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# Existence for the $\alpha$-patch model and the QG sharp front in Sobolev spaces

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**Abstract**

We consider a family of contour dynamics equations depending on a parameter $\alpha$ with $0 < \alpha \leqslant 1$. The vortex patch problem of the 2-D Euler equation is obtained taking $\alpha \to 0$, and the case $\alpha = 1$ corresponds to a sharp front of the QG equation. We prove local-in-time existence for the family of equations in Sobolev spaces.


*Keywords:* Incompressible flow; Contour dynamics; Quasi-geostrophic equations; Local-in-time existence

## 1. Introduction

The 2-D QG equation provides particular solutions of the evolution of the temperature from a general quasi-geostrophic system for atmospheric and oceanic flows. This equation is derived considering small Rossby and Ekman numbers and constant potential vorticity (see [12] for more details). It reads

$$\theta_t(x,t) + u(x,t)\cdot\nabla\theta(x,t) = 0, \quad x \in \mathbb{R}^2,$$
$$\theta(x,0) = \theta_0(x), \tag{1}$$

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where $\theta$ is the temperature of the fluid. The incompressible velocity $u$ is expressed by means of the stream function as follows

$$u = \nabla^{\perp}\psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi),$$

and the relation between the stream function and the temperature is given by

$$\theta = -(-\Delta)^{1/2}\psi.$$

This system has been considered in frontogenesis, where the dynamics of hot and cold fluids is studied together with the formation and the evolution of fronts (see [3,4,7,11]).

From a mathematical point of view, this equation has been presented as a two-dimensional model of the 3-D Euler equation due to their strong analogies (see [3]), being the formation of singularities for a regular initial data an open problem (see [3,5,6]). Nevertheless the QG equation has global in time weak solutions due to an extra cancellation (see [13]). A few sparse results are known about weak solutions of the 2-D and 3-D Euler equation in its primitive-variable form.

An outstanding kind of weak solutions for the QG equation are those in which the temperature takes two different values in complementary domains, modelling the evolution of a sharp front as follows

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t), \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases} \tag{2}$$

In this work we study a problem similar to the 2-D vortex patch problem, where the vorticity of the 2-D Euler equation is given by a characteristic function of a domain, and the regularity of the free boundary of the domain is considered. For this equation the vorticity satisfies

$$\begin{aligned} &w_t(x,t) + u(x,t)\cdot\nabla w(x,t) = 0, \quad x \in \mathbb{R}^2, \\ &w(x,0) = w_0(x), \end{aligned} \tag{3}$$

in a weak sense, and the velocity is given by the Biot–Savart law or analogously

$$u = \nabla^{\perp}\psi \quad \text{and} \quad w = \Delta\psi.$$

Chemin [2] proved global-in-time regularity for the free boundary using paradifferential calculus. A simpler proof can be found in [1] due to Bertozzi and Constantin.

We point out that in the QG equation, the velocity is determined from the temperature by singular integral operators (see [15]) as follows

$$u = (-R_2\theta, R_1\theta), \tag{4}$$

where $R_1$ and $R_2$ are the Riesz transforms, making the system more singular than (3).

Rodrigo [14] proposed the problem of the evolution of a sharp front for the QG equation. He derived the velocity on the free boundary in the normal direction, and proved local-existence and uniqueness for a periodic $C^{\infty}$ front, i.e.

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \{f(x_1,t) > x_2\}, \\ \theta_2, & \{f(x_1,t) \leqslant x_2\}, \end{cases}$$

with $f(x_1, t)$ periodic, using the Nash–Moser iteration.

In this paper we study a family of contour dynamics equation given by weak solutions of the following system

$$\theta_t + u \cdot \nabla\theta = 0, \quad x \in \mathbb{R}^2,$$
$$u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{1-\alpha/2}\psi, \quad 0 < \alpha \leqslant 1, \tag{5}$$

where the active scalar $\theta(x, t)$ satisfies (2). We notice that the case $\alpha = 0$ is the 2-D vortex patch problem, and $\alpha = 1$ corresponds to the sharp front for the QG equation.

This system was introduced by Córdoba, Fontelos, Mancho and Rodrigo in [8], where they present a proof of local existence for a periodic $C^\infty$ front, and show evidence of singularities in finite time. The singular scenario is due to the point-wise collapse of two patches.

Here we give a proof of local existence of the system (5) where the solution satisfies (2), with the boundary $\partial\Omega(t)$ given by the curve

$$\partial\Omega(t) = \{x(\gamma, t) = (x_1(\gamma, t), x_2(\gamma, t)): \gamma \in [-\pi, \pi]\},$$

and $x(\gamma, t)$ belongs to a Sobolev space. In the cases $0 < \alpha < 1$ we show uniqueness.

It is well known (see [9] and [14]) that in these kind of contour dynamics equations, the velocity in the tangential direction only moves the particles on the boundary. Therefore we do not alter the shape of the contour if we change the tangential component of the velocity; i.e., we are changing the parametrization. In the most singular case, $\alpha = 1$ or the QG equation, we need to change the velocity in the tangential direction in order to get existence in the Sobolev spaces. We take a tangential velocity in such a way that $|\partial_\gamma x(\gamma, t)|$ satisfies

$$|\partial_\gamma x(\gamma, t)|^2 = A(t),$$

and does not depend on $\gamma$. We would like to cite the work of Hou, Lowengrub and Shelley [9] in which this idea was used to study a contour dynamics problem.

We notice that in order to get a nonsingular normal velocity of the curve for $0 < \alpha \leqslant 1$ (see [8] and [14]), we need a one-to-one curve, and parameterized in such a way that

$$|\partial_\gamma x(\gamma, t)|^2 > 0.$$

Rigorously, we need that

$$\frac{|x(\gamma, t) - x(\gamma - \eta, t)|}{|\eta|} > 0, \quad \forall \gamma, \eta \in [-\pi, \pi], \tag{6}$$

therefore we give initial data satisfying this property, and we prove that this condition is satisfied locally in time. It is evident from the numerical simulations in [8], that one needs to take into account the evolution of this quantity.

## 2. The contour equation

In this section we deduce the family of contour equations in term of the free boundary $x(\gamma,t)$. We consider the equations given by the system (1), with the velocity satisfying

$$u(x,t)=\nabla^{\perp}\psi(x,t), \tag{7}$$

for the stream function it follows

$$\theta=-(-\Delta)^{1-\alpha/2}\psi, \tag{8}$$

and the active scalar fulfills

$$\theta(x_1,x_2,t)=\begin{cases}\theta_1, & \Omega(t),\\ \theta_2, & \mathbb{R}^2\setminus\Omega(t).\end{cases} \tag{9}$$

The boundary of $\Omega(t)$ is given by the curve

$$\partial\Omega(t)=\big\{x(\gamma,t)=\big(x_1(\gamma,t),x_2(\gamma,t)\big)\colon \gamma\in[-\pi,\pi]=\mathbb{T}\big\},$$

where $x(\gamma,t)$ is one-to-one. Due to the identity (9), we see that

$$\nabla^{\perp}\theta=(\theta_1-\theta_2)\partial_\gamma x(\gamma,t)\delta\big(x-x(\gamma,t)\big),$$

where $\delta$ is the Dirac distribution. Using (7) and (8), we have

$$u=-(-\Delta)^{\alpha/2-1}\nabla^{\perp}\theta.$$

The integral operators, $-(-\Delta)^{\alpha/2-1}$ are Riesz potentials (see [15]), so that using the last two identities we obtain that

$$u(x,t)=-\frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma-\eta,t)}{|x-x(\gamma-\eta,t)|^\alpha}\,d\eta, \tag{10}$$

for $x\neq x(\gamma,t)$, and $\Theta_\alpha=(\theta_1-\theta_2)\Gamma(\alpha/2)/2^{1-\alpha}\Gamma(2-\alpha/2)$. We notice that for $\alpha=1$, if $x\to x(\gamma,t)$, then the integral in (10) is divergent. As we have showed before, we are interested in the normal velocity of the systems. Using the identity (10), and taking the limit as follows

$$u(x,t)\cdot\partial_\gamma^{\perp}x(\gamma,t),\quad x\to x(\gamma,t), \tag{11}$$

we obtain

$$u\big(x(\gamma,t),t\big)\cdot\partial_\gamma^{\perp}x(\gamma,t)=-\frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^{\perp}x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}\,d\eta. \tag{12}$$

This identity is well defined for $0<\alpha\leqslant 1$ and a one-to-one curve $x(\gamma,t)$. Due to the fact that tangential velocity does not change the shape of the boundary, we fix the contour $\alpha$-patch equations as follows

$$x_t(\gamma,t)=\frac{\Theta_\alpha}{2\pi}\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}\,d\eta,\quad 0<\alpha\leqslant 1,$$
$$x(\gamma,0)=x_0(\gamma). \tag{13}$$

Seeing Eq. (10), we show that the velocity in QG presents a logarithmic divergence in the tangential direction on the boundary. Nevertheless it belongs to $L^p(\mathbb{R}^2)$ for $1<p<\infty$, and to the bounded mean oscillation space (see [15] for the definition of the BMO space). In QG the velocity is given by (4), and writing the temperature in the following way

$$\theta(x,t)=(\theta_1-\theta_2)\chi_{\Omega(t)}(x)+\theta_2,$$

we see that

$$u(x,t)=(\theta_1-\theta_2)\big(-R_2(\chi_{\Omega(t)}),R_1(\chi_{\Omega(t)})\big).$$

Using that $\chi_{\Omega(t)}\in L^p(\mathbb{R}^2)$ for $1\leqslant p\leqslant\infty$, we conclude the argument. In particular the energy of the system is conserved due to the fact that $\|u\|_{L^2}(t)=|\theta_1-\theta_2||\Omega(t)|^{1/2}$, and the area of $\Omega(t)$ is constant in time.

## 3. Weak solutions for the $\alpha$-system

In this section we show that if $\theta(x,t)$ is defined by (9) and the curve $x(\gamma,t)$ is convected by the normal velocity (12), then $\theta(x,t)$ is a weak solution of the system (5) and conversely. We give the definition of weak solutions below.

