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Strong approximations of stochastic differential equations with jumps

Nicola Bruti-Liberati[a], Eckhard Platen[b,*]

[a] School of Finance and Economics, Australia
[b] Department of Mathematical Sciences, School of Finance and Economics, University of Technology, Sydney, PO Box 123, Broadway, NSW 2007, Australia

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Abstract

This paper is a survey of strong discrete time approximations of jump-diffusion processes described by stochastic differential equations (SDEs). It also presents new results on strong discrete time approximations for the specific case of pure jump SDEs.

Strong approximations based on jump-adapted time discretizations, which produce no discretization error in the case of pure jump processes, are analyzed. The computational complexity of these approximations is proportional to the jump intensity. By exploiting a stochastic expansion for pure jump processes, higher order discrete time approximations, whose computational complexity is not dependent on the jump intensity, are proposed. For the specific case of pure jump SDEs, the strong order of convergence of strong Taylor schemes is established under weaker conditions than those currently known in the literature.

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1. Introduction

As one tries to build more realistic models in economics, finance, biology, the social sciences, chemistry, physics and other areas, stochastic effects need to be taken into account. In certain areas, such as finance, the uncertainty in the dynamics is, in fact, the essential phenomenon that needs to be modeled. Event-driven dynamics become more and more important in most fields of application and lead to stochastic differential equations (SDEs) with jumps. In finance one needs to properly model credit events like defaults and credit rating changes, see for instance [18]. Also, the short rate, typically set by a central bank, jumps after random waiting times, usually by a quarter of a percent up or down, see [1]. See also [9,4] for jump-diffusion models in finance. In chemistry the reactions of single molecules or coupled reactions also yield stochastic models, see [7,8,36]. In [36], for instance, discrete time approximations for the solution of models of in vivo reactions are discussed. A class of so-called Poisson Runge–Kutta methods are introduced, by approximating the jump process with a Wiener process. The numerical schemes developed in the current

* Corresponding author.
E-mail addresses: nicola.brutiliberati@uts.edu.au (N. Bruti-Liberati), eckhard.platen@uts.edu.au (E. Platen).

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paper, which directly discretize the pure jump SDEs, can be used in these applications. Also in population genetics and other biological sciences, where the number of individual items evolves randomly through reactions, birth and death phenomena or migrations, pure jump processes arise. Simulation methods are in many cases the only practical numerical techniques that allow the study of solutions of certain high dimensional nonlinear systems of SDEs with jumps.

Much progress has been made on the simulation of continuous solutions of SDEs driven by Wiener processes only, see [19]. However, as already mentioned above, advanced modeling needs to capture the effects of event-driven uncertainty. Consequently, one has to simulate solutions of SDEs with jumps.

In the current paper we analyze strong discrete time approximations of SDEs with jumps. We consider time discretizations with maximum time step size $\Delta \in (0, \Delta_0)$, where $\Delta_0 \in (0, 1)$. A discrete time approximation Y is said to converge with strong order γ if there exist constants C and $\Delta_0 \in (0, 1)$, such that

$$\varepsilon(\Delta) := \sqrt{E(|X_T - Y_T|^2)} \leqslant C\Delta^{\gamma}, \tag{1}$$

for all $\Delta \in (0, \Delta_0)$. Here X_T is the solution of the given SDE at a terminal time $T \in [0, \infty)$ and Y_T is the corresponding value of the approximation.

The criterion (1) allows us to classify different discrete time approximations by their strong orders of convergence. These approximations are constructed to converge pathwise to the solution $X = \{X_t, t \in [0, T]\}$ of the SDE in question and are, therefore, suited for problems such as scenario simulation and filtering.

In the first part of this paper we give a review of the existing literature on the strong approximation of SDEs driven by Wiener processes and Poisson random measures. This leads, in general, to quite complicated higher order numerical schemes, as one needs to compute multiple stochastic integrals with respect to time, Wiener processes and Poisson random measures, see [2]. Jump-adapted approximations, introduced in [30], drastically reduce the complexity of numerical schemes, as they avoid multiple stochastic integrals with respect to the Poisson measure. However, for the case of SDEs driven by high intensity jump processes, simulations based on jump-adapted approximations may become unfeasible, as their computational complexity is proportional to the intensity. Therefore, for such processes, one needs to include multiple stochastic integrals with respect to the Poisson measure to obtain higher order numerical schemes. The literature in this area is still rather limited. In [30] a convergence theorem for strong Taylor schemes of any order of strong convergence $\gamma \in \{0.5, 1, 1.5, \ldots\}$ is presented and jump-adapted approximations are introduced. The works of Maghsoodi [25,26] present an analysis of discrete time approximations up to strong order $\gamma = 1.5$. Gardoǹ [6] presents a convergence theorem for strong Taylor schemes of any given strong order $\gamma \in \{0.5, 1, 1.5, \ldots\}$, for SDEs driven by Wiener processes and homogeneous Poisson processes. As mentioned above, [36] introduce a class of so-called Poisson Runge–Kutta methods for the approximation of pure jump SDEs. Higham and Kloeden [11,12] propose a class of implicit schemes with strong order $\gamma = 0.5$ for SDEs driven by Wiener processes and homogeneous Poisson processes. They also provide a detailed analysis of numerical stability properties. The paper [2] presents a convergence theorem for strong approximations of any order $\gamma \in \{0.5, 1, 1.5, \ldots\}$, including derivative-free, implicit and jump-adapted approximations.

For applications, such as derivative pricing and the computation of moments or expected utilities, one needs to estimate the expected value of a function of X_T. In this case only a probability approximation is sought and the so-called weak approximations are sufficient. For weak approximations of SDEs with jumps we refer to [28,24,21,10,3]. For time discrete approximations constructed to satisfy different criteria, we refer to [37,27,22,23]. Finally, we refer the reader to [20,34,15–17] for strong and weak approximations of SDEs driven by more general semimartingales.

