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On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains

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Abstract

We study tangential vector fields on the boundary of a bounded Lipschitz domain Ω in $\mathbb{R}^3$. Our attention is focused on the definition of suitable Hilbert spaces corresponding to fractional Sobolev regularities and also on the construction of tangential differential operators on the non-smooth manifold. The theory is applied to the characterization of tangential traces for the space $\mathbf{H}(\mathbf{curl}, \Omega)$. Hodge decompositions are provided for the corresponding trace spaces, and an integration by parts formula is proved.

1. Introduction

Let Ω be a bounded open Lipschitz domain with a connected boundary Γ. Standard Sobolev spaces $H^s(\Omega)$ for any $s \in \mathbb{R}$ and $H^t(\Gamma)$ for $t \in [-1, 1]$ are defined on the domain Ω and on its boundary Γ, respectively (see [17,20]). Moreover, we set

$$\mathbf{H}^s(\Omega) = \left(H^s(\Omega)\right)^3, \qquad \mathbf{H}^t(\Gamma) = \left(H^t(\Gamma)\right)^3, \qquad \mathbf{L}^2(\Gamma) = \mathbf{H}^0(\Gamma),$$
$$\mathbf{H}(\mathbf{curl}, \Omega) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\right\},$$
$$\mathbf{H}(\mathrm{div}, \Omega) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathrm{div}\,\mathbf{u} \in L^2(\Omega)\right\}.$$

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The characterization of tangential traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ is an important tool in the analysis of boundary value problems for Maxwell's equations. For smooth domains Ω, it is well known [1,10,21] that this space coincides with $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. It admits several different equivalent descriptions, its dual space is known to be $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, and Hodge decompositions have been proved.

For non-smooth domains Ω, not all the different descriptions make sense, and if they do, they are not always equivalent. For the analysis of the boundary value problems, in particular in connection with boundary integral methods, one still would like to have such intrinsic descriptions, including characterizations of the dual space and Hodge decompositions (see [8]).

For the case of polyhedral domains Ω, a theory has been developed in [5–7]. For the case of Lipschitz domains, a characterization of $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ has been given by Tartar in [23], and the surjectivity of the tangential trace mapping was shown. In the present paper, we give other definitions of $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ on Lipschitz boundaries and, using Tartar's result, we show their equivalence. We also characterize the dual space and prove Hodge decompositions.

Let us mention the different ways of characterizing the tangential trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$ in the case of smooth domains and the difficulties which appear on non-smooth ones. For any regular vector field $\mathbf{u}$ in Ω, we define the tangential trace $\gamma_\tau(\mathbf{u}) = \mathbf{u} \wedge \mathbf{n}|_\Gamma$ and the projection on the tangential plane $\pi_\tau(\mathbf{u}) = \mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})|_\Gamma$, where $\mathbf{n}$ denotes the outward unit vector normal to Γ.

The Sobolev space $TH^{1/2}(\Gamma)$ of tangential vector fields of order 1/2 on the surface Γ can be defined in at least five different ways:

(i) $TH^{1/2}(\Gamma) = \pi_\tau(\mathbf{H}^1(\Omega))$;
(ii) $TH^{1/2}(\Gamma) = \gamma_\tau(\mathbf{H}^1(\Omega))$;
(iii) $TH^{1/2}(\Gamma) = \{\mathbf{v}: \Gamma \to \mathbb{R}^3 \mid \mathbf{v} = (v_1, v_2, v_3)^t \in (H^{1/2}(\Gamma))^3, \mathbf{v}\cdot\mathbf{n} = 0\}$.

The condition of tangentiality $\mathbf{v}\cdot\mathbf{n} = 0$ can be formulated in less obvious equivalent ways:

(iv) $$TH^{1/2}(\Gamma) = \left\{\mathbf{v} \in \mathbf{H}^{1/2}(\Gamma) \;\middle|\; \int_\Gamma \mathbf{v}\cdot\nabla\phi = 0\ \forall\phi \in H^2(\Omega)\cap H_0^1(\Omega)\right\}. \tag{1}$$

In addition, one can introduce local coordinate systems $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ with two linearly independent smooth tangential vector fields $\mathbf{e}_1, \mathbf{e}_2$ and write

(v) $TH^{1/2}(\Gamma) = \{\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \mid \alpha_1, \alpha_2 \in H^{1/2}(\Gamma)\}$.

Note that (v) defines a space of "tangent fields" on Γ, i.e., of sections of the tangent bundle $T\Gamma$ of Γ, characterized by a certain regularity, whereas (iii) and (iv) define subspaces of 3D vector fields living on Γ, i.e., of sections of the tangent bundle $T\mathbb{R}^3$ of $\mathbb{R}^3$, restricted to Γ. For smooth domains, these two points of view are obviously equivalent, but both of them will be useful for different purposes: The "3D field" aspect corresponds more directly to the definition of traces, whereas the "tangent field" aspect is needed for the definition of surface differential operators and Hodge decompositions.

On an arbitrary Lipschitz boundary, the only level of Sobolev regularity where these two aspects are obviously equivalent is the level of L^2 regularity. One of the principal themes of the present paper is to study this equivalence for Sobolev regularity indices $\pm 1/2$. Some results concerning Sobolev index $3/2$ have recently been obtained in [15] and [9].

Even for piecewise smooth domains, (i), (ii), (iii), and (v) will, in general, give *four* different spaces (see [6]).

The tangential trace of $\mathbf{H}(\mathbf{curl}, \Omega)$ can be defined by using the Green formula for $C^1(\bar{\Omega})$ functions $\mathbf{u}, \mathbf{v}$,

$$\int_{\Omega} (\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u})\, d\Omega = \int_{\Gamma} \gamma_\tau(\mathbf{u}) \cdot \mathbf{v}\, d\Gamma, \tag{2}$$

which extends by continuity to $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$, $\mathbf{v} \in \mathbf{H}^1(\Omega)$. From the surjectivity of the trace mapping $\gamma : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ it follows that

$$\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\Gamma)$$

is well-defined and continuous, where $\mathbf{H}^{-1/2}(\Gamma)$ is defined as the dual space of $\mathbf{H}^{1/2}(\Gamma)$. Here, $\mathbf{L}^2(\Gamma)$ is taken as pivot space.

Thus $\gamma_\tau(\mathbf{u}) \in TH^{-1/2}(\Gamma)$, where $TH^{-1/2}(\Gamma)$ for smooth domains can be defined analogously to (i)–(v), and additionally, as the dual space of $TH^{1/2}(\Gamma)$.

Some of these definitions of $TH^{-1/2}(\Gamma)$ neither make sense any more for Lipschitz domains nor even for polyhedral domains. The normal unit vector is discontinuous, only $\mathbf{n} \in L^\infty(\Gamma)$ for Lipschitz domains, hence the scalar product $\mathbf{v} \cdot \mathbf{n}$ is not defined for $\mathbf{v} \in \mathbf{H}^{-1/2}(\Gamma)$, so that the condition $\mathbf{v} \cdot \mathbf{n} = 0$ does not make sense. Similarly, the construction of a tangential field by its components in local coordinates, i.e., corresponding to (v), $\mathbf{v} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ does not make sense for $\alpha_1, \alpha_2 \in H^{-1/2}(\Gamma)$.

The situation gets even more problematic when we look at the tangential trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$, i.e., $\gamma_\tau(\mathbf{H}(\mathbf{curl}, \Omega)) \subseteq \mathbf{H}^{-1/2}(\Gamma)$. Since $\mathbf{curl}\,\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$, the Green formula for $\mathbf{w} = \mathbf{curl}\,\mathbf{u}$ and $\phi \in H^1(\Omega)$,

$$\int_{\Omega} \mathbf{w} \cdot \nabla\phi\, d\Omega = \int_{\Gamma} \mathbf{n} \cdot \mathbf{w}\phi\, d\Gamma,$$

allows to define $\mathbf{n} \cdot \mathbf{curl}\,\mathbf{u} \in H^{-1/2}(\Gamma)$. So far, this makes sense for any Lipschitz domain. For smooth domains, $\mathbf{n} \cdot \mathbf{curl}\,\mathbf{u}$ can be expressed in local coordinates and one finds

$$\mathbf{n} \cdot \mathbf{curl}\,\mathbf{u} = \mathrm{div}_\Gamma\big(\gamma_\tau(\mathbf{u})\big) = \frac{1}{\sqrt{g}}\big(\partial_2(\sqrt{g}\alpha_1) - \partial_1(\sqrt{g}\alpha_2)\big), \tag{3}$$

where $g = \det\{G\}$, $G = \{g_{ik}\}_{ik=1,2}$, $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ and ∂_i stands for the partial derivative with respect to $\mathbf{e}_i$.

For non-smooth domains, (3) needs a careful reinterpretation if one wants to retain the result that $\mathbf{n} \cdot \mathbf{curl}\,\mathbf{u}$ is indeed obtained by the action of a tangential differential operator on the tangent field $\gamma_\tau(\mathbf{u})$.

On a smooth domain, one sees that

$$\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \tag{4}$$

is continuous where

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \left\{ \mathbf{v} \in TH^{-1/2}(\Gamma) \mid \mathrm{div}_\Gamma \mathbf{v} \in H^{-1/2}(\Gamma) \right\}. \tag{5}$$

The result of Paquet [21] shows that the mapping (4) is surjective for a smooth domain, and the result of Tartar ([23], see Section 7) shows that it is surjective for a Lipschitz domain, if $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is defined as

$$\begin{aligned} &\big\{ \mathbf{v} \in H^{-1/2}(\Gamma)^3 \mid \exists \eta \in H^{-1/2}(\Gamma): \\ &\quad {}_{\mathbf{H}^{-1/2}(\Gamma)}\big\langle \mathbf{v}, \gamma(\nabla\phi) \big\rangle_{\mathbf{H}^{1/2}(\Gamma)} = {}_{H^{-1/2}(\Gamma)}\langle \eta, \phi \rangle_{H^{1/2}(\Gamma)}\ \forall \phi \in H^2(\Omega) \big\}. \end{aligned} \tag{6}$$

Note that the tangentiality of $\mathbf{v}$ is contained in (6) in the weak sense of (1). The condition

$$\eta = -\,\mathrm{div}_\Gamma \mathbf{v}$$

is implied in a very weak sense. This makes it difficult to study the dual space of (6), to show Hodge decompositions, or even to understand the relation of this space with the pivot space

$$\mathbf{L}_t^2(\Gamma) := \left\{ \mathbf{u} \in \mathbf{L}^2(\Gamma) \mid \mathbf{u} \cdot \mathbf{n} = 0 \right\}. \tag{7}$$

For example, with $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma) = \{\mathbf{v} \in \mathbf{L}_t^2(\Gamma) \mid \mathrm{div}_\Gamma \mathbf{v} \in L^2(\Gamma)\}$, do the inclusions

$$\mathbf{H}(\mathrm{div}_\Gamma, \Gamma) \subseteq \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \qquad \pi_\tau\big(\mathbf{H}^1(\Omega)\big) \subseteq \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

hold?

