularity of the fractured manifold and so of the solution prevents us to have an optimal formulation for the shape functional, notably the shape gradient boundary expression. This suggests the introduction of a regularization in order to estimate the non-Lipschitzian open set by a family with parameter of piecewise smooth (and so Lipschitzian) open subsets via a family of *envelopes* surrounding the *fracture*. That allows us to get rid of the lack of regularity due to the fracture. Hence, we get the associated family of parametrized shape gradients. We establish a continuity result to the tangential Neumann problem with respect to the considered parameter smooth family.

Therefore, the shape gradient $dJ(\omega \backslash \sigma, V)$ turns out to be characterized by a distributed gradient supported on the closure of the *fracture* $\bar{\sigma}$ and the boundary γ, its expression is given as a sum of a distributed term on γ, a jump distributed term in $L^1(\sigma)$ plus a Dirac measures at the two extremities s_i.

4. Case of a Lipschitzian manifold

Definition 4.1. We define a Lipschitzian open subset ω in Γ assuming its relative boundary $\partial_\Gamma \omega = \gamma$ being Lipschitz continuous in the following sense:

There exists an into mapping $\lambda : [0,1] \to \mathbb{R}^3$ such that $\lambda \in Lip(]0,1[;\mathbb{R}^3)$, $\lambda([0,1]) = \gamma$ and $\lambda(0) = \lambda(1)$. λ is a parametrization of γ.

Throughout this section we assume ω to be a such Lipschitz open subset which is locally in Γ on one side of its boundary. The normal field v exists almost every where (for the $\mathscr{H}^1$ Hausdorff measure, see [11]) on γ.

We consider the tangential Neumann problem.

$$\mathscr{P}\begin{cases} -\Delta_\Gamma \Phi = f & \text{in } \omega, \\ \dfrac{\partial \Phi}{\partial v} = 0 & \text{on } \partial\omega. \end{cases}$$

Remark 4.1. Since the boundary $\partial\omega$ is Lipschitzian, the Green formula holds.

4.1. Shape analysis

4.1.1. Moving domain

We consider the parameter family of open subsets ω_t generated by the family of flows $T_t(V)$ associated with the vector field V. Thanks to the condition satisfied by V, the family of boundaries γ_t of ω_t is moving on the surface Γ.

From a heuristic point of view, looking for the shape sensitivity consists in observing the perturbation effect on the solution defined in $T_t(V)(\omega) = \omega_t$ when $t \to 0$. For this we perturb the domain ω by the transformation T_t; it follows that

$$\mathscr{P}_t\begin{cases} -\Delta_\Gamma \Phi_t = f_t & \text{in } \omega_t, \\ \dfrac{\partial \Phi_t}{\partial v_t} = 0 & \text{on } \partial\omega_t. \end{cases}$$

Remark 4.2. It is clear that, $\forall t \in [0, \delta[$, there exists a unique solution Φ_t of the perturbed problem under the condition $\int_{\omega_t} f_t = 0$, this motivates the choice of f_t in the following lemma.

Lemma 4.1. *Let F belongs to $H^{\frac{1}{2}+\delta}(D)$ such that $F_{|\omega} = f$ and $f_t = F_{|\omega_t} - \frac{1}{|\omega|}\int_{\omega_t} F$.*

Then the mapping: $t \to f^t = f_t \circ T_t$ is weakly differentiable in $H^{\delta-1}(\omega)$.

Furthermore: $\frac{f^t - f}{t} \to \nabla_\Gamma F.V(0)$ *weakly in* $H^{-1}(\omega)$. *The proof can be found in* [16].

Lemma 4.2. *We refer to* [12] *to introduce the Green's formula associated with a manifold* ω *having a boundary* γ. *Let* E *and* φ *be regular functions, it follows that*

$$-\int_\omega div_\Gamma\, E.\varphi = \int_\omega \{E.\nabla_\Gamma \varphi - H\varphi \langle E, n \rangle_{\mathbb{R}^N}\} - \int_\gamma \varphi \langle E, v \rangle,$$

where $H = \Delta b_\Omega|_\Gamma$ *is the mean curvature of the manifold* Γ.

4.2. Material derivative

We are interested in the sequel in establishing the existence of the material derivative $\dot{\Phi}$.

4.2.1. Existence of the material derivative of the state

In the following we intend to deal with the differentiability of the map: $t \to \Phi_t \circ T_t$ at zero.

Theorem 4.1. *The map*: $t \to \Phi_t \circ T_t$ *is differentiable at zero and its derivative* $\dot{\Phi} = \lim_{t \to 0} \frac{\Phi_t \circ T_t - \Phi}{t}$, *in* $H^1_*(\omega)$, *satisfies the equation*

$$\begin{aligned} \int_\omega \nabla_\Gamma \dot{\Phi} \nabla_\Gamma \psi = & \int_\omega \langle 2\varepsilon(V) \nabla_\Gamma \Phi, \nabla_\Gamma \psi \rangle \\ & - \int_\omega \langle div_\Gamma\, V(0) \nabla_\Gamma \Phi, \nabla_\Gamma \psi \rangle \\ & + \int_\omega div_\Gamma (f.V(0)) \psi. \end{aligned} \tag{4}$$

Remark 4.3. A direct way to get the existence and characterization of $\dot{\Phi} = \frac{\partial}{\partial t}(\Phi_t \circ T_t)|_{t=0}$ is to apply the implicit function theorem. This way, we would directly get the result concerning the material derivative if the right-hand side $F_{|_\Gamma}$ of the equation is assumed more regular than $L^2(\Gamma)$. Here, that $F_{|_\Gamma}$ belongs to $L^2(\Gamma)$ does not imply the strong convergence in $H^{-1}(\Gamma)$ of the quotient $\frac{f^t - f}{t}$, as would be required to apply the implicit function theorem. This lack of regularity requires a delicate proof for the existence of $\dot{\Phi}$. In [16] there are counter examples for which one can not expect the mapping to be strongly differentiable in $H^{-1}(\Gamma)$ for any f in $L^2(\Gamma)$.

Lemma 4.3. *Let $\delta > 0$ be a given real number. The mapping $t \to DT_t^{-1}$ is differentiable on $]0, \delta[$ and we have $\forall t \in]0, \delta[$, $\exists \alpha \in]0, 1[$ such that*

$$DT_t^{-1} = Id - t({}^*DT_{\alpha t}^{-1}(V) \circ T_{\alpha t}^{-1}.V(\alpha t)).$$

Lemma 4.4. *Let $\delta > 0$ be a given real number and $j(t)$ the associated Jacobian to the flow $T_t(V)$.*

The application $t \in [0, \delta[\to j(t) \in C^{k-1}$, $k \geqslant 1$ is differentiable and

$$j'(0) = div V(0) - (DV(0)n, n) = div_\Gamma \, V(0).$$

Proof of Lemma 4.4. The boundary Jacobian $j(t) = det(DT_t)||{}^*DT_t^{-1}.n||$ is differentiable for transformations $T_t(V)$ in $C^1([0, \delta[, C^2(\bar{D}, \mathbb{R}^N))$ and we have

$$\left.\frac{\partial||{}^*DT_t^{-1}.n||}{\partial t}\right|_{t=0} = -(DV(0)n, n)$$

$$\left.\frac{\partial det(DT_t)}{\partial t}\right|_{t=0} = div \, V(0). \qquad \square$$

4.2.2. Proof of Theorem 4.1

On the one hand, via Green's formula given in Lemma 4.2, the weak formulation of the perturbed problem is given by

$$\int_{\omega_t} \nabla_\Gamma \Phi_t \nabla_\Gamma \varphi_t = \int_{\omega_t} f_t \varphi_t; \quad \forall \varphi_t \in H^1_*(\omega_t) \tag{5}$$

whether return to the fixed domain:

$$\int_{\omega_t} \nabla_\Gamma \Phi_t \nabla_\Gamma \varphi_t = \int_{\omega} (\nabla_\Gamma \Phi_t) \circ T_t.(\nabla_\Gamma \varphi_t) \circ T_t \, j(t) \tag{6}$$

whence

$$(\nabla_\Gamma \Phi_t) \circ T_t = \nabla(\Phi_t \circ p) \circ T_t|_\Gamma = {}^*(DT_t)^{-1} \nabla(\Phi_t \circ p \circ T_t)|_\Gamma. \tag{7}$$

We notice, via a suitable choice of test functions φ_t, that only the tangential component of the vector $\nabla(\varphi_t \circ T_t)$ does not vanish.

In fact, let $\varphi_t = \psi \circ T_t^{-1}$ where ψ belongs to $H^{\frac{3}{2}}(D)$, so its trace on Γ is in $H^1(\Gamma)$, with $\frac{\partial \psi}{\partial n} = \langle \nabla \psi, n \rangle = 0$.

We note that

$$(\nabla \varphi_t) \circ T_t|_\Gamma = {}^*(DT_t)^{-1} \nabla \psi|_\Gamma = {}^*(DT_t)^{-1} \nabla_\Gamma \psi$$

then, due to $\langle \nabla(\Phi_t \circ p), n \rangle = 0$, we get

$$\langle (DT_t)^{-1}.^*(DT_t)^{-1}\nabla(\Phi_t \circ p \circ T_t), n \rangle = 0$$

this means that the vector $(DT_t)^{-1}.^*(DT_t)^{-1}\nabla(\Phi_t \circ p \circ T_t)$ is tangential.

For the sake of brevity, let us use this mere change of functions:

$$\theta^t = j(t)\Phi_t \circ p \circ T_t$$

which yields

$$\int_{\omega_t} \nabla_\Gamma \Phi_t \nabla_\Gamma \varphi_t = \int_\omega \langle D(t).\nabla(\theta^t), \nabla_\Gamma(\psi) \rangle, \tag{8}$$

where

$$j(t) = det(DT_t)||^*(DT_t)^{-1}.n||$$

and

$$D(t) = (DT_t)^{-1}.^*(DT_t)^{-1}.$$

On the other hand

$$\int_{\omega_t} f_t \varphi_t = \int_\omega f^t \psi j(t). \tag{9}$$

Definition 4.2. If the $\lim_{t\to 0} \frac{\theta^t - \theta}{t}$ exists strongly in $H^1_*(\omega)$ (denoted $\dot{\theta}$) we say that θ has a material derivative in the direction of the vector field V.

The sequel will be devoted to proving the existence of the material derivative $\dot{\theta}$ which provides the state with one.

Weak material derivative. Let $z^t = \frac{\theta^t - \theta}{t}$ in $H^1_*(\omega)$, which satisfies for all $\psi \in H^1(\omega)$:

$$\int_\omega \nabla_\Gamma z^t \nabla_\Gamma \psi = -\int_\omega \left\langle \frac{D(t) - I}{t} \nabla(\theta^t), \nabla_\Gamma(\psi) \right\rangle + \int_\omega \frac{f^t j(t) - f}{t} \psi. \tag{10}$$

By embedding the test function $\psi = z^t - l$ in (10) where l verifies

$$\int_\omega l = \int_\omega z^t, \quad \frac{\partial z^t}{\partial n} = \frac{\partial l}{\partial n} \quad \text{and} \quad \nabla_\Gamma l = \alpha \nabla_\Gamma z^t \text{ with } \alpha \neq 1$$

then we get

$$\int_\omega \nabla_\Gamma z^t \nabla_\Gamma z^t = \int_\omega \nabla_\Gamma z^t \nabla_\Gamma l - \int_\omega \left\langle \frac{D(t)-I}{t} \nabla(\theta^t), \nabla_\Gamma (z^t - l) \right\rangle + \int_\omega \frac{f^t j(t) - f}{t} (z^t - l) \tag{11}$$

which enables us to point out the following estimate, there exists a constant c independent on the parameter t such that

$$||\nabla_\Gamma z^t||_{L^2(\omega)} \leqslant c.$$

It follows that z^t is bounded in $H^1_*(\omega)$, so by a compacity argument one can extract a subsequence still denoted by z^t which converges weakly in the same space. Let $\dot\theta$ be this weak limit, it fulfills the below equation:

$$\int_\omega \nabla_\Gamma \dot\theta \nabla_\Gamma \psi = -\int_\omega \langle D'(0)\nabla_\Gamma \theta, \nabla_\Gamma \psi \rangle + \int_\omega [fj'(0) + \nabla F V(0)]\psi. \tag{12}$$

Obviously $\dot\theta$ is unique so the whole sequence z^t is weakly convergent to $\dot\theta$ in the space $H^1_*(\omega)$.