In the second part of the current paper we present some new results on the discrete time approximation of pure jump SDEs. The piecewise constant nature of their solutions simplifies the resulting numerical schemes. For instance, jump-adapted approximations, constructed on time discretizations including all jump times, produce no discretization error in this case. Therefore, in the case of low to medium jump intensities one can construct efficient schemes without discretization error. In the case of high intensity jump processes, jump-adapted schemes are often not feasible. However, we will demonstrate that one can derive higher order discrete time approximations whose complexities turn out to be significantly lower than that of numerical approximations of jump diffusions. In the case of SDEs driven by a Poisson process, the generation of the multiple stochastic integrals required for higher order approximations is straightforward, since it involves only one Poisson distributed random variable in each time step. Moreover, the simple structure of pure jump SDEs permits us to illustrate the use of a stochastic expansion in the derivation of higher order approximations.

Finally, we show that higher strong orders of convergence of discrete time approximations for SDEs driven purely by a Poisson random measure can be derived under much weaker conditions than those typically required by convergence theorems for jump diffusions.

2. Jump-diffusion dynamics

After having given in the Introduction an overview of the literature on the strong numerical approximation of jump-diffusion SDEs, we provide in this section a more detailed survey and introduction.

2.1. Jump-diffusion model

Let us consider a filtered probability space $(\Omega, \mathscr{A}_T, \underline{\mathscr{A}}, P)$ satisfying the usual conditions and a mark space $(\mathscr{E}, \mathrm{B}(\mathscr{E}))$ with $\mathscr{E} \subseteq \mathbb{R}^r \backslash \{0\}$, for $r \in \{1, 2, \ldots\}$. On the mark space $\mathscr{E} \times [0, T]$ we define an $\underline{\mathscr{A}}$-adapted Poisson random measure $p_\phi(\mathrm{d}v \times \mathrm{d}t)$, where $v \in \mathscr{E}^n$, with intensity measure

$$q_\phi(\mathrm{d}v \times \mathrm{d}t) = \phi(\mathrm{d}v)\,\mathrm{d}t.$$

We also require finite intensity $\lambda = \phi(\mathscr{E}) < \infty$. By $p_\phi = \{p_\phi(t) := p_\phi(\mathscr{E} \times [0, t]), t \in [0, T]\}$ we denote the stochastic process that counts the number of jumps until some given time. Moreover, the Poisson random measure $p_\phi(\mathrm{d}v \times \mathrm{d}t)$ generates a sequence of pairs $\{(\tau_i, \xi_i), i \in \{1, 2, \ldots, p_\phi(T)\}\}$. Here $\{\tau_i : \Omega \to \mathbb{R}_+, i \in \{1, 2, \ldots, p_\phi(T)\}\}$ is a sequence of increasing nonnegative random variables representing the jump times of a standard Poisson process with intensity λ, and $\{\xi_i : \Omega \to \mathscr{E}, i \in \{1, 2, \ldots, p_\phi(T)\}\}$ is a sequence of independent identically distributed (i.i.d.) random variables, where ξ_i is distributed according to $\phi(\mathrm{d}u)/\phi(\mathscr{E})$. We can interpret τ_i as the time of the ith jump event and the mark ξ_i as its amplitude.

We consider the following d-dimensional jump-diffusion SDE

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t + \int_{\mathscr{E}} c(t, X_{t-}, v)\, p_\phi(\mathrm{d}v \times \mathrm{d}t), \tag{2}$$

for $t \in [0, T]$, with $X_0 \in \mathbb{R}^d$, and $W = \{W_t = (W_t^1, \ldots, W_t^m)^\top, t \in [0, T]\}$ an $\underline{\mathscr{A}}$-adapted m-dimensional Wiener process. Here $a(t, x)$ and $c(t, x, v)$ are d-dimensional vectors of measurable real valued functions on $[0, T] \times \mathbb{R}^d$ and on $[0, T] \times \mathbb{R}^d \times \mathscr{E}$, respectively. Moreover, $b(t, x)$ is a $d \times m$-matrix of measurable real valued functions on $[0, T] \times \mathbb{R}^d$. Let us note that for a given vector a we adopt the notation a^i to denote its ith component. Similarly, by $b^{i,j}$ we denote the component of the ith row and jth column of a given matrix b. We denote the almost-sure left-hand limit at time t of $X = \{X_t, t \in [0, T]\}$ by $X_{t-} = \lim_{s \Uparrow t} X_t$. For a detailed presentation of jump-diffusion models we refer to [35,29].

We require that the coefficients of the SDE (2) are measurable and satisfy the Lipschitz conditions

$$\begin{aligned} &|a(t, x) - a(t, y)|^2 \leqslant C_1 |x - y|^2, \quad |b(t, x) - b(t, y)|^2 \leqslant C_2 |x - y|^2, \\ &\int_{\mathscr{E}} |c(t, x, v) - c(t, y, v)|^2 \phi(\mathrm{d}v) \leqslant C_3 |x - y|^2, \end{aligned} \tag{3}$$

for every $t \in [0, T]$ and $x, y \in \mathbb{R}^d$, and the linear growth conditions

$$\begin{aligned} &|a(t, x)|^2 \leqslant K_1(1 + |x|^2), \quad |b(t, x)|^2 \leqslant K_2(1 + |x|^2), \\ &\int_{\mathscr{E}} |c(t, x, v)|^2 \phi(\mathrm{d}v) \leqslant K_3(1 + |x|^2), \end{aligned} \tag{4}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^d$. A unique strong solution of the SDE (2) exists under conditions (3) and (4), see [14,29].

The specification of the jump integral as an Itô stochastic integral with respect to a Poisson measure allows high flexibility in modeling. For instance, models with state-dependent intensities can be easily included in this framework, see [2].