The two descriptions (5) and (6) are typical examples of the two points of view mentioned above: (5) considers tangent fields and (6) 3D fields.

In a more general framework of differential forms on Lipschitz domains in arbitrary dimensions, related questions are considered in [18]. This is part of a project of studying boundary value problems for generalized Maxwell equations, see [19].

We have seen that one difficulty arises from the vector $\mathbf{n}$ which, being discontinuous, is not a multiplier in the spaces $\mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma)$. As a replacement for $\mathbf{n}$, one can consider more regular normal vector fields, for example the traces of

$$\ker(\gamma_\tau) \cap \mathbf{H}^1(\Omega) = \left\{ \mathbf{u} \in \mathbf{H}^1(\Omega) \mid \gamma_\tau(\mathbf{u}) = 0 \text{ on } \Gamma \right\},$$

which would correspond to $H^{1/2}(\Gamma)$ normal vector fields. One has to be careful, however, when using this space, because it can be very small.

On a polyhedron, the vanishing of the tangential component of a vector field on the whole boundary implies that at the edges *all* components vanish. For the class of Lipschitz domains, Filonov in [13] has an example of a domain which is even of class $C^{3/2}$, for which the vanishing of the tangential components of $\mathbf{H}^1(\Omega)$ vector fields implies that all components vanish on the whole boundary:

$$\ker(\gamma_\tau) \cap \mathbf{H}^1(\Omega) \equiv \mathbf{H}^1_0(\Omega).$$

In this case, there exists no non-trivial $H^{1/2}(\Gamma)$ normal vector field. We shall give a short description of Filonov's construction in Section 6.

The outline of the paper is as follows: In Section 2, some spaces of tangential vector fields on Lipschitz domains are defined arising from natural definitions of tangential traces of $\mathbf{H}^1$ vector fields. In particular, three spaces, V'_γ, V'_π, and V'_0 are defined which play the role of $TH^{-1/2}(\Gamma)$ on smooth boundaries. In Section 3, we define and analyze tangential differential operators acting in these spaces. In Section 4, the ranges of π_τ and γ_τ are characterized in our functional context. In Section 5, the validity of Hodge decompositions is proved. Sections 6 and 7 are appendices: In Section 6, we report some details related to Filonov's example of a "regular pathological domain," and in Section 7, we present Tartar's proof of the surjectivity of the tangential trace map onto the space defined in (6).

2. Tangential trace spaces for $\mathbf{H}^1(\Omega)$

In the following, we set $V = \mathbf{H}^{1/2}(\Gamma)$ and $V' = \mathbf{H}^{-1/2}(\Gamma)$. Moreover, we adopt the point of view that the subspace $\mathbf{L}^2_t(\Gamma)$ of $\mathbf{L}^2(\Gamma)$ defined in (7) is considered as a space of two dimensional tangent fields.

Definition 2.1. The "tangential components trace" mapping $\pi_\tau : \mathcal{D}(\bar{\Omega})^3 \to \mathbf{L}^2_t(\Gamma)$ and the "tangential trace" mapping $\gamma_\tau : \mathcal{D}(\bar{\Omega})^3 \to \mathbf{L}^2_t(\Gamma)$ are defined as $\mathbf{u} \mapsto \mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})|_\Gamma$ and $\mathbf{u} \mapsto \mathbf{u} \wedge \mathbf{n}|_\Gamma$, respectively.

Let $\gamma : \mathbf{H}^1(\Omega) \to V$ be the standard (vector) trace operator and γ^{-1} one of its right inverses. We will also use the notation π_τ (respectively γ_τ) for the composite operator $\pi_\tau \circ \gamma^{-1}$ (respectively $\gamma_\tau \circ \gamma^{-1}$) which acts only on traces.

By density of $\mathcal{D}(\bar{\Omega})^3|_\Gamma$ into $\mathbf{L}^2(\Gamma)$, these operators can be extended to linear continuous operators in $\mathbf{L}^2(\Gamma)$. Moreover, it is easy to see that

$$\ker(\pi_\tau) = \ker(\gamma_\tau) \quad \text{in } \mathbf{L}^2(\Gamma). \tag{8}$$

We define:

Definition 2.2. Let $V_\gamma := \gamma_\tau(V)$ and $V_\pi := \pi_\tau(V)$.

V_γ and V_π are Hilbert spaces endowed with the norms that assure the continuity of the operators γ_τ and π_τ, respectively. We set

$$\|\boldsymbol{\lambda}\|_{V_\gamma} = \inf_{\mathbf{u}\in V}\left\{\|\mathbf{u}\|_V\colon\ \gamma_\tau(\mathbf{u}) = \boldsymbol{\lambda}\right\}, \tag{9}$$

$$\|\boldsymbol{\lambda}\|_{V_\pi} = \inf_{\mathbf{u}\in V}\left\{\|\mathbf{u}\|_V\colon\ \pi_\tau(\mathbf{u}) = \boldsymbol{\lambda}\right\}. \tag{10}$$

These spaces will be the bases of our construction. Note that the density of V in $\mathbf{L}^2(\Gamma)$ means that V_γ and V_π are dense subspaces of $\mathbf{L}^2_t(\Gamma)$. These spaces as well as their dual spaces V'_γ and V'_π can therefore be considered as spaces of tangent fields of regularity $1/2$ and $-1/2$, respectively.

If the surface Γ was regular, then

$$V_\gamma = V_\pi = TH^{1/2}(\Gamma) \quad \text{and} \quad V'_\gamma = V'_\pi = TH^{-1/2}(\Gamma), \tag{11}$$

where $TH^{1/2}(\Gamma)$ and $TH^{-1/2}(\Gamma)$ denote the standard Hilbertian Sobolev spaces of tangential vector fields of order $1/2$ and $-1/2$, respectively. Already in the case of piecewise regular surfaces, the spaces V_γ and V_π are different (see [6]). In the following we show that actually the equalities in (11) can be replaced by suitable isomorphisms.

Let $i_\pi : \mathbf{L}^2_t(\Gamma) \to \mathbf{L}^2(\Gamma)$ and $i_\gamma : \mathbf{L}^2_t(\Gamma) \to \mathbf{L}^2(\Gamma)$ be the adjoint operators of π_τ and γ_τ, respectively. These operators are the identifications of tangent fields with 3D vector fields mentioned above. It is important to realize that they are different identifications. Thanks to the Lipschitz assumption, a local system of orthonormal coordinates $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \mathbf{n})$ can be defined at almost every $\mathbf{x} \in \Gamma$. Here, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are two orthonormal vectors belonging to the tangent plane for almost every $\mathbf{x} \in \Gamma$, while $\mathbf{n}$ is the outer normal to Ω. Of course, the vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ can also be considered as "tangent fields" (sections of the tangent bundle) and, for the sake of clarity, we denote by $\tilde{\boldsymbol{\tau}}_1$ and $\tilde{\boldsymbol{\tau}}_2$ this basis of tangent fields.

This means that

$$\pi_\tau(\mathbf{u}) = (\mathbf{u}|_\Gamma \cdot \boldsymbol{\tau}_1)\tilde{\boldsymbol{\tau}}_1 + (\mathbf{u}|_\Gamma \cdot \boldsymbol{\tau}_2)\tilde{\boldsymbol{\tau}}_2, \tag{12}$$

$$\gamma_\tau(\mathbf{u}) = (\mathbf{u}|_\Gamma \cdot \boldsymbol{\tau}_2)\tilde{\boldsymbol{\tau}}_1 - (\mathbf{u}|_\Gamma \cdot \boldsymbol{\tau}_1)\tilde{\boldsymbol{\tau}}_2. \tag{13}$$

Accordingly, the operator i_π simply associates to a vector in $\mathbf{L}^2_t(\Gamma)$, the vector in $\mathbf{L}^2(\Gamma)$ with the same tangential component and zero normal component. On the other hand, the operator i_γ rotates the tangential component:

$$\mathbf{u} \in \mathbf{L}^2_t(\Gamma), \qquad \mathbf{u} = u_1\tilde{\boldsymbol{\tau}}_1 + u_2\tilde{\boldsymbol{\tau}}_2, \qquad \begin{cases} i_\pi(\mathbf{u}) = u_1\boldsymbol{\tau}_1 + u_2\boldsymbol{\tau}_2, \\ i_\gamma(\mathbf{u}) = -u_2\boldsymbol{\tau}_1 + u_1\boldsymbol{\tau}_2. \end{cases} \tag{14}$$

These operators can be extended in the following way:

$$i_\pi : V'_\pi \to \left(\ker(\pi_\tau) \cap V\right)^0, \qquad i_\gamma : V'_\gamma \to \left(\ker(\gamma_\tau) \cap V\right)^0, \tag{15}$$

where $\cdot^0$ denotes the polar set (or "annihilator," see [4,24]). Note that because of (8), the two range spaces in (15) coincide. Moreover, the operators defined in (15) are isomorphisms: thanks to Definition 2.2, the ranges V_π and V_γ of π_τ and γ_τ are closed.

By using (14), it is natural to define a rotation operator acting on $\mathbf{L}_t^2(\Gamma)$ fields as follows: $r : \mathbf{L}_t^2(\Gamma) \to \mathbf{L}_t^2(\Gamma)$, $r := i_\pi^{-1} \circ i_\gamma$. The operator r corresponds to the geometric operation $\mathbf{n} \wedge \cdot$. Now, by using (15) and recalling the definition of V_π and V_γ, we immediately have that the operator r can be restricted and extended in the following way:

$$r : V_\pi \to V_\gamma \quad \text{and} \quad r : V'_\pi \to V'_\gamma. \tag{16}$$

Finally, for any choice of spaces, r is invertible and $r^{-1} = r^\star = -r$; for any $\mathbf{u} \in \mathbf{L}^2(\Gamma)$, we have

$$\gamma_\tau(\mathbf{u}) = -r\big(\pi_\tau(\mathbf{u})\big) \quad \text{and} \quad \pi_\tau(\mathbf{u}) = r\big(\gamma_\tau(\mathbf{u})\big). \tag{17}$$

It is important to underline that by our simple functional analytic argument, in (16) we have defined the rotation operator also between our two spaces of order $-1/2$, V'_π and V'_γ. This is a generalization of the geometric operation $\mathbf{n} \wedge \cdot$ which will be useful in the following.

We have seen that the spaces V'_π and V'_γ are two (in general different) incarnations of the space of tangent fields of regularity $-1/2$. In (15), we have the isomorphic inclusion of V'_π and V'_γ into the same subspace of a dual space of 3D vector fields of regularity 1/2 which, by this duality, can be interpreted as a space of 3D vector fields of regularity $-1/2$. This space admits two other natural definitions, and in the following lemma we show that these definitions are, in fact, equivalent.