**Definition 3.1.** The active scalar $\theta$ is a weak solution of the $\alpha$-system if for any function $\varphi\in C_c^\infty(\mathbb{R}^2\times(0,T))$, we have

$$\int\limits_0^T\int\limits_{\mathbb{R}^2}\theta(x,t)\big(\partial_t\varphi(x,t)+u(x,t)\cdot\nabla\varphi(x,t)\big)\,dx\,dt=0, \tag{14}$$

where the incompressible velocity $u$ is given by (7), and the stream function satisfies (8).

**Proposition 3.2.** *If $\theta(x,t)$ is defined by* (9)*, and the curve $x(\gamma,t)$ satisfies* (6) *and* (12)*, then $\theta(x,t)$ is a weak solution of the $\alpha$-system. Furthermore, if $\theta(x,t)$ is a weak solution of the $\alpha$-system given by* (9)*, and $x(\gamma,t)$ satisfies* (6)*, then $x(\gamma,t)$ verifies* (12)*.*

**Proof.** Let $\theta(x,t)$ be a weak solution of the $\alpha$-system defined by (9). If we consider the surface $\{(x_1(\gamma,t),x_2(\gamma,t),t)\colon\ \gamma\in\mathbb{T},t\in[0,T]\}$, integrating by parts it follows:

$$I=\int\limits_0^T\int\limits_{\mathbb{R}^2}\theta(x,t)\partial_t\varphi(x,t)\,dx\,dt$$

$$= \theta_1 \int_0^T \int_{\Omega(t)} \partial_t \varphi(x,t)\,dx\,dt + \theta_2 \int_0^T \int_{\mathbb{R}^2\setminus\Omega(t)} \partial_t \varphi(x,t)\,dx\,dt$$

$$= -(\theta_1-\theta_2) \int_0^T \int_{\mathbb{T}} \varphi\big(x(\gamma,t),t\big) x_t(\gamma,t)\cdot \partial_\gamma^\perp x(\gamma,t)\,d\gamma\,dt.$$

On the other hand, we obtain

$$J = \int_0^T \int_{\mathbb{R}^2} \theta u\cdot\nabla\varphi\,dx\,dt = \theta_1 \int_0^T \int_{\Omega} u\cdot\nabla\varphi\,dx\,dt + \theta_2 \int_0^T \int_{\mathbb{R}^2\setminus\Omega} u\cdot\nabla\varphi\,dx\,dt.$$

Taking

$$\Omega_1^\varepsilon(t) = \big\{x\in\Omega\colon\ dist\big(x,\Omega(t)\big)\geqslant\varepsilon\big\},$$

and

$$\Omega_2^\varepsilon(t) = \big\{x\in\mathbb{R}^2\setminus\Omega\colon\ dist\big(x,\mathbb{R}^2\setminus\Omega(t)\big)\geqslant\varepsilon\big\},$$

we have that $J^\varepsilon\to J$ if $\varepsilon\to 0$, where $J^\varepsilon$ is given by

$$J^\varepsilon = \theta_1 \int_0^T \int_{\Omega_1^\varepsilon(t)} u\cdot\nabla\varphi\,dx\,dt + \theta_2 \int_0^T \int_{\Omega_2^\varepsilon(t)} u\cdot\nabla\varphi\,dx\,dt.$$

Integrating by parts in $J^\varepsilon$, using that the velocity is divergence free, and taking the limit as in (11), we obtain

$$J = (\theta_1-\theta_2) \int_0^T \int_{\mathbb{T}} \varphi\big(x(\gamma,t),t\big) u\big(x(\gamma,t),t\big)\cdot\partial_\gamma^\perp x(\gamma,t)\,d\gamma\,dt$$

$$= -(\theta_1-\theta_2)\frac{\Theta_\alpha}{2\pi} \int_0^T \int_{\mathbb{T}} \varphi\big(x(\gamma,t),t\big)\bigg(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha} d\eta\bigg)\,d\gamma\,dt.$$

We have that $I+J=0$ using (14), and it follows

$$\int_0^T \int_{\mathbb{T}} f(\gamma,t)\bigg(x_t(\gamma,t)\cdot\partial_\gamma^\perp x(\gamma,t) + \frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}\,d\eta\bigg)\,d\gamma\,dt = 0,$$

for $f(\gamma,t)$ periodic in $\gamma$. We see that (12) is satisfied. Following the same arguments it is easy to check that if $x(\gamma,t)$ satisfies (12), then $\theta$ is a weak solution given by (9).

## 4. Local well-posedness for $0 < \alpha < 1$

In this section we prove the existence and uniqueness for the contour equation in the cases $0 < \alpha < 1$. We denote the Sobolev spaces by $H^k(\mathbb{T})$, with norms

$$\|x\|_{H^k}^2 = \|x\|_{L^2}^2 + \left\|\partial_\gamma^k x\right\|_{L^2}^2,$$

and the spaces $C^k(\mathbb{T})$ with

$$\|x\|_{C^k} = \max_{j \leqslant k} \left\|\partial_\gamma^j x\right\|_{L^\infty}.$$

We need that the curve satisfies

$$\frac{|x(\gamma,t) - x(\gamma-\eta,t)|}{|\eta|} > 0, \quad \forall \gamma, \eta \in [-\pi, \pi], \tag{15}$$

and we define

$$F(x)(\gamma,\eta,t) = \frac{|\eta|}{|x(\gamma,t) - x(\gamma-\eta,t)|} \quad \forall \gamma, \eta \in [-\pi, \pi], \tag{16}$$

with

$$F(x)(\gamma,0,t) = \frac{1}{|\partial_\gamma x(\gamma,t)|}.$$

The following theorem is the main result of the section.

**Theorem 4.1.** *Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geqslant 3$ with $F(x_0)(\gamma,\eta) < \infty$. Then there exists a time $T > 0$ so that there is a unique solution to* (13) *for $0 < \alpha < 1$ in $C^1([0,T]; H^k(\mathbb{T}))$, with $x(\gamma,0) = x_0(\gamma)$.*

**Proof.** We can choose $\Theta_\alpha = 2\pi$ without loss of generality, obtaining the following equation

$$\begin{aligned} x_t(\gamma,t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|^\alpha}\, d\eta, \quad 0 < \alpha < 1, \\ x(\gamma,0) &= x_0(\gamma). \end{aligned} \tag{17}$$

We present the proof for $k = 3$, being analogous for $k > 3$. We use energy estimates (see [10] for more details). We ignore the time dependence to simplify the notation. Considering the quantity

$$\begin{aligned} \int_{\mathbb{T}} x(\gamma) \cdot x_t(\gamma)\, d\gamma &= \int_{\mathbb{T}} \int_{\mathbb{T}} x(\gamma) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma) - x(\eta)|^\alpha}\, d\eta\, d\gamma \\ &= -\int_{\mathbb{T}} \int_{\mathbb{T}} x(\eta) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma) - x(\eta)|^\alpha}\, d\eta\, d\gamma \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(x(\gamma)-x(\eta))\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\eta))}{|x(\gamma)-x(\eta)|^\alpha}\,d\eta\,d\gamma \\
&= \frac{1}{2(2-\alpha)}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma \big|x(\gamma)-x(\gamma-\eta)\big|^{2-\alpha}\,d\gamma\,d\eta \\
&= 0, \qquad (18)
\end{aligned}
$$

we obtain

$$
\frac{d}{dt}\|x\|_{L^2}(t) = 0. \qquad (19)
$$

We decompose as follows

$$
\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3 x_t(\gamma)\,d\gamma = I_1+I_2+I_3+I_4,
$$

where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\frac{\partial_\gamma^4 x(\gamma)-\partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^\alpha}\,d\eta\,d\gamma, \\
I_2 &= 3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)\big)\partial_\gamma\big(\big|x(\gamma)-x(\gamma-\eta)\big|^{-\alpha}\big)\,d\eta\,d\gamma, \\
I_3 &= 3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big)\partial_\gamma^2\big(\big|x(\gamma)-x(\gamma-\eta)\big|^{-\alpha}\big)\,d\eta\,d\gamma, \\
I_4 &= \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\partial_\gamma^3\big(\big|x(\gamma)-x(\gamma-\eta)\big|^{-\alpha}\big)\,d\eta\,d\gamma.
\end{aligned}
$$

Operating as in (18), the term $I_1$ becomes

$$
\begin{aligned}
I_1 &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \big(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)\big)\cdot\frac{\partial_\gamma^4 x(\gamma)-\partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^\alpha}\,d\eta\,d\gamma \\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\partial_\gamma|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^\alpha}\,d\eta\,d\gamma \\
&= \frac{\alpha}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)|^2(x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}\,d\eta\,d\gamma.
\end{aligned}
$$

One finds that

$$I_1 \leqslant \frac{\alpha}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)|^2|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+1}}\,d\eta\,d\gamma,$$

and due to the inequality $|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)||\eta|^{-1} \leqslant \|x\|_{C^2}$, it follows that

$$\begin{aligned} I_1 &\leqslant \frac{\alpha}{4}\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}} |\eta|^{-\alpha}\big|F(x)(\gamma,\eta)\big|^{1+\alpha}\big|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)\big|^2\,d\eta\,d\gamma \\ &\leqslant \frac{1}{2}\big\|F(x)\big\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}\big(\big|\partial_\gamma^3 x(\gamma)\big|^2+\big|\partial_\gamma^3 x(\gamma-\eta)\big|^2\big)\,d\gamma\,d\eta \\ &\leqslant \big\|F(x)\big\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\big\|\partial_\gamma^3 x\big\|_{L^2}^2\int_{\mathbb{T}}|\eta|^{-\alpha}\,d\eta \\ &\leqslant C_\alpha\big\|F(x)\big\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\big\|\partial_\gamma^3 x\big\|_{L^2}^2. \end{aligned} \tag{20}$$