2.2. Strong approximations

Discrete time approximations for the multi-dimensional SDE (2) can be constructed based on the Wagner–Platen expansion and the convergence Theorem 1 presented in Section 2.4. However, for ease of presentation, we illustrate, in the following, discrete time approximations for the autonomous one-dimensional jump-diffusion SDE

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t + \int_{\mathscr{E}} c(X_{t-}, v)\,p_\phi(\mathrm{d}v \times \mathrm{d}t), \tag{5}$$

for $t \in [0, T]$, with $X_0 \in \mathbb{R}$. Here $W = \{W_t, t \in [0, T]\}$ is an $\underline{\mathscr{A}}$-adapted one-dimensional Wiener process and $p_\phi(\mathrm{d}v \times \mathrm{d}t)$ an $\underline{\mathscr{A}}$-adapted Poisson measure. The corresponding mark space $\mathscr{E} \subseteq \mathbb{R}\backslash\{0\}$ is one-dimensional and the intensity measure is $\phi(\mathrm{d}v)\,\mathrm{d}t = \lambda f(v)\,\mathrm{d}v\,\mathrm{d}t$, where $f(\cdot)$ is a given probability density function. The SDE (5) can be written in integral form as

$$X_t = X_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s + \sum_{i=1}^{p_\phi(t)} c(X_{\tau_i -}, \xi_i),$$

where $\{(\tau_i, \xi_i), i \in \{1, 2, \ldots, p_\phi(t)\}\}$ is the double sequence of jump times and marks generated by the Poisson random measure p_ϕ with $p_\phi(t) = p_\phi(\mathscr{E} \times [0, t])$ for $t \in [0, T]$.

We construct a time discretization $0 = t_0 < t_1 < \cdots < t_{n_T} = T$, where for all $t \in [0, T]$ the index

$$n_t = \max\{n \in \{0, 1, \ldots\} : t_n \leqslant t\} \tag{6}$$

denotes the last discretization point before the time t. For simplicity, we choose an equidistant time discretization with $t_n = n\Delta$, for $n \in \{0, 1, \ldots, T/\Delta\}$, where $\Delta \in (0, 1)$ is the time step size.

The strong schemes to be presented involve, in general, multiple stochastic integrals with respect to time, Wiener process and the Poisson random measure. Usually, the required multiple stochastic integrals are quite complex, especially for higher order schemes. However, for problems such as filtering, these multiple stochastic integrals can be constructed from the given data. In view of this kind of applications it is important to develop and study higher order schemes that involve multiple stochastic integrals. We will show in Section 3 that, in the specific case of pure jump SDEs driven by one Poisson process, higher order strong schemes require only the generation of a Poisson random variable in each time step. Other special cases may arise that reduce the complexity of the multiple stochastic integrals involved.

The simplest scheme is the *Euler scheme*

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta W_n + \int_{t_n}^{t_{n+1}} \int_{\mathscr{E}} c(Y_n, v)\,p_\phi(\mathrm{d}v \times \mathrm{d}s) \\ &= Y_n + a\Delta + b\Delta W_n + \sum_{i=p_\phi(t_n)+1}^{p_\phi(t_{n+1})} c(Y_n, \xi_i), \end{aligned} \tag{7}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with initial value $Y_0 = X_0$. Note that we have used the abbreviations $a = a(Y_n)$ and $b = b(Y_n)$. Also in the sequel, when no misunderstanding is possible, for any coefficient function g, along with its derivatives, we will write $g = g(Y_n)$. We denote by $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ the $\mathscr{N}(0, \Delta)$ Gaussian distributed nth increment of the Wiener process W and by $p_\phi(s) = p_\phi(\mathscr{E} \times [0, s])$ a Poisson distributed random variable with mean λs representing the number of jumps of the random measure up to time s. The Euler scheme (7) achieves, in general, strong order $\gamma = 0.5$.

In the case of a mark-independent jump coefficient, which means $c(x, v) = c(x)$, the Euler scheme reduces to

$$Y_{n+1} = Y_n + a\Delta + b\Delta W_n + c\Delta p_n,$$

where $\Delta p_n = p_\phi(t_{n+1}) - p_\phi(t_n)$ is a Poisson distributed random variable with mean $\lambda\Delta$.

When higher accuracy is sought, one should construct approximations with higher order of strong convergence. By including further terms from the Wagner–Platen expansion, which is the stochastic extension of the Taylor formula, see [30,31], we obtain the *order 1.0 strong Taylor scheme*

$$
\begin{aligned}
Y_{n+1} = Y_n + a\Delta + b\Delta W_n + \int_{t_n}^{t_{n+1}} \int_{\mathscr{E}} c(Y_n, v)\, p_\phi(\mathrm{d}v \times \mathrm{d}s) \\
&+ bb' \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} \mathrm{d}W(s_1)\, \mathrm{d}W(s_2) \\
&+ \int_{t_n}^{t_{n+1}} \int_{\mathscr{E}} \int_{t_n}^{s_2} bc'(Y_n, v)\, \mathrm{d}W(s_1)\, p_\phi(\mathrm{d}v \times \mathrm{d}s_2) \\
&+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} \int_{\mathscr{E}} \{b(Y_n + c(Y_n, v)) - b\}\, p_\phi(\mathrm{d}v \times \mathrm{d}s_1)\, \mathrm{d}W(s_2) \\
&+ \int_{t_n}^{t_{n+1}} \int_{\mathscr{E}} \int_{t_n}^{s_2} \int_{\mathscr{E}} \{c(Y_n + c(Y_n, v_1), v_2) - c(Y_n, v_2)\} \\
&\times p_\phi(\mathrm{d}v_1 \times \mathrm{d}s_1)\, p_\phi(\mathrm{d}v_2 \times \mathrm{d}s_2), \qquad (8)
\end{aligned}
$$

where

$$
b'(x) := \frac{\mathrm{d}b(x)}{\mathrm{d}x} \quad \text{and} \quad c'(x, v) := \frac{\partial c(x, v)}{\partial x}.
$$

The scheme (8) achieves, in general, strong order $\gamma = 1.0$. This scheme was presented in [30]. Other schemes of strong order $\gamma = 1.0$ are studied in [25,26,6].