Lemma 2.3. *Let*

$$V'_0 := \big(\ker(\pi_\tau) \cap V\big)^0 = \big\{\boldsymbol{\xi} \in V' \mid {}_{V'}\langle \boldsymbol{\xi}, \boldsymbol{\varphi}\rangle_V = 0 \ \forall \boldsymbol{\varphi} \in \ker(\pi_\tau) \cap V\big\}.$$

The following holds:

$$V'_0 = \overline{i_\pi\big(\mathbf{L}_t^2(\Gamma)\big)}^{V'} = \overline{i_\gamma\big(\mathbf{L}_t^2(\Gamma)\big)}^{V'}, \tag{18}$$

where $\overline{\cdot}^{V'}$ *denotes the closure of the space with respect to the topology induced by* V'. *Let* γ *be the standard trace operator acting on vectors* $\gamma : \mathbf{H}^1(\Omega) \to V$. *Then there holds*

$$V'_0 = \big\{\boldsymbol{\xi} \in V' \mid {}_{V'}\big\langle \boldsymbol{\xi}, \gamma(\nabla\phi)\big\rangle_V = 0 \ \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega)\big\}. \tag{19}$$

Proof. We start by showing (18). It is enough to prove the first equality since $i_\pi(\mathbf{L}_t^2(\Gamma)) = i_\gamma(\mathbf{L}_t^2(\Gamma))$. Let $V'_1 := \overline{i_\pi(\mathbf{L}_t^2(\Gamma))}^{V'}$. We first prove the inclusion $V'_1 \subseteq V'_0$. Since V'_0 is closed in V', it suffices to show that $i_\pi(\mathbf{L}_t^2(\Gamma)) \subseteq V'_0$. Let $\boldsymbol{\xi} \in i_\pi(\mathbf{L}_t^2(\Gamma))$. For any $\mathbf{v} \in \ker(\pi_\tau) \cap V$ we have ${}_{V'}\langle \boldsymbol{\xi}, \mathbf{v}\rangle_V = \int_\Gamma \boldsymbol{\xi} \cdot \mathbf{v} = 0$. Thus $\boldsymbol{\xi} \in (\ker(\pi_\tau) \cap V)^0$. In order to show the converse inclusion, we proceed by a duality argument. Let $\mathbf{v} \in V$ be such that ${}_{V'}\langle \boldsymbol{\xi}, \mathbf{v}\rangle_V = 0$ for any $\boldsymbol{\xi} \in i_\pi(\mathbf{L}_t^2(\Gamma))$.

This means that for any $\boldsymbol{\eta} \in \mathbf{L}^2_t(\Gamma)$, we have ${}_{V'_\pi}\langle \boldsymbol{\eta}, \pi_\tau(\mathbf{v})\rangle_{V_\pi} = {}_{V'}\langle i_\pi(\boldsymbol{\eta}), \mathbf{v}\rangle_V = 0$. Thus $\mathbf{v} \in \ker(\pi_\tau) \cap V$.

Now, we pass to the proof of (19). We shall show the dual equality

$$\ker(\gamma_\tau) \cap V = \{\gamma(\nabla\phi) \mid \phi \in H^2(\Omega) \cap H^1_0(\Omega)\}.$$

It is easy to see that for $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$ we have $\gamma_\tau(\nabla\phi) = 0$. Using (2) we obtain

$$\int_\Gamma \gamma_\tau(\nabla\phi) \cdot \boldsymbol{\xi} = \int_\Omega \nabla\phi \cdot \mathbf{curl}\, \boldsymbol{\xi} = \int_\Gamma \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{n}\phi = 0 \quad \forall \boldsymbol{\xi} \in \mathcal{D}(\bar{\Omega})^3, \tag{20}$$

which implies $\gamma_\tau(\nabla\phi) = 0$.

The converse inclusion is obtained by a vector potential argument on the domain Ω, similar to arguments in [2,3,11]. Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that $\gamma_\tau(\mathbf{u}) = 0$. We denote by $\mathcal{O} \subseteq \mathbb{R}^3$ a regular domain such that $\bar{\Omega} \subset \mathcal{O}$. We denote by $\tilde{\mathbf{u}}$ the extension of $\mathbf{u}$ by 0 outside of $\bar{\Omega}$. The function $\tilde{\mathbf{u}}$ belongs to $\mathbf{H}(\mathbf{curl}, \mathcal{O})$ and it is not hard to see that

$$\widetilde{\mathbf{curl}(\mathbf{u})} = \mathbf{curl}(\tilde{\mathbf{u}}).$$

There exists then a function $\boldsymbol{\xi} \in \mathbf{H}^1(\mathcal{O})$ and a function $p \in H^1(\mathcal{O})$ such that $\tilde{\mathbf{u}} = \boldsymbol{\xi} + \nabla p$. This implies in particular $p \in H^2(\Omega)$. Now, since $\tilde{\mathbf{u}} = 0$ in $\mathcal{O} \setminus \bar{\Omega}$, we obtain $\boldsymbol{\xi} = -\nabla p$ which shows $p \in H^2(\mathcal{O} \setminus \bar{\Omega})$. The function p can now be extended from $\mathcal{O} \setminus \bar{\Omega}$ to Ω preserving its H^2 regularity [20], and we denote by p_R this extension. We have then that $\mathbf{u} = (\boldsymbol{\xi} + \nabla p_R) + (\nabla p - \nabla p_R)$ in Ω where $\boldsymbol{\xi} + \nabla p_R \in H^1_0(\Omega)^3$ and $p - p_R \in H^1_0(\Omega)$, since $p|_\Gamma = p_R|_\Gamma$ and $\nabla p_R + \boldsymbol{\xi} \in H^1(\mathcal{O})$ with $\boldsymbol{\xi} = -\nabla p_R$ on $\mathcal{O}$ by construction. Finally, this means that $\gamma(\mathbf{u}) = \gamma(\nabla\phi)$ with $\phi = p - p_R \in H^2(\Omega) \cap H^1_0(\Omega)$. □

Remark 2.4. From (19) and Filonov's example, we see that there exist Lipschitz (even $C^{3/2}$) domains for which $V'_0 \equiv V'$. In this case, (18) implies that $\mathbf{L}^2_t(\Gamma)$ is dense in $\mathbf{H}^{-1/2}(\Gamma)$, and that in fact the latter space is isomorphic to the "tangential" spaces $V_\pi{}'$ and $V_\gamma{}'$ which thus, loosely speaking, do not show much "tangentiality" any more.

3. Tangential differential operators

The spaces $H^s(\Gamma)$ for any $s \in [-1, 1]$ have an intrinsic definition (by localization) on the Lipschitz surface Γ due to their invariance with respect to Lipschitz transformations. Moreover, the spaces $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ are in duality with $L^2(\Gamma)$ as pivot space. We denote by $\langle\cdot,\cdot\rangle_{s,\Gamma}$ the corresponding duality pairing.

Following Něcas [20], we introduce local coordinates. Let Δ_j be the closed 2D unit square $\Delta = \{0 \leqslant x_{j1}, x_{j2} \leqslant 1\}$ associated to a system of coordinates

(x_{j1}, x_{j2}, x_{j3}). There exist M open, regular and connected subsets of Γ, say $\{\gamma_j\}_j$ such that $\bigcup_j \overline{\gamma_j} = \Gamma$, and M Lipschitz functions $a_j : \Delta_j \to \mathbb{R}$ such that $\overline{\gamma_j} = \{(x_{j1}, x_{j2}, a_j(x_{j1}, x_{j2})) \mid (x_{j1}, x_{j2}) \in \Delta_j\}$. Finally, we denote by $A_j : \mathbb{R}^2 \to \mathbb{R}^3$ the mapping $(x_{j1}, x_{j2}) \mapsto (x_{j1}, x_{j2}, a_j(x_{j1}, x_{j2}))$.

The spaces $H^s(\Gamma)$, $s = 0, 1$, are separable Hilbert spaces endowed with the following norms:

$$\|u\|^2_{0,\Gamma} = \sum_{j=1}^{M} \|u \circ A_j\|^2_{0,\Delta_j}, \qquad \|u\|^2_{1,\Gamma} = \sum_{j=1}^{M} \|u \circ A_j\|^2_{1,\Delta_j}.$$

Different maps give rise to equivalent norms. The parameterizations A_j induce, in a natural way, two tangent vectors on γ_j, namely $\mathbf{e}_1 = (1, 0, \partial_1 a_j(1, 0))$, $\mathbf{e}_2 = (0, 1, \partial_2 a_j(0, 1))$, which are not orthogonal. We set $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ for $i, k = 1, 2$, and $G = \{g_{ik}\}$ the corresponding positive definite Gram matrix. We set $G^{-1} = \{g^{ik}\}$ and $g = \det\{G\}$. As in the case of the regular domains, the dual base of tangential vectors reads $\mathbf{e}^i = \sum_{k=1}^{2} g^{ik} \mathbf{e}_k$.

Definition 3.1. We define $\nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$ and $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$ by ($\varphi \in H^1(\Gamma)$, $j = 1, \ldots, M$)

$$\begin{aligned} (\nabla_\Gamma \varphi)|_{\gamma_j} &= \partial_1(\varphi \circ A_j)\pi_\tau(\mathbf{e}^1) + \partial_2(\varphi \circ A_j)\pi_\tau(\mathbf{e}^2), \\ (\mathbf{curl}_\Gamma\, \varphi)|_{\gamma_j} &= \frac{1}{\sqrt{g}}\big(\partial_2(\phi \circ A_j)\pi_\tau(\mathbf{e}_1) - \partial_1(\phi \circ A_j)\pi_\tau(\mathbf{e}_2)\big) \\ &= -r(\nabla_\Gamma \varphi)|_{\gamma_j}. \end{aligned} \tag{21}$$

The invariance of $H^1(\Gamma)$ with respect to the choice of local parameterizations ensures that the definition (21) is independent of the choice of $\{A_j\}_j$ (see [20]).

Remark 3.2. The vectors $\mathbf{e}_i$ and $\mathbf{e}^i$, $i = 1, 2$, are defined as 3D vector fields living on Γ. The vectors $\pi_\tau(\mathbf{e}^i)$ and $\pi_\tau(\mathbf{e}_i)$, $i = 1, 2$, are the corresponding "tangent fields" on Γ, i.e., sections of the tangent bundle $T\Gamma$ of Γ.

Proposition 3.3. *The operators* $\nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$ *and* $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$ *are linear and continuous.*

Their adjoint operators $\operatorname{div}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}(\Gamma)$ and $\operatorname{curl}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}(\Gamma)$, respectively, are then defined by the following dualities:

$$\begin{aligned} \langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \varphi\rangle_{1,\Gamma} &= -\int_\Gamma \boldsymbol{\lambda} \cdot \nabla_\Gamma \varphi \, d\Gamma, \\ \langle \operatorname{curl}_\Gamma \boldsymbol{\lambda}, \varphi\rangle_{1,\Gamma} &= \int_\Gamma \boldsymbol{\lambda} \cdot \mathbf{curl}_\Gamma\, \varphi \, d\Gamma. \end{aligned} \tag{22}$$

By using the dualities (22), it is easy to see that $\mathrm{curl}_\Gamma(\boldsymbol{\lambda}) = -\operatorname{div}_\Gamma(r(\boldsymbol{\lambda}))$ and conversely, $\operatorname{div}_\Gamma(\boldsymbol{\lambda}) = \mathrm{curl}_\Gamma(r(\boldsymbol{\lambda}))$ for any $\boldsymbol{\lambda} \in \mathbf{L}^2_t(\Gamma)$.