As before, we can obtain $I_2 = -6I_1$, so that

$$I_2 \leqslant C_\alpha\big\|F(x)\big\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\big\|\partial_\gamma^3 x\big\|_{L^2}^2. \tag{21}$$

In order to estimate the term $I_3$, we consider $I_3 = J_1 + J_2 + J_3$, where

$$J_1 = -3\alpha\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big)\frac{A(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}\,d\eta\,d\gamma,$$

$$J_2 = -3\alpha\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big)\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}\,d\eta\,d\gamma,$$

$$J_3 = 3\alpha(2+\alpha)\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big)\frac{(B(\gamma,\eta))^2}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+4}}\,d\eta\,d\gamma,$$

with

$$A(\gamma,\eta) = \big(x(\gamma)-x(\gamma-\eta)\big)\cdot\big(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big),$$

and

$$B(\gamma,\eta) = \big(x(\gamma)-x(\gamma-\eta)\big)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big).$$

The identity

$$\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta) = \eta\int_0^1 \partial_\gamma^3 x\big(\gamma+(s-1)\eta\big)\,ds, \tag{22}$$

yields

$$J_1 \leqslant 3\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}} |\eta| \frac{(|\partial_\gamma^2 x(\gamma)| + |\partial_\gamma^2 x(\gamma-\eta)|)|\partial_\gamma^3 x(\gamma)||\partial_\gamma^3 x(\gamma+(s-1)\eta)|}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+1}}\,d\gamma\,d\eta\,ds$$

$$\leqslant 3\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\int_0^1\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}\big(|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x\big(\gamma+(s-1)\eta\big)|^2\big)\,d\gamma\,d\eta\,ds$$

$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\|\partial_\gamma^3 x\|_{L^2}^2.$$

Using (22), we have for $J_2$

$$J_2 = -3\alpha\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}\big|F(x)(\gamma,\eta)\big|^{2+\alpha}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{\eta}\frac{\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3 x(\gamma+(s-1)\eta)}{|\eta|^\alpha}\,d\gamma\,d\eta\,ds$$

$$\leqslant 3\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\int_0^1\int_{\mathbb{T}}|\eta|^{1-\alpha}\int_{\mathbb{T}}\big(|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x\big(\gamma+(s-1)\eta\big)|^2\big)\,d\gamma\,d\eta\,ds$$

$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\|\partial_\gamma^3 x\|_{L^2}^2.$$

The term $J_3$ is estimated by

$$J_3 \leqslant 9\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}|\eta|\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2|\partial_\gamma^3 x(\gamma)||\partial_\gamma^3 x(\gamma+(s-1)\eta)|}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}\,d\gamma\,d\eta\,ds$$

$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\|\partial_\gamma^3 x\|_{L^2}^2.$$

Finally, we obtain

$$I_3 \leqslant C_\alpha\big(\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2} + \|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\big)\|\partial_\gamma^3 x\|_{L^2}^2. \tag{23}$$

We decompose the term $I_4 = J_4 + J_5 + J_6 + J_7 + J_8$ as follows

$$J_4 = -\alpha\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{C(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}\,d\eta\,d\gamma,$$

$$J_5 = -3\alpha\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{D(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}\,d\eta\,d\gamma,$$

$$J_6 = 5\alpha(\alpha+2)\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{A(\gamma,\eta)B(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+4}}\,d\eta\,d\gamma,$$

$$J_7 = 5\alpha(\alpha+2)\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{B(\gamma,\eta)|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+4}}\,d\eta\,d\gamma,$$

$$J_8 = -2\alpha(\alpha+2)(\alpha+4)\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{(B(\gamma,\eta))^3}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+6}}\,d\eta\,d\gamma,$$

with

$$C(\gamma,\eta) = \big(x(\gamma)-x(\gamma-\eta)\big)\cdot\big(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)\big),$$
$$D(\gamma,\eta) = \big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\cdot\big(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big).$$

The most singular term is $J_4$, in such a way that

$$J_4 \leqslant \|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}\big|\partial_\gamma^3 x(\gamma)\big|\big|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)\big|\,d\gamma\,d\eta$$
$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\big\|\partial_\gamma^3 x\big\|_{L^2}^2.$$

For $J_5$, we have

$$J_5 \leqslant 3\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}\big|\partial_\gamma^3 x(\gamma)\big|\big|\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big|\,d\gamma\,d\eta$$
$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\big\|\partial_\gamma^2 x\big\|_{L^2}\big\|\partial_\gamma^3 x\big\|_{L^2}.$$

In a similar way, we obtain

$$J_6 \leqslant 15\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}\big|\partial_\gamma^3 x(\gamma)\big|\big|\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big|\,d\gamma\,d\eta$$
$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\big\|\partial_\gamma^2 x\big\|_{L^2}\big\|\partial_\gamma^3 x\big\|_{L^2},$$

and

$$J_7 \leqslant 15\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}\big|\partial_\gamma^3 x(\gamma)\big|\big|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big|\,d\gamma\,d\eta$$
$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3\|\partial_\gamma x\|_{L^2}\big\|\partial_\gamma^3 x\big\|_{L^2}.$$

For the term $J_8$, we get

$$J_8 \leqslant 30\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}\big|\partial_\gamma^3 x(\gamma)\big|\big|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big|\,d\gamma\,d\eta$$
$$\leqslant C_\alpha\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3\big\|\partial_\gamma x\big\|_{L^2}\big\|\partial_\gamma^3 x\big\|_{L^2},$$

so that

$$I_4 \leqslant C_\alpha\big(\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2} + \|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2 + \|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3\big)\|x\|_{H^3}^2. \tag{24}$$

The inequalities (20), (21), (23) and (24) yield

$$\frac{d}{dt}\|\partial_\gamma^3 x\|_{L^2}^2(t) \leqslant C_\alpha\|F(x)\|_{L^\infty}^{3+\alpha}(t)\|x\|_{C^2}^3(t)\|x\|_{H^3}^2(t).$$

Due to the identity $\|x\|_{H^3}^2 = \|x\|_{L^2}^2 + \|\partial_\gamma^3 x\|_{L^2}^2$ and (19), we have

$$\frac{d}{dt}\|x\|_{H^3}(t) \leqslant C_\alpha\|F(x)\|_{L^\infty}^{3+\alpha}(t)\|x\|_{C^2}^3(t)\|x\|_{H^3}(t).$$

Finally, using Sobolev inequalities, we obtain

$$\frac{d}{dt}\|x\|_{H^3}(t) \leqslant C_\alpha\|F(x)\|_{L^\infty}^{3+\alpha}(t)\|x\|_{H^3}^4(t). \tag{25}$$

Notice that if we use energy methods at this point of the proof (see [10] to get the comprehensive argument), we need to regularize Eq. (17) as follows:

$$\begin{aligned} x_t^\varepsilon(\gamma,t) &= \phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^\varepsilon(\gamma,t) - \phi_\varepsilon * x^\varepsilon(\gamma-\eta,t))}{|x^\varepsilon(\gamma,t) - x^\varepsilon(\gamma-\eta,t)|^\alpha}\,d\eta, \\ x^\varepsilon(\gamma,0) &= x_0(\gamma), \end{aligned} \tag{26}$$

where $\phi_\varepsilon$ is a regular approximation to the identity. If the inequality (15) is satisfied initially, due to the properties of the regular approximations to the identity, we get a Picard system as follows

$$\begin{aligned} x_t^\varepsilon(\gamma,t) &= G^\varepsilon\big(x^\varepsilon(\gamma,t)\big), \\ x^\varepsilon(\gamma,0) &= x_0(\gamma), \end{aligned}$$

where $G^\varepsilon$ is Lipschitz. Therefore, for any $\varepsilon > 0$, we obtain a time of existence $t_\varepsilon$ where (15) is fulfilled. In order to have a time of existence for the system (26), independent of $\varepsilon$, we need to find energy estimates with bounds independent of $\varepsilon$. Next, by letting $\varepsilon \to 0$, we get solutions of the original equation. In this particular case, we have

$$\frac{d}{dt}\|x^\varepsilon\|_{H^3}(t) \leqslant C_\alpha\|F(x^\varepsilon)\|_{L^\infty}^{3+\alpha}(t)\|x^\varepsilon\|_{H^3}^4(t),$$

and if we let $\varepsilon \to 0$, it is possible that $\|F(x^\varepsilon)\|_{L^\infty} \to \infty$. In fact, we have an energy estimate that depends on $\varepsilon$, and so the argument fails. We cannot suppose that if the initial data fulfils (15), then there exists a time $t > 0$ independent of $\varepsilon$ in which (15) is satisfied, because just at this moment of the proof we do not have a well-posed system when $\varepsilon \to 0$, as the Lipschitz constant of $G^\varepsilon$ goes to infinity when $\varepsilon \to 0$.