For applications such as scenario or hedge simulation, it is a separate problem how to efficiently generate the multiple stochastic integrals appearing in the order 1.0 strong Taylor scheme. It is worth to explicitly consider the case of mark-independent jump size, where the order 1.0 strong Taylor scheme reduces to

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\Delta + b\Delta W_n + c\Delta p_n + bb' I_{(1,1)} + bc' I_{(1,-1)} \\
&\quad + \{b(Y_n + c) - b\} I_{(-1,1)} + \{c(Y_n + c) - c\} I_{(-1,-1)},
\end{aligned}
$$

with multiple stochastic integrals

$$
I_{(1,1)} := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} \mathrm{d}W(s_1)\, \mathrm{d}W(s_2) = \frac{1}{2}\{(\Delta W_n)^2 - \Delta\},
$$

$$
I_{(1,-1)} := \int_{t_n}^{t_{n+1}} \int_{\mathscr{E}} \int_{t_n}^{s_2} \mathrm{d}W(s_1)\, p_\phi(\mathrm{d}v \times \mathrm{d}s_2) = \sum_{i=p_\phi(t_n)+1}^{p_\phi(t_{n+1})} W_{\tau_i} - \Delta p_n W_{t_n},
$$

$$
I_{(-1,1)} := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} \int_{\mathscr{E}} p_\phi(\mathrm{d}v \times \mathrm{d}s_1)\, \mathrm{d}W(s_2) = \Delta p_n \Delta W_n - I_{(1,-1)},
$$

$$
\begin{aligned}
I_{(-1,-1)} &:= \int_{t_n}^{t_{n+1}} \int_{\mathscr{E}} \int_{t_n}^{s_2} \int_{\mathscr{E}} p_\phi(\mathrm{d}v_1 \times \mathrm{d}s_1)\, p_\phi(\mathrm{d}v_2 \times \mathrm{d}s_2) \\
&= \tfrac{1}{2}\{(\Delta p_n)^2 - \Delta p_n\}.
\end{aligned}
$$

Even in the case of mark-independent jump size the order 1.0 strong scheme is quite complex. For this reason we present in the next section a different type of discrete time approximations, the so-called jump-adapted approximations. These approximations are much easier to implement and generally more efficient. Moreover, in Section 3, we will see that for a class of pure jump SDEs the order 1.0 strong Taylor scheme and other higher order schemes involve multiple stochastic integrals that can be easily generated.

It is useful to develop higher order schemes that do not involve the computation of derivatives of the coefficient functions. By replacing the derivatives with corresponding difference ratios one obtains the, so-called, derivative-free schemes, see [19] for diffusions and [2] for jump diffusions.

When considering an SDE with multiplicative noise as test equation, explicit schemes have been shown to have narrow regions of numerical stability, see [13] for diffusions and [11,12] for jump diffusions. Implicit schemes have, in general, wider regions of stability and are, thus, important in the approximation of stiff SDEs, involving different time scales. We refer to [11,12,2] for the derivation of implicit discrete time approximations of jump-diffusion SDEs.

2.3. Jump-adapted approximations

In this section we present the, so-called, jump-adapted approximations introduced in [30]. We consider a *jump-adapted time discretization* $0 = t_0 < t_1 < \cdots < t_M = T$, which is constructed by a superposition of the jump times $\{\tau_1, \tau_2, \ldots\}$ of the Poisson measure p_ϕ and an equidistant time discretization with step size Δ, as given in Section 2.2. Let us note that by construction the jump-adapted time discretization includes all jump times of the Poisson random measure and its maximum step size is equal to or less than the constant Δ. Therefore, the solution X of the SDE (5) jumps only at discretization points. In this section we set $Y_{t_n} = Y_n$ and we define

$$Y_{t_{n+1}-} = \lim_{s \Uparrow t_{n+1}} Y_s,$$

in the almost sure limit.

The *jump-adapted Euler scheme* is given by

$$Y_{t_{n+1}-} = Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} \tag{9}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1}-} + \int_{\mathscr{E}} c(Y_{t_{n+1}-}, v)\, p_\phi(\mathrm{d}v \times \{t_{n+1}\}), \tag{10}$$

where $\Delta_{t_n} = t_{n+1} - t_n$ and $\Delta W_{t_n} = W_{t_{n+1}} - W_{t_n} \sim \mathscr{N}(0, \Delta_{t_n})$. The approximation of the diffusion between discretization points is performed by (9), while the impact of jumps is simulated by (10). If t_{n+1} is a jump time, then $\int_{\mathscr{E}} p_\phi(\mathrm{d}v \times \{t_{n+1}\}) = 1$ and

$$\int_{\mathscr{E}} c(Y_{t_{n+1}-}, v)\, p_\phi(\mathrm{d}v \times \{t_{n+1}\}) = c(Y_{t_{n+1}-}, \xi_{p_\phi(t_{n+1})}),$$

while if t_{n+1} is not a jump time $Y_{t_{n+1}} = Y_{t_{n+1}-}$, as $\int_{\mathscr{E}} p_\phi(\mathrm{d}v \times \{t_{n+1}\}) = 0$.

The strong order of convergence of the scheme (9)–(10) is $\gamma = 0.5$.

By replacing the diffusive part (9) with a Milstein scheme as it is known for diffusions, we obtain the *jump-adapted order* 1.0 *strong Taylor scheme*

$$Y_{t_{n+1}-} = Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} + \frac{bb'}{2}((\Delta W_{t_n})^2 - \Delta_{t_n}) \tag{11}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1}-} + \int_{\mathscr{E}} c(Y_{t_{n+1}-}, v)\, p_\phi(\mathrm{d}v \times \{t_{n+1}\}), \tag{12}$$

which achieves strong order of convergence $\gamma = 1.0$. Scheme (11)–(12) was introduced in [30]. Similar jump-adapted schemes of strong order of convergence $\gamma = 1.0$ are proposed in [25,26].

By comparing the jump-adapted order 1.0 strong Taylor scheme (11)–(12) to the order 1.0 strong Taylor scheme (8), one notices that the former is much simpler. This simplicity is a characteristic property of jump-adapted schemes, since they do not involve multiple stochastic integrals with respect to the Poisson random measure. However, their computational complexity heavily depends on the intensity of the Poisson random measure. For high intensities many time steps have to be included in the dicretization and many operations have to be performed. Therefore, while in many practical applications jump-adapted schemes are very efficient, for SDEs driven by jump processes with high intensity, strong schemes based on non jump-adapted time discretizations can be more efficient.