Strong material derivative. Via the same choice of test function we prove the convergence in norm, in fact

$$\begin{aligned}\lim_{t\to 0} ||\nabla_\Gamma z^t||^2_{L^2(\omega)} &= \int_\omega \nabla_\Gamma \dot\theta \nabla_\Gamma l - \int_\omega \langle D'(0)\nabla(\theta), \nabla_\Gamma(\dot\theta - l)\rangle \\ &\quad + \int_\omega [fj'(0) + \nabla F.V(0)]\,(\dot\theta - l) \\ &= ||\nabla_\Gamma \dot\theta||^2_{L^2(\omega)}.\end{aligned} \tag{13}$$

We conclude that $\frac{\theta^t - \theta}{t} \to \dot\theta$ strongly in $H^1_*(\omega)$, which provides the existence of the state $\dot\Phi$ in $H^1_*(\omega)$, indeed

$$\dot\theta = j'(0)\Phi \circ p + j(0)\dot\Phi \circ p.$$

Then by introducing this last identity in Eq. (12) we deduce that $\dot\Phi$ satisfies the following equation:

$$\begin{aligned}\int_\omega \nabla_\Gamma \dot\Phi \nabla_\Gamma \psi &= -\int_\omega \langle D'(0)\nabla_\Gamma \Phi, \nabla_\Gamma \psi \rangle \\ &\quad - \int_\omega \langle j'(0)\nabla_\Gamma \Phi, \nabla_\Gamma \psi \rangle \\ &\quad + \int_\omega [fj'(0) + \nabla_\Gamma F.V(0)]\psi \, \forall \psi \in H^1_*(\omega).\end{aligned} \tag{14}$$

Thus, the equation verified by $\dot{\Phi}$ is rewritten as

$$a(\dot{\Phi}, \psi) = l(\psi) \quad \forall \psi \in H^1_*(\omega),$$

where a is the coercive bilinear form given as follows:

$$a(\dot{\Phi}, \psi) = \int_\omega \nabla_\Gamma \dot{\Phi} \nabla_\Gamma \psi \tag{15}$$

and l is the following linear form:

$$\begin{aligned} l(\psi) = \int_\omega & - D'(0) \nabla_\Gamma \Phi \nabla_\Gamma \psi - \int_\omega j'(0) \nabla_\Gamma \Phi \nabla_\Gamma \psi \\ & + \int_\omega [fj'(0) + \nabla_\Gamma F.V(0)]\psi, \end{aligned} \tag{16}$$

where the expressions of $D'(0)$ and $j'(0)$ are given by Lemmas 4.3, 4.4:

$$-D'(0) = \{DV(0) + {}^*DV(0)\} = 2\varepsilon(V).$$

$\varepsilon(V)$ is the symmetrized of DV.

$$\nabla_\Gamma f.V(0) + f\, div_\Gamma\, V(0) = div_\Gamma (fV(0))$$

which achieves the proof of Theorem 4.1. □

4.3. Shape gradient distributed expression

According to the existence of the material derivative, we are able to provide the shape gradient $dJ(\omega, V)$.

Proposition 4.1. *The distributed expression of the shape gradient is given by*:

$$\begin{aligned} dJ(\omega, V) = & \int_\omega \left[\frac{1}{2} (\Phi - \Phi_d)^2 - \nabla_\Gamma \Phi \nabla_\Gamma P \right] div_\Gamma\, V(0) - \int_\omega (\Phi - \Phi_d) \nabla_\Gamma \Phi_d . V(0) \\ & + \int_\omega 2 \langle \varepsilon(V) \nabla_\Gamma \Phi, \nabla_\Gamma P \rangle + \int_\omega div_\Gamma (f\, V(0)) P, \end{aligned} \tag{17}$$

where P is the adjoint state.

Proof of Proposition 4.1. A mere change of variable in the cost functional expressed in ω_t leads to

$$J(\omega_t) = \frac{1}{2} \int_{\omega_t} (\Phi_t - \Phi_d)^2 = \frac{1}{2} \int_\omega (\Phi_t \circ T_t - \Phi_d)^2 j(t)$$

which yields

$$dJ(\omega, V) = \int_\omega (\Phi - \Phi_d)(\dot{\Phi} - \nabla_\Gamma \Phi_d.V(0)) + \frac{1}{2}\int_\omega (\Phi - \Phi_d)^2\, div_\Gamma\, V(0). \quad (18)$$

In order to eliminate the material derivative $\dot{\Phi}$ from the last expression, we use the following adjoint problem:

$$\begin{cases} -\Delta_\Gamma P = (\Phi - \Phi_d) & \text{in } \omega, \\ \dfrac{\partial P}{\partial v} = 0 & \text{sur } \partial\omega, \end{cases} \quad (19)$$

where P the adjoint of the state Φ, belongs to $H^1_*(\omega)$.

Then thanks to the conjugate form and Green's formula, it follows that

$$\begin{aligned} dJ(\omega, V) &= a(\dot{\Phi}, P) + \frac{1}{2}\int_\omega (\Phi - \Phi_d)^2\, div_\Gamma\, V(0) - \int_\omega (\Phi - \Phi_d)\nabla_\Gamma \Phi_d.V(0) \\ &= l(P) + \frac{1}{2}\int_\omega (\Phi - \Phi_d)^2\, div_\Gamma\, V(0) - \int_\omega (\Phi - \Phi_d)\nabla_\Gamma \Phi_d.V(0). \quad (20) \end{aligned}$$

Thus, via the expression of the linear form l we deduce the announced proposition. □

5. Shape gradient boundary expression

In this section we deal with the shape gradient boundary expression in the case of a piecewise smooth manifold. Which requires some technicalities, indeed, we will use a differentiation result provided by the *min–max* theory through a hidden boundary regularity of the state. Let ω be a piecewise smooth open subset of the manifold Γ containing m singularities s_i. We consider the same problem $\mathscr{P}$ and also the moving one $\mathscr{P}_t$. We note that all the results established in the Lipschitzian case hold, mainly Theorem 4.1 and Proposition 4.1.

5.1. Hidden boundary regularity

The hidden boundary regularity of the state Φ will be provided by the *extractor* method. Therefore, we introduce the *min–max* theory in order to avoid differentiating the state equation, it consists in establishing the saddle-points of the *Lagrangian* related to the state-adjoint coupled problem (of which Φ and P are solutions). We will need the hypothesis of a Dirichlet condition, let ($\mathscr{H}_1 : \Phi = 0$ on γ_0 with $\gamma_0 \subset \gamma$).

5.1.1. Extractor method

We begin by announcing the fundamental result.

Theorem 5.1. *We assume that Γ is a C^2 manifold and $\partial\omega$ a piecewise smooth curve. The state Φ has a hidden boundary regularity on $\partial\omega$.*

Indeed

$$\nabla_\Gamma \Phi \in L^2(\partial\omega).$$

We start with this technical lemma:

Lemma 5.1. *The set $C^2(\bar{\omega})$ is dense in $H^1_\Delta(\omega)$ where*

$$H^1_\Delta(\omega) = \{\varphi \in H^1(\omega);\ \textit{such that}\ \Delta_\Gamma \varphi \in L^2(\omega)\}.$$

Proof of Theorem 5.1. Let W be a vector field belonging to $C^1(\Gamma, \mathbb{R}^2)$ and satisfying the hypothesis of Theorem 1, such that $W.n = 0$; we associate with W the flow $T_s(W)$, s is a parameter lays in $[0, \delta[$. Thus, the tangential extractor related to W of the function sequence $\psi_n \in C^1(\bar{\omega})$ such that $(\psi_n, -\Delta_\Gamma \psi_n) \to (\Phi, f)$ strongly in $H^1(\omega) \times L^2(\omega)$, is given by

$$\begin{aligned}
\mathscr{E}_W(\psi_n) &= \frac{d}{ds}\left(\int_{T_s\omega} |\nabla_\Gamma(\psi_n \circ T_s^{-1})|^2 \right)_{|_{s=0}} \\
&= \int_\omega \langle [D'(0) + j'(0)I]\nabla_\Gamma \psi_n, \nabla_\Gamma \psi_n \rangle \\
&= \int_{\partial\omega} |\nabla_\Gamma \psi_n|^2 \langle W, v \rangle - 2 \int_\omega \Delta_\Gamma \psi_n \nabla_\Gamma \psi_n . W
\end{aligned}$$

and so

$$\begin{aligned}
\int_{\partial\omega} |\nabla_\Gamma \psi_n|^2 \langle W, v \rangle = &\int_\omega \langle [D'(0) + j'(0)I]\nabla_\Gamma \psi_n, \nabla_\Gamma \psi_n \rangle \\
&+ 2 \int_\omega \Delta_\Gamma \psi_n \nabla_\Gamma \psi_n . W.
\end{aligned}$$

We may choose W such that $0 < \alpha < W.v < \beta$ on $\partial\omega$, indeed we use here the fact that there is only a finite number of singular points. The mapping

$$\xi \in L^2 : \xi \to \left(\int_{\partial\omega} |\xi|^2 \langle W, v \rangle \right)^{\frac{1}{2}}$$

is a norm equivalent to the usual norm of $L^2(\partial\omega)$ which is weakly *lsc* one.

Therefore there exists $M > 0$ such that $\int_{\partial\omega} |\nabla_\Gamma \psi_n|^2 < M$; from the weak compacity of the closed ball in $L^2(\partial\omega)$, exists a subsequence $\overrightarrow{\xi}_{n_k} = \nabla_\Gamma \psi_{n_k}$ converging weakly to $\overrightarrow{\xi}$ in $L^2(\partial\omega)$.

It is required to prove that $\overrightarrow{\xi}$ is exactly $\nabla_\Gamma \Phi$. It is enough to use an integration by part result existing in [14], indeed let $\pi \in \mathbb{D}(\partial\omega)$ with $\{s_1, \dots, s_m\} \subset (\mathrm{supp}\pi)^c$, so

$$\int_{\partial\omega} \nabla_\Gamma \psi_{n_k} \nabla_\Gamma \pi = \int_{\partial\omega} \psi_{n_k} \mathscr{F}(\pi),$$

where $\mathscr{F}(\pi)$ is the adequate expression existing in [14]. So we also compute the limit with k, then under integration by part argument, one easily checks that:

$$\int_{\partial\omega} \overrightarrow{\xi}.\nabla_\Gamma \pi = \int_{\partial\omega} \nabla_\Gamma \Phi \nabla_\Gamma \pi \quad \forall \pi \in \mathbb{D}(\partial\omega)$$

which yields

$$\overrightarrow{\xi} = \nabla_\Gamma \Phi$$

thus, we conclude to the existence of a boundary hidden regularity of the state Φ on $\partial\omega$. Let

$$\nabla_\Gamma \Phi \in L^2(\partial\omega)$$

and achieve the proof. □

5.2. *The min–max theory*

We look for the boundary expression to the last shape gradient, given in Proposition 4.1, by the derivation method of the functional expressed in a *min–max* (this means that we consider the state equation as a constraint).

In order to apply this method the state Φ and the adjoint P have to be more regular than the variational regularity H^1 used in the last section. In fact, we just need the regularity of Φ and P only in neighborhoods of the points where the vector field V does not vanish. The boundary γ being piecewise so we have a finite number of singular points $s_1, \dots, s_m$. On these points Φ and P are not enough regular nevertheless, we can apply the *min–max* method where the points s_i are not moving. That is why we build a family of vector field V_n, through the vector field V, which vanishes in a neighborhood B_n of the singular points. Therefore, we get the boundary expression by *min–max* then we pass to the limit with n. For this we need to build the neighborhood B_n and the field V_n satisfying the below lemma. This lemma is the tool which allows us to pass to the limit in the shape gradient expression. Then we shall consider the convex set $\mathscr{K}^t$ undertaking the hidden boundary regularity provided by the *extractor*.

Lemma 5.2. *The vector field V being given with ($\mathscr{H}_2$; $V = 0$ on γ_0), let B_n be an union of m-neighborhoods B_n^i of the singularities s_i; $i = 1, \dots, m$.*

There exists a family of vector field V_n belongs to $W^{1,\infty}(\Gamma)$ such that: $V_n = 0$ on B_n. V_n is given by

$$V_n(x) = V(x) - \left(\sum_i y_n^i V \right)(x)$$

notice that V_n verifies also $\mathscr{H}_2$.

Where B_n^i is a neighborhood of s_i "of size $\frac{1}{n}$" in the sense developed below, y_n^i is a function with support $B^i_{\sqrt{n}}$ containing B_n^i such that

$$y_n^i \geqslant \chi_{B_n^i}$$

and $\chi_{B_n^i}$ is the characteristic function associated to B_n^i.

We give hereafter explicitly $y_n^i \circ (\xi^i)^{-1}$, where ξ^i is an associated projection

$$\xi^i : B^i_{\sqrt{n}} \to T_{s_i}\Gamma$$

such that $\xi^i(B_n^i)$ is the ball of $T_{s_i}\Gamma$ of which the ray is equal to $\frac{1}{n}$, where $T_{s_i}\Gamma$ being the tangent space to the manifold Γ on s_i.

The function $z_n^i = y_n^i \circ (\xi^i)^{-1}$ defined in the ball $\xi^i(B^i_{\sqrt{n}})$ can be chosen as follows:

$$z_n^i = \left(\frac{n}{1 - \sqrt{n}} \right) \rho - \frac{\sqrt{n}}{1 - \sqrt{n}}$$

therefore

$$y_n^i = \begin{cases} 1 & \text{in } B_n^i \\ z_n^i \circ \xi^i & \text{in } B^i_{\sqrt{n}} \end{cases}$$

Lemma 5.3. *We have the following convergence results:*

(i) *The vector field V_n is star-weakly convergent to V in $L^\infty(\omega)$*
(ii) *The vector field V_n converges all most everywhere to V in $W^{1,\infty}$.*

Proof of Lemma 5.3. The proof is a direct consequence from the fact that the function y_n^i is star-weakly convergent to zero in $L^\infty(\omega)$. □

Let us denote by $\omega_{t,n} = T_t(V_n)(\omega)$ the family of open subset generated by the flow $T_t(V_n)$ associated with the vector field V_n.