In order to solve this problem, we consider the evolution of the quantity $\|F(x)\|_{L^\infty}$. Taking $p > 2$, it follows that

$$\frac{d}{dt}\|F(x)\|^p_{L^p}(t)=\frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\eta|}{|x(\gamma,t)-x(\gamma-\eta,t)|}\right)^p d\gamma\,d\eta$$
$$=-p\int_{\mathbb{T}}\int_{\mathbb{T}}|\eta|^p\frac{(x(\gamma,t)-x(\gamma-\eta,t))\cdot(x_t(\gamma,t)-x_t(\gamma-\eta,t))}{|x(\gamma,t)-x(\gamma-\eta,t)|^{p+2}}\,d\gamma\,d\eta$$
$$\leqslant p\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\eta|}{|x(\gamma,t)-x(\gamma-\eta,t)|}\right)^{p+1}\frac{|x_t(\gamma,t)-x_t(\gamma-\eta,t)|}{|\eta|}\,d\gamma\,d\eta.$$

We have

$$x_t(\gamma)-x_t(\gamma-\eta)=\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma)-x(\gamma-\xi)|^\alpha}\,d\xi-\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma-\eta)-\partial_\gamma x(\gamma-\eta-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}\,d\xi$$
$$=\int_{\mathbb{T}}\left(\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma)-x(\gamma-\xi)|^\alpha}-\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}\right)d\xi$$
$$+\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)+\partial_\gamma x(\gamma-\eta-\xi)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}\,d\xi$$
$$=I_5+I_6.$$

In order to estimate the term $I_5$, we consider the function $f(a)=a^\alpha$. For $a,b>0$, we have

$$\left|a^\alpha-b^\alpha\right|=\alpha\left|\int_0^1\big(sa+(1-s)b\big)^{\alpha-1}(a-b)\,ds\right|\leqslant\alpha\big(\min\{a,b\}\big)^{\alpha-1}|a-b|. \tag{27}$$

One finds that

$$I_5\leqslant\int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)|\,||x(\gamma)-x(\gamma-\xi)|^\alpha-|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha|}{|x(\gamma)-x(\gamma-\xi)|^\alpha|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}\,d\xi$$
$$\leqslant\|F(x)\|^{2\alpha}_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{1-\alpha}\left|\left|\frac{x(\gamma)-x(\gamma-\xi)}{\xi}\right|^\alpha-\left|\frac{x(\gamma-\eta)-x(\gamma-\eta-\xi)}{\xi}\right|^\alpha\right|d\xi.$$

Using (27), we get

$$I_5\leqslant\alpha\|F(x)\|^{1+\alpha}_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{1-\alpha}\left|\left|\frac{x(\gamma)-x(\gamma-\xi)}{\xi}\right|-\left|\frac{x(\gamma-\eta)-x(\gamma-\eta-\xi)}{\xi}\right|\right|d\xi$$
$$\leqslant\alpha\|F(x)\|^{1+\alpha}_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{-\alpha}\big(\left|x(\gamma)-x(\gamma-\eta)\right|+\left|x(\gamma-\xi)-x(\gamma-\eta-\xi)\right|\big)\,d\xi$$

$$\leqslant 2\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2}^2 |\eta| \int_{\mathbb{T}} |\xi|^{-\alpha}\, d\xi$$

$$\leqslant C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2}^2 |\eta|.$$

For $I_6$, we obtain

$$I_6 \leqslant \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)| + |\partial_\gamma x(\gamma-\eta-\xi) - \partial_\gamma x(\gamma-\xi)|}{|x(\gamma-\eta) - x(\gamma-\eta-\xi)|^\alpha}\, d\xi$$

$$\leqslant C_\alpha \|F(x)\|_{L^\infty}^{\alpha} \|x\|_{C^2} |\eta|.$$

The last two estimates show that

$$\frac{d}{dt}\|F(x)\|_{L^p}^p(t) \leqslant pC_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{1+\alpha}(t) \int_{\mathbb{T}^2} \big(F(x)(\gamma,\eta,t)\big)^{p+1}\, d\gamma\, d\eta$$

$$\leqslant pC_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{2+\alpha}(t) \|F(x)\|_{L^p}^p(t),$$

and therefore

$$\frac{d}{dt}\|F(x)\|_{L^p}(t) \leqslant C_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{2+\alpha}(t) \|F(x)\|_{L^p}(t).$$

Integrating in time it follows that

$$\|F(x)\|_{L^p}(t+h) \leqslant \|F(x)\|_{L^p}(t) \exp\left(C_\alpha \int_t^{t+h} \|x\|_{C^2}^2(s) \|F(x)\|_{L^\infty}^{2+\alpha}(s)\, ds\right),$$

and taking $p \to \infty$ we obtain

$$\|F(x)\|_{L^\infty}(t+h) \leqslant \|F(x)\|_{L^\infty}(t) \exp\left(C_\alpha \int_t^{t+h} \|x\|_{C^2}^2(s) \|F(x)\|_{L^\infty}^{2+\alpha}(s)\, ds\right).$$

In order to estimate the derivative of the quantity $\|F(x)\|_{L^\infty}(t)$, we use the last inequality, so that

$$\frac{d}{dt}\|F(x)\|_{L^\infty}(t) = \lim_{h\to 0}\big(\|F(x)\|_{L^\infty}(t+h) - \|F(x)\|_{L^\infty}(t)\big)h^{-1}$$

$$\leqslant \|F(x)\|_{L^\infty}(t) \lim_{h\to 0}\left(\exp\left(C_\alpha \int_t^{t+h} \|x\|_{C^2}^2(s) \|F(x)\|_{L^\infty}^{2+\alpha}(s)\, ds\right) - 1\right)h^{-1}$$

$$\leqslant C_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{3+\alpha}(t).$$

Applying Sobolev inequalities we conclude that

$$\frac{d}{dt}\|F(x)\|_{L^\infty}(t) \leqslant C_\alpha \|x\|^2_{H^3}(t)\|F(x)\|^{3+\alpha}_{L^\infty}(t). \tag{28}$$

This estimate does not give a global in time bound for $\|F(x)\|_{L^\infty}(t)$ in terms of norms of $x(\gamma, t)$, but adding the estimate (28) to (25), we have

$$\frac{d}{dt}\big(\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t)\big) \leqslant C_\alpha \|F(x)\|^{3+\alpha}_{L^\infty}(t)\|x\|^4_{H^3}(t),$$

and finally

$$\frac{d}{dt}\big(\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t)\big) \leqslant C_\alpha\big(\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t)\big)^{7+\alpha}. \tag{29}$$

Integrating, we get

$$\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t) \leqslant \frac{\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty}}{(1 - tC_\alpha(\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty})^{6+\alpha})^{\frac{1}{6+\alpha}}},$$

where $C_\alpha$ depends on $\alpha$. Using the regularized problem (26), the same estimate is obtained with $x^\varepsilon$ in place of $x$. Therefore we have found a time of existence independent of $\varepsilon$, and letting $\varepsilon \to 0$, the existence result follows.

Let $x$ and $y$ be two solutions of Eq. (17) with $x(\gamma, 0) = y(\gamma, 0)$, and $z = x - y$. One has that

$$\begin{aligned}
\int_{\mathbb{T}} z(\gamma)\cdot z_t(\gamma)\,d\gamma &= \int_{\mathbb{T}}\int_{\mathbb{T}} z(\gamma)\cdot\left(\frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)}{|x(\gamma) - x(\gamma-\eta)|^\alpha} - \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)}{|y(\gamma) - y(\gamma-\eta)|^\alpha}\right)d\eta\,d\gamma \\
&\quad + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{z(\gamma)\cdot(\partial_\gamma z(\gamma) - \partial_\gamma z(\gamma-\eta))}{|y(\gamma) - y(\gamma-\eta)|^\alpha}\,d\eta\,d\gamma \\
&= I_7 + I_8.
\end{aligned}$$

The term $I_7$ is estimated using (27) by

$$\begin{aligned}
I_7 &\leqslant \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|z(\gamma)||\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)||\,|x(\gamma) - x(\gamma-\eta)|^\alpha - |y(\gamma) - y(\gamma-\eta)|^\alpha|}{|x(\gamma) - x(\gamma-\eta)|^\alpha |y(\gamma) - y(\gamma-\eta)|^\alpha}\,d\eta\,d\gamma \\
&\leqslant \|F(x)\|^\alpha_{L^\infty}\|F(y)\|^\alpha_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}} |\eta|^{1-\alpha}|z(\gamma)|\left|\left|\frac{x(\gamma) - x(\gamma-\eta)}{\eta}\right|^\alpha \right. \\
&\quad \left. - \left|\frac{y(\gamma) - y(\gamma-\eta)}{\eta}\right|^\alpha\right| d\eta\,d\gamma \\
&\leqslant \|F(x)\|_{L^\infty}\|F(y)\|_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}} |\eta|^{1-\alpha}|z(\gamma)|\left|\left|\frac{x(\gamma) - x(\gamma-\eta)}{\eta}\right|\right.
\end{aligned}$$

$$
\begin{aligned}
&- \left|\frac{y(\gamma)-y(\gamma-\eta)}{\eta}\right|\Bigg| \, d\eta \, d\gamma \\
&\leqslant \big\|F(x)\big\|_{L^\infty}\big\|F(y)\big\|_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}|\eta|^{-\alpha}\big|z(\gamma)\big|\big|z(\gamma)-z(\gamma-\eta)\big|\,d\eta\,d\gamma \\
&\leqslant C_\alpha\big\|F(x)\big\|_{L^\infty}\big\|F(y)\big\|_{L^\infty}\|x\|_{C^2}\|z\|_{L^2}^2.
\end{aligned}
$$

Integration by parts in $I_8$ yields

$$
\begin{aligned}
I_8 &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{(z(\gamma)-z(\gamma-\eta))\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\gamma-\eta))}{|y(\gamma)-y(\gamma-\eta)|^\alpha}\,d\eta\,d\gamma \\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\partial_\gamma(|z(\gamma)-z(\gamma-\eta)|^2)}{|y(\gamma)-y(\gamma-\eta)|^\alpha}\,d\eta\,d\gamma \\
&= \frac{\alpha}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|z(\gamma)-z(\gamma-\eta)|^2(y(\gamma)-y(\gamma-\eta))\cdot(\partial_\gamma y(\gamma)-\partial_\gamma y(\gamma-\eta))}{|y(\gamma)-y(\gamma-\eta)|^{\alpha+2}}\,d\eta\,d\gamma \\
&\leqslant C_\alpha\big\|F(y)\big\|_{L^\infty}^{1+\alpha}\|y\|_{C^2}\|z\|_{L^2}^2.
\end{aligned}
$$

Finally we obtain

$$
\frac{d}{dt}\|z\|_{L^2}^2(t) \leqslant C\big(\alpha, x, F(x), y, F(y)\big)\|z\|_{L^2}^2(t),
$$

and using Gronwall inequality we conclude that $z=0$. □

## 5. Existence for $\alpha = 1$; the QG sharp front

In this section we prove the existence for the QG sharp front in Sobolev spaces. We give the norm of the Holder space $C^{k,\frac{1}{2}}(\mathbb{T})$ by

$$
\|x\|_{C^{k,\frac{1}{2}}} = \|x\|_{C^k} + \max_{\gamma,\eta\in\mathbb{T}}\frac{|\partial_\gamma^k x(\gamma)-\partial_\gamma^k x(\gamma-\eta)|}{|\eta|^{1/2}}.
$$