It is possible to derive higher order jump-adapted schemes by using higher order strong schemes for the approximation of the diffusion part, see [19]. Moreover, by using derivative-free or implicit schemes, as known for diffusions, in the diffusive part of the jump-adapted scheme, one obtains jump-adapted derivative-free or jump-adapted implicit schemes, respectively, see [2].

2.4. Convergence theorem for jump diffusions

In this section we present a convergence theorem for strong Taylor approximations of order $\gamma \in \{0.5, 1, \ldots\}$ which establishes the strong orders of convergence of the approximations presented in Section 2.2. For the proof of a similar theorem for jump-adapted strong Taylor approximations, which covers the schemes of Section 2.3, we refer to [2].

The key to the construction and analysis of strong Taylor approximations is the Wagner–Platen expansion of the solution of the SDE (2), see [30–32]. In Section 3.5 we will illustrate the use of the Wagner–Platen expansion in the derivation of higher order strong schemes for one-dimensional SDEs driven by a Poisson process.

It is convenient to rewrite the SDE (2) in terms of the *compensated Poisson measure*

$$\widetilde{p}_{\phi}(\mathrm{d}v \times \mathrm{d}t) := p_{\phi}(\mathrm{d}v \times \mathrm{d}t) - \phi(\mathrm{d}v)\,\mathrm{d}t,$$

as

$$\mathrm{d}X_t = \widetilde{a}(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t + \int_{\mathscr{E}} c(t, X_{t-}, v)\widetilde{p}_{\phi}(\mathrm{d}v \times \mathrm{d}t), \tag{13}$$

where

$$\widetilde{a}(t, x) := a(t, x) + \int_{\mathscr{E}} c(t, x, v)\phi(\mathrm{d}v),$$

for $t \in [0, T]$ and $x \in \mathbb{R}^d$.

We introduce a compact notation to express multiple stochastic integrals and corresponding stochastic expansions. Due to the generality of the results, the following part will be quite technical. For the first reading of this survey the reader may jump to Section 3. For a more detailed explanation and examples of this notation we refer to [2]. We call a row vector $\alpha = (j_1, j_2, \ldots, j_l)$, where $j_i \in \{-1, 0, 1, \ldots, m\}$ for $i \in \{1, 2, \ldots, l\}$, a *multi-index* of length $l := l(\alpha) \in \{1, 2, \ldots\}$. Here m is the number of Wiener processes in the SDE (13). For $m \in \mathscr{N}$ we denote the set of all multi-indices α by

$$\mathscr{M}_m = \{(j_1, \ldots, j_l) : j_i \in \{-1, 0, 1, 2, \ldots, m\}, \quad i \in \{1, 2, \ldots, l\} \text{ for } l \in \mathscr{N}\} \cup \{v\},$$

where v is the multi-index of length zero.

In this notation $n(\alpha)$ is the number of components of a multi-index α that are equal to 0 and $s(\alpha)$ the number of components of a multi-index α that equal -1. Moreover, we write $\alpha-$ for the multi-index obtained by deleting the last component of α and $-\alpha$ for the multi-index obtained by deleting the first component of α.

We define the following sets of predictable stochastic processes $g = \{g(t), t \in [0, T]\}$, which will appear as integrands of the multiple stochastic integrals in the stochastic expansions,

$$\mathscr{H}_v = \left\{ g : \sup_{t \in [0,T]} E(|g(t, \omega)|) < \infty \right\},$$

$$\mathscr{H}_{(0)} = \left\{ g : E\left(\int_0^T |g(s, \omega)|\,\mathrm{d}s \right) < \infty \right\},$$

$$\mathscr{H}_{(-1)} = \left\{ g : E\left(\int_0^T \int_{\mathscr{E}} |g(s, v, \omega)|^2 \phi(\mathrm{d}v)\,\mathrm{d}s \right) < \infty \right\},$$

$$\mathscr{H}_{(j)} = \left\{ g : E\left(\int_0^T |g(s, \omega)|^2 \mathrm{d}s \right) < \infty \right\},$$

for $j \in \{1, 2, \ldots, m\}$. We will define below the set $\mathscr{H}_\alpha$ for a multi-index $\alpha \in \mathscr{M}_m$ with $l(\alpha) > 1$.

Let ρ and τ be two stopping times with $0 \leqslant \rho \leqslant \tau \leqslant T$ a.s. For any multi-index $\alpha \in \mathscr{M}_m$ and predictable process $g(\cdot) \in \mathscr{H}_\alpha$ we define the multiple stochastic integral $I_\alpha[g(\cdot)]_{\rho,\tau}$ recursively by

$$I_\alpha[g(\cdot)]_{\rho,\tau} := \begin{cases} g(\tau) & \text{when } l = 0 \text{ and } \alpha = v, \\ \int_\rho^\tau I_{\alpha-}[g(\cdot)]_{\rho,z}\,\mathrm{d}z & \text{when } l \geqslant 1 \text{ and } j_l = 0, \\ \int_\rho^\tau I_{\alpha-}[g(\cdot)]_{\rho,z}\,\mathrm{d}W_z^{j_l} & \text{when } l \geqslant 1 \text{ and } j_l \in \{1, \ldots, m\}, \\ \int_\rho^\tau \int_{\mathscr{E}} I_{\alpha-}[g(\cdot)]_{\rho,z-}\,\widetilde{p}_\phi(\mathrm{d}v \times \mathrm{d}z) & \text{when } l \geqslant 1 \text{ and } j_l = -1, \end{cases}$$

where $g(\cdot) = g(\cdot, v)$, with $v \in \mathscr{E}^{s(\alpha)}$. For simplicity, when it is not strictly necessary, we will omit the dependence of the integrand process g on one or more of the components $v^1, \ldots, v^{s(\alpha)}$ of the vector v expressing the marks of the Poisson measure.