In the sequel, we introduce the Lagrangian saddle-points related to the coupled state-adjoint problem in order to derive a *min–max* formulation for the shape gradient.

5.2.1. Lagrangian and saddle-points

We refer to [2,13] for characterizing the saddle-points.

Proposition 5.1. *It is known that*

(φ, ψ) is a saddle-point of the Lagrangian if and only if (φ, ψ) is a solution of the coupled state-adjoint problem.

We come to

Lemma 5.4. *Let τ_t be the tangent vector to $(\partial\omega)$ and set*

$$\mathscr{K}^t = \{\varphi_t \in H^1(\Gamma) \cap H^2_{\text{loc}}(\Gamma \backslash B_n), \nabla_\Gamma \varphi_t.\tau_t \in L^2(\partial\omega_t)\varphi_t = 0 \ on \ \gamma_0\}$$

then the functional $J(\omega_{t,n}, V_n)$ is the solution of the min–max problem:

$$\begin{aligned} J(\omega_{t,n}) &= \frac{1}{2}\int_{\omega_{t,n}} (\Phi_{t,n} - \Phi_d)^2 \\ &= \min_{\varphi_t \in \mathscr{K}^t} \max_{\psi_t \in \mathscr{K}^t} \mathscr{L}^t(\varphi_t, \psi_t) \end{aligned}$$

with $\mathscr{L}^t$ the associated Lagrangian given by

$$\mathscr{L}^t(\varphi_t, \psi_t) = \frac{1}{2}\int_{\omega_{t,n}} (\varphi_t - \Phi_d)^2 + \int_{\omega_{t,n}} [\nabla_\Gamma \varphi_t \nabla_\Gamma \psi_t - f\psi_t].$$

In order to work over a fixed space, we carry out a classical change of functions. Let $\varphi = \varphi_t \circ T_t$ and $\psi = \psi_t \circ T_t$; it yields

Lemma 5.5. *Let $\mathscr{K}$ be the fixed space, then*

$$J(\omega_{t,n}) = \min_{\varphi \in \mathscr{K}} \max_{\psi \in \mathscr{K}} L^t(\varphi, \psi),$$

where

$$L^t(\varphi, \psi) = \frac{1}{2}\int_{\omega_{t,n}} (\varphi \circ T_t^{-1} - \Phi_d)^2 + \int_{\omega_{t,n}} [\nabla_\Gamma(\varphi \circ T_t^{-1})\nabla_\Gamma(\psi \circ T_t^{-1}) - f\psi \circ T_t^{-1}]$$

and

$$\mathscr{K} = \{\varphi \in H^1(\Gamma) \cap H^2_{\text{loc}}(\Gamma \backslash B_n), \nabla_\Gamma \varphi.\tau \in L^2(\partial\omega)\varphi = 0 \text{ on } \gamma_0\}.$$

5.2.2. Min–max differentiation

In order to obtain the boundary expression of the shape gradient of the functional J, we may use the following important theorem.

Theorem 5.2. *By applying min–max differentiation result (see [2]). The shape gradient has the following form:*

$$dJ(\omega, V_n) = \frac{\partial}{\partial t} L^t(\Phi,\ P)_{|_{t=0}}.$$

For the proof we refer to [6,10].

This leads

Lemma 5.6. *According to the previous results, the mapping: $t \in [0, \delta[\to \varphi \circ T_t^{-1}(V_n) \in H^1(\Gamma) \cap H^2_{\text{loc}}(\Gamma \backslash B_n)$ is continuous and differentiable in $H^1(\Gamma)$.*

Moreover

$$\lim_{t \to 0} \left\| \frac{\varphi \circ T_t^{-1}(V_n) - \varphi}{t} - (-\nabla_\Gamma \varphi . V_n(0)) \right\|_{H^1(\Gamma)} = 0.$$

Proof of Lemma 5.6. Under the regularity of φ and the continuity of the flow T_t^{-1}, the continuity of the previous mapping is obvious.

Concerning the differentiability, it will be deduced also from the same argument. Notice that

$$\varphi \circ T_t^{-1}(x) = \varphi(x) + \int_0^1 \nabla_\Gamma \varphi(x + s(T_t^{-1}(x) - x)).(T_t^{-1}(x) - x)\, ds$$

it follows that

$$\begin{aligned} \frac{1}{t}(\varphi \circ T_t^{-1}(x) - \varphi(x)) + (\nabla_\Gamma \varphi . V_n(0)) = & \int_0^1 [\nabla_\Gamma \varphi(x + s(T_t^{-1}(x) - x))] \\ & \times \frac{(T_t^{-1}(x) - x)}{t} ds - \int_0^1 \nabla_\Gamma \varphi(x) . \frac{(T_t^{-1}(x) - x)}{t} ds \\ & + \nabla_\Gamma \varphi . \left[\frac{(T_t^{-1}(x) - x)}{t} + V_n(0, x) \right]. \end{aligned}$$

We come to investigate the following limit when t tends to 0:

$$\lim_{t \to 0} \left(I_t = \left\| \frac{1}{t}(\varphi \circ T_t^{-1}(x) - \varphi(x)) + (\nabla_\Gamma \varphi . V_n(0)) \right\|_{H^1(\Gamma)} \right).$$

First, let us begin by the $L^2(\Gamma)$ norm. We denote it with I_t^1.

It follows that

$$(I_t^1)^2 \leqslant 2 \int_\Gamma \left| \left\{ \int_0^1 [\nabla_\Gamma \varphi(x + s(T_t^{-1}(x) - x)) - \nabla_\Gamma \varphi(x)]. \frac{(T_t^{-1}(x) - x)}{t} ds \right\} \right|^2 d\Gamma$$
$$+ 2 \int_\Gamma \left| \nabla_\Gamma \varphi. \left[\frac{(T_t^{-1}(x) - x)}{t} + V_n(0, x) \right] \right|^2 d\Gamma.$$

It is clear, according to the previous hypothesis and from the continuity of the mapping $t \to \nabla_\Gamma \varphi. \left[\frac{(T_t^{-1}(x)-x)}{t} + V_n(0, x) \right]$ in $H^1(\Gamma)$, that the second term tends to 0 with t. Then by applying Hôlder's inequality and Lebesgue's theorem, it arises that the first term is overestimated by

$$c \int_0^1 \left\{ \int_\Gamma \left| [\nabla_\Gamma \varphi_\varepsilon (x + s(T_t^{-1}(x) - x)) - \nabla_\Gamma \varphi(x)]. \frac{(T_t^{-1}(x) - x)}{t} \right|^2 d\Gamma \right\} ds.$$

Let $h(s,t) = \int_\Gamma |[\nabla_\Gamma \varphi(x + s(T_t^{-1}(x) - x)) - \nabla_\Gamma \varphi(x)]. \frac{(T_t^{-1}(x)-x)}{t}|^2 \, d\Gamma$, we remark that $h(s,t) \leqslant h(1,t) \; \forall s$.

Therefore $\lim_{t\to 0} I_t^1 = 0$.

Also the semi-norm $|.|_{1,\Gamma}$ denoted by I_t^2 converges to zero with t. In fact, it is clear that

$$(I_t^2)^2 \leqslant 2 \int_\Gamma \left| \nabla_\Gamma \left\{ \int_0^1 [\nabla_\Gamma \varphi_\varepsilon (x + s(T_t^{-1}(x) - x)) - \nabla_\Gamma \varphi(x)]. \frac{(T_t^{-1}(x) - x)}{t} ds \right\} \right|^2 d\Gamma$$
$$+ 2 \int_\Gamma \left| \nabla_\Gamma \left\{ \nabla_\Gamma \varphi_\varepsilon . \left[\frac{(T_t^{-1}(x) - x)}{t} + V_n(0, x) \right] \right\} \right|^2 d\Gamma.$$

Remark 5.1. It can be seen that we integrate entirely into Γ; indeed, since $\frac{(T_t^{-1}(x)-x)}{t} \simeq - V_n(0)$ when t tends to zero and due to the regularity of φ outside the singularities s_i, the integration domain is reduced to $(\Gamma \backslash B_n)$, which validates the previous expressions.

By using the same arguments as previously, it comes that I_t^2 converges to zero with t. This implies the convergence of I_t to zero with t.

Which achieves the proof of the lemma. □

5.3. Shape gradient boundary expression

We begin by giving a first boundary expression of the shape gradient through the *min–max* differentiation result, the second one will be given subsequently in the shape boundary derivative section. We characterize the shape gradient boundary expression as a distributed term on the manifold's boundary and a pointwise terms on the singularities s_i.

5.3.1. First expression

The following technical lemma given in [12] will be useful.

Lemma 5.7. *We consider a mapping*

$$t \in [0, \delta[\to u(t) = u_t \in H^1(\omega).$$

We suppose that $u(.)$ *is differentiable in* $H^1(\omega)$*. Then*

$$\frac{\partial}{\partial t}\left(\int_{\omega_t} u_t \right)\Big|_{t=0} = \int_\omega u'_\Gamma(\omega, V) + \int_\omega Hu\langle V(0), n\rangle + \int_{\partial\omega} u\langle V(0), \nu\rangle,$$

where u'_Γ *is the shape boundary derivative.*

By using the *min–max* differentiation result and the previous lemma, we get

$$\begin{aligned} dJ(\omega, V_n) = &\int_{\partial\omega} \left[\frac{1}{2}(\Phi - \Phi_d)^2 + \nabla_\Gamma\Phi\nabla_\Gamma P - fP\right]\langle V_n(0), \nu\rangle \\ &+ \int_\omega (\Phi - \Phi_d)(-\nabla_\Gamma\Phi V_n(0)) + \nabla_\Gamma P\nabla_\Gamma(-\nabla_\Gamma\Phi V_n(0)) \\ &+ \int_\omega \nabla_\Gamma\Phi\nabla_\Gamma(-\nabla_\Gamma P V_n(0)) - f(-\nabla_\Gamma P V_n(0)). \end{aligned}$$

The two last terms vanish since they represent the weak formulation of Φ and P in the test function $(-\nabla_\Gamma P V_n(0))$ and $(-\nabla_\Gamma\Phi V_n(0))$ which are vanishing on γ_0 with $V_n(0)$. We deduce the following lemma;

Lemma 5.8.

$$dJ(\omega, V_n) = \int_{\partial\omega} \left[\frac{1}{2}(\Phi - \Phi_d)^2 + \nabla_\Gamma\Phi\nabla_\Gamma P - fP\right]\langle V_n(0), \nu\rangle$$

5.3.2. Limit in the boundary expression

According to the hidden boundary regularity provided by the tangential extractor the function $\nabla_\Gamma\Phi\nabla_\Gamma P$ belongs to $L^1(\partial\omega)$. Since $V_n(0) \to V(0)$ in $L^\infty(\partial\omega)$ weak star-topology. Moreover $V_n(0)$ and $V(0)$ are in $L^\infty(\partial\omega)$. Hence, we deduce this result.

Lemma 5.9. *It easy to see that*

$$\lim_{n\uparrow\infty} \int_{\partial\omega} \nabla_\Gamma \Phi \nabla_\Gamma P \langle V_n(0), v\rangle = \int_{\partial\omega} \nabla_\Gamma \Phi \nabla_\Gamma P \langle V(0), v\rangle .$$

Since we have

$$dJ(\omega, V) = dJ(\omega, V_n) + dJ\left(\omega, \sum_i y_n^i V\right)$$

then

$$\lim_{n\uparrow\infty} dJ\left(\omega, \sum_i y_n^i V\right) = dJ(\omega, V) - \lim_{n\uparrow\infty} dJ(\omega, V_n)$$

so also

$$dJ\left(\omega, \sum_i y_n^i V\right) = dJ\left(\omega, \sum_i y_n^i [V - V(s_i)]\right) + dJ\left(\omega, \sum_i y_n^i V(s_i)\right)$$

thus

Lemma 5.10. *Accordingly*

$$\lim_{n\uparrow\infty} dJ\left(\omega, \sum_i y_n^i [V - V(s_i)]\right) = 0, \quad n\uparrow\infty .$$

Proof of Lemma 5.10. The proof is recovered from the distributed expression given in Proposition 4.1. In deed we have terms such as $\int_\omega \langle \varepsilon_\Gamma(y_n^i (V - V(s_i)))\nabla_\Gamma \Phi, \nabla_\Gamma P\rangle$, on the one hand from the construction of y_n^i the support of $y_n^i(V - V(s_i))$ is contained in $B_{\sqrt{n}}$ then it can be seen that $y_n^i(V - V(s_i))$ as well as $D_\Gamma[y_n^i(V - V(s_i))]$, $\varepsilon_\Gamma(y_n^i(V - V(s_i))), div_\Gamma(y_n^i(V - V(s_i)))$ converge to zero almost everywhere. On the other hand, y_n^i can be chosen (its behaviour in infinity with n) such that $||D_\Gamma[y_n^i(V - V(s_i))]||$ is uniformly bounded on ω : it exists $M > 0$;

$$D_\Gamma[y_n^i(V - V(s_i))]..D_\Gamma[y_n^i(V - V(s_i))] \leqslant M$$

then we get

$$|\langle \varepsilon_\Gamma(y_n^i(V - V(s_i))\nabla_\Gamma \Phi, \nabla_\Gamma P\rangle| \leqslant M \nabla_\Gamma \Phi \nabla_\Gamma P$$

therefrom by dominated convergence theorem the limit of the integral is zero. □

Lemma 5.11. *In view of the result of last section we deduce that the sequence of pointwise terms*

$$dJ\left(\omega, \sum_i y_n^i V(s_i)\right) = \sum_i \langle G_\omega^{i,n}, V(s_i) \rangle$$

has a limit, when $n \uparrow \infty$, *which is independent on the choice of the sequence* y_n^i. $G_\omega^{i,n}$ *is a vector given by the shape gradient distributed expression* (17):

$$G_\omega^{i,n} = \int_\omega \left[\frac{1}{2}(\Phi - \Phi_d)^2 + fP - \nabla_\Gamma \Phi \nabla_\Gamma P\right] \nabla_\Gamma y_n^i - \int_\omega [(\Phi - \Phi_d)\nabla_\Gamma \Phi_d + \nabla f] y_n^i + \int_\omega \langle \nabla_\Gamma y_n^i, \nabla_\Gamma P \rangle \nabla_\Gamma \Phi + \int_\omega \langle \nabla_\Gamma y_n^i, \nabla_\Gamma \Phi \rangle \nabla_\Gamma P. \tag{21}$$

As a consequence of these lemmas, it is easy to check that the shape gradient boundary expression is splitting in two terms; a continuous term and a pointwise one.