In the case of $\alpha = 1$, we have the following equation

$$
\begin{aligned}
x_t(\gamma,t) &= \frac{\theta_2-\theta_1}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta, \\
x(\gamma,0) &= x_0(\gamma).
\end{aligned}
\tag{30}
$$

We can take $\theta_2-\theta_1 = 2\pi$ without loss of generality. This equation loses two derivatives, therefore the technique applied in the last section does not work. Recall that we are trying to solve the QG equation in a weak sense, so we can modify the system (30) in the tangential direction without changing the shape of the front, as long as the curve satisfies

$$x_t(\gamma,t)\cdot\partial_\gamma^\perp x(\gamma,t) = -\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta.$$

We showed in Section 3 that the temperature $\theta(x,t)$ given by (9) is a weak solution of the QG equation. We propose to modify Eq. (30) as follows

$$\begin{aligned} x_t(\gamma,t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta + \lambda(\gamma,t)\partial_\gamma x(\gamma,t), \\ x(\gamma,0) &= x_0(\gamma). \end{aligned} \tag{31}$$

We have introduced the parameter $\lambda(\gamma,t)$ in order to get an extra cancellation in such a way that

$$\partial_\gamma x(\gamma,t)\cdot\partial_\gamma^2 x(\gamma,t) = 0. \tag{32}$$

Given an initial datum satisfying (15), we can reparameterize to obtain $|\partial_\gamma x(\gamma,0)|^2 = 1$, and therefore (32) is fulfilled at $t=0$. We cannot have $|\partial_\gamma x(\gamma,t)|^2 = 1$ for all time, but

$$\left|\partial_\gamma x(\gamma,t)\right|^2 = A(t). \tag{33}$$

We have

$$\begin{aligned} A'(t) &= 2\partial_\gamma x(\gamma,t)\cdot\partial_\gamma x_t(\gamma,t) \\ &= 2\partial_\gamma x(\gamma,t)\cdot\partial_\gamma\left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\right) + 2\partial_\gamma\lambda(\gamma,t)A(t), \end{aligned}$$

so that

$$\partial_\gamma\lambda(\gamma,t) = \frac{A'(t)}{2A(t)} - \frac{1}{A(t)}\partial_\gamma x(\gamma,t)\cdot\partial_\gamma\left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\right). \tag{34}$$

Because $\lambda(\gamma,t)$ has to be periodic, we obtain

$$\frac{A'(t)}{2A(t)} = \frac{1}{2\pi A(t)}\int_{\mathbb{T}} \partial_\gamma x(\gamma,t)\cdot\partial_\gamma\left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\right)d\gamma. \tag{35}$$

Using (35) in (34), and integrating in $\gamma$, one gets the following formula for $\lambda(\gamma,t)$

$$\lambda(\gamma,t)=\frac{\gamma+\pi}{2\pi}\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)}{|\partial_\gamma x(\gamma,t)|^2}\cdot\partial_\gamma\left(\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\right)d\gamma$$
$$-\int\limits_{-\pi}^{\gamma}\frac{\partial_\gamma x(\eta,t)}{|\partial_\gamma x(\eta,t)|^2}\cdot\partial_\eta\left(\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\eta,t)-\partial_\gamma x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|}\,d\xi\right)d\eta, \tag{36}$$

taking $\lambda(-\pi,t)=\lambda(\pi,t)=0$. If we consider solutions of Eq. (31) with $\lambda(\gamma,t)$ given by (36), it is easy to check that

$$\frac{d}{dt}\left|\partial_\gamma x(\gamma,t)\right|^2=\lambda(\gamma,t)\partial_\gamma\left|\partial_\gamma x(\gamma,t)\right|^2+\mu(t)\left|\partial_\gamma x(\gamma,t)\right|^2,$$

where

$$\mu(t)=\frac{1}{\pi}\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)}{|\partial_\gamma x(\gamma,t)|^2}\cdot\partial_\gamma\left(\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\right)d\gamma.$$

Solving this linear partial differential equation, if (32) is satisfied initially, one finds that the unique solution is given by

$$\left|\partial_\gamma x(\gamma,t)\right|^2=\left|\partial_\gamma x(\gamma,0)\right|^2+\frac{1}{\pi}\int\limits_0^t\int\limits_{\mathbb{T}}\partial_\gamma x(\gamma,s)\cdot\partial_\gamma\left(\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,s)-\partial_\gamma x(\gamma-\eta,s)}{|x(\gamma,s)-x(\gamma-\eta,s)|}\,d\eta\right)d\gamma\,ds.$$

Therefore we obtain (33).

The main result of this section is the following theorem.

**Theorem 5.1.** *Let $x_0(\gamma)\in H^k(\mathbb{T})$ for $k\geqslant 3$ with $F(x_0)(\gamma,\eta)<\infty$. Then there exists a time $T>0$ so that there is a solution to* (31) *in $C^1([0,T];H^k(\mathbb{T}))$ with $x(\gamma,0)=x_0(\gamma)$ and $\lambda(\gamma,t)$ given by* (36)*.*

**Proof.** We let $k=3$, the proof for $k>3$ being analogous. We have showed that (33) is satisfied if $x(\gamma,t)$ is a solution to (31). We can rewrite $\lambda(\gamma,t)$ as follows

$$\lambda(\gamma,t)=\frac{\gamma+\pi}{2\pi A(t)}\int\limits_{\mathbb{T}}\partial_\gamma x(\gamma,t)\cdot\partial_\gamma\left(\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\right)d\gamma$$
$$-\frac{1}{A(t)}\int\limits_{-\pi}^{\gamma}\partial_\gamma x(\eta,t)\cdot\partial_\eta\left(\int\limits_{\mathbb{T}}\frac{\partial_\gamma x(\eta,t)-\partial_\gamma x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|}\,d\xi\right)d\eta. \tag{37}$$

We obtain

$$\int\limits_{\mathbb{T}}x(\gamma)\cdot x_t(\gamma)\,d\gamma=\int\limits_{\mathbb{T}}\int\limits_{\mathbb{T}}x(\gamma)\cdot\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\,d\gamma+\int\limits_{\mathbb{T}}\lambda(\gamma)x(\gamma)\cdot\partial_\gamma x(\gamma)\,d\gamma$$
$$=I_1+I_2.$$

One finds that $I_1 = 0$, since

$$\begin{aligned} I_1 &= \int_{\mathbb{T}}\int_{\mathbb{T}} x(\gamma)\cdot\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\eta)}{|x(\gamma)-x(\eta)|}\,d\eta\,d\gamma = -\int_{\mathbb{T}}\int_{\mathbb{T}} x(\eta)\cdot\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\eta)}{|x(\gamma)-x(\eta)|}\,d\eta\,d\gamma \\ &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(x(\gamma)-x(\eta))\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\eta))}{|x(\gamma)-x(\eta)|}\,d\eta\,d\gamma = \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma\big|x(\gamma)-x(\gamma-\eta)\big|\,d\gamma\,d\eta \\ &= 0. \end{aligned}$$

For the term $I_2$, one obtains that $I_2 \leqslant \|\lambda\|_{L^\infty}\|x\|_{L^2}\|\partial_\gamma x\|_{L^2}$, and

$$\begin{aligned} \|\lambda\|_{L^\infty} &\leqslant \frac{2}{A(t)}\int_{\mathbb{T}}\big|\partial_\gamma x(\gamma)\big|\left|\partial_\gamma\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\right|d\gamma \\ &\leqslant \frac{2}{A(t)}\int_{\mathbb{T}}\big|\partial_\gamma x(\gamma)\big|\int_{\mathbb{T}}\frac{|\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)|}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\,d\gamma \\ &\quad+\frac{2}{A(t)}\int_{\mathbb{T}}\big|\partial_\gamma x(\gamma)\big|\int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^2}\,d\eta\,d\gamma = J_1+J_2. \end{aligned}$$

Due to $1/A(t) \leqslant \|F(x)\|^2_{L^\infty}(t)$, we have

$$J_1 \leqslant 2\big\|F(x)\big\|^3_{L^\infty}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}\big|\partial_\gamma^3 x\big(\gamma+(s-1)\eta\big)\big|\big|\partial_\gamma x(\gamma)\big|\,d\gamma\,d\eta\,ds \leqslant 2\big\|F(x)\big\|^3_{L^\infty}\|x\|^2_{H^3},$$

and

$$J_2 \leqslant 2\big\|F(x)\big\|^4_{L^\infty}\|x\|_{C^1}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}\big|\partial_\gamma^2 x\big(\gamma+(s-1)\eta\big)\big|^2\,d\gamma\,d\eta\,ds \leqslant 2\big\|F(x)\big\|^4_{L^\infty}\|x\|^3_{H^3}.$$

Therefore we obtain that

$$\frac{d}{dt}\|x\|^2_{L^2}(t) \leqslant C\big\|F(x)\big\|^4_{L^\infty}(t)\|x\|^5_{H^3}(t). \tag{38}$$

We decompose as follows

$$\begin{aligned} \int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3 x_t(\gamma)\,d\gamma &= \int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3\left(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\right)d\gamma \\ &\quad+\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3\big(\lambda(\gamma)\partial_\gamma x(\gamma)\big)\,d\gamma \\ &= I_3+I_4. \end{aligned}$$

We take $I_3 = J_3 + J_4 + J_5 + J_6$ where

$$J_3 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\, d\eta\, d\gamma,$$

$$J_4 = 3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \big(\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)\big)\partial_\gamma\big(\big|x(\gamma)-x(\gamma-\eta)\big|^{-1}\big)\, d\eta\, d\gamma,$$

$$J_5 = 3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \big(\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta)\big)\partial_\gamma^2\big(\big|x(\gamma)-x(\gamma-\eta)\big|^{-1}\big)\, d\eta\, d\gamma,$$

$$J_6 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \big(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)\big)\partial_\gamma^3\big(\big|x(\gamma)-x(\gamma-\eta)\big|^{-1}\big)\, d\eta\, d\gamma.$$