For every multi-index $\alpha = (j_1, \ldots, j_l) \in \mathscr{M}_m$ with $l(\alpha) > 1$, the sets $\mathscr{H}_\alpha$ are defined recursively as the sets of predictable stochastic processes $g = \{g(t), t \geqslant 0\}$ such that the integral process $\{I_{\alpha-}[g(\cdot)]_{\rho,t}, t \in [0, T]\}$ satisfies $I_{\alpha-}[g(\cdot)]_{\rho,\cdot} \in \mathscr{H}_{(j_l)}$.

Furthermore, $\mathscr{L}^0$ is the set of functions $f(t, x, u) : [0, T] \times \mathbb{R}^d \times \mathscr{E}^{s(\alpha)} \longrightarrow \mathbb{R}^d$ for which

$$f(t, x + c(t, x, v), u) - f(t, x, u) - \sum_{i=1}^d c^i(t, x, v) \frac{\partial}{\partial x^i} f(t, x, u)$$

is $\phi(\mathrm{d}v)$-integrable for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $u \in \mathscr{E}^{s(\alpha)}$ and $f(\cdot, \cdot, u) \in \mathscr{C}^{1,2}$. Note that, according to the notation defined in Section 2.1, we denote by c^i the ith component of the jump coefficient vector c. With $\mathscr{L}^k$, $k \in \{1, \ldots, m\}$, we denote the set of functions $f(t, x, u)$ with partial derivatives $(\partial/\partial x^i) f(t, x, u)$, $i \in \{1, \ldots, d\}$ and with $\mathscr{L}^{-1}$ the set of functions for which

$$\{f(t, x + c(t, x, v), u) - f(t, x, u)\}^2$$

is $\phi(\mathrm{d}v)$-integrable for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $u \in \mathscr{E}^{s(\alpha)}$.

For a function $f(t, x, u) \in \mathscr{L}^k$, with $k \in \{-1, 0, 1, \ldots, m\}$, we define the following operators:

$$\begin{aligned} L^{(0)} f(t, x, u) := {} & \frac{\partial}{\partial t} f(t, x, u) + \sum_{i=1}^d a^i(t, x) \frac{\partial}{\partial x^i} f(t, x, u) \\ & + \frac{1}{2} \sum_{i,r=1}^d \sum_{j=1}^m b^{i,j}(t, x) b^{r,j}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} f(t, x, u) \\ & + \int_{\mathscr{E}} \{f(t, x + c(t, x, v), u) - f(t, x, u)\} \phi(\mathrm{d}v), \end{aligned}$$

$$L^{(k)} f(t, x, u) := \sum_{i=1}^d b^{i,k}(t, x) \frac{\partial}{\partial x^i} f(t, x, u) \quad \text{for } k \in \{1, \ldots, m\}$$

and

$$L^{(-1)} f(t, x, u) := f(t, x + c\,(t, x, v)\,, u) - f(t, x, u), \tag{14}$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $u \in \mathscr{E}^{s(\alpha)}$. Here the operator in (14) adds a new dependence on the component $v \in \mathscr{E}$, which we do not explicitly express in our notation for ease of presentation.

For all $\alpha = (j_1, \ldots, j_{l(\alpha)}) \in \mathscr{M}_m$ and a function $f : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, we define recursively the *Itô coefficient functions*

$$f_\alpha(t, x, u) := \begin{cases} f(t, x) & \text{for } l(\alpha) = 0, \\ f(t, \widetilde{a}(t, x)) & \text{for } l(\alpha) = 1, \;\; j_1 = 0, \\ f(t, b^{j_1}(t, x)) & \text{for } l(\alpha) = 1, \;\; j_1 \in \{1, \ldots, m\}, \\ f(t, c(t, x, u)) & \text{for } l(\alpha) = 1, \;\; j_1 = -1, \\ L^{(j_1)} f_{-\alpha}(t, x, u^1, \ldots, u^{s(-\alpha)}) & \text{for } l(\alpha) \geqslant 2, \;\; j_1 \in \{-1, 0, \ldots, m\}. \end{cases}$$

Here $b^{j_1}(t, x)$ denotes the d-dimensional vector of real valued functions on $[0, T] \times \mathbb{R}^d$ obtained by extracting the j_1th column from the matrix $b(t, x)$. With $u^1, \ldots, u^{s(-\alpha)}$ we denote the components of the vector $u \in \mathscr{E}^{s(-\alpha)}$. We assume that the coefficients of the SDE (2) and the function f satisfy the smoothness and integrability conditions needed for the operators in $L^{(j_1)}$, with $j_1 \in \{-1, 0, \ldots, m\}$, to be well defined.

Finally, we specify some particular sets of multi-indices. A subset $\mathscr{A} \in \mathscr{M}_m$ is called *hierarchical* if $\mathscr{A}$ is non-empty, the multi-indices in $\mathscr{A}$ are uniformly bounded in length, which means $\sup_{\alpha \in \mathscr{A}} l(\alpha) < \infty$, and if $-\alpha \in \mathscr{A}$ for each $\alpha \in \mathscr{A} \backslash \{v\}$. Moreover, we define the *remainder* set $\mathscr{B}(\mathscr{A})$ of $\mathscr{A}$ by

$$\mathscr{B}(\mathscr{A}) = \{\alpha \in \mathscr{M}_m \backslash \mathscr{A} : -\alpha \in \mathscr{A}\}.$$

Given a hierarchical set $\mathscr{A} \in \mathscr{M}_m$, two stopping times ρ and τ with $0 \leqslant \rho \leqslant \tau \leqslant T$ a.s. and a function $f : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, we obtain the Wagner–Platen expansion

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathscr{A}} I_\alpha[f_\alpha(\rho, X_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathscr{B}(\mathscr{A})} I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\rho,\tau}.$$

Here the function f and the coefficients of the SDE (2) are assumed to be sufficiently smooth and integrable such that the coefficient functions f_α are well defined and all the multiple stochastic integrals exist.

By choosing as function f the identity functions $f(t, x) = x$ we obtain a representation of the process $X = \{X_t, t \in [0, T]\}$ as solution of the SDE (2) by the Wagner–Platen expansion

$$X_\tau = \sum_{\alpha \in \mathscr{A}} I_\alpha[f_\alpha(\rho, X_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathscr{B}(\mathscr{A})} I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\rho,\tau}. \tag{15}$$

In (15) we have suppressed in the notation the dependence of f_α on $u \in \mathscr{E}^{s(\alpha)}$.