Proposition 5.2. *We have*

$$dJ(\omega, V) = \int_{\partial\omega} \left[\frac{1}{2}(\Phi - \Phi_d)^2 + \nabla_\Gamma \Phi \nabla_\Gamma P - fP\right] \langle V(0), v \rangle + \sum_i \langle G_\omega^i, V(s_i) \rangle_{\mathbb{R}^N},$$

where

$$G_\omega^i = \lim_{n \uparrow \infty} G_\omega^{i,n},$$

$$G_\omega^i = \lim_{n \uparrow \infty} \left\{ \int_\omega \left[\frac{1}{2}(\Phi - \Phi_d)^2 + fP - \nabla_\Gamma \Phi \nabla_\Gamma P\right] \nabla_\Gamma y_n^i + \int_\omega \langle \nabla_\Gamma y_n^i, \nabla_\Gamma P \rangle \nabla_\Gamma \Phi + \int_\omega \langle \nabla_\Gamma y_n^i, \nabla_\Gamma \Phi \rangle \nabla_\Gamma P \right\}. \tag{22}$$

It is relatively easy to establish the hereafter proposition.

Proposition 5.3. *There are two possible cases concerning the pointwise term.*

(i) *If the singularity order of the solutions* Φ *and* P *in a neighborhood of* s_i *is equal to* $\frac{1}{2}$ *then* G_ω^i *does not vanish* (*it corresponds to the flat case*, (*see also* [15])).
(ii) *If the previous order is different of* $\frac{1}{2}$ *then* G_ω^i *vanishes.*

6. Fractured manifold

The lack of regularity of the open set $(\omega\backslash\sigma)$ and so of the solution Φ prevents us to have an optimal formulation for the shape functional $J(\omega\backslash\sigma)$, notably the shape gradient boundary expression. This suggests the introduction of a regularization in order to estimate the non Lipschitzian open set $(\omega\backslash\sigma)$ by a family with parameter of piecewise smooth (and so Lipschitzian) open subsets $(\omega\backslash\sigma)_\varepsilon$. We thus get the associated family of parametrized shape gradients. Thereafter, we establish a continuity result for the Neumann problem with respect to the parameter. Therefrom, we recover the shape gradient distributed expression, as for the shape gradient boundary expression splits up into a distributed term on γ, a jump distributed term in $L^1(\sigma)$ plus a Dirac measure at the end points s_i.

6.1. Regularized problem

We *regularize* the domain $(\omega\backslash\sigma)$ by using a family, with parameter, of singular *envelopes* e_ε with extremities s_1, s_2 and surrounding the fracture σ, which will be defined subsequently. We denote by $(\omega\backslash\sigma)_\varepsilon$ the obtained regular open subset (the complementary of e_ε in $(\omega\backslash\sigma)$) in which we formulate the following homogeneous tangential problem.

$$(\mathcal{N}T)_\varepsilon \begin{cases} -\Delta_\Gamma \Phi_\varepsilon = f & \text{in } (\omega\backslash\sigma)_\varepsilon, \\ \dfrac{\partial \Phi_\varepsilon}{\partial \nu_\varepsilon} = 0 & \text{on } \partial(\omega\backslash\sigma)_\varepsilon. \end{cases}$$

Remark 6.1. The family, with parameter ε, of open subsets $(\omega\backslash\sigma)_\varepsilon$ is Lipschitzian. As a consequence Green's formula holds.

6.2. Shape analysis

Since we are over a piecewise smooth subset $(\omega\backslash\sigma)_\varepsilon$, all the previous results concerning the shape analysis given in Section 4 hold.

6.2.1. Material derivative

Let us consider the moving problem on $(\omega\backslash\sigma)_{\varepsilon,t}$ at each fixed ε:

$$(\mathcal{N}T)_{\varepsilon,t} \begin{cases} -\Delta_\Gamma \Phi_{\varepsilon,t} = f_t & \text{in } (\omega\backslash\sigma)_{\varepsilon,t}, \\ \dfrac{\partial \Phi_{\varepsilon,t}}{\partial \nu_{\varepsilon,t}} = 0 & \text{on } \partial(\omega\backslash\sigma)_{\varepsilon,t}. \end{cases}$$

We apply Theorem 4.1 and derive the following result.

Proposition 6.1. *At each fixed* ε, *the map*: $t \to \Phi_{\varepsilon,t} \circ T_t$ *is differentiable at zero and its derivative* $\dot{\Phi}_\varepsilon = \lim_{t\to 0} \frac{\Phi_{\varepsilon,t}\circ T_t - \Phi_\varepsilon}{t}$, *in* $H^1_*((\omega\backslash\sigma)_\varepsilon)$, *satisfies the equation*

$$\int_{(\omega\backslash\sigma)_\varepsilon} \nabla_\Gamma \dot{\Phi}_\varepsilon \nabla_\Gamma \psi = \int_{(\omega\backslash\sigma)_\varepsilon} \langle 2\varepsilon(V)\nabla_\Gamma \Phi_\varepsilon, \nabla_\Gamma \psi \rangle - \int_{(\omega\backslash\sigma)_\varepsilon} \langle div_\Gamma\, V(0)\nabla_\Gamma \Phi_\varepsilon, \nabla_\Gamma \psi \rangle + \int_{(\omega\backslash\sigma)_\varepsilon} [fj'(0) + \nabla_\Gamma F V(0)]\psi. \tag{23}$$

6.2.2. Shape gradient

We recall these results from previous sections.

Proposition 6.2. *The distributed expression of the shape gradient is given for each fixed* ε *by*:

$$dJ((\omega\backslash\sigma)_\varepsilon, V) = \int_{(\omega\backslash\sigma)_\varepsilon} \left[\frac{1}{2}(\Phi_\varepsilon - \Phi_d)^2 - \nabla_\Gamma \Phi_\varepsilon \nabla_\Gamma P_\varepsilon\right] div_\Gamma V(0) - \int_{(\omega\backslash\sigma)_\varepsilon} (\Phi_\varepsilon - \Phi_d)\nabla_\Gamma \Phi_d . V(0) + \int_{(\omega\backslash\sigma)_\varepsilon} 2\varepsilon(V)\nabla_\Gamma \Phi_\varepsilon \nabla_\Gamma P_\varepsilon + \int_{(\omega\backslash\sigma)_\varepsilon} div_\Gamma(f\ V(0))P_\varepsilon \tag{24}$$

and also

Proposition 6.3. *We have, for each fixed* ε, *the shape gradient boundary derivative.*

$$dJ((\omega\backslash\sigma)_\varepsilon, V) = \int_{\partial(\omega\backslash\sigma)_\varepsilon} \left[\frac{1}{2}(\Phi_\varepsilon - \Phi_d)^2 + \nabla_\Gamma \Phi_\varepsilon \nabla_\Gamma P_\varepsilon - fP_\varepsilon\right] \langle V(0), \nu_\varepsilon \rangle + \sum_i \langle G^i_{(\omega\backslash\sigma)_\varepsilon}, V(s_i) \rangle_{\mathbb{R}^N}, \tag{25}$$

where $G^i_{(\omega\backslash\sigma)_\varepsilon}$ *is a vector having an expression analogous to that of* G^i_ω *but written in* $(\omega\backslash\sigma)_\varepsilon$:

$$G^i_{(\omega\backslash\sigma)_\varepsilon} = \lim_{n\uparrow\infty} \left\{ \int_{(\omega\backslash\sigma)_\varepsilon} \left[\frac{1}{2}(\Phi_\varepsilon - \Phi_d)^2 + fP_\varepsilon - \nabla_\Gamma \Phi_\varepsilon \nabla_\Gamma P_\varepsilon\right] \nabla_\Gamma y^i_n + \int_{(\omega\backslash\sigma)_\varepsilon} \langle \nabla_\Gamma y^i_n, \nabla_\Gamma P_\varepsilon \rangle \nabla_\Gamma \Phi_\varepsilon + \int_{(\omega\backslash\sigma)_\varepsilon} \langle \nabla_\Gamma y^i_n, \nabla_\Gamma \Phi_\varepsilon \rangle \nabla_\Gamma P_\varepsilon \right\}$$

that element is independent on the choice of the function y^i_n.

6.3. Continuity of the Neumann problem

In this section, we study the behaviour of the shape gradient with respect to the parameter ε. It's obvious that we have to prove a strong convergence result. In order to get it, we will need to get the continuity of the Neumann tangential problem with respect to the open subset e_ε which will be chosen hereafter.

6.3.1. A priori estimate

Lemma 6.1. *We have the following estimates*:

$$||1_{(\omega\backslash\sigma)_\varepsilon}\nabla_\Gamma\Phi_\varepsilon||_{L^2(\omega\backslash\sigma)} \leqslant \lambda_\varepsilon^{\frac{-1}{2}}||f||_{L^2(\omega\backslash\sigma)}$$

$$||1_{(\omega\backslash\sigma)_\varepsilon}\Phi_\varepsilon||_{L^2(\omega\backslash\sigma)} \leqslant \lambda_\varepsilon^{\frac{-3}{2}}||f||_{L^2(\omega\backslash\sigma)}$$

where λ_ε is the first eigenvalue of the Laplace–Beltrami operator.

Proof of Lemma 6.1. By using Green's formula we establish the weak formulation associated with the regular problem, so

$$\int_{(\omega\backslash\sigma)_\varepsilon} \nabla_\Gamma\Phi_\varepsilon\nabla_\Gamma\varphi = \int_{(\omega\backslash\sigma)_\varepsilon} f\varphi \quad \forall\varphi\in H^1_*((\omega\backslash\sigma)_\varepsilon)$$

it yields

$$\int_{\omega\backslash\sigma} |1_{(\omega\backslash\sigma)_\varepsilon}\nabla_\Gamma\Phi_\varepsilon|^2 \leqslant ||f||_{L^2(\omega\backslash\sigma)}||\Phi_\varepsilon||_{L^2((\omega\backslash\sigma)_\varepsilon)}.$$

Thanks to Poincare's inequality given by the space $H^1_*((\omega\backslash\sigma)_\varepsilon)$, we come to the result. □

Because of the dependence of the second term on ε we are not able to get an uniform estimate. A particular choice of the *envelope* e_ε enables us to overcome this difficulty.

6.3.2. Choice of the Envelope e_ε

The *envelope* e_ε will be the open subset whose boundary is the convict of the fracture σ by the T_t at $t=\varepsilon$ associated to the non autonomous vector field $E_\sigma = (E^+_\sigma, E^-_\sigma)$. The field E_σ satisfies the following conditions:

$E^+_\sigma \in C^k(\overline{\omega\backslash\sigma_+})$, $E^-_\sigma \in C^k(\overline{\omega\backslash\sigma_-})E_\sigma.n = 0$, $[E_\sigma]\neq 0$ on σ and $E^+_\sigma.v > 0$ therefore $E^-_\sigma.v < 0$ on σ. Et $E_\sigma(s_i) = 0$, where s_i are the extremities of the fracture σ. So also $\overline{(\partial e_\varepsilon)}_+ = T_\varepsilon(E^+_\sigma)(\sigma_+)$, et $\overline{(\partial e_\varepsilon)}_- = T_\varepsilon(E^-_\sigma)(\sigma_-)$. We denote by $\partial e_\varepsilon = \overline{(\partial e_\varepsilon)}_+ \cup \overline{(\partial e_\varepsilon)}_- =$

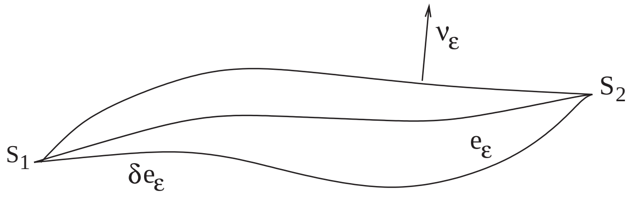

Fig. 2. Building of the *Envelope*.

$T_\varepsilon(E_\sigma)(\sigma)$. It is clear that $\partial(e_\varepsilon)_+$ and $\partial(e_\varepsilon)_-$ are two C^∞-manifolds, which enables us to control the first eigenvalue of the Laplace–Beltrami operator (Fig. 2).