We now study the ranges of suitable restrictions and extensions of the operators ∇_Γ, $\mathbf{curl}_\Gamma$ and their adjoints.

Proposition 3.4. *Let $H^{3/2}(\Gamma) = \gamma(H^2(\Omega))$. The restrictions of ∇_Γ and $\mathbf{curl}_\Gamma$ verify*

$$\nabla_\Gamma \varphi = \pi_\tau(\nabla\varphi), \quad \mathbf{curl}_\Gamma\, \varphi = -r(\nabla_\Gamma\varphi) = \gamma_\tau(\nabla\varphi) \quad \forall\varphi \in H^2(\Omega). \tag{23}$$

Moreover, $\nabla_\Gamma : H^{3/2}(\Gamma) \to V_\pi$ and $\mathbf{curl}_\Gamma : H^{3/2}(\Gamma) \to V_\gamma$ are linear and continuous.

Proof. The proof is straightforward. Using (20), we know that, for any $\varphi \in H^2(\Omega)$, the quantity $\pi_\tau(\nabla\varphi)$ depends only on the trace $\gamma(\varphi)$ on Γ.

The rest of the proof follows from (12) and Definition 2.2. □

Definition 3.5. Let $H^{-3/2}(\Gamma)$ be the dual space of $H^{3/2}(\Gamma)$ with $L^2(\Gamma)$ as pivot space. We define $\operatorname{div}_\Gamma : V'_\pi \to H^{-3/2}(\Gamma)$ and $\mathrm{curl}_\Gamma : V'_\gamma \to H^{-3/2}(\Gamma)$ by the dualities

$$\langle \operatorname{div}_\Gamma \boldsymbol{\lambda}, \varphi\rangle_{3/2,\Gamma} = -{}_{V'_\pi}\langle \boldsymbol{\lambda}, \nabla_\Gamma\varphi\rangle_{V_\pi}, \quad \boldsymbol{\lambda} \in V'_\pi,\ \varphi \in H^2(\Omega), \tag{24}$$

$$\langle \mathrm{curl}_\Gamma \boldsymbol{\lambda}, \varphi\rangle_{3/2,\Gamma} = {}_{V'_\gamma}\langle \boldsymbol{\lambda}, \mathbf{curl}_\Gamma\, \varphi\rangle_{V_\gamma}, \quad \boldsymbol{\lambda} \in V'_\gamma,\ \varphi \in H^2(\Omega), \tag{25}$$

where $\langle\cdot,\cdot\rangle_{3/2,\Gamma}$ denotes the duality pairing between $H^{-3/2}(\Gamma)$ and $H^{3/2}(\Gamma)$ while ${}_{V'_\pi}\langle\cdot,\cdot\rangle_{V_\pi}$ (${}_{V'_\gamma}\langle\cdot,\cdot\rangle_{V_\gamma}$ respectively) denotes the duality pairing between V'_π (V'_γ respectively) and V_π (V_γ respectively).

Again, by a duality argument and using the rotation operator r defined in (16), the following hold: $\forall\boldsymbol{\lambda} \in V'_\gamma$, $\forall\boldsymbol{\psi} \in V'_\pi$

$$\mathrm{curl}_\Gamma(\boldsymbol{\lambda}) = -\operatorname{div}_\Gamma\bigl(r(\boldsymbol{\lambda})\bigr) \quad \text{and} \quad \operatorname{div}_\Gamma(\boldsymbol{\psi}) = \mathrm{curl}_\Gamma\bigl(r(\boldsymbol{\psi})\bigr). \tag{26}$$

Next, we want to define suitable extensions of the operators ∇_Γ and $\mathbf{curl}_\Gamma$. To this aim, we note that the following integration by parts formula can be easily proved by a density argument (a complete derivation can be found in [6] in the case of polyhedra):

$$\begin{aligned} &\forall\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega), \quad \forall\mathbf{v} \in \mathbf{H}^1(\Omega): \\ &\int_\Omega \{\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} - \mathbf{curl}\,\mathbf{u}\cdot\mathbf{v}\}\, d\Omega = {}_{V'_\pi}\bigl\langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v})\bigr\rangle_{V_\pi} \\ &\qquad\qquad = -{}_{V'_\gamma}\bigl\langle \pi_\tau(\mathbf{u}), \gamma_\tau(\mathbf{v})\bigr\rangle_{V_\gamma}. \end{aligned} \tag{27}$$

Since the operators $\pi_\tau : \mathbf{H}^1(\Omega) \to V_\pi$ and $\gamma_\tau : \mathbf{H}^1(\Omega) \to V_\gamma$ are surjective, by (27), we obtain that the operators

$$\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to V_\pi' \quad \text{and} \quad \pi_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to V_\gamma'$$

are continuous.

Moreover, from (27), we also deduce that for any $\varphi \in H^1(\Omega)$, the maps $\varphi \mapsto \pi_\tau(\nabla\varphi)$ and $\varphi \mapsto \gamma_\tau(\nabla\varphi)$ are linear, continuous and depend only on the trace of φ on the boundary Γ. The following then holds:

Proposition 3.6. *The operators ∇_Γ and $\mathbf{curl}_\Gamma$ defined in* (23) *can be extended to $H^{1/2}(\Gamma)$. Moreover, $\nabla_\Gamma : H^{1/2}(\Gamma) \to V_\gamma'$ and $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to V_\pi'$ are linear and continuous. Analogously, the adjoint operators introduced in the Definition* 3.5 *are also linear and continuous for the following choice of spaces*: $\mathrm{div}_\Gamma : V_\gamma \to H^{-1/2}(\Gamma)$ *and* $\mathrm{curl}_\Gamma : V_\pi \to H^{-1/2}(\Gamma)$. *The equalities* (26) *still hold for any $\boldsymbol{\lambda} \in V_\pi$ and $\boldsymbol{\psi} \in V_\gamma$.*

Corollary 3.7. *In $H^{1/2}(\Gamma)$, there holds* $\ker(\nabla_\Gamma) = \ker(\mathbf{curl}_\Gamma) = \mathbb{R}$.

Proof. We simply prove that $\ker(\nabla_\Gamma) = \mathbb{R}$ since the other equality is then straightforward. Let $p \in H^1(\Omega)$ be such that $\nabla_\Gamma p = 0$. Using (27), we immediately obtain

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \nabla p = 0 \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega).$$

Integrating by parts, since $\mathrm{div}(\mathbf{curl}\,\mathbf{u}) = 0$, we obtain

$$\langle \mathbf{curl}\,\mathbf{u} \cdot \mathbf{n}, p \rangle_{1/2,\Gamma} = 0 \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega). \tag{28}$$

In order to deduce from (28) that p must be constant, we need to prove that the set $\{\mathbf{curl}\,\mathbf{u} \cdot \mathbf{n} \mid \mathbf{u} \in \mathbf{H}^1(\Omega)\}$ coincides with $H_\star^{-1/2}(\Gamma) = \{\xi \in H^{-1/2}(\Gamma) \mid \langle \xi, 1\rangle_{1/2,\Gamma} = 0\}$. Let $\xi \in H_\star^{-1/2}(\Gamma)$. We first take a function $\mathbf{w} \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\mathbf{w} \cdot \mathbf{n} = \xi$. Now there exists a function $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ such that $\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{w}$. The existence of such functions $\mathbf{w}$ and $\mathbf{v}$ is proved in [16]. From Lemma 3.5 in [2] follows the existence of a function $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{w} - \mathbf{v} = \mathbf{curl}\,\mathbf{u}$. We have then $\xi = \mathbf{w} \cdot \mathbf{n} = \mathbf{curl}\,\mathbf{u} \cdot \mathbf{n}$. □

Finally, let $\Delta_\Gamma : H^1(\Gamma) \to H^{-1}(\Gamma)$ be the Laplace–Beltrami operator defined by $p \mapsto \mathrm{div}_\Gamma(\nabla_\Gamma p)$. Of course, Δ_Γ is linear and continuous. By using (23) and (26), it is immediate to see that $\mathrm{div}_\Gamma(\nabla_\Gamma p) \equiv -\mathrm{curl}_\Gamma(\mathbf{curl}_\Gamma\, p)$ for any $p \in H^1(\Gamma)$.

4. Traces for $\mathbf{H}(\mathbf{curl}, \boldsymbol{\Omega})$

We already know that $\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to V_\pi'$ and $\pi_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to V_\gamma'$ are linear and continuous. In the following we describe the ranges of these operators.

Using (27), we proceed as in [23] and/or [6] to obtain $\forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$

$$\operatorname{div}_\Gamma\big(\gamma_\tau(\mathbf{u})\big) = \mathbf{curl}\,\mathbf{u} \cdot \mathbf{n} \in H^{-1/2}(\Gamma),$$
$$\big\|\operatorname{div}_\Gamma\big(\gamma_\tau(\mathbf{u})\big)\big\|_{-1/2,\Gamma} \leqslant C\|\mathbf{u}\|_{0,\mathrm{curl}}. \tag{29}$$

By the same argument, but making use of the second duality in the right-hand side of (27), we obtain $\forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$

$$\operatorname{curl}_\Gamma\big(\pi_\tau(\mathbf{u})\big) = \mathbf{curl}\,\mathbf{u} \cdot \mathbf{n} \in H^{-1/2}(\Gamma),$$
$$\big\|\operatorname{curl}_\Gamma\big(\pi_\tau(\mathbf{u})\big)\big\|_{-1/2,\Gamma} \leqslant C\|\mathbf{u}\|_{0,\mathrm{curl}}. \tag{30}$$

Remark that (30) can be directly obtained by using (17) and (26).