For the proof of the Wagner–Platen expansion for jump-diffusion processes, we refer to [30,31]. In Section 3.4 we will show the Wagner–Platen expansion for pure jump processes by an iterative application of the Itô formula.

For every $\gamma \in \{0.5, 1, 1.5, 2, \ldots\}$ we define the hierarchical set

$$\mathscr{A}_\gamma = \{\alpha \in \mathscr{M} : l(\alpha) + n(\alpha) \leqslant 2\gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + \tfrac{1}{2}\}.$$

For an equidistant time discretization with maximum step size $\Delta \in (0, 1)$, as that introduced in Section 2.2, we define the *order γ strong Taylor scheme* by the vector equation

$$\begin{aligned} Y_{n+1}^\Delta &= Y_n^\Delta + \sum_{\alpha \in \mathscr{A}_\gamma \backslash \{v\}} I_\alpha[f_\alpha(t_n, Y_n^\Delta)]_{t_n, t_{n+1}} \\ &= \sum_{\alpha \in \mathscr{A}_\gamma} I_\alpha[f_\alpha(t_n, Y_n^\Delta)]_{t_n, t_{n+1}}, \end{aligned} \tag{16}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$.

We present a convergence theorem which establishes the order of strong convergence of the strong Taylor scheme (16).

Theorem 1. *For given* $\gamma \in \{0.5, 1, 1.5, 2, \ldots\}$, *let* $Y^{\Delta} = \{Y_n^{\Delta}, n \in \{0, 1, \ldots, n_T\}\}$ *be the order* γ *strong Taylor scheme* (16) *corresponding to a time discretization with maximum step size* $\Delta \in (0, 1)$.

We assume that

$$E(|X_0|^2) < \infty \quad \textit{and} \quad \sqrt{E(|X_0 - Y_0^{\Delta}|^2)} \leqslant K_1 \Delta^{\gamma}.$$

Moreover, suppose that the coefficient functions f_α *satisfy the following conditions*:

For $\alpha \in \mathscr{A}_\gamma$, $t \in [0, T]$, $u \in \mathscr{E}^{s(\alpha)}$ *and* $x, y \in \mathbb{R}^d$ *the coefficient function* f_α *satisfies the Lipschitz-type condition*

$$|f_\alpha(t, x, u) - f_\alpha(t, y, u)| \leqslant K_1(u)|x - y|,$$

where $K_1(u)^2$ *is* $\phi(\mathrm{d}u)$*-integrable.*

For all $\alpha \in \mathscr{A}_\gamma \cup \mathscr{B}(\mathscr{A}_\gamma)$ *we assume*

$$f_{-\alpha} \in \mathscr{C}^{1,2} \quad \textit{and} \qquad f_\alpha \in \mathscr{H}_\alpha,$$

and for $\alpha \in \mathscr{A}_\gamma \cup \mathscr{B}(\mathscr{A}_\gamma)$, $t \in [0, T]$, $u \in \mathscr{E}^{s(\alpha)}$ *and* $x \in \mathbb{R}^d$, *we require*

$$|f_\alpha(t, x, u)|^2 \leqslant K_2(u)(1 + |x|^2),$$

where $K_2(u)$ *is* $\phi(\mathrm{d}u)$*-integrable.*

Then the estimate

$$\sqrt{E\left(\max_{0 \leqslant n \leqslant n_T} |X_n - Y_n^{\Delta}|^2 | \mathscr{A}_0\right)} \leqslant K_3 \Delta^{\gamma}$$

holds, where the constant K_3 *does not depend on* Δ.

The proof of this theorem is given in [2]. We also refer to [6] for the case of SDEs driven by Wiener processes and homogeneous Poisson processes.

3. Pure jump dynamics

In this section we consider the case of pure jump SDEs. We discuss differences to the previous results that arise due to the piecewise constant nature of the dynamics. By exploiting the simpler structure of the stochastic expansion in the pure jump case, we will also illustrate its use in the derivation of higher order strong schemes. Finally, we will show that the strong orders of convergence are established under weaker assumptions than those required by the convergence Theorem 1 for jump diffusions.

3.1. Pure jump model

Let us consider a counting process $N = \{N_t, t \in [0, T]\}$, which is right-continuous with left-hand limits and counts the arrival of certain events. Most of the following analysis applies for rather general counting processes. However, for simplicity, we take N to be a *Poisson process* with constant *intensity* $\lambda \in (0, \infty)$ that starts at time $t = 0$ in $N_0 = 0$. It is defined on a filtered probability space $(\Omega, \mathscr{A}_T, \underline{\mathscr{A}}, P)$ with $\underline{\mathscr{A}} = (\mathscr{A}_t)_{t \in [0,T]}$ satisfying the usual conditions.

The Poisson process N generates an increasing sequence $(\tau_n)_{n \in \{1,2,\ldots,N_T\}}$ of jump times. For any right-continuous process $Z = \{Z_t, t \in [0, T]\}$ we define its *jump size* ΔZ_t at time t as the difference $\Delta Z_t = Z_t - Z_{t-}$, for $t \in [0, T]$, where Z_{t-} denotes the left-hand limit of Z at time t. Thus, we can write $N_t = \sum_{s \in (0,t]} \Delta N_s$ for $t \in [0, T]$.

For a pure jump process $X = \{X_t, t \in [0, T]\}$ that is driven by the Poisson process N we assume that its value X_t at time t satisfies the SDE

$$\mathrm{d}X_t = c(t, X_{t-})\,\mathrm{d}N_t \tag{17}$$

for $t \in [0, T]$ with deterministic initial value $X_0 \in \mathbb{R}$. The *jump coefficient* $c : [0, T] \times \mathbb{R} \to \mathbb{R}$ is assumed to be Borel measurable, Lipschitz continuous, such that

$$|c(t, x) - c(t, y)| \leqslant K|x - y|$$

and to satisfy the growth condition

$$|c(t,x)|^2 \leqslant K(1+|x|^2)$$

for $t \in [0,T]$ and $x, y \in \mathbb{R}$ with some constant $K \in (0,\infty)$. According to [14,29] there exists a unique, right-continuous solution of the SDE (17).