6.3.3. Boundeness of the first eigenvalue

We begin by giving this result:

Proposition 6.4. *Let λ_ε be the first eigenvalue of the Laplace–Beltrami operator*, *i.e.*

$$\lambda_\varepsilon = inf\left\{c_\varepsilon; c_\varepsilon \int_{(\omega\backslash\sigma)_\varepsilon} v_\varepsilon^2 \leqslant \int_{(\omega\backslash\sigma)_\varepsilon} |\nabla_\Gamma v_\varepsilon|^2 \forall v_\varepsilon \in H_*^1((\omega\backslash\sigma)_\varepsilon)\right\}$$

so

(i) *there exists $\varphi_\varepsilon \in H_*^1((\omega\backslash\sigma)_\varepsilon)$, $\int_{(\omega\backslash\sigma)_\varepsilon}(\varphi_\varepsilon)^2 = 1$ such that*

$$\lambda_\varepsilon = \int_{(\omega\backslash\sigma)_\varepsilon} |\nabla_\Gamma \varphi_\varepsilon|^2.$$

(ii) *Under hypothesis $\mathscr{H}_1$, for all ε, λ_ε is underestimated via λ. Where λ is the first eigenvalue of the Laplace–Beltrami operator formulated in $(\omega\backslash\sigma)$.*

In order to prove the above result we have to specify the domain's topology.

Remark 6.2. The open subset $(\omega\backslash\sigma)_\varepsilon$ converges with ε to $(\omega\backslash\sigma)$ for the Hausdorff complementary topology endowed with the metric (see also [1,3])

$$d_{H^c}(\omega_1, \omega_2) = d_H(\bar{\omega}\backslash\omega_1, \bar{\omega}\backslash\omega_2),$$

where

$$d_H(K_1, K_2) = \max\left\{\sup_{x\in K_1} \inf_{y\in K_2} |x-y|, \sup_{y\in K_2} \inf_{x\in K_1} |x-y|\right\}.$$

is the Hausdorff distance between two closed subsets of the open set ω.

We say that $(\omega\backslash\sigma)_\varepsilon$ converges in the measure sense to $(\omega\backslash\sigma)$ if the corresponding characteristic functions converge strongly in $L^1(\omega)$.

Proof of Proposition 6.4. (i) It is sufficient to consult [17].

(ii) The proof is deduced directly from hypothesis $\mathscr{H}_1$ and the following equivalence

$$\varphi_\varepsilon \in H^1_{\gamma_0}((\omega\backslash\sigma)_\varepsilon) \;\Leftrightarrow\; \varphi_\varepsilon \circ T_\varepsilon \in H^1_{\gamma_0}(\omega\backslash\sigma)$$

with $j(\varepsilon) \to 1$ when $\varepsilon \downarrow 0$. □

Corollary 6.1. *The sequences* $1_{(\omega\backslash\sigma)_\varepsilon}\Phi_\varepsilon$ *et* $1_{(\omega\backslash\sigma)_\varepsilon}\nabla_\Gamma\Phi_\varepsilon$ *are uniformly bounded in* $L^2(\omega\backslash\sigma)$ *with respect to* ε.

6.3.4. Strong convergence

The last proposition enables us to obtain the following result.

Proposition 6.5. *The sequences* $1_{(\omega\backslash\sigma)_\varepsilon}\Phi_\varepsilon$ *and* $1_{(\omega\backslash\sigma)_\varepsilon}\nabla_\Gamma\Phi_\varepsilon$ *converge strongly respectively to* $1_{(\omega\backslash\sigma)}\Phi$ *and* $1_{(\omega\backslash\sigma)}\nabla_\Gamma\Phi$.

Proof of Proposition 6.5. Thanks to Corollary 6.1, a compacity argument yields to extract two subsequences denoted still further $1_{(\omega\backslash\sigma)_\varepsilon}\Phi_\varepsilon$ and $1_{(\omega\backslash\sigma)_\varepsilon}\nabla_\Gamma\Phi_\varepsilon$ converging in $L^2(\omega\backslash\sigma)$ respectively to μ and $\overrightarrow{\theta}$.

In order to prove that $\overrightarrow{\theta} = \nabla_\Gamma\mu$, we will adopt the compactivor property which consists in that the open set $(\omega\backslash\sigma)_\varepsilon$ soaks up all compacts in the open set $(\omega\backslash\sigma)$.

Indeed, for any compact $K \subset \omega\backslash\sigma$, $\exists\, n_k$ such that $\forall n \geqslant n_K$ we have $K \subset (\omega\backslash\sigma)_{\varepsilon_n}$.

Let $\overrightarrow{\varphi} \in \mathbb{D}(\omega\backslash\sigma)$ whose support is K; then $\exists n_K$ such that for any $n \geqslant n_K$, $\varphi \in ID((\omega\backslash\sigma)_{\varepsilon n})$, which yields

$$\begin{aligned}\int_{\omega\backslash\sigma}\overrightarrow{\theta}.\overrightarrow{\varphi} &= \lim_{\varepsilon\to 0}\int_{\omega\backslash\sigma} 1_{(\omega\backslash\sigma)_\varepsilon}\nabla_\Gamma\Phi_\varepsilon\overrightarrow{\varphi} = \lim_{\varepsilon\to 0}\int_{(\omega\backslash\sigma)_\varepsilon}\nabla_\Gamma\Phi_\varepsilon\overrightarrow{\varphi}\\ &= \lim_{\varepsilon\to 0} -\int_{(\omega\backslash\sigma)_\varepsilon}\Phi_{\varepsilon_n}\, div_\Gamma\,\overrightarrow{\varphi} = \lim_{\varepsilon\to 0} -\int_{\omega\backslash\sigma} 1_{(\omega\backslash\sigma)_\varepsilon}\Phi_\varepsilon\, div_\Gamma\,\overrightarrow{\varphi}\\ &= -\int_{\omega\backslash\sigma}\mu\, div_\Gamma\,\overrightarrow{\varphi} = \int_{\omega\backslash\sigma}\nabla_\Gamma\mu.\overrightarrow{\varphi}.\end{aligned} \tag{26}$$

We set that

$$\overrightarrow{\theta} = \nabla_\Gamma\mu.$$

We should get the problem of which μ is solution. It arises by passing to the limit in the weak formulation having Φ_ε for solution. So

$$\int_{\omega\backslash\sigma} \nabla_\Gamma \mu . \nabla_\Gamma \varphi = \int_{\omega\backslash\sigma} f\varphi \forall \varphi \in H^1_*(\omega\backslash\sigma).$$

It follows that μ is the solution of the homogeneous Neumann problem posed in $\omega\backslash\sigma$. Therefore,by uniqueness, μ is equal to Φ.

Thus, all the sequence Φ_ε converges weakly to μ, so also to Φ.

As for strong convergence, it will be obtained also from the weak formulation. Indeed, let Φ_ε be the test function. Then

$$\int_{(\omega\backslash\sigma)_\varepsilon} |\nabla_\Gamma \Phi_\varepsilon|^2 = \int_{(\omega\backslash\sigma)_\varepsilon} f\Phi_\varepsilon$$

but the right-hand side converges to $\int_{\omega\backslash\sigma} f\Phi$, which is equal to $\int_{\omega\backslash\sigma} |\nabla_\Gamma \Phi|^2$. Hence the convergence in norms in $H^1_*(\omega\backslash\sigma)$ and so the strong convergence. Thus, the proof is achieved. □

Remark 6.3. The Neumann problem is continuous with respect to the perturbation $T_\varepsilon(E_\sigma)$.

Corollary 6.2. *We have the same convergence result for the adjoint problem of which P_ε is solution. Let P be the corresponding limit.*

6.4. Shape gradient convergence

Given the previous results, we are interested in computing the limit of the shape gradient $dJ((\omega\backslash\sigma)_\varepsilon, V)$ when ε tends to zero. This result will be provided from the continuity of the tangential Neumann problem.

Proposition 6.6. *The distributed gradient expression converges and its limit is given by*

$$\begin{aligned}\lim_{\varepsilon\to 0} dJ((\omega\backslash\sigma)_\varepsilon, V) = &\int_{\omega\backslash\sigma} 2\varepsilon(V)\nabla_\Gamma\Phi\nabla_\Gamma P + \int_{\omega\backslash\sigma} div_\Gamma\ V(0)\nabla_\Gamma\Phi\nabla_\Gamma P\\ &+ \int_{\omega\backslash\sigma} (\Phi-\Phi_d)^2\, div_\Gamma\ V(0) - \int_{\omega\backslash\sigma} (\Phi-\Phi_d)\nabla_\Gamma\Phi_d . V(0)\\ &+ \int_{\omega\backslash\sigma} div_\Gamma(f\ V)P\end{aligned}$$

and we have

$$dJ(\omega\backslash\sigma, V) = \lim_{\varepsilon\to 0} dJ((\omega\backslash\sigma)_\varepsilon, V).$$

Proof of Proposition 6.6. Obviously we have the Hausdorff convergence of the open subset $(\omega\backslash\sigma)_\varepsilon$ to $\omega\backslash\sigma$ with ε. From the previous continuity we can check:

$$\lim_{\varepsilon\to 0} l(P_\varepsilon) = \int_{\omega\backslash\sigma} div_\Gamma\, V(0)\nabla_\Gamma\Phi\nabla_\Gamma P + \int_{\omega\backslash\sigma} 2\varepsilon(V)\nabla_\Gamma\Phi\nabla_\Gamma P + \int_{\omega\backslash\sigma} div_\Gamma(f\, V(0))P,$$

$$\lim_{\varepsilon\to 0} \int_{(\omega\backslash\sigma)_\varepsilon} (\Phi_\varepsilon - \Phi_d)^2\, div_\Gamma\, V(0) = \int_{\omega\backslash\sigma} (\Phi - \Phi_d)^2\, div_\Gamma\, V(0),$$

$$\lim_{\varepsilon\to 0} \int_{(\omega\backslash\sigma)_\varepsilon} (\Phi_\varepsilon - \Phi_d)\nabla_\Gamma\Phi_d.V(0) = \int_{\omega\backslash\sigma} (\Phi - \Phi_d)\nabla_\Gamma\Phi_d.V(0)$$

then thanks to the homogeneous boundary Neumann condition on $\partial(\omega\backslash\sigma)$ the material derivative of the state Φ exists and so we deduce the continuity result for the shape gradient with respect to ε. □

6.5. Jump through the crack

We have the splitting $\partial(\omega\backslash\sigma)_\varepsilon = \gamma \cup \overline{(\partial e_\varepsilon)_+} \cup \overline{(\partial e_\varepsilon)_-}$; then, by passing to the limit in the shape gradient $dJ((\omega\backslash\sigma)_\varepsilon, V)$ with ε we provide the shape gradient boundary expression.

Proposition 6.7. *Let*

$$g_\varepsilon = \left\{ [\nabla_\Gamma\Phi^\varepsilon\nabla_\Gamma P^\varepsilon] + \frac{1}{2}[(\Phi - \Phi_d)^2] - f[P] \right\}$$

then we have

$$\begin{aligned} dJ(\omega\backslash\sigma, V) = \lim_{\varepsilon\to 0} &\int_\sigma g_\varepsilon \langle V(0), \nu\rangle\, d\sigma + \frac{1}{2}\int_\gamma (\Phi - \Phi_d)^2 \langle V(0), \nu\rangle\, d\gamma \\ &- \int_\gamma fP\langle V(0), \nu\rangle\, d\gamma + \int_\gamma \nabla_\Gamma\Phi\nabla_\Gamma P\langle V(0), \nu\rangle\, d\gamma \\ &+ \sum_i \langle G^i_{(\omega\backslash\sigma)}, V(s_i)\rangle_{\mathbb{R}^N} \end{aligned} \tag{27}$$

with

$$[\nabla_\Gamma\Phi^\varepsilon\nabla_\Gamma P^\varepsilon]_\sigma = \nabla_\Gamma\Phi^\varepsilon_+\nabla_\Gamma P^\varepsilon_+ - \nabla_\Gamma\Phi^\varepsilon_-\nabla_\Gamma P^\varepsilon_-,$$

where

$$\Phi^\varepsilon_\pm = \Phi_\varepsilon\circ T_\varepsilon(E_\pm)P^\varepsilon_\pm = P_\varepsilon\circ T_\varepsilon(E_\pm)$$

and

$$G^i_{(\omega\backslash\sigma)} = \lim_{\varepsilon\downarrow 0} G^i_{(\omega\backslash\sigma)_\varepsilon}.$$

Proof of Proposition 6.7. It is enough to notice that: $v_\varepsilon \circ T_\varepsilon = ||{}^*D_\Gamma T_\varepsilon^{-1}.v||^{-1}\,{}^*D_\Gamma T_\varepsilon^{-1}.v$ and to remark, when ε tends to 0, that:

(i) $j(\varepsilon) = \det(DT_\varepsilon)||{}^*D_\Gamma T_\varepsilon^{-1}.v|| \to 1$ in $L^\infty(\sigma)$,
(ii) $v_\varepsilon \circ T_\varepsilon(E^+, E^-) \to (v^+, v^-)$ in $L^\infty(\sigma)$,
(iii) $D(\varepsilon) \to Id$ in $L^\infty(\sigma)$. □

Remark 6.4. Proposition 6.6 provides that the shape gradient is independent on the choice of the vector field $E_\sigma = (E_+, E_-)$ building the *envelope* e_ε. Indeed, when $V(0)$ vanishes in neighborhoods of γ and s_i, expression (27) may be given by

$$dJ(\omega\backslash\sigma, V) = \lim_{\varepsilon\to 0} \int_\sigma g_\varepsilon \langle V(0), v\rangle \, d\sigma.$$

As $\langle V(0), v\rangle$ belongs to $C^0(\sigma)$, g_ε converges weakly star to g in the measure space on σ. Moreover g is independent on the construction. Hence we get the main result of this section:

Theorem 6.1. *The functional $J(\omega\backslash\sigma)$ has a shape gradient at σ. The preventable defined elements g and G^i are independent on the construction. Its boundary expression is given by*

$$dJ(\omega\backslash\sigma, V) = \langle G, V(0)\rangle_{\mathscr{D}'(\Gamma, T\Gamma)\times\mathscr{D}(\Gamma, T\Gamma)}$$

with

$$G = \gamma_\gamma^*(hv) + \gamma_\sigma^*(gv) + \sum_i G^i_{(\omega\backslash\sigma)}\delta_{s_i},$$

where

$$h = fP + \frac{1}{2}(\Phi - \Phi_d)^2 + \nabla_\Gamma \Phi \nabla_\Gamma P$$

and γ^* is the adjoint of the trace operator on the corresponding boundary; let S be a boundary included in Γ, we get

$$\gamma_{|_S} : \mathscr{D}(\Gamma, T\Gamma) \to \mathscr{D}(S, T\Gamma).$$

Remark 6.5. The shape gradient $dJ(\omega\backslash\sigma, V)$ turns out to be characterized by a distributed gradient supported on the closure of the *fracture* $\bar{\sigma}$ and the boundary γ, its expression is given as a sum of a distributed term on γ, a jump distributed term in $L^1(\sigma)$ plus a Dirac measures at the two extremities s_i.