The term $J_3$ can be written as

$$\begin{aligned}
J_3 &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \big(\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)\big) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\, d\eta\, d\gamma \\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\partial_\gamma |\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|}\, d\eta\, d\gamma \\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)|^2 (x(\gamma)-x(\gamma-\eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3}\, d\eta\, d\gamma.
\end{aligned}$$

Defining

$$B(\gamma,\eta) = \big(x(\gamma)-x(\gamma-\eta)\big) \cdot \big(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)\big),$$

by (32), we see that

$$J_3 = \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \big|F(x)(\gamma,\eta)\big|^3 \big|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)\big|^2 \frac{B(\gamma,\eta)\eta^{-2} - \partial_\gamma x(\gamma)\cdot\partial_\gamma^2 x(\gamma)}{|\eta|}\, d\eta\, d\gamma.$$

Using

$$\left|\frac{B(\gamma,\eta)\eta^{-2} - \partial_\gamma x(\gamma)\cdot\partial_\gamma^2 x(\gamma)}{\eta}\right| \leqslant 2\|x\|^2_{C^{2,\frac{1}{2}}} |\eta|^{-1/2},$$

we see that

$$J_3 \leqslant \big\|F(x)\big\|^3_{L^\infty} \|x\|^2_{C^{2,\frac{1}{2}}} \int_{\mathbb{T}} |\eta|^{-1/2} \int_{\mathbb{T}} \big(\big|\partial_\gamma^3 x(\gamma)\big|^2 + \big|\partial_\gamma^3 x(\gamma-\eta)\big|^2\big)\, d\gamma\, d\eta$$

$$\leqslant C\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\frac{1}{2}}}\|\partial^3_\gamma x\|^2_{L^2}$$
$$\leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}. \tag{39}$$

We obtain that $J_4 = -6J_3$, which gives

$$J_4 \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}. \tag{40}$$

In order to estimate the term $J_5$, we consider $J_5 = K_1 + K_2 + K_3$, where

$$K_1 = -3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot\big(\partial^2_\gamma x(\gamma) - \partial^2_\gamma x(\gamma-\eta)\big)\frac{C(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\,d\gamma,$$
$$K_2 = -3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot\big(\partial^2_\gamma x(\gamma) - \partial^2_\gamma x(\gamma-\eta)\big)\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\,d\gamma,$$
$$K_3 = 9\int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot\big(\partial^2_\gamma x(\gamma) - \partial^2_\gamma x(\gamma-\eta)\big)\frac{(B(\gamma,\eta))^2}{|x(\gamma)-x(\gamma-\eta)|^5}\,d\eta\,d\gamma,$$

and

$$C(\gamma,\eta) = \big(x(\gamma)-x(\gamma-\eta)\big)\cdot\big(\partial^2_\gamma x(\gamma) - \partial^2_\gamma x(\gamma-\eta)\big).$$

The inequality

$$\big|\partial^2_\gamma x(\gamma) - \partial^2_\gamma x(\gamma-\eta)\big||\eta|^{-1/2} \leqslant \|x\|_{C^{2,\frac{1}{2}}}, \tag{41}$$

yields

$$K_1 \leqslant 3\|F(x)\|^2_{L^\infty}\|x\|_{C^{2,\frac{1}{2}}}\int_0^1\int_{\mathbb{T}}|\eta|^{-1/2}\int_{\mathbb{T}}\big|\partial^3_\gamma x(\gamma)\big|\big|\partial^3_\gamma x\big(\gamma+(s-1)\eta\big)\big|\,d\gamma\,d\eta\,ds$$
$$\leqslant C\|F(x)\|^2_{L^\infty}\|x\|^3_{H^3}.$$

As before, we have for $K_2$ that

$$K_2 \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\|\partial^3_\gamma x\|^2_{L^2} \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

The term $K_3$ is estimated by

$$K_3 \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\|\partial^3_\gamma x\|^2_{L^2} \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

Finally, we obtain

$$J_5 \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}. \tag{42}$$

Decomposing the term $J_6 = K_4 + K_5 + K_6 + K_7 + K_8$ as

$$K_4 = -\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{D(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\,d\gamma,$$

$$K_5 = -3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{E(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\,d\gamma,$$

$$K_6 = 15\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{B(\gamma,\eta)C(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^5}\,d\eta\,d\gamma,$$

$$K_7 = 15\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{B(\gamma,\eta)|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^5}\,d\eta\,d\gamma,$$

$$K_8 = -30\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{(B(\gamma,\eta))^3}{|x(\gamma)-x(\gamma-\eta)|^7}\,d\eta\,d\gamma,$$

where

$$D(\gamma,\eta) = \big(x(\gamma)-x(\gamma-\eta)\big)\cdot\big(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)\big),$$
$$E(\gamma,\eta) = \big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\cdot\big(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)\big),$$

we obtain

$$K_5 \leqslant 3\|F(x)\|_{L^\infty}^3\|x\|_{C^2}^2\|\partial_\gamma^3 x\|_{L^2}^2 \leqslant 3\|F(x)\|_{L^\infty}^3\|x\|_{H^3}^4,$$
$$K_6 \leqslant 15\|F(x)\|_{L^\infty}^3\|x\|_{C^2}^2\|\partial_\gamma^3 x\|_{L^2}^2 \leqslant 15\|F(x)\|_{L^\infty}^3\|x\|_{H^3}^4,$$
$$K_7 \leqslant 15\|F(x)\|_{L^\infty}^4\|x\|_{C^2}^3\|\partial_\gamma^3 x\|_{L^2}\|\partial_\gamma^2 x\|_{L^2} \leqslant 15\|F(x)\|_{L^\infty}^4\|x\|_{H^3}^5,$$

and

$$K_8 \leqslant 30\|F(x)\|_{L^\infty}^4\|x\|_{C^2}^3\|\partial_\gamma^3 x\|_{L^2}\|\partial_\gamma^2 x\|_{L^2} \leqslant 30\|F(x)\|_{L^\infty}^4\|x\|_{H^3}^5.$$

For the most singular term, we have

$$\begin{aligned}K_4 &= \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{\eta\partial_\gamma x(\gamma)\cdot(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta))-D(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\,d\gamma\\
&\quad-\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{\eta\partial_\gamma x(\gamma)\cdot(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\,d\gamma\\
&= L_1 + L_2,\end{aligned}$$

so that

$$L_1 \leqslant \|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\partial^3_\gamma x(\gamma)\right|\left|\partial^3_\gamma x(\gamma)-\partial^3_\gamma x(\gamma-\eta)\right|d\gamma\, d\eta \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

Decomposing the $L_2$ term, we see that

$$\begin{aligned} L_2 = {} & \int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\frac{\eta(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\cdot\partial^3_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\, d\gamma \\ & - \int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\eta \\ & \times \frac{\partial_\gamma x(\gamma)\cdot\partial^3_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\cdot\partial^3_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\, d\gamma \\ = {} & M_1+M_2. \end{aligned}$$

We estimate the $M_1$ term as

$$M_1 \leqslant \|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\partial^3_\gamma x(\gamma)\right|\left|\partial^3_\gamma x(\gamma-\eta)\right|d\gamma\, d\eta \leqslant \|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

Taking the derivative in (32), we see that $\partial_\gamma x(\gamma)\cdot\partial^3_\gamma x(\gamma) = -|\partial^2_\gamma x(\gamma)|^2$, and we rewrite

$$M_2 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big)\eta\frac{|\partial^2_\gamma x(\gamma)|^2-|\partial^2_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\, d\gamma.$$

The inequality

$$\left|\left|\partial^2_\gamma x(\gamma)\right|^2-\left|\partial^2_\gamma x(\gamma-\eta)\right|^2\right| \leqslant 2\|x\|_{C^2}|\eta|\int_0^1\left|\partial^3_\gamma x\big(\gamma+(s-1)\eta\big)\right|ds, \tag{43}$$

yields

$$M_2 \leqslant 2\|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\partial^3_\gamma x(\gamma)\right|\left|\partial^3_\gamma x\big(\gamma+(s-1)\eta\big)\right|d\gamma\, d\eta\, ds \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

Recalling that $K_4 = L_1+L_2 = L_1+M_1+M_2 \leqslant C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}$, we see that

$$J_6 \leqslant C\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3}. \tag{44}$$

Due to (39), (40), (42) and (44), we obtain

$$I_3 \leqslant C\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3}. \tag{45}$$

We write $I_4 = J_7 + J_8 + J_9 + J_{10}$, where

$$J_7 = \int_{\mathbb{T}} \lambda(\gamma)\partial^3_\gamma x(\gamma)\cdot\partial^4_\gamma x(\gamma)\,d\gamma, \qquad J_8 = 3\int_{\mathbb{T}} \partial_\gamma\lambda(\gamma)\big|\partial^3_\gamma x(\gamma)\big|^2\,d\gamma,$$

$$J_9 = 3\int_{\mathbb{T}} \partial^2_\gamma\lambda(\gamma)\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\,d\gamma, \qquad J_{10} = \int_{\mathbb{T}} \partial^3_\gamma\lambda(\gamma)\partial^3_\gamma x(\gamma)\cdot\partial_\gamma x(\gamma)\,d\gamma.$$

Integrating by parts in the term $J_7$ we have

$$J_7 = -\frac{1}{2}\int_{\mathbb{T}} \partial_\gamma\lambda(\gamma)\big|\partial^3_\gamma x(\gamma)\big|^2 d\gamma \leqslant \frac{1}{2}\|\partial_\gamma\lambda\|_{L^\infty}\big\|\partial^3_\gamma x\big\|^2_{L^2}.$$