We now state one of our main results:

Theorem 4.1. *Let*

$$\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) := \big\{\boldsymbol{\lambda} \in V_\pi' \mid \operatorname{div}_\Gamma(\boldsymbol{\lambda}) \in H^{-1/2}(\Gamma)\big\}, \tag{31}$$
$$\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma) := \big\{\boldsymbol{\lambda} \in V_\gamma' \mid \operatorname{curl}_\Gamma(\boldsymbol{\lambda}) \in H^{-1/2}(\Gamma)\big\}. \tag{32}$$

The operators $\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ *and* $\pi_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$ *are linear, continuous, and surjective.*

Proof. The continuity of the trace operator π_τ (respectively γ_τ) is a direct consequence of (27) and (29) ((30) respectively). The proof of the surjectivity, on the other hand, is based on the proof given by Tartar in [23]. (For the sake of completeness, we present this proof in Section 7.) Let

$$T := \big\{\boldsymbol{\xi} \in V' \mid \exists \eta \in H^{-1/2}(\Gamma)\colon \forall \phi \in H^2(\Omega),$$
$${}_{V'}\big\langle \boldsymbol{\xi}, \gamma(\nabla\phi)\big\rangle_V = \langle \eta, \phi\rangle_{1/2,\Gamma}\big\}. \tag{33}$$

In [23], the tangential trace operator is defined as $\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to T$, $\mathbf{u} \mapsto \mathbf{n} \wedge \mathbf{u}$, and it is proven to be surjective by a localization argument. Here, our setting is different: the ranges of the operators π_τ and γ_τ defined in the previous sections are Hilbert spaces of tangent fields. We show that the mapping i_π defined in (15) is indeed an isomorphism between T and $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$, i.e.,

$$i_\pi\big(\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)\big) \equiv T.$$

Let $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$. In particular, $\boldsymbol{\lambda}$ belongs to V_π'. From (15), we see that $\boldsymbol{\xi} := i_\pi(\boldsymbol{\lambda}) \in (\ker(\pi_\tau) \cap V)^0$. This space was characterized in Lemma 2.3 and from (19), we know that

$${}_{V'}\big\langle \boldsymbol{\xi}, \gamma(\nabla\phi)\big\rangle_V = 0 \quad \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega).$$

Further, since i_π defined in (15) is an isomorphism, we can actually exhibit the quantity $\eta \in H^{-1/2}(\Gamma)$ appearing in the definition (33). Indeed, for any $\phi \in H^2(\Omega)$,

$$\begin{aligned} {}_{V'}\langle \boldsymbol{\xi}, \gamma(\nabla\phi)\rangle_V &= {}_{V'_\pi}\langle \boldsymbol{\lambda}, \nabla_\Gamma \phi\rangle_{V_\pi} = -\langle \operatorname{div}_\Gamma(\boldsymbol{\lambda}), \phi\rangle_{1/2,\Gamma} \\ &= -\langle \operatorname{div}_\Gamma\big(i_\pi^{-1}(\boldsymbol{\xi})\big), \phi\rangle_{1/2,\Gamma}, \end{aligned} \tag{34}$$

which proves that $i_\pi(\boldsymbol{\lambda}) \in T$ for any $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$. On the other hand, let $\boldsymbol{\xi} \in T$. Then from the definition of T, we have

$${}_{V'}\langle \boldsymbol{\xi}, \gamma(\nabla\phi)\rangle_V = 0 \quad \forall \phi \in H^2(\Omega) \cap H^1_0(\Omega).$$

Using again the characterization given in (19), we deduce that $\boldsymbol{\xi} \in (\ker(\pi_\tau) \cap V)^0 \equiv i_\pi(V'_\pi)$. Thus, there exists a unique $\boldsymbol{\lambda} \in V'_\pi$ such that $i_\pi(\boldsymbol{\lambda}) = \boldsymbol{\xi}$. We deduce from (34) that $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$.

The proof of the surjectivity for the operator π_τ is now easy. Let $\boldsymbol{\psi} \in \mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$. Since the rotation operator r defined in (16) is an isomorphism, there exists a $\boldsymbol{\lambda} \in V'_\pi$ such that $\boldsymbol{\psi} = r(\boldsymbol{\lambda})$. Moreover, from (26) we see that

$$\operatorname{div}_\Gamma \boldsymbol{\lambda} = \operatorname{curl}_\Gamma\big(r(\boldsymbol{\lambda})\big) = \operatorname{curl}_\Gamma \boldsymbol{\psi} \in H^{-1/2}(\Gamma),$$

which implies $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$. Since γ_τ has already been proven to be surjective, let $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ be such that $\gamma_\tau(\mathbf{u}) = \boldsymbol{\lambda}$. Using (17), we see that $\pi_\tau(\mathbf{u}) = r(\gamma_\tau(\mathbf{u})) = \boldsymbol{\psi}$. □

5. Hodge decomposition of tangential vector fields

In this section we focus our attention on the construction of an Hodge decomposition for the spaces of traces $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and $\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$ similar to the ones introduced in [12] for regular surfaces and in [7] for polyhedra. From now on we assume for simplicity that Ω is connected and simply connected. The extension of the following results to general non-connected domains is straightforward, while the generalization to non-simply connected geometries would require some additional work.

We first establish the validity of an integration by parts formula based on (27), but which holds for any field in $\mathbf{H}(\mathbf{curl}, \Omega)$. A different interpretation of the boundary term can be found in [22].

We recall the following decomposition of $\mathbf{H}(\mathbf{curl}, \Omega)$, see, e.g., [2]:

$$\forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega),\ \exists \boldsymbol{\Phi} \in \mathbf{H}^1(\Omega),\ p \in H^1(\Omega) \quad \text{such that} \quad \mathbf{u} = \boldsymbol{\Phi} + \nabla p.$$

Now, let $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$ be decomposed as $\mathbf{u} = \boldsymbol{\Phi} + \nabla p$ and $\mathbf{v} = \boldsymbol{\Psi} + \nabla q$ with $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathbf{H}^1(\Omega)$, $p, q \in H^1(\Omega)$. We then have

$$\int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}=\int_\Omega \mathbf{curl}\,\boldsymbol{\Phi}\cdot\boldsymbol{\Psi}-\boldsymbol{\Phi}\cdot\mathbf{curl}\,\boldsymbol{\Psi} + \int_\Omega \mathbf{curl}\,\boldsymbol{\Phi}\cdot\nabla q-\int_\Omega \mathbf{curl}\,\boldsymbol{\Psi}\cdot\nabla p.$$

Applying (27) three times, we obtain

$$\int_\Omega \{\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}-\mathbf{curl}\,\mathbf{u}\cdot\mathbf{v}\} = {}_\gamma\big\langle\gamma_\tau(\mathbf{u}),\pi_\tau(\mathbf{v})\big\rangle_\pi, \tag{35}$$

where the boundary term can be defined as

$$\begin{aligned}
&{}_\gamma\big\langle\gamma_\tau(\mathbf{u}),\pi_\tau(\mathbf{v})\big\rangle_\pi \\
&\quad=\int_\Gamma \gamma_\tau(\boldsymbol{\Phi})\cdot\pi_\tau(\boldsymbol{\Psi}) + {}_{V_\gamma'}\big\langle\nabla_\Gamma q,\gamma_\tau(\boldsymbol{\Psi})\big\rangle_{V_\gamma} + {}_{V_\pi'}\big\langle\mathbf{curl}_\Gamma\, p,\pi_\tau(\boldsymbol{\Psi})\big\rangle_{V_\pi} \\
&\quad=\int_\Gamma \gamma_\tau(\boldsymbol{\Phi})\cdot\pi_\tau(\boldsymbol{\Psi}) - \big\langle\operatorname{div}_\Gamma\gamma_\tau(\boldsymbol{\Phi}),q\big\rangle_{1/2,\Gamma} + \big\langle\operatorname{curl}_\Gamma\pi_\tau(\boldsymbol{\Psi}),p\big\rangle_{1/2,\Gamma}.
\end{aligned} \tag{36}$$

Thanks to the surjectivity of the trace operators proved in Theorem 4.1, the relation (36) defines a duality between $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ and $\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)$ with $\mathbf{L}_t^2(\Gamma)$ as pivot space. This definition as well as the integration by parts (35) are somewhat unsatisfactory since the duality on the boundary Γ is defined by means of a decomposition in "regular" and "singular" parts on Ω and not by means of an intrinsic characterization of the spaces $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ and $\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)$.

We will now prove some properties of the differential operators defined in Section 3.

Theorem 5.1. *The following equalities hold*:

$$\begin{aligned}
&\ker(\operatorname{curl}_\Gamma)\cap V_\gamma' = \nabla_\Gamma\big(H^{1/2}(\Gamma)\big),\\
&\ker(\operatorname{curl}_\Gamma)\cap \mathbf{L}_t^2(\Gamma) = \nabla_\Gamma H^1(\Gamma).
\end{aligned} \tag{37}$$

Proof. We concentrate on the first equality. We first prove that

$$\nabla_\Gamma\big(H^{1/2}(\Gamma)\big)\subseteq\ker(\operatorname{curl}_\Gamma)\cap V_\gamma'.$$

We have to show that $\operatorname{curl}_\Gamma(\nabla_\Gamma p)=0$ for any $p\in H^{1/2}(\Gamma)$. Indeed, using (27), we get for any $\phi\in H^2(\Omega)$

$$\big\langle\operatorname{curl}_\Gamma(\nabla_\Gamma p),\phi\big\rangle_{3/2,\Gamma} = {}_{V_\gamma'}\langle\nabla_\Gamma p,\mathbf{curl}_\Gamma\,\phi\rangle_{V_\gamma} = {}_{V_\gamma'}\big\langle\nabla_\Gamma p,\gamma_\tau(\nabla\phi)\big\rangle_{V_\gamma}\equiv 0.$$

We pass to prove that $\nabla_\Gamma(H^{1/2}(\Gamma))\supseteq\ker(\operatorname{curl}_\Gamma)\cap V_\gamma'$. Let $\boldsymbol{\lambda}\in V_\gamma'$ be such that $\operatorname{curl}_\Gamma\boldsymbol{\lambda}=0$. Then in particular, $\boldsymbol{\lambda}\in H^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)$ which means that there exists a $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that $\pi_\tau(\mathbf{u}) = \boldsymbol{\lambda}$. Moreover, $\mathbf{curl}\,\mathbf{u} \cdot \mathbf{n} = \mathrm{curl}_\Gamma(\boldsymbol{\lambda}) = 0$. It is then known, see, e.g., [2], that $\mathbf{u}$ can be written as $\mathbf{u} = \boldsymbol{\Phi} + \nabla p$, with $\boldsymbol{\Phi} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $p \in H^1(\Omega)$. This implies that $\boldsymbol{\lambda} = \pi_\tau(\mathbf{u}) = \pi_\tau(\nabla p) = \nabla_\Gamma p$. Thus, using Proposition 3.6, we have shown that for any $\boldsymbol{\lambda} \in V_\gamma'$, $\mathrm{curl}_\Gamma \boldsymbol{\lambda} = 0$ implies that there exists a $p \in H^{1/2}(\Gamma)$ such that $\boldsymbol{\lambda} = \nabla_\Gamma p$.