7. Study of the shape boundary derivative

In this section we deal with the shape boundary derivative and the existence of an optimal domain. The shape boundary derivative provides, withal, the derivatives with respect to the surface Γ of cost functionals governed by the state Φ. Indeed, according to the identity $\Phi'_\Gamma = \dot{\Phi}|_\Gamma - \nabla_\Gamma \Phi V(0)$ (see [16]), it transpires that Φ'_Γ is less regular than the material derivative $\dot{\Phi}$ of the state Φ which requires technicalities. On the one hand, we consider the smooth case. We characterize the shape boundary derivative Φ'_Γ of the state as the solution of a non-homogeneous elliptic tangential problem.

Thereafter, we extend the previous result to the piecewise smooth case. On the other hand, we relax the gradient tangential, normal component, of the shape boundary derivative in the fractured case.

Finally, we prove the necessary optimality condition of the initial domain and we establish the existence of an optimal domain by using the Kuratowski continuity of the Sobolev spaces.

8. Shape boundary derivative

In this section we deal with the shape boundary derivative in different cases. We deal with the smooth case in a general setting where the flow mapping does not preserve the manifold Γ; this means that the vector field is not tangent to Γ. One of the main results in this case is the characterization of the shape boundary derivative as the solution to a tangential elliptic problem linked to the Laplace–Beltrami operator. Thereafter, it consists in extending the previous characterization to the piecewise smooth case. We end by giving a relaxation for the normal trace of the shape boundary derivative in the fractured case.

Definition 8.1. The shape boundary derivative Φ'_Γ is the element $(\frac{\partial}{\partial t} Y(0))|_\Gamma$ where Y is any smooth extension of Φ verifying:

(i) $Y \in C^1([0, \delta[, H^{\frac{3}{2}}(D) \cap H^1_*(D))$,
(ii) $Y(0, .)|_\Gamma = \Phi(\Gamma)$,
(iii) $\frac{\partial}{\partial n} Y(0) = 0$ on Γ.

From [12], we know that $(\frac{\partial}{\partial t} Y(0))|_\Gamma$ is independent on the choice of such extension Y.

Thus, we have the following proposition

Proposition 8.1. *The shape boundary derivative* Φ'_Γ, *if it exists, is given in* [16] *by this relation*

$$\Phi'_\Gamma = \dot{\Phi}|_\Gamma - \nabla_\Gamma \Phi . V(0), \tag{28}$$

where $\dot{\Phi}_\Gamma$ *is the restriction of the material derivative onto* Γ.

8.1. Smooth case

We deal with this case in a general setting where the flow mapping does not preserve the manifold Γ; this means that the vector field is not tangent to Γ (i.e. $\langle V(0), n \rangle \neq 0$), then $\Gamma_t = T_t(\Gamma)$. Let ω be a manifold from the surface Γ with C^2-regularity. The main result in this case is to characterize the shape boundary derivative as the solution to a tangential elliptic problem linked to the Laplace–Beltrami operator.

Remark 8.1. The regularity of the solution of problem $\mathscr{P}$ is, at least, $H^2(\omega)$. Such regularity is enough to exhibit the shape boundary derivative.

Theorem 8.1. *Let* Φ_t *be the solution of problem* $\mathscr{P}_t$ *with second member* $f_t = F_{|_{\Gamma_t}} \in H^{\frac{1}{2}+\delta}(D)$. *The shape boundary derivative* Φ'_Γ *exists in* $H^1(\omega)$ *and is the solution of the following elliptic problem*:

$$\begin{cases} -\Delta_\Gamma \Phi'_\Gamma = -div_\Gamma[(2D^2b - H)\nabla_\Gamma \Phi \langle V(0), n \rangle] \\ \qquad\qquad + \left(\dfrac{\partial F}{\partial n} + Hf \right) \langle V(0), n \rangle \ \text{in } \omega, \\ \dfrac{\partial \Phi'_\Gamma}{\partial \nu} = (f - div_\gamma \nabla_\gamma \Phi) \langle V(0), \nu \rangle + (\nabla_\gamma \Phi . \tau) \langle \nabla_\gamma V(0). \tau, \nu \rangle \\ \qquad\qquad + k \langle \nu, \nu_F \rangle (\nabla_\gamma \Phi . \tau) \langle V(0), \tau \rangle \ \text{on } \gamma, \end{cases}$$

where Φ *is the solution to problem* $\mathscr{P}$ *in* $H^1_*(\omega)$ *and* H *is the mean curvature of the surface* Γ, $H = \frac{1}{R_1} + \frac{1}{R_2}$ *with* R_i^{-1} *are the principal curvatures—or eigenvalues different to zero of the curvature matrix* D^2b. *Then* k *is the curvature of the curve* $\partial\omega$, ν_F *is the unitary normal field of the Frenet trihedral and* τ *is the tangent vector to* $\partial\omega$ *which forms with* ν *and* n *a local trihedral.*

Remark 8.2. In dimension $N = 3$, $D^2b - \frac{H}{2}Id$ denotes the deviatoric part of the curvature tensor. In [4], many intrinsic models of shell are formulated with the same type of second order tangential operator. In [6] we find the intrinsic derivative with respect to the domain related to the solution of the elastic thin shells equations with respect to the mean surface. When ω_t is kept in Γ during the perturbation process, we take $\langle V(0), n \rangle = 0$ on Γ so also $\Delta_\Gamma \Phi'_\Gamma = 0$ in ω; this problem generalizes the case without curvature [16].

8.2. Proof of Theorem 8.1

The proof requires many technical lemmas.

Lemma 8.1 (see Desaint and Zolesio [12]). *We characterize the shape boundary derivative: of $\nabla_{\Gamma_t}\phi$ as follows:*

$$(\nabla_{\Gamma_t}\phi)'_\Gamma = -D^2 b.\nabla_\Gamma \phi \langle V(0), n \rangle + \nabla_\Gamma \phi.\nabla_\Gamma(\langle V(0), n \rangle)n.$$

Lemma 8.2 (see Desaint and Zolesio [12]). *We establish a relation between the shape derivative and the shape boundary one:*

$$(\Phi_t \circ p_t)' = (\Phi_t \circ p_t)' \circ p + b \langle D_\Gamma \Phi \circ p, m(V) \rangle$$

which provides

$$\nabla(\Phi_t \circ p_t)' = \nabla(\Phi'_\Gamma \circ p) + \nabla[\langle D_\Gamma \Phi \circ p, m(V) \rangle]$$

with

$$m(V) = b^* DV.\nabla b + \nabla(b \langle n \circ p, V \circ p - V \rangle)$$

it easy to see that $m(V)$ vanishes on Γ.

Lemma 8.3 (see Delfour and Zolésio [11]). *Let $Z(\Gamma)$ be an element of $H^1(\Gamma, \mathbb{R})$ and $y(\Omega)$ in $H^2(\Omega, \mathbb{R})$, in the case when $Z(\Gamma) = y(\Omega)|_\Gamma$ we have*

$$Z(\Gamma, V)' = y_N(\Omega, v)' + \frac{\partial y}{\partial n}(\Omega) \langle V(0), n \rangle,$$

where $y_N(\Omega, V)' = \dot{y}(\Omega, V) - \langle \nabla y(\Omega), V(0) \rangle$,

Lemma 8.4. *Green's formula supplies the weak formulation linked to problem $\mathscr{P}_t$. For all φ in $H^1(\omega_t)$*

$$\int_{\omega_t} \nabla_{\Gamma_t} \Phi_t \nabla_{\Gamma_t} \varphi = \int_{\omega_t} f_t \varphi.$$

Lemma 8.5. *Let us choose test functions defined on the whole space* $\mathbb{R}^3$ *fulfilling* $\frac{\partial\varphi}{\partial n} = 0$; since $[\nabla_{\Gamma_t}\varphi]' = 0$, *then Lemma* 5.7 *provides*

$$\frac{\partial}{\partial t}\left(\int_{\omega_t} \nabla_{\Gamma_t}\Phi_t\nabla_{\Gamma_t}\varphi\right)_{|_{t=0}} = \int_\omega \nabla[(\Phi_t\circ p_t)']\nabla_\Gamma\varphi + \int_\omega \nabla(\Phi\circ p)[\nabla_{\Gamma_t}\varphi]' + \int_\omega\left(\frac{\partial}{\partial n}[\nabla(\Phi\circ p).\nabla\varphi] + H\nabla_\Gamma\Phi\nabla_\Gamma\varphi\right)\langle V(0), n\rangle + \int_\gamma \nabla_\Gamma\Phi\nabla_\Gamma\varphi\langle V(0), \nu\rangle.$$

Remark 8.3. The structure theorem guarantees that we can choose a velocity field other than V; the only constraint is that it has to have the same normal component as V to supply the same final result for the shape boundary derivative. Then if we consider $V\circ p$, where p is the projection mapping onto Γ, instead of V, the shape boundary derivative result will remain. This simplifies the expression of $m(V)$ which becomes $m(V\circ p) = b^*D(V\circ p).\nabla b$.

Lemma 8.6. *Accordingly*

$$\langle\nabla[\langle(\nabla_\Gamma\Phi\circ p, m(V)\rangle]_{|_\Gamma}, \nabla_\Gamma\varphi\rangle = 0.$$

Proof of Lemma 8.1. It is easy to see that

$$\nabla[\langle(\nabla_\Gamma\Phi\circ p, m(V)\rangle] = {}^*Dm(V).\nabla_\Gamma\Phi + {}^*D_\Gamma(\nabla_\Gamma\Phi).m(V)$$

the expression of $Dm_{|_\Gamma}$ is given in [12], it follows on one hand

$$Dm(V)_{|_\Gamma}.\nabla_\Gamma\Phi = n^*n.DV.\nabla_\Gamma\Phi - 2n^*n\varepsilon(V)n^*n.\nabla_\Gamma\Phi$$
$$= n^*n.D_\Gamma(V).\nabla_\Gamma\Phi$$

on the other hand

$$({}^*D_\Gamma(\nabla_\Gamma\Phi).m(V)) = {}^*D_\Gamma(\nabla_\Gamma\Phi)_{|_\Gamma}.m(V)_{|_\Gamma} = 0$$

therefore

$$\langle\nabla[\langle(\nabla_\Gamma\Phi\circ p, m(V)\rangle]_{|_\Gamma}, \nabla_\Gamma\varphi\rangle = \langle{}^*Dm(V).\nabla_\Gamma\Phi, \nabla_\Gamma\varphi\rangle$$
$$= \langle n^*n.D_\Gamma V.\nabla_\Gamma\Phi, \nabla_\Gamma\varphi\rangle$$
$$= \langle D_\Gamma V.\nabla_\Gamma\Phi, n^*n\nabla_\Gamma\varphi\rangle$$
$$= 0. \qquad \square$$

Lemma 8.7. *It follows that*

$$\frac{\partial}{\partial t}\left(\int_{\omega_t} \nabla_{\Gamma_t}\Phi_t\nabla_{\Gamma_t}\varphi\right)_{|_{t=0}} = \int_\omega \nabla_\Gamma \Phi'_\Gamma \nabla_\Gamma\varphi + \int_\omega \frac{\partial}{\partial n}[\nabla(\Phi\circ p).\nabla\varphi]\langle V(0),n\rangle$$
$$+ \int_\omega H\nabla_\Gamma\Phi\nabla_\Gamma\varphi\langle V(0),n\rangle$$
$$+ \int_\gamma \nabla_\Gamma\Phi\nabla_\Gamma\varphi\langle V(0),\nu\rangle$$

then

$$\frac{\partial}{\partial t}\left(\int_{\omega_t} f_t\varphi\right)\Big|_{t=0} = \int_\omega f'_\Gamma\varphi + \int_\gamma Hf\varphi\langle V(0),n\rangle + \int_\gamma f\varphi\langle V(0),\nu\rangle$$

with $f'_\Gamma = \frac{\partial F}{\partial n}\langle V(0),n\rangle$.