Using (37), we see that

$$\begin{aligned}
\partial_\gamma\lambda(\gamma,t) &= \frac{1}{2\pi A(t)}\int_{\mathbb{T}} \partial_\gamma x(\gamma,t)\cdot\partial_\gamma\bigg(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\bigg)\,d\gamma \\
&\quad - \frac{1}{A(t)}\partial_\gamma x(\gamma,t)\cdot\partial_\gamma\bigg(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\bigg) \\
&= K_9 + K_{10}.
\end{aligned} \tag{46}$$

The term $K_9$ is estimated in the same way as $J_1$ and $J_2$, so that

$$K_9 \leqslant \|F(x)\|^4_{L^\infty}\|x\|^3_{H^3}.$$

We have for $K_{10}$ that

$$\begin{aligned}
K_{10} &\leqslant \frac{\|x\|_{C^2}}{A(t)}\int_{\mathbb{T}}\bigg(\frac{|\partial^2_\gamma x(\gamma,t)-\partial^2_\gamma x(\gamma-\eta,t)|}{|x(\gamma,t)-x(\gamma-\eta,t)|} + \frac{|\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)|^2}{|x(\gamma,t)-x(\gamma-\eta,t)|^2}\bigg)\,d\eta \\
&\leqslant 2\|F(x)\|^4_{L^\infty}\|x\|^3_{C^{2,\frac{1}{2}}}\int_{\mathbb{T}} |\eta|^{-1/2}\,d\eta \\
&\leqslant C\|F(x)\|^4_{L^\infty}\|x\|^3_{H^3},
\end{aligned}$$

and therefore

$$J_7 \leqslant C\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3}. \tag{47}$$

Due to the identity $J_8 = -6J_7$, one finds that

$$J_8 \leqslant C\big\|F(x)\big\|^4_{L^\infty}\|x\|^5_{H^3}. \tag{48}$$

Using

$$\partial^2_\gamma\lambda(\gamma,t) = -\frac{1}{A(t)}\partial^2_\gamma x(\gamma,t)\cdot\partial_\gamma\bigg(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\bigg) - \frac{1}{A(t)}\partial_\gamma x(\gamma,t)\cdot\partial^2_\gamma\bigg(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}\,d\eta\bigg),$$

one sees that

$$\begin{aligned} J_9 &= -\frac{1}{A(t)}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\partial^2_\gamma x(\gamma)\cdot\partial_\gamma\bigg(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\bigg)d\gamma \\ &\quad - \frac{1}{A(t)}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\partial_\gamma x(\gamma)\cdot\partial^2_\gamma\bigg(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\bigg)d\gamma \\ &= L_3 + L_4. \end{aligned}$$

Therefore

$$\begin{aligned} L_3 &\leqslant \frac{\|x\|^2_{C^2}}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}\big|\partial^3_\gamma x(\gamma)\big|\bigg(\frac{|\partial^2_\gamma x(\gamma,t)-\partial^2_\gamma x(\gamma-\eta,t)|}{|x(\gamma,t)-x(\gamma-\eta,t)|}+\frac{|\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)|^2}{|x(\gamma,t)-x(\gamma-\eta,t)|^2}\bigg)d\eta\,d\gamma \\ &\leqslant \big\|F(x)\big\|^4_{L^\infty}\|x\|^3_{C^2}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}\big|\partial^3_\gamma x(\gamma)\big|\big(\big|\partial^3_\gamma x\big(\gamma+(t-1)\eta\big)\big|+\big|\partial^2_\gamma x\big(\gamma+(t-1)\eta\big)\big|\big)\,d\gamma\,d\eta\,ds \\ &\leqslant C\big\|F(x)\big\|^4_{L^\infty}\|x\|^5_{H^3}. \end{aligned}$$

Moreover

$$\begin{aligned} L_4 &= -\frac{1}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\partial_\gamma x(\gamma)\cdot\frac{\partial^3_\gamma x(\gamma)-\partial^3_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\,d\gamma \\ &\quad + \frac{2}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\partial_\gamma x(\gamma)\cdot\frac{(\partial^2_\gamma x(\gamma)-\partial^2_\gamma x(\gamma-\eta))B(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}\,d\eta\,d\gamma \\ &\quad - \frac{1}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\partial_\gamma x(\gamma)\cdot\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big) \\ &\quad \times\partial^2_\gamma\big(\big|x(\gamma)-x(\gamma-\eta)\big|^{-1}\big)\,d\eta\,d\gamma \\ &= M_3+M_4+M_5. \end{aligned}$$

The terms $M_4$ and $M_5$ are estimated as before, so that

$$M_4 + M_5 \leqslant C \| F(x) \|^5_{L^\infty} \|x\|^6_{H^3}.$$

The most singular term is $M_3$, but we find that

$$\begin{aligned} M_3 &= \frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial^3_\gamma x(\gamma) \cdot \partial^2_\gamma x(\gamma) \partial^3_\gamma x(\gamma - \eta) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} \, d\eta \, d\gamma \\ &\quad - \frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial^3_\gamma x(\gamma) \cdot \partial^2_\gamma x(\gamma) \frac{\partial^3_\gamma x(\gamma) \cdot \partial_\gamma x(\gamma) - \partial^3_\gamma x(\gamma - \eta) \cdot \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} \, d\eta \, d\gamma \\ &= N_1 + N_2. \end{aligned}$$

We obtain

$$N_1 \leqslant \| F(x) \|^3_{L^\infty} \|x\|^2_{C^2} \| \partial^3_\gamma x \|^2_{L^2} \leqslant \| F(x) \|^3_{L^\infty} \|x\|^4_{H^3},$$

and using (32)

$$N_2 = \frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial^3_\gamma x(\gamma) \cdot \partial^2_\gamma x(\gamma) \frac{|\partial^2_\gamma x(\gamma)|^2 - |\partial^2_\gamma x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|} \, d\eta \, d\gamma.$$

Due to (43), we conclude that

$$N_2 \leqslant 2 \| F(x) \|^3_{L^\infty} \|x\|^2_{C^2} \| \partial^3_\gamma x \|^2_{L^2} \leqslant 2 \| F(x) \|^3_{L^\infty} \|x\|^4_{H^3}.$$

We have $J_9 = L_3 + L_4 = L_3 + M_3 + M_4 + M_5 = L_3 + N_1 + N_2 + M_4 + M_5$, so that

$$J_9 \leqslant \| F(x) \|^5_{L^\infty} \|x\|^6_{H^3}. \tag{49}$$

The identity (32) yields

$$J_{10} = -\int_{\mathbb{T}} \partial^3_\gamma \lambda(\gamma) \left| \partial^2_\gamma x(\gamma) \right|^2 d\gamma = 2 \int_{\mathbb{T}} \partial^2_\gamma \lambda(\gamma) \partial^3_\gamma x(\gamma) \cdot \partial^2_\gamma x(\gamma) \, d\gamma = \frac{2}{3} J_9,$$

and therefore

$$J_{10} \leqslant \| F(x) \|^5_{L^\infty} \|x\|^6_{H^3}. \tag{50}$$

Due to the inequalities (47)–(49) and (50), we get

$$I_4 \leqslant C \| F(x) \|^5_{L^\infty} \|x\|^6_{H^3}.$$

Using (45) and the last estimate, we have

$$\frac{d}{dt}\left\|\partial_\gamma^3 x\right\|_{L^2}^2(t) \leqslant C\left\|F(x)\right\|_{L^\infty}^5(t)\|x\|_{H^3}^6(t).$$

This inequality and (38) bound the evolution of the Sobolev norms of the curve as follows

$$\frac{d}{dt}\|x\|_{H^3}(t) \leqslant C\left\|F(x)\right\|_{L^\infty}^5(t)\|x\|_{H^3}^5(t). \tag{51}$$

We continue the argument considering the evolution of the quantity $\|F(x)\|_{L^\infty}(t)$. Taking $p > 2$, we see that

$$\frac{d}{dt}\left\|F(x)\right\|_{L^p}^p(t) \leqslant p\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\eta|}{|x(\gamma,t)-x(\gamma-\eta,t)|}\right)^{p+1}\frac{|x_t(\gamma,t)-x_t(\gamma-\eta,t)|}{|\eta|}\,d\gamma\,d\eta.$$

We have

$$\begin{aligned} x_t(\gamma)-x_t(\gamma-\eta) &= \int_{\mathbb{T}}\left(\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma)-x(\gamma-\xi)|}-\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|}\right)d\xi \\ &\quad+\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)+\partial_\gamma x(\gamma-\eta-\xi)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|}\,d\xi \\ &\quad+\big(\lambda(\gamma)-\lambda(\gamma-\eta)\big)\partial_\gamma x(\gamma)+\lambda(\gamma-\eta)\big(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\big) \\ &= I_5+I_6+I_7+I_8. \end{aligned}$$

Now,

$$\begin{aligned} I_5 &\leqslant \int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)|\,\big||x(\gamma)-x(\gamma-\xi)|-|x(\gamma-\eta)-x(\gamma-\eta-\xi)|\big|}{|x(\gamma)-x(\gamma-\xi)|\,|x(\gamma-\eta)-x(\gamma-\eta-\xi)|}\,d\xi \\ &\leqslant \left\|F(x)\right\|_{L^\infty}^2\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{-1}\big|x(\gamma)-x(\gamma-\eta)-\big(x(\gamma-\xi)-x(\gamma-\eta-\xi)\big)\big|\,d\xi \\ &\leqslant \left\|F(x)\right\|_{L^\infty}^2\|x\|_{C^2}|\eta|\int_0^1\int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma+(s-1)\eta)-\partial_\gamma x(\gamma+(s-1)\eta-\xi)|}{|\xi|}\,d\xi\,ds \\ &\leqslant 2\pi\left\|F(x)\right\|_{L^\infty}^2\|x\|_{C^2}^2|\eta|. \end{aligned}$$