To provide a simple, still illustrative, example, let us consider the linear SDE

$$\mathrm{d}X_t = X_{t-}\psi\,\mathrm{d}N_t \tag{18}$$

for $t \in [0,T]$ with $X_0 > 0$ and constant $\psi \in \mathbb{R}$. Here the jump coefficient has the form $c(t,x) = x\psi$. By application of the Itô formula one can demonstrate that the solution $X = \{X_t, t \in [0,T]\}$ of the SDE (18) is a pure jump process with explicit representation

$$X_t = X_0 \exp\{N_t \ln(\psi+1)\} = X_0(\psi+1)^{N_t} \tag{19}$$

for $t \in [0,T]$.

3.2. Jump-adapted approximations

We consider a *jump-adapted time discretization* $0 = t_0 < t_1 < \cdots < t_{n_T} = T$, where n_T is defined in (6) and the sequence $t_1 < \cdots < t_{n_T-1}$ equals that of the jump times $\tau_1 < \cdots < \tau_{N_T}$ of the Poisson process N. On this jump-adapted time grid we construct the *jump-adapted Euler scheme* by the algorithm

$$Y_{n+1} = Y_n + c\Delta N_n, \tag{20}$$

for $n \in \{0,1,\ldots,n_T-1\}$, with initial value $Y_0 = X_0$, where $\Delta N_n = N_{t_{n+1}} - N_{t_n}$ is the nth increment of the Poisson process N. Between discretization times the right-continuous process Y is set to be piecewise constant. Note that here and in the sequel, when no misunderstanding is possible, we use the abbreviation $c = c(t_n, Y_n)$.

Since the discretization points are constructed exactly at the jump times of N and the simulation of the increments $N_{t_{i+1}} - N_{t_i} = 1$ of N is exact, the jump-adapted Euler scheme (20) produces no discretization error. Let us emphasize that this is a particular feature of jump-adapted schemes when applied to pure jump SDEs. In the case of jump-diffusion SDEs, the jump-adapted schemes in Section 2.3 produce typically a discretization error.

For the implementation of the scheme (20) one needs to compute the jump times $\tau_n, n \in \{1,2,\ldots,N_T\}$, and has then to apply Eq. (20) recursively for every $n \in \{0,1,2,\ldots,n_T-1\}$. One can obtain the jump times via the corresponding waiting times between two consecutive jumps by sampling from an exponential distribution with parameter λ.

The computational effort when running the algorithm (20) is heavily dependent on the intensity λ of the jump process. Indeed, for large intensities the average number of steps and thus of operations is proportional to the intensity λ. Below we will introduce alternative methods suitable for large intensities.

3.3. Euler approximation

In the this section we develop discrete time strong approximations whose computational complexity is independent of the jump intensity level.

We consider an equidistant time discretization with time step size $\Delta \in (0,1)$ as in Section 2.2. The simplest strong Taylor approximation $Y = \{Y_t, t \in [0,T]\}$ is the *Euler scheme*, which is given by

$$Y_{n+1} = Y_n + c\Delta N_n \tag{21}$$

for $n \in \{0,1,\ldots,n_T-1\}$ with initial value $Y_0 = X_0$ and $\Delta N_n = N_{t_{n+1}} - N_{t_n}$. Between discretization times the right-continuous process Y is assumed to be piecewise constant.

By comparing the scheme (21) with the algorithm (20), we notice that the difference in the schemes consists in the time discretization. We emphasize that the average number of operations and, thus, the computational complexity of the Euler scheme (21) is independent of the jump intensity. Therefore, a simulation based on the Euler scheme (21) is feasible also in the case of jump processes with high intensity. However, while the jump-adapted Euler scheme (20) produces no discretization error, the accuracy of the Euler scheme (21) depends on the size of the time step Δ.

For the linear SDE (18) the Euler scheme (21) has the form

$$Y_{n+1} = Y_n + Y_n \psi \Delta N_n = Y_n(1 + \psi \Delta N_n) \tag{22}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with $Y_0 = X_0$. Since the equidistant time discretization does not include the jump times of the underlying Poisson process, we have an approximation error. Such error has similarity to that of discrete time approximations of SDEs that are driven purely by Wiener processes, as described, for instance, in [19].

Theorem 1 shows that the Euler approximation (21) achieves strong order of convergence $\gamma = 0.5$. This raises the question of constructing higher order discrete time approximations for the case of pure jump SDEs. This problem can be approached by exploiting stochastic Taylor expansions similar to the Wagner–Platen formula, also known as Itô–Taylor formula, which has been described and applied in many ways in [19] for diffusion SDEs. The stochastic Taylor expansion for pure jump SDEs that we will describe below is a particular case of the Wagner–Platen expansion (15) for jump diffusions and the stochastic Taylor formula for semimartingales derived in [31].

3.4. Stochastic Taylor expansions

Since the use of stochastic Taylor expansions for jump processes is not common in the literature let us at first illustrate the structure of a stochastic Taylor formula for a simple example. For any measurable function $f : \mathbb{R} \to \mathbb{R}$ and a given adapted counting process $N = \{N_t, t \in [0, T]\}$ we have the representation

$$f(N_t) = f(N_0) + \sum_{s \in (0,t]} \Delta f(N_s) \tag{23}$$

for all $t \in [0, T]$. We can formally write Eq. (23) in the form of an SDE

$$\mathrm{d} f(N_t) = (f(N_{t-} + 1) - f(N_{t-}))\, \mathrm{d}N_t$$

for $t \in [0, T]$. This equation can also be obtained from the Itô formula for semimartingales, see [33], for the case with jumps.

Obviously, the following difference expression $\widetilde{\Delta}_N f(N_{s-})$ defines a measurable function, as long as

$$\widetilde{\Delta}_N f(N) = f(N + 1) - f(N) \tag{24}$$

is a measurable function of N. By using this function we can rewrite (23) in the form