Now, in order to prove the second equality in (37), we only need to show that $\nabla_\Gamma(H^1(\Gamma))$ is closed in $\mathbf{L}_t^2(\Gamma)$. Let $\{\boldsymbol{\varphi}_n\}_{n\in\mathbb{N}}$ be a sequence in $\nabla_\Gamma(H^1(\Gamma))$ which converges in $\mathbf{L}_t^2(\Gamma)$ to a function $\boldsymbol{\varphi} \in \mathbf{L}_t^2(\Gamma)$. There exists then a sequence $\{p_n\}_{n\in\mathbb{N}} \subset H^1(\Gamma)$ such that $\boldsymbol{\varphi}_n = \nabla_\Gamma p_n$ for any n. From the first part of the proof, we know that p_n converges to a function p which is a priori only in $H^{1/2}(\Gamma)$. Moreover, using the definition of the gradient and the local maps A_j, we see that, for any $j = 1, \ldots, M$,

$$p_n \circ A_j - \int_{\Delta_j} p_n \circ A_j \to \xi_j, \quad \xi_j \in H^1(\Delta_j).$$

Now, since $p_n \to p$ in $H^{1/2}(\Gamma)$, then $\int_{\Delta_j} p_n \circ A_j \to m_j \in \mathbb{R}$. By uniqueness of the limit and invariance of the space $H^1(\Gamma)$ under Lipschitz change of coordinates, $p \circ A_j \in H^1(\Delta_j)$ for any j and this implies $p \in H^1(\Gamma)$. □

Remark 5.2. Using the closed graph theorem [4] and Corollary 3.7, we obtain a priori estimates

$$\|p\|_{H^{1/2}(\Gamma)/\mathbb{R}} \leqslant C\|\nabla_\Gamma p\|_{V_\pi'}, \qquad \|p\|_{H^1(\Gamma)/\mathbb{R}} \leqslant C\|\nabla_\Gamma p\|_{\mathbf{L}_t^2(\Gamma)}.$$

Corollary 5.3. *The following holds*:

$$\ker(\mathrm{div}_\Gamma) \cap V_\pi' = \mathbf{curl}_\Gamma\big(H^{1/2}(\Gamma)\big),$$
$$\ker(\mathrm{div}_\Gamma) \cap \mathbf{L}_t^2(\Gamma) = \mathbf{curl}_\Gamma\big(H^1(\Gamma)\big).$$

Proof. The result comes immediately from Theorem 5.1 by applying the rotation operator $r = i_\pi^{-1} \circ i_\gamma$. □

For any $0 \leqslant s \leqslant 1$, set

$$H_\star^{-s}(\Gamma) = \big\{u \in H^{-s}(\Gamma) \mid \langle u, 1\rangle_{s,\Gamma} = 0\big\}.$$

Corollary 5.4. *The operators* $\mathrm{div}_\Gamma : \mathbf{L}_t^2(\Gamma) \to H_\star^{-1}(\Gamma)$ *and* $\mathrm{curl}_\Gamma : \mathbf{L}_t^2(\Gamma) \to H_\star^{-1}(\Gamma)$ *and their restrictions* $\mathrm{div}_\Gamma : V_\gamma \to H_\star^{-1/2}(\Gamma)$ *and* $\mathrm{curl}_\Gamma : V_\pi \to H_\star^{-1/2}(\Gamma)$ *are surjective.*

Set

$$\mathcal{H}(\Gamma) := \big\{p \in H^1(\Gamma)/\mathbb{R} \mid \Delta_\Gamma p \in H_\star^{-1/2}(\Gamma)\big\}.$$

We are now in the position to state the main result of this section:

Theorem 5.5. *The following decompositions hold*:

$$\mathbf{L}_t^2(\Gamma) = \nabla_\Gamma\big(H^1(\Gamma)\big) \overset{\perp}{\oplus} \mathbf{curl}_\Gamma\big(H^1(\Gamma)\big), \tag{38}$$

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = V_\gamma + \mathbf{curl}_\Gamma\big(H^{1/2}(\Gamma)\big), \tag{39}$$

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = V_\pi + \nabla_\Gamma\big(H^{1/2}(\Gamma)\big). \tag{40}$$

Also, the following decompositions are direct:

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \nabla_\Gamma\big(\mathcal{H}(\Gamma)\big) \oplus \mathbf{curl}_\Gamma\big(H^{1/2}(\Gamma)\big), \tag{41}$$

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma\big(\mathcal{H}(\Gamma)\big) \oplus \nabla_\Gamma\big(H^{1/2}(\Gamma)\big). \tag{42}$$

Proof. Let us prove (38) first. Given $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$, we solve the following problem:

$$\begin{aligned}&\text{Find} \quad p \in H^1(\Gamma)/\mathbb{R}\\ &\text{such that} \quad \int_\Gamma \nabla_\Gamma p \cdot \nabla_\Gamma \phi = \int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \phi \quad \forall \phi \in H^1(\Gamma)/\mathbb{R}.\end{aligned}$$

Thanks to Theorem 5.1 and Corollary 3.7, we see that this problem admits a unique solution $p \in H^1(\Gamma)/\mathbb{R}$. Now, of course, $\mathrm{div}_\Gamma(\mathbf{u} - \nabla_\Gamma p) = 0$ and, again by using Theorem 5.1, Corollary 5.3, there exists a unique $q \in H^1(\Gamma)/\mathbb{R}$ such that $\mathbf{u} = \nabla_\Gamma p + \mathbf{curl}_\Gamma q$.

We focus now our attention on (39). Let $\mathbf{u} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. Thanks to the surjectivity of the div_Γ operator, there exists a function $\boldsymbol{\psi} \in V_\gamma$ such that

$$\mathrm{div}_\Gamma \mathbf{u} = \mathrm{div}_\Gamma \boldsymbol{\psi}. \tag{43}$$

On the other hand, using Corollary 5.3, since $\mathrm{div}_\Gamma(\mathbf{u} - \boldsymbol{\psi}) = 0$, there exists a unique $\beta \in H^{1/2}(\Gamma)/\mathbb{R}$ such that

$$\mathbf{u} = \boldsymbol{\psi} + \mathbf{curl}_\Gamma \beta, \quad \boldsymbol{\psi} \in V_\gamma,\ \beta \in H^{1/2}(\Gamma)/\mathbb{R}. \tag{44}$$

The decomposition (39) is thus proved and (40) can be proved in the same way. Note that these decompositions are neither orthogonal nor direct.

Now, we focus our attention on (41). We know that, for $\boldsymbol{\psi} \in V_\gamma \subseteq \mathbf{L}_t^2(\Gamma)$, (38) gives us $p, q \in H^1(\Gamma)/\mathbb{R}$ such that $\boldsymbol{\psi} = \nabla_\Gamma p + \mathbf{curl}_\Gamma q$. Applying the tangential divergence to this equation, we find $\mathrm{div}_\Gamma \boldsymbol{\psi} = \Delta_\Gamma p \in H^{-1/2}(\Gamma)$, hence $p \in \mathcal{H}(\Gamma)$. Replacing then the function $\boldsymbol{\psi}$ in (44) by this decomposition, we obtain

$$\mathbf{u} = \nabla_\Gamma p + \mathbf{curl}_\Gamma(q + \beta), \quad p \in \mathcal{H}(\Gamma) \text{ and } q + \beta \in H^{1/2}(\Gamma)/\mathbb{R}. \tag{45}$$

The fact that this decomposition is direct follows easily from Theorem 5.1 and Corollary 5.3. Finally, (42) is an immediate consequence of (41) applying the rotation operator r defined in (16). □

Now a duality can be defined between $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$.

Lemma 5.6. *Let* $\mathbf{u} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and* $\mathbf{v} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *be decomposed as* $\mathbf{u} = \nabla_\Gamma \alpha_u + \mathbf{curl}_\Gamma\, \beta_u$, $\mathbf{v} = \nabla_\Gamma \beta_v + \mathbf{curl}_\Gamma\, \alpha_v$ *with* $\beta_u, \beta_v \in H^{1/2}(\Gamma)$ *and* $\alpha_u, \alpha_v \in \mathcal{H}(\Gamma)$. *We have*

$$ {}_\gamma\langle \mathbf{u}, \mathbf{v}\rangle_\pi := -\langle \Delta_\Gamma \alpha_u, \beta_v\rangle_{1/2,\Gamma} + \langle \Delta_\Gamma \alpha_v, \beta_u\rangle_{1/2,\Gamma} \tag{46} $$

and the integration by parts formula (35) *is consistent with this definition.*

The pivot space in the duality (46) is $\mathbf{L}^2_t(\Gamma)$ and this can be shown by an easy density argument. Let $\{\beta_u^n\}_{n\in\mathbb{N}} \subseteq H^1(\Gamma)$ and $\{\beta_v^n\}_{n\in\mathbb{N}} \subseteq H^1(\Gamma)$ be two sequences such that $\beta_u^n \to \beta_u$ and $\beta_v^n \to \beta_v$ in $H^{1/2}(\Gamma)$. Moreover, let $\mathbf{u}^n = \nabla_\Gamma \alpha_u + \mathbf{curl}_\Gamma\, \beta_u^n$ and $\mathbf{v}^n = \nabla_\Gamma \beta_v^n + \mathbf{curl}_\Gamma\, \alpha_v$.

We have then $\mathbf{u}^n, \mathbf{v}^n \in \mathbf{L}^2_t(\Gamma)$ and

$$ \int_\Gamma \mathbf{u}^n \cdot \mathbf{v}^n = \int_\Gamma \nabla_\Gamma \alpha_u \cdot \nabla_\Gamma \beta_v^n + \int_\Gamma \mathbf{curl}_\Gamma\, \beta_u^n \cdot \mathbf{curl}_\Gamma\, \alpha_v \quad \forall n \in \mathbb{N}, \tag{47} $$

since $\int_\Gamma \nabla_\Gamma \alpha_u \cdot \mathbf{curl}_\Gamma\, \alpha_v = \int_\Gamma \nabla_\Gamma \beta_v^n \cdot \mathbf{curl}_\Gamma\, \beta_u^n \equiv 0$. Using definition (22) in both terms in the right-hand side of (47) and recalling that $\Delta_\Gamma = -\mathrm{curl}_\Gamma\, \mathbf{curl}_\Gamma = \mathrm{div}_\Gamma\, \nabla_\Gamma$, we have

$$ \begin{aligned} \int_\Gamma \mathbf{u}^n \cdot \mathbf{v}^n &= -\langle \Delta_\Gamma \alpha_u, \beta_v^n\rangle_{1,\Gamma} + \langle \Delta_\Gamma \alpha_v, \beta_u^n\rangle_{1,\Gamma} \\ &= -\langle \Delta_\Gamma \alpha_u, \beta_v^n\rangle_{1/2,\Gamma} + \langle \Delta_\Gamma \alpha_v, \beta_u^n\rangle_{1/2,\Gamma} \end{aligned} $$

where the last equality comes from the fact that $\alpha_u, \alpha_v \in \mathcal{H}(\Gamma)$. Now, letting n going to infinity, we obtain (46).

6. Filonov's example

In this section we consider an example of a "pathological domain" which was introduced and first studied in [13]. We report here the main steps of the construction of this domain and we focus our attention on the impact of its properties on both standard functional spaces and the ones introduced and analyzed in the previous sections.

For $q \in \mathbb{N}$, let us define $f(x) = \sum_{k=1}^{\infty} q^{-k} \sin(q^{2k} x)$ for $x \in \mathbb{R}$. In [13] and [14], Filonov shows the following:

- For any $q > 1$: $f \in C^{1/2}$, $f(0) = f(2\pi) = 0$, $|f(x)| < 1$, $\int_0^{2\pi} f(x)\, dx = 0$.

- Let q be sufficiently large. Then for any $a, b \in H^{1/2}([0, 2\pi])$, the equation $a = fb$ implies $a = b = 0$ (see Theorem 4.1 in [14]). This "separation property" is proved by means of Lemma 4.2 in [14] which states

$$\forall x \in [0, 2\pi] \quad \int_0^{2\pi} \frac{|f(x) - f(y)|^2}{|x - y|^2}\, dy = +\infty.$$

Using polar coordinates (r, φ) in $\mathbb{R}^2$, we set $F(\varphi) = 1 + \int_0^{\varphi} f(t)\, dt$ and

$$\omega := \bigl\{(r, \varphi) \in \mathbb{R}^2 \mid r < F(\varphi)\bigr\}.$$

In [14], it is proved that ω has a $C^{3/2}$ boundary and that for any vector $\mathbf{v} \in H^1(\omega)^2$ the vanishing of the normal component on the boundary implies also the vanishing of its tangential component.