For $I_6$ we see that

$$I_6 \leqslant \left\|F(x)\right\|_{L^\infty}|\eta|\int_0^1\int_{\mathbb{T}}\frac{|\partial_\gamma^2 x(\gamma+(s-1)\eta)-\partial_\gamma^2 x(\gamma+(s-1)\eta-\xi)|}{|\xi|}\,d\xi\,ds$$

$$\leqslant \|F(x)\|_{L^\infty}\|x\|_{C^{2,\frac{1}{2}}}|\eta|\int_0^1\int_{\mathbb{T}}|\xi|^{-1/2}\,d\xi\,ds$$
$$\leqslant C\|F(x)\|_{L^\infty}\|x\|_{C^{2,\frac{1}{2}}}|\eta|.$$

For $I_7$, we have

$$I_7 \leqslant \frac{2\|x\|_{C^2}}{A(t)}|\eta|\max_\gamma\left|\partial_\gamma x(\gamma)\right|\left|\partial_\gamma\left(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\right)d\eta\right|$$
$$\leqslant 2\|F(x)\|_{L^\infty}^2\|x\|_{C^2}^2|\eta|\max_\gamma\left(\int_{\mathbb{T}}\frac{|\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)|}{|x(\gamma)-x(\gamma-\eta)|}\,d\eta\right.$$
$$\left.+\int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^2}\,d\eta\right)$$
$$\leqslant 4\|F(x)\|_{L^\infty}^4\|x\|_{H^3}^4|\eta|.$$

Estimating $\|\lambda\|_{L^\infty}$ as before, we easily get

$$I_8 \leqslant \|\lambda\|_{L^\infty}\|x\|_{C^2}|\eta| \leqslant 4\|F(x)\|_{L^\infty}^4\|x\|_{H^3}^4|\eta|.$$

The last four estimates show that

$$\frac{d}{dt}\|F(x)\|_{L^p}(t) \leqslant C\|x\|_{H^3}^4(t)\|F(x)\|_{L^\infty}^5(t)\|F(x)\|_{L^p}(t),$$

so that, by integrating in time and taking $p \to \infty$, we obtain

$$\|F(x)\|_{L^\infty}(t+h) \leqslant \|F(x)\|_{L^\infty}(t)\exp\left(C\int_t^{t+h}\|x\|_{H^3}^4(s)\|F(x)\|_{L^\infty}^5(s)\,ds\right).$$

As in the previous section,

$$\frac{d}{dt}\|F(x)\|_{L^\infty}(t) \leqslant C\|x\|_{H^3}^4(t)\|F(x)\|_{L^\infty}^6(t)$$

so that, due to (51) and the above estimate, we see that

$$\frac{d}{dt}\left(\|x\|_{H^3}(t)+\|F(x)\|_{L^\infty}(t)\right) \leqslant C\left(\|x\|_{H^3}(t)+\|F(x)\|_{L^\infty}(t)\right)^{10}.$$

Integrating,

$$\|x\|_{H^3}(t)+\|F(x)\|_{L^\infty}(t) \leqslant \frac{\|x_0\|_{H^3}+\|F(x_0)\|_{L^\infty}}{(1-tC(\|x_0\|_{H^3}+\|F(x_0)\|_{L^\infty})^9)^{\frac{1}{9}}},$$

where $C$ is a constant.

We have used the equality (32) to obtain the a priori estimates. In order to get the solution of (31), we have to choose an appropriate regularized problem preserving (32). We propose the system

$$x_t^{\varepsilon,\delta}(\gamma,t)=\phi_\varepsilon*\int_{\mathbb{T}}\frac{\partial_\gamma(\phi_\varepsilon*x^{\varepsilon,\delta}(\gamma,t)-\phi_\varepsilon*x^{\varepsilon,\delta}(\gamma-\eta,t))}{|x^{\varepsilon,\delta}(\gamma,t)-x^{\varepsilon,\delta}(\gamma-\eta,t)|+\delta}\,d\eta+\lambda^{\varepsilon,\delta}(\gamma,t)\partial_\gamma x^{\varepsilon,\delta}(\gamma,t),$$
$$x^{\varepsilon,\delta}(\gamma,0)=x_0(\gamma), \tag{52}$$

with

$$\begin{aligned}\lambda^{\varepsilon,\delta}&(\gamma,t)\\ &=\frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x^{\varepsilon,\delta}(\gamma,t)}{|\partial_\gamma x^{\varepsilon,\delta}(\gamma,t)|^2}\cdot\partial_\gamma\left(\phi_\varepsilon*\int_{\mathbb{T}}\frac{\partial_\gamma(\phi_\varepsilon*x^{\varepsilon,\delta}(\gamma,t)-\phi_\varepsilon*x^{\varepsilon,\delta}(\gamma-\eta,t))}{|x^{\varepsilon,\delta}(\gamma,t)-x^{\varepsilon,\delta}(\gamma-\eta,t)|+\delta}\,d\eta\right)d\gamma\\ &\quad-\int_{-\pi}^{\gamma}\frac{\partial_\gamma x^{\varepsilon,\delta}(\eta,t)}{|\partial_\gamma x^{\varepsilon,\delta}(\eta,t)|^2}\cdot\partial_\eta\left(\phi_\varepsilon*\int_{\mathbb{T}}\frac{\partial_\gamma(\phi_\varepsilon*x^{\varepsilon,\delta}(\eta,t)-\phi_\varepsilon*x^{\varepsilon,\delta}(\eta-\xi,t))}{|x^{\varepsilon,\delta}(\eta,t)-x^{\varepsilon,\delta}(\eta-\xi,t)|+\delta}\,d\xi\right)d\eta.\end{aligned}$$

We can obtain energy estimates for the system (52) depending on $\varepsilon$ and $\delta$, but without using (32), and therefore we obtain existence of (52). As long as the solution exists, we have that

$$\partial_\gamma x^{\varepsilon,\delta}(\gamma,t)\cdot\partial_\gamma^2 x^{\varepsilon,\delta}(\gamma,t)=0.$$

Using this property of the solution, we obtain energy estimates that depend only on $\delta$, and taking $\varepsilon\to 0$ we get a solution of the following equation

$$x_t^{\delta}(\gamma,t)=\int_{\mathbb{T}}\frac{\partial_\gamma x^{\delta}(\gamma,t)-\partial_\gamma x^{\delta}(\gamma-\eta,t))}{|x^{\delta}(\gamma,t)-x^{\delta}(\gamma-\eta,t)|+\delta}\,d\eta+\lambda^{\delta}(\gamma,t)\partial_\gamma x^{\delta}(\gamma,t),$$
$$x^{\delta}(\gamma,0)=x_0(\gamma), \tag{53}$$

with

$$\begin{aligned}\lambda^{\delta}(\gamma,t)&=\frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x^{\delta}(\gamma,t)}{|\partial_\gamma x^{\delta}(\gamma,t)|^2}\cdot\partial_\gamma\left(\int_{\mathbb{T}}\frac{\partial_\gamma x^{\delta}(\gamma,t)-\partial_\gamma x^{\delta}(\gamma-\eta,t)}{|x^{\delta}(\gamma,t)-x^{\delta}(\gamma-\eta,t)|+\delta}\,d\eta\right)d\gamma\\ &\quad-\int_{-\pi}^{\gamma}\frac{\partial_\gamma x^{\delta}(\eta,t)}{|\partial_\gamma x^{\delta}(\eta,t)|^2}\cdot\partial_\eta\left(\int_{\mathbb{T}}\frac{\partial_\gamma x^{\delta}(\eta,t)-\partial_\gamma x^{\delta}(\eta-\xi,t)}{|x^{\delta}(\eta,t)-x^{\delta}(\eta-\xi,t)|+\delta}\,d\xi\right)d\eta.\end{aligned}$$

Again we have that the solutions of this system satisfy

$$\partial_\gamma x^{\delta}(\gamma,t)\cdot\partial_\gamma^2 x^{\delta}(\gamma,t)=0,$$

and taking advantage of this, we find energy estimates independent of $\delta$. Letting $\delta$ tend to 0, we conclude the existence result. □

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## References

[1] A.L. Bertozzi, P. Constantin, Global regularity for vortex patches, Comm. Math. Phys. 152 (1) (1993) 19–28.
[2] J.Y. Chemin, Persistence of geometric structures in two-dimensional incompressible fluids, Ann. Sci. Ecole. Norm. Sup. 26 (4) (1993) 517–542.
[3] P. Constantin, A.J. Majda, E. Tabak, Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar, Nonlinearity 7 (1994) 1495–1533.
[4] P. Constantin, Q. Nie, N. Schorghofer, Nonsingular surface quasi-geostrophic flow, Phys. Lett. A 241 (3) (1998) 168–172.
[5] D. Córdoba, Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, Ann. of Math. 148 (1998) 1135–1152.
[6] D. Córdoba, C. Fefferman, Scalars convected by a two-dimensional incompressible flow, Comm. Pure Appl. Math. 55 (2002) 255–260.
[7] D. Córdoba, C. Fefferman, J.L. Rodrigo, Almost sharp fronts for the surface quasi-geostrophic equation, Proc. Natl. Acad. Sci. USA 101 (2004) 2687–2691.
[8] D. Córdoba, M.A. Fontelos, A.M. Mancho, J.L. Rodrigo, Evidence of singularities for a family of contour dynamics equations, Proc. Natl. Acad. Sci. USA 102 (2005) 5949–5952.
[9] T.Y. Hou, J.S. Lowengrub, M.J. Shelley, The long-time motion of vortex sheets with surface tension, Phys. Fluids 9 (7) (1997) 1933–1954.
[10] A.J. Majda, A.L. Bertozzi, Vorticity and the Mathematical Theory of Incompressible Fluid Flow, Cambridge Press, 2002.
[11] K. Ohkitani, M. Yamada, Inviscid and inviscid-limit behavior of a surface quasigeostrophic flow, Phys. Fluids 9 (1997) 876–882.
[12] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.
[13] S. Resnick, Dynamical problems in nonlinear advective partial differential equations, PhD thesis, University of Chicago, Chicago, 1995.
[14] J.L. Rodrigo, On the Evolution of Sharp Fronts for the Quasi-Geostrophic Equation, Comm. Pure Appl. Math. 58 (2005) 0821–0866.
[15] E. Stein, Harmonic Analysis, Princeton University Press, Princeton, NJ, 1993.