We will expand the last expression term by term.

Lemma 8.8. *Green's formula yields*

$$\int_\omega \nabla_\Gamma\Phi'\nabla_\Gamma\varphi = -\int_\omega \Delta_\Gamma\Phi'\varphi + \int_\gamma \frac{\partial\Phi'}{\partial\nu}\varphi$$
$$\int_\omega \nabla_\Gamma\Phi\nabla_\Gamma\varphi\langle V(0),n\rangle H = -\int_\omega div_\Gamma(\nabla_\Gamma\Phi\langle V(0),n\rangle H)\varphi$$
$$+ \int_\gamma H\langle\nabla_\Gamma\Phi,\nu\rangle\ \langle V(0),\nu\rangle\varphi.$$

Lemma 8.9 (see Desaint and Zolesio [12]). *Given $\psi = \varphi\circ p$, the normal derivative is as follows*:

$$\frac{\partial}{\partial n}[\nabla(\Phi\circ p).\nabla\psi]_{|_\Gamma} = -2\langle D^2b\nabla_\Gamma\Phi, \nabla_\Gamma\varphi\rangle \quad on\ \Gamma.$$

We come to

Lemma 8.10.

$$\int_\omega 2\langle D^2b.\nabla_\Gamma\Phi\nabla_\Gamma\varphi\rangle\ \langle V(0),n\rangle = -\int_\omega 2\,div_\Gamma(D^2b.\nabla_\Gamma\Phi\langle V(0),n\rangle)\varphi$$
$$+ \int_\omega 2H\langle V(0),n\rangle\ \langle D^2b\nabla_\Gamma\Phi,n\rangle\varphi$$
$$- \int_\gamma \langle 2D^2b\nabla_\Gamma\Phi,\nu\rangle\ \langle V(0),n\rangle\varphi,$$

whereas $D^2b(x).n(x) = 0$ then $\int_\omega 2H\langle V(0),n\rangle\ \langle D^2b\nabla_\Gamma\Phi,n\rangle\varphi = 0$.

Let s be the curvilinear abscissa of the curve γ; in the forthcoming, we shall adopt the following notation:

$$\frac{\partial \varphi}{\partial s} = \nabla_\Gamma \varphi.\tau,$$

$$div_\gamma(\varphi) = \frac{\partial \varphi}{\partial s}.\tau.$$

Proposition 8.2. *The term defined on* γ *is given as follows*:

$$\begin{aligned}\int_\gamma \nabla_\Gamma \Phi \nabla_\Gamma \varphi \langle V(0), \nu\rangle = &- \int_\gamma div_\gamma \nabla_\gamma \Phi \langle V(0), \nu\rangle \varphi \\ &+ \int_\gamma (\nabla_\gamma \Phi.\tau)\langle \nabla_\gamma V(0).\tau, \nu\rangle \varphi \\ &+ \int_\gamma k\langle \nu, \nu_F\rangle\, (\nabla_\gamma \Phi.\tau)\, \langle V(0), \tau\rangle \\ &+ \int_\gamma \langle D^2 b.\nu, \tau\rangle (\nabla_\gamma \Phi.\tau)\, \langle V(0), n\rangle.\end{aligned}$$

The proof will be obvious via the subsequent lemmas.

Remark 8.4. We have

$$\nabla_\Gamma \Phi = (\nabla_\Gamma \Phi.\nu).\nu + (\nabla_\Gamma \Phi.\tau).\tau$$

since $\nabla_\Gamma \Phi.\nu = 0$ then $\nabla_\Gamma \Phi = (\nabla_\Gamma \Phi.\tau).\tau = (\nabla_\gamma \Phi.\tau).\tau = \frac{\partial \Phi}{\partial s}.\tau$, one can check

$$\int_\gamma \nabla_\Gamma \Phi \nabla_\Gamma \varphi \langle V(0), \nu\rangle = \int_\gamma \left(\frac{\partial \Phi}{\partial s}.\tau\right).\left(\frac{\partial \varphi}{\partial s}.\tau\right)\langle V(0), \nu\rangle.$$

In order to integrate by part the last expression, we may use the parametrization by arclength for the curve γ; let (I, g) be such a parametrization where I is an open interval in $\mathbb{R}$ and for all y in I, $g(y) = s$. Let $\tau = \partial_y g$ be the tangential field to γ on a such point s. Hence,

Lemma 8.11.

$$\begin{aligned}\int_\gamma \left(\frac{\partial \Phi}{\partial s}.\tau\right).\left(\frac{\partial \varphi}{\partial s}.\tau\right)\langle V(0), \nu\rangle &= \int_I \left(\frac{\partial (\Phi\circ g)}{\partial y}\right).\left(\frac{\partial (\varphi\circ g)}{\partial y}\right)\langle V(0)\circ g, \nu\rangle\, |\partial_y g| \\ &= - \int_I \frac{\partial}{\partial y}\left[\frac{\partial (\Phi\circ g)}{\partial y}\langle V(0)\circ g, \nu\rangle\right]\varphi\circ g\end{aligned}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial(\Phi\circ g)}{\partial y}\langle V(0)\circ g, v\rangle\right] = \frac{\partial(\Phi\circ g)}{\partial y}\langle \partial_g V(0, g(y)).\partial_y g, v\rangle + \frac{\partial(\Phi\circ g)}{\partial y}\langle V(0)\circ g, \partial_s v\rangle + \frac{\partial^2(\Phi\circ g)}{\partial y^2}\langle V(0)\circ g, v\rangle.$$

Remark 8.5. Our local trihedral (τ, v, n) is different from the Frenet one (we can distinguish the difference if we consider the curve γ as a parallel of a sphere) which forbids us to apply the Frenet formulas.

We have the following lemmas.

Lemma 8.12.

$$\frac{\partial^2(\Phi\circ g)}{\partial y^2} = \left(\frac{\partial^2\Phi}{\partial s^2}.\tau\right).\tau + \frac{\partial(\Phi\circ g)}{\partial s}.\partial_y^2 g = div_\gamma(\nabla_\gamma\Phi),$$

$$\frac{\partial(\Phi\circ g)}{\partial y} = \nabla_\gamma\Phi.\tau,$$

$$\partial_g V(0, g(y)).\partial_y g = \nabla_\gamma V(0).\tau.$$

Lemma 8.13.

$$\langle \partial_s v, \tau\rangle = -k\langle v, v_F\rangle.$$

Proof of Lemma. Indeed, since $||v|| = 1$ we have $\langle \partial_s v, v\rangle = 0$ then as $\langle v, n\rangle = 0$ it follows $\langle \partial_s v, n\rangle = -\langle v, \partial_s n\rangle$ but $\partial_s n = \partial_s(\nabla b\circ p\circ g) = {}^*D^2 b\circ g.\tau$, we deduce $\langle \partial_s v, n\rangle = -\langle D^2 b\circ g.v, \tau\rangle$ so also, with $\langle v, \tau\rangle = 0$ we obtain $\langle \partial_s v, \tau\rangle = -\langle v, \partial_s\tau\rangle$, but $\partial_s\tau = \partial_s^2 g = kv_F$, where k is the curvature of the curve γ and v_F is the Frenet trihedral normal, which achieves the result. □

Proof of Theorem 8.1. Lemmas 8.7, 8.8 and 8.10 provide the voluminal equation in ω then the boundary condition on γ is supplied by Lemmas 8.8, 8.10 and Proposition 8.2.

Thus, the result is proved. □

8.3. Piecewise smooth case

8.3.1. Existence of the shape boundary derivative

In this case, we will extend Theorem 8.1 characterizing the shape boundary derivative provided by the smooth case (ω is C^2) to the piecewise smooth one (ω is C^2 by part).

Theorem 8.2. *Assume that Γ is a C^2 manifold and γ is a piecewise C^2 smooth curve, then for any vector field V in $\mathscr{V}$*

(a) *the gradient of the shape boundary derivative has a boundary regularity*:

$$\frac{\partial \Phi'_\Gamma}{\partial \nu} \text{ belongs to } H^{\frac{-1}{2}}(\partial\omega).$$

(b) *the shape boundary derivative is the solution to the following problem*:

$$\begin{cases} -\Delta_\Gamma \Phi'_\Gamma = 0 \text{ in } \omega, \\ \dfrac{\partial \Phi'_\Gamma}{\partial \nu} = (f - div_\gamma \nabla_\gamma \Phi) \langle V(0), \nu \rangle + (\nabla_\gamma \Phi.\tau) \langle \nabla_\gamma V(0).\tau, \nu \rangle \\ \qquad + k \langle \nu, \nu_F \rangle (\nabla_\gamma \Phi.\tau) \langle V(0), \tau \rangle \text{ on } \partial\omega. \end{cases} \tag{29}$$

Remark 8.6. Indeed, in this case, because of the shape boundary derivative's lack of regularity, the Neumann boundary condition on $\partial\omega$ has no sense a priori.

In order to prove the last result and to overcome this difficulty, we shall use on one hand the established first shape gradient boundary expression for the functional cost provided by the *min–max* theory given in Proposition 5.2. On the other hand, from Lemma 5.7 and by using the adjoint state we get a second shape gradient boundary expression in which the Neumann boundary condition appears. Thus, by uniqueness of the shape gradient we relax the boundary condition on γ.

8.3.2. Second boundary expression of the shape gradient

In the following, we attempt to show another expression of the boundary shape gradient of the functional cost $J(\omega)$.

Proposition 8.3. *The shape gradient boundary expression is given by*

$$dJ(\omega, V) = \frac{1}{2} \int_{\partial\omega} (\Phi - \Phi_d)^2 \langle V(0), \nu \rangle - \int_{\partial\omega} P \langle \nabla_\Gamma \Phi'_\Gamma, \nu \rangle. \tag{30}$$

Remark 8.7. A priori the boundary term $\int_{\partial\omega} P \langle \nabla_\Gamma \Phi'_\Gamma, \nu \rangle$ has no sense because of the regularity lack of the shape boundary derivative Φ'_Γ in the neighborhood of singularities.

Proof of Proposition 8.3. Under Lemma 5.7 and Green's formula, we come to

$$dJ(\omega, V) = \int_{\omega} (\Phi - \Phi_d)\Phi'_{\Gamma} + \frac{1}{2}\int_{\omega} H(\Phi - \Phi_d)^2 \langle V(0), n \rangle + \frac{1}{2}\int_{\partial\omega} (\Phi - \Phi_d)^2 \langle V(0), \nu \rangle. \tag{31}$$

We use as before the method of adjoint state in order to get rid of the boundary shape derivative Φ'_{Γ} from the last expression.

Hence

$$dJ(\omega, V) = \int_{\omega} \Phi'_{\Gamma}\Delta_{\Gamma}P + \frac{1}{2}\int_{\omega} H(\Phi - \Phi_d)^2 \langle V(0), n \rangle + \frac{1}{2}\int_{\partial\omega} (\Phi - \Phi_d)^2 \langle V(0), \nu \rangle \tag{32}$$

by Green's formula, we come to

$$\begin{aligned} dJ(\omega, V) = &\int_{\omega} P\Delta_{\Gamma}\Phi'_{\Gamma} + \frac{1}{2}\int_{\omega} H(\Phi - \Phi_d)^2 \langle V(0), n \rangle \\ &+ \frac{1}{2}\int_{\partial\omega} (\Phi - \Phi_d)^2 \langle V(0), \nu \rangle - \int_{\partial\omega} P\langle \nabla_{\Gamma}\Phi'_{\Gamma}, \nu \rangle \\ &+ \int_{\partial\omega} \Phi'_{\Gamma}\langle \nabla_{\Gamma}P, \nu \rangle \end{aligned}$$

but since $\langle \nabla_{\Gamma}P, \nu \rangle = 0$, $\Delta_{\Gamma}\Phi'_{\Gamma} = 0$ in ω and $(V, n) = 0$ on Γ the proof is achieved. □

8.3.3. Proof of the Theorem 8.2

Thanks to the last results, Proposition 5.2 and Proposition 8.3, one can easily check the following relaxation of the boundary condition of the shape boundary derivative.

Lemma 8.14. *The shape boundary derivative Φ'_{Γ} is regular and is given as follows, for all P in $H^1(\omega)$ (i.e. for all Φ_d in $H^1(\omega)$),*

$$\int_{\partial\omega} P\langle \nabla_{\Gamma}\Phi'_{\Gamma}, \nu \rangle = \int_{\partial\omega} \{-\nabla_{\Gamma}\Phi\nabla_{\Gamma}P + fP\}\langle V(0), \nu \rangle - \langle G^i_{\omega}, V(s_i) \rangle_{\mathbb{R}^N}$$

so also,

$$\frac{\partial\Phi'_{\Gamma}}{\partial\nu} \in H^{\frac{-1}{2}}(\partial\omega).$$

Thus, according to the previous results we come to the proof of Theorem 8.2. Indeed, by embedding relation (28) in Eq. (4) and via the above lemma it will be enough to refer to Theorem 8.1, and so the proof is achieved. □

8.4. Fractured case

The fractured manifold $\omega\backslash\sigma$ is regularized by the family with parameter of piecewise smooth domain $(\omega\backslash\sigma)_\varepsilon$. Therefore, we apply Theorem 8.2 and we get at each fixed ε.