Let $\boldsymbol{\tau}$ be the counterclockwise tangent vector to $\partial\omega$. We will use the following equivalent form of Filonov's result:

Lemma 6.1. *For any* $\mathbf{v} \in H^1(\omega)^2$ *such that* $\mathbf{v}|_\Gamma \cdot \boldsymbol{\tau} = 0$ *on* Γ, *we have* $\gamma(\mathbf{v}) = 0$.

This result can be extended to higher dimensional domains. Here we are interested in the three-dimensional case in particular. We use the domain as constructed by Filonov, but we concentrate on properties different from the ones considered in [14]. Let (r, φ, z) be cylindrical coordinates in $\mathbb{R}^3$. Then the domain Ω is defined by

$$\Omega := \left\{(r, \varphi, z) \in \mathbb{R}^3\colon\ \frac{r^2}{F^2(\varphi)} + z^2 < 1\right\}.$$

The following result generalizes Lemma 6.1.

Theorem 6.2. *For the domain* Ω, *there holds*

$$\bigl\{\mathbf{u} \in \mathbf{H}^1(\Omega) \mid \gamma_\tau(\mathbf{u}) = 0\bigr\} = \mathbf{H}_0^1(\Omega).$$

Proof. The boundary $\partial\Omega$ is the set $\{(r, \varphi, z) \in \mathbb{R}^3 \mid r^2 + (z^2 - 1)F^2(\varphi) = 0\}$. It is easy to see that an exterior normal vector Ω is given by $\mathbf{n} = (n_1, n_2, n_3)$ with

$$n_1 = -\frac{(z^2 - 1)}{r} F(\varphi)\bigl(F(\varphi)\cos\varphi + f(\varphi)\sin\varphi\bigr),$$
$$n_2 = \frac{(z^2 - 1)}{r} F(\varphi)\bigl(f(\varphi)\cos\varphi - F(\varphi)\sin\varphi\bigr),$$
$$n_3 = zF^2(\varphi).$$

We consider the two independent tangent vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ defined as follows:

$$\boldsymbol{\tau}_1 = (-n_2, n_1, 0), \qquad \boldsymbol{\tau}_2 = (\alpha n_1, \alpha n_2, 1) \quad \text{with } \alpha = -\frac{n_3}{\sqrt{n_1^2 + n_2^2}}.$$

These vectors are well defined for any z, $|z| < 1$. Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$. It is easy to see that the condition $\gamma_\tau(\mathbf{u}) = 0$ corresponds to $\mathbf{u} \cdot \boldsymbol{\tau}_1 = \mathbf{u} \cdot \boldsymbol{\tau}_2 = 0$.

We denote by u_i, $i = 1, 2, 3$, the Cartesian components of $\mathbf{u}$. Now for almost every $\bar{z} \in]-1, 1[$, we have $u_i(x, y, \bar{z}) \in H^1(\Omega_{\bar{z}})$, $\Omega_{\bar{z}} = \Omega \cap \{z = \bar{z}\}$, $i = 1, 2, 3$. For fixed $\bar{z}$, the condition $\mathbf{u} \cdot \boldsymbol{\tau}_1 = 0$ corresponds exactly to the condition in Proposition 6.1 for the two-dimensional domain $\Omega_{\bar{z}}$. We conclude that $u_1 = u_2 = 0$ on $\partial\Omega$. Finally, the condition $\mathbf{u} \cdot \boldsymbol{\tau}_2 = 0$ implies that also $u_3 = 0$ on Γ. Hence $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$. □

Corollary 6.3. *The domain $\Omega \in \mathbb{R}^3$ constructed above has the following properties*:

(i) *Ω has a $C^{3/2}$ boundary*;
(ii) $H^2(\Omega) \cap H^1_0(\Omega) = H^2_0(\Omega)$;
(iii) *The spaces V_0' and $V' = \mathbf{H}^{-1/2}(\Gamma)$ considered in Lemma* 2.3 *satisfy* $V_0' = V'$;
(iv) *Let $\mathbf{L}^2_n(\Gamma)$ be defined by*

$$\mathbf{L}^2(\Gamma) = \mathbf{L}^2_t(\Gamma) \overset{\perp}{\oplus} \mathbf{L}^2_n(\Gamma).$$

For any $\mathbf{u} \in \mathbf{L}^2_n(\Gamma)$, there exists a sequence $\mathbf{u}_k \in \mathbf{L}^2_t(\Gamma)$, $k \in \mathbb{N}$, such that $\mathbf{u}_k \to \mathbf{u}$ in $\mathbf{H}^{-1/2}(\Gamma)$.

Proof. The regularity of the domain Ω is straightforward. Let $u \in H^2(\Omega) \cap H^1_0(\Omega)$. We have $\nabla u \in \mathbf{H}^1(\Omega)$, and from (20) we know that $\pi_\tau(\nabla u) = 0$. Theorem 6.2 leads to $\nabla u \in \mathbf{H}^1_0(\Omega)$ and thus $u \in H^2_0(\Omega)$.

Furthermore, the space $\ker(\pi_\tau) \cap V$ is reduced to zero, and therefore by definition $V_0' = V'$. Lemma 2.3 implies that $i_\pi(\mathbf{L}^2_t(\Gamma))$ is dense in V'. The last statement is then straightforward. □

7. Tartar's surjectivity result

In the case of a Lipschitz boundary, we know of only one way to prove that the tangential trace map γ_τ maps the space $\mathbf{H}(\mathbf{curl}, \Omega)$ *onto* the space $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, namely to use, as we did in the proof of Theorem 4.1, Tartar's explicit construction of a right inverse of the trace map given in [23]. For the sake of completeness, we present Tartar's construction in this section.

Since we did not apply this surjectivity result prior to the proof of Theorem 4.1, we may use what we learned before that point. In particular, we know that Tartar's space T as defined in (33) is isomorphic to our space $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, and that tangential traces of vector fields in $\mathbf{H}(\mathbf{curl}, \Omega)$ belong to $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, hence to T. In view of the discussion in Section 2, however, we will not consider this isomorphism to be an identification, and we therefore have to distinguish between the tangential trace map γ_τ and the trace map as considered by Tartar.

We denote the latter by γ_T. It is defined like γ_τ by the Green formula (2) as an extension of the trace mapping $u \mapsto \gamma(\mathbf{u} \wedge \mathbf{n})$ from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}(\mathbf{curl}, \Omega)$, where we use γ to denote the scalar trace mapping, applied here to the three Cartesian components of a 3D vector field.

Thus we have for $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$

$$\int_\Omega (\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u})\, d\mathbf{x} = \langle \gamma_T \mathbf{u}, \gamma \mathbf{v} \rangle_{1/2,\Gamma}, \tag{48}$$

and this formula defines γ_T as a continuous operator

$$\gamma_T : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\Gamma).$$

Theorem 7.1 (Tartar). *Let Ω be a domain with Lipschitz boundary Γ as above. Then γ_T maps $\mathbf{H}(\mathbf{curl}, \Omega)$ onto the space T defined by*

$$T = \Big\{ \boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma) \mid \exists \eta \in H^{-1/2}(\Gamma): \\ \forall \phi \in H^2(\Omega),\ \big\langle \boldsymbol{\xi}, \gamma(\nabla\phi)\big\rangle_{1/2,\Gamma} = \langle \eta, \gamma\phi \rangle_{1/2,\Gamma} \Big\}. \tag{49}$$

Proof. We only need to show the surjectivity of γ_T. Thus, for $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma)$ satisfying (49) with some $\eta \in H^{-1/2}(\Gamma)$, we have to construct $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that $\boldsymbol{\xi} = \gamma_T \mathbf{u}$. This will be done in four steps.

The first step consists of localization. In order to apply a partition of unity argument, one has to note that for any sufficiently smooth function θ there holds $\theta\boldsymbol{\xi} \in T$, because $\theta\boldsymbol{\xi}$ satisfies (49) with η replaced by $\theta\eta - \boldsymbol{\xi} \cdot \nabla\theta \in H^{-1/2}(\Gamma)$.

This allows us to assume for the following that the support of $\boldsymbol{\xi}$ is sufficiently small so that in a neighborhood of this support, Γ can be represented by a Lipschitz graph. Without loss of generality, we can therefore assume that we are in the following situation:

$$F : \mathbb{R}^2 \to \mathbb{R} \quad \text{is uniformly Lipschitz;}$$

$$\Omega = \big\{ \mathbf{x} \in \mathbb{R}^3 \mid x_3 > F(x_1, x_2) \big\}, \qquad \Gamma = \big\{ \mathbf{x} \in \mathbb{R}^3 \mid x_3 = F(x_1, x_2) \big\};$$

$$\boldsymbol{\xi} \in \big(H^{-1/2}(\Gamma)\big)^3 \ \text{ and } \ \eta \in H^{-1/2}(\Gamma) \ \text{ have compact support and satisfy}$$

$$\sum_{i=1}^{3} \langle \xi_i, \gamma\partial_i\phi \rangle_{1/2,\Gamma} = \langle \eta, \gamma\phi \rangle_{1/2,\Gamma} \quad \text{for all } \phi \in H^2(\Omega). \tag{50}$$

Here $\langle\cdot,\cdot\rangle_{1/2,\Gamma}$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, and we write ∂_i for the partial derivative with respect to x_i.

The second step consists of expressing the duality in (50) in the parameter space $\mathbb{R}^2$. The operator

$$\Pi: f \mapsto \Pi f, \qquad (\Pi f)\big(x_1, x_2, F(x_1, x_2)\big) = f(x_1, x_2)$$

is an isomorphism from $L^2(\mathbb{R}^2)$ to $L^2(\Gamma)$ and from $H^1(\mathbb{R}^2)$ to $H^1(\Gamma)$, hence by interpolation from $H^s(\mathbb{R}^2)$ to $H^s(\Gamma)$ for all $s \in [0,1]$. The adjoint operator Π^* is therefore an isomorphism

$$\Pi^*: H^{-s}(\Gamma) \to H^{-s}(\mathbb{R}^2) \quad \forall s \in [0,1].$$

If we choose the test function ϕ in (50) as a tensor product

$$\phi(x_1, x_2, x_3) = f(x_1, x_2)g(x_3)$$

where $f \in H^2(\mathbb{R}^2)$ and $g \in H^2(\mathbb{R})$ have compact support and $g \equiv 1$ on a neighborhood of $\{x_3 \mid (x_1, x_2, x_3) \in \operatorname{supp} \boldsymbol{\xi}\}$, then on a neighborhood of $\operatorname{supp} \boldsymbol{\xi}$ we have $\gamma\phi = \Pi f$ and $\gamma\partial_i\phi = \Pi\partial_i f$ $(i = 1, 2)$, $\gamma\partial_3\phi = 0$. We obtain from (50)

$$\sum_{i=1}^{2}\langle \Pi^*\xi_i, \partial_i f\rangle_{1/2,\mathbb{R}^2} = \sum_{i=1}^{2}\langle \xi_i, \Pi\partial_i f\rangle_{1/2,\Gamma} = \sum_{i=1}^{3}\langle \xi_i, \gamma\partial_i\phi\rangle_{1/2,\Gamma}$$
$$= \langle \eta, \gamma\phi\rangle_{1/2,\Gamma} = \langle \eta, \Pi f\rangle_{1/2,\Gamma} = \langle \Pi^*\eta, f\rangle_{1/2,\Gamma}.$$

This holds in particular for all $f \in C_0^\infty(\mathbb{R}^2)$. Hence we have in the sense of distributions on $\mathbb{R}^2$

$$-\sum_{i=1}^{2}\partial_i\Pi^*\xi_i = \Pi^*\eta.$$

This means that the distribution $\boldsymbol{\lambda} = (\Pi^*\xi_1, \Pi^*\xi_2) \in (H^{-1/2}(\mathbb{R}^2))^2$ satisfies $\operatorname{div}_{\mathbb{R}^2}\boldsymbol{\lambda} = -\Pi^*\eta \in H^{-1/2}(\mathbb{R}^2)$ and therefore belongs to $\mathbf{H}^{-1/2}(\operatorname{div}, \mathbb{R}^2)$.

In the third step, the vector field $\mathbf{u}$ is constructed from $\boldsymbol{\lambda}$. To this end, one notes that $\boldsymbol{\lambda}$ can be represented as

$$\boldsymbol{\lambda} = \mathbf{p} + \mathbf{curl}_{\mathbb{R}^2}\, q \tag{51}$$

where $\mathbf{p} \in (H^{1/2}(\mathbb{R}^2))^2$ and $q \in H^{1/2}(\mathbb{R}^2)$ have compact support.

Indeed, by Fourier transform or by solving $\Delta_{\mathbb{R}^2}\varphi_0 = \operatorname{div}_{\mathbb{R}^2}\boldsymbol{\lambda}$ and taking $\mathbf{p}_0 = \nabla_{\mathbb{R}^2}\varphi_0$, we obtain a Hodge decomposition $\boldsymbol{\lambda} = \mathbf{p}_0 + \mathbf{curl}_{\mathbb{R}^2}\, q_0$, where $\mathbf{p}_0 \in H^{1/2}$ and $q_0 \in H^{1/2}$ in a neighborhood of $\operatorname{supp}\boldsymbol{\lambda}$. Multiplying by $\theta \in C_0^\infty(\mathbb{R}^2)$ with $\theta \equiv 1$ on $\operatorname{supp}\boldsymbol{\lambda}$, we get (51) with $\mathbf{p} = \theta\mathbf{p}_0 - q_0\,\mathbf{curl}_{\mathbb{R}^2}\,\theta$ and $q = \theta q_0$. For the three $H^{1/2}(\Gamma)$ functions $b_1 = \Pi p_2$, $b_2 = -\Pi p_1$, $w = -\Pi q$ we now choose liftings $B_1, B_2, W \in H^1(\mathbb{R}^3)$ such that $\gamma B_i = b_i$ and $\gamma W = w$ on Γ. With $B_3 = 0$ and $\mathbf{B} = (B_1, B_2, B_3)$ we define $\mathbf{u} = \mathbf{B} + \nabla W$. It is clear that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$.

In the fourth step, we show $\gamma_T\mathbf{u} = \boldsymbol{\xi}$ on Γ. We choose a test function $\boldsymbol{\varphi}$ of the form $\boldsymbol{\varphi} = (\varphi_1, 0, 0)$ with $\varphi_1(x_1, x_2, x_3) = f(x_1, x_2)g(x_3)$, $f \in H^2(\mathbb{R}^2)$ with compact support and $g \in C_0^\infty(\mathbb{R})$, $g \equiv 1$ on a neighborhood of $\{x_3 = F(x_1, x_2) \mid (x_1, x_2) \in \operatorname{supp} f\}$. With the definition (48) we obtain

$$\begin{aligned}
\big\langle (\gamma_T \mathbf{u})_1, \gamma\varphi_1 \big\rangle_{1/2,\Gamma} &= \int_\Omega (\mathbf{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} - \boldsymbol{\varphi} \cdot \mathbf{curl}\, \mathbf{u})\, d\mathbf{x} \\
&= \int_\Omega (\mathbf{B} \cdot \mathbf{curl}\, \boldsymbol{\varphi} - \boldsymbol{\varphi} \cdot \mathbf{curl}\, \mathbf{B} + \nabla W \cdot \mathbf{curl}\, \boldsymbol{\varphi})\, d\mathbf{x} \\
&= \int_\Gamma \big(n_3 \gamma B_2 \gamma \varphi_1 + \gamma W (n_2 \gamma \partial_3 \varphi_1 - n_3 \gamma \partial_2 \varphi_1)\big)\, d\sigma \\
&= \int_\Gamma (\gamma B_2 \gamma \varphi_1 - \gamma W \gamma \partial_2 \varphi_1) n_3\, d\sigma \\
&= \int_\Gamma (-\Pi p_1 \Pi f + \Pi q \Pi \partial_2 f) n_3\, d\sigma.
\end{aligned}$$

Here the outward normal vector $\mathbf{n} = (n_1, n_2, n_3)$ and the surface measure $d\sigma$ satisfy $n_3 = -(1 + |\nabla F|^2)^{-1/2}$ and $d\sigma = \sqrt{1 + |\nabla F|^2}\, dx_1\, dx_2$. We obtain

$$\big\langle (\gamma_T \mathbf{u})_1, \Pi f \big\rangle_{1/2,\Gamma} = \int_{\mathbb{R}^2} (p_1 f - q \partial_2 f)\, dx_1\, dx_2.$$

This holds for all $f \in C_0^\infty(\mathbb{R}^2)$, hence

$$\Pi^*(\gamma_T \mathbf{u})_1 = p_1 + \partial_2 q = (\mathbf{p} + \mathbf{curl}_{\mathbb{R}^2}\, q)_1 = \lambda_1 = \Pi^* \xi_1.$$

Since Π^* is an isomorphism, we get $(\gamma_T \mathbf{u})_1 = \xi_1$. A similar computation gives $(\gamma_T \mathbf{u})_2 = \xi_2$.

The proof is finished by showing $(\gamma_T \mathbf{u})_3 = \xi_3$ which follows from an argument displaying the tangential nature of the elements of the space T. Let $\boldsymbol{\psi} = \gamma_T \mathbf{u} - \boldsymbol{\xi}$. We have seen that $\boldsymbol{\psi} \in T$ and $\psi_1 = \psi_2 = 0$. We show that this implies $\psi_3 = 0$.

As a test function in the relation (49) we choose

$$\phi(x_1, x_2, x_3) = f(x_1, x_2) g(x_3) \big(x_3 - G(x_1, x_2)\big),$$

where f and g are as before and $G \in H^2(\mathbb{R}^2)$ has compact support. This gives

$$\langle \psi_3, \Pi f \rangle = \langle \psi_3, \gamma \partial_3 \phi \rangle = \langle \boldsymbol{\psi}, \gamma \nabla \phi \rangle = \langle \eta, \gamma \phi \rangle = \big\langle \eta, \Pi f (F - G) \big\rangle.$$

By first varying G, one obtains $\eta \Pi f = 0$ for all f, hence $\eta = 0$, hence finally $\psi_3 = 0$. □

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References

[1] A. Alonso, A. Valli, Some remarks on the characterization of the space of tangential traces of $\mathbf{H}(\mathrm{rot};\Omega)$ and the construction of an extension operator, Manuscripta Math. 89 (1996) 159–178.
[2] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (1998) 823–864.
[3] M. Birman, M. Solomyak, L^2-theory of the Maxwell operator in arbitrary domains, Russian Math. Surveys 42 (1987) 75–96.
[4] H. Brezis, Analyse Fonctionelle, Masson, Paris, 1983.
[5] A. Buffa, Hodge decompositions on the boundary of non-smooth domains: the multi-connected case, Math. Models Methods Appl. Sci. 11 (2001) 1491–1504.
[6] A. Buffa, P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. Part I: an integration by parts formula in Lipschitz polyhedra, Math. Methods Appl. Sci. 24 (2001) 9–30.
[7] A. Buffa, P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications, Math. Methods Appl. Sci. 24 (2001) 31–48.
[8] A. Buffa, M. Costabel, C. Schwab, Boundary element methods for Maxwell's equations on non-smooth domains, Numer. Math. 92 (2002) 679–710.
[9] A. Buffa, G. Geymonat, On traces for $W^{2,p}(\Omega)$ in Lipschitz domains, C. R. Acad. Sci. Paris Sér. I Math. 332 (2001) 699–704.
[10] M. Cessenat, Mathematical Methods in Electromagnetism, Linear Theory and Applications, in: Series in Advances in Mathematics for Applied Sciences, Vol. 41, World Scientific, River Edge, NJ, 1996.
[11] M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, Math. Methods Appl. Sci. 12 (1990) 365–368.
[12] A. de la Bourdonnaye, Décomposition de $H_{\mathrm{div}}^{-1/2}(\Gamma)$ et nature de l'opérateur de Steklov–Poincaré du problème extérieur de l'électromagnétisme, C. R. Acad. Sci. Paris Sér. I 316 (1993) 369–372.
[13] N. Filonov, Système de Maxwell dans des domaines singuliers, Ph.D. Thesis, Université de Bordeaux 1 (1996).
[14] N. Filonov, Principal singularities of the magnetic field in resonators with boundary of given smoothness, St. Petersburg Math. J. 9 (1998) 379–390.
[15] G. Geymonat, F. Krasucki, On the existence of the Airy function in Lipschitz domains. Application to the traces of H^2, C. R. Acad. Sci. Paris Sér. I 330 (2000) 355–360.
[16] V. Girault, P.-A. Raviart, Finite Element Methods for Navier–Stokes Equations, in: Springer Series in Computational Mathematics, Vol. 5, Springer, Berlin, 1986.
[17] D. Jerison, C.E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.
[18] D. Mitrea, M. Mitrea, Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds, J. Funct. Anal. 190 (2002) 339–417.
[19] D. Mitrea, M. Mitrea, J. Pipher, Vector potential theory on nonsmooth domains in $\mathbb{R}^3$ and applications to electromagnetic scattering, J. Fourier Anal. Appl. 3 (1997) 131–192.
[20] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques, Masson, Paris, 1967.
[21] L. Paquet, Problèmes mixtes pour le système de Maxwell, Ann. Fac. Sci. Toulouse Math. 4 (1982) 103–141.
[22] D. Sheen, A generalized Green's theorem, Appl. Math. Lett. 5 (1992) 95–98.
[23] L. Tartar, On the characterization of traces of a Sobolev space used for Maxwell's equation, in: A.-Y. Le Roux (Ed.), Proceedings of a Meeting Held in Bordeaux, in Honour of Michel Artola, November, 1997.
[24] K. Yosida, Functional Analysis, Springer, Berlin, 1995; Reprint of the sixth (1980) edition.