Proposition 8.4. *The shape boundary derivative is solution to the following problem*:

$$\begin{cases} -\Delta_\Gamma \Phi'_{\varepsilon\Gamma} = 0 \ \ in \ (\omega\backslash\sigma)_\varepsilon \\ \dfrac{\partial \Phi'_{\varepsilon\Gamma}}{\partial v_\varepsilon} = (f - div_\gamma \nabla_\gamma \Phi_\varepsilon)\langle V(0), v_\varepsilon\rangle + (\nabla_\gamma \Phi_\varepsilon.\tau_\varepsilon)\langle \nabla_\gamma V(0).\tau_\varepsilon, v_\varepsilon\rangle \\ \qquad\qquad + k\langle v_\varepsilon, v_{\varepsilon F}\rangle(\nabla_\gamma \Phi_\varepsilon.\tau_\varepsilon)\langle V(0), \tau_\varepsilon\rangle \ \ on \ \partial(\omega\backslash\sigma)_\varepsilon \end{cases} \tag{33}$$

and also,

Proposition 8.5. *We have, at each* ε;

$$dJ((\omega\backslash\sigma)_\varepsilon, V) = \frac{1}{2}\int_{\partial(\omega\backslash\sigma)_\varepsilon} (\Phi_\varepsilon - \Phi_d)^2 \langle V(0), v_\varepsilon\rangle - \int_{\partial(\omega\backslash\sigma)_\varepsilon} P_\varepsilon \langle \nabla_\Gamma \Phi'_{\varepsilon\Gamma}, v_\varepsilon\rangle \tag{34}$$

therefore,

$$\begin{aligned} \int_{\partial(\omega\backslash\sigma)_\varepsilon} P_\varepsilon \langle \nabla_\Gamma \Phi'_{\varepsilon\Gamma}, v_\varepsilon\rangle = & -\int_{\partial(\omega\backslash\sigma)_\varepsilon} \nabla_\Gamma \Phi \nabla_\Gamma P \langle V(0), v\rangle \, d\gamma \\ & + \int_{\partial(\omega\backslash\sigma)_\varepsilon} fp\langle V(0), v\rangle - \sum_i \langle G^i_{(\omega\backslash\sigma)_\varepsilon}, V(s_i)\rangle_{\mathbb{R}^N}. \end{aligned}$$

Thanks to the continuity of the Neumann tangential problem with respect to the *envelope* e_ε we supply this lemma.

Lemma 8.15. *There exist* ξ *in* $H^{\frac{1}{2}}(\sigma)$ *such that* $[P_\varepsilon \circ T_\varepsilon]_\sigma$ *converges to* ξ.

Proof of Lemma 8.15. Let φ be in $H^1(\omega\backslash\sigma)$, then

$$\int_{\omega\backslash\sigma} 1_{(\omega\backslash\sigma)_\varepsilon} \nabla_\Gamma P_\varepsilon \nabla_\Gamma \varphi = \int_{(\omega\backslash\sigma)_\varepsilon} \nabla_\Gamma P_\varepsilon \nabla_\Gamma \varphi.$$

Green's formula yields

$$\int_{(\omega\backslash\sigma)_\varepsilon} \nabla_\Gamma P_\varepsilon \nabla_\Gamma \varphi + \int_{(\omega\backslash\sigma)_\varepsilon} P_\varepsilon \Delta_\Gamma \varphi = \int_{\partial(\omega\backslash\sigma)_\varepsilon} P_\varepsilon \langle \nabla_\Gamma \varphi, v_\varepsilon \rangle$$

the continuity of the Neumann problem result provides that the left hand side converges with respect to ε. Hence the right-hand one converges. Indeed, by a change of variable we get

$$\int_{\partial e_\varepsilon} P_\varepsilon \langle \nabla_\Gamma \varphi, v_\varepsilon \rangle = \int_\sigma [P_\varepsilon \circ T_\varepsilon \langle \nabla_\Gamma \varphi \circ T_\varepsilon, v_\varepsilon \circ T_\varepsilon \rangle] j(\varepsilon)$$

therefore, the right-hand side converges for all φ in $H^1(\omega\backslash\sigma)$ so also for any $\langle \nabla_\Gamma \varphi \circ T_\varepsilon, v_\varepsilon \rangle$ in $H^{-\frac{1}{2}}(\sigma)$. Which provides the proof of the lemma. □

Therewith, we have an optimal relaxation of the normal component of the gradient tangential of the shape boundary derivative $\Phi'_{\varepsilon\Gamma}$.

Proposition 8.6. *Let T_ε be the flow mapping associated with the vector field E_σ. For any ξ in $H^{\frac{1}{2}}(\sigma)$ (i.e. for any Φ_d in $H^1(\omega\backslash\sigma)$), we have*

$$\lim_{\varepsilon\to 0} \int_\sigma [\xi \langle \nabla_\Gamma(\Phi'_{\varepsilon\Gamma} \circ T_\varepsilon), v \rangle] \, d\sigma = -\int_\gamma \nabla_\Gamma \Phi \nabla_\Gamma P \langle V(0), v \rangle \, d\gamma$$
$$- \int_\gamma fP \langle V(0), v \rangle - \int_\sigma g \langle V(0), v \rangle \, d\sigma$$
$$+ \sum_i \langle G^i_{(\omega\backslash\sigma)}, V(s_i) \rangle_{\mathbb{R}^N} \tag{35}$$

and so $\nabla_\Gamma \Phi'_{\varepsilon\Gamma}.v_\varepsilon$ converges toward a function $q = (q^+, q^-)$ in $H^{\frac{-1}{2}}(\sigma)$.

The proof is a direct consequence of Eq. (34) and Lemma 8.15.

Remark 8.8. As far as we know, because of the lack of regularity, we are not able to confirm whether the function q is a shape derivative of the state Φ.

9. Existence of an optimal domain

9.1. A necessary optimality condition

For $\alpha > 0$, we consider the following penalization of the functional J:

$$J_\alpha(\omega\backslash\sigma) = J(\omega\backslash\sigma) + \frac{\alpha}{2} \min_{\sigma_p \in \Sigma^p} ||\sigma^p||^2_{H^2(0,1)},$$

where Σ^p is the set of parametrizations for the curve σ linked to the open interval $(0,1)$.

$$\Sigma^p = \{\sigma^p \in H^2(0,1); \sigma^p(0) = s_1, \sigma^p(1) = s_2; (\sigma^p)' \neq 0\}$$

Assume that $\omega \backslash \sigma_*$ is an optimal admissible domains, i.e.

$$J_\alpha(\omega \backslash \sigma_*) = \min_{\sigma \in \Sigma} J_\alpha(\omega \backslash \sigma),$$

where

$$\Sigma = \{\sigma = \sigma^p(0,1)\}$$

Σ is compact with the Hausdorff topology.

Proposition 9.1. *An admissible domain $\omega \backslash \sigma_*$ (with respect to fracture σ_*) within the set Σ is optimal if and only if, for any field V in $\mathscr{V}$, (see [2])*

$$dJ_\alpha(\omega \backslash \sigma_*, V) = 0$$

9.2. Existence of an optimal domain

Let σ_n converge towards σ_* with respect to the Hausdorff topology, i.e. $\omega \backslash \sigma_n$ converges towards $\omega \backslash \sigma_*$ with respect to the Hausdorff complementary topology. We will prove the Kuratowski continuity of the Sobolev space, (see [1,5]).

9.2.1. Kuratowski continuity of the Sobolev space

Theorem 9.1. *Given φ a test function belonging to $H^1_*(\omega \backslash \sigma)$, then there exists φ_n belongs to $H^1_*(\omega \backslash \sigma_n)$, such that*

$$1_{(\omega \backslash \sigma_n)} \varphi_n \to 1_{(\omega \backslash \sigma_*)} \varphi \text{ in } L^2(\omega),$$

$$1_{(\omega \backslash \sigma_n)} \nabla_\Gamma \varphi_n \to 1_{(\omega \backslash \sigma_*)} \nabla_\Gamma \varphi \text{ in } L^2(\omega).$$

Proposition 9.2. *Let σ_n be a sequence in Σ which converges toward σ in Σ. There exists h_{max} such for all $0 < h < h_{max}$, there exists $N > 0$ such that*

(i) *for any $n > N$, σ_n lays in a tubular neighbourhood U_h of thickness $2h$.*
(ii) *b, the oriented distance function to Ω is defined in U_h and belongs to $C^2(U_h)$, the projection p is also well-defined in U_h and belongs to $C^1(U_h)$.*
(iii) *b_n, the oriented distance function to Ω_n, is defined in U_h and belongs to $C^2(U_h)$, the projection p_n is also well-defined in U_h and belongs to $C^1(U_h)$.*
(iv) *b_n converges towards b in $C^2(U_h)$, and p_n converges towards p in $C^1(U_h)$.*

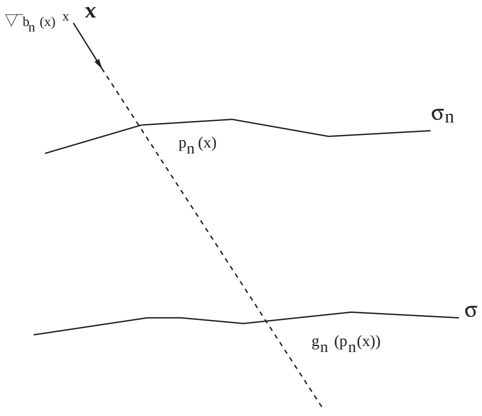

Fig. 3. Building of g_n.

It follows for any $n > N$ that

Proposition 9.3. *For any x in σ_n, the intersection between the normal to σ_n at x and σ is reduced to a single point denoted $g_n(x)$. Moreover the mapping g_n is a C^1-diffeomorphism from σ_n to σ and we have* (Fig. 3)

$$\forall\ x \in \sigma_n, \quad g_n(x) = x + b_n \circ g_n(x).\nabla b_n(x) \in \sigma.$$

This formulation will allow us to build the test functions.

Proof of Theorem 9.1. Assume that the Jacobian of the diffeomorphism g_n fulfills the hypothesis: $j(n) = 1$. Therefore, it's enough to choose $\varphi_n = \varphi \circ g_n$ and to adapt the same technicalities as for the proof of Proposition 6.4. □

9.2.2. Continuity of the cost functional with respect to the domain

Proposition 9.4. *The mapping $\omega \backslash \sigma \to J_\alpha(\omega \backslash \sigma)$ is lower semi-continuous.*

In order to establish the proof, the following proposition will be needed.

Proposition 9.5. *Let Φ_n be the solution to the Neumann problem in $(\omega \backslash \sigma_n)$ and $1_{(\omega \backslash \sigma_n)}$ its characteristic. Then Φ_n converges strongly to Φ solution to the Neumann problem.*

Proof of Proposition 9.5. Using the same arguments than in the proof of Proposition 6.4, we can bound in $L^2(\omega)$ the sequences $1_{(\omega\backslash\sigma_n)}\Phi_n$ and $1_{(\omega\backslash\sigma_n)}\nabla_\Gamma\Phi_n$. Then, basically, the compactivor property and Theorem 9.1 provide the result. □

Proof of Proposition 9.4. Let σ_n be any minimizing sequence of J_α in Σ. On the one hand, according to Proposition 9.5 the functional $J(\omega\backslash\sigma_n)$ converges to $J(\omega\backslash\sigma)$. On the other hand, let σ_n^p be the parametrization related to the curve σ_n realizing the minimum, since $||\sigma_n^p||_{H^2}\leqslant c$ one can extract a subsequence denoted also σ_n^p which converges weakly towards a such λ in $H^2(0,1)$, then it converges strongly to λ in $C^1(0,1)$, therefore $\lambda(0)=\sigma_n^p(0)=s_1$ and $\lambda(1)=\sigma_n^p(1)=s_2$ so λ is an immersion: $(\lambda'\neq 0)$. We have to prove that λ is the minimum-parametrization for σ. Indeed, when $n\geqslant N$, $\sigma_n^p=g_n^{-1}\circ\sigma^p$ where σ^p is any parametrization for σ, hence $g_n^{-1}\circ\sigma^p$ converges strongly towards λ in $C^1(0,1)$, whereas g_n^{-1} converges towards Id, it follows that $g_n^{-1}\circ\sigma^p$ converges strongly towards σ^p, whence by uniqueness $\lambda=\sigma^p$, therefore σ^p is a parametrization for the curve σ. By the truth that the norm H^2 is l.s.c., we come to

$$||\sigma^p||_{H^2}^2\leqslant \liminf_{n\uparrow\infty}\ ||\sigma_n^p||_{H^2}^2.$$

Hence the curve σ minimizes the functional J_α.

Thus the lower semi-continuity result is proved. □

10. Conclusion

We have investiguated the Laplace–Beltrami operator in a fractured manifold. The boundary expression of the shape gradient of a cost functional governed by the cracked manifold is provided. The shape boundary derivative is characterized in smooth and non-smooth cases. The techniques used allow us to deal with the situation in which the fracture needs not to be smooth and to extend the results to lager classes of operators in the dimensional n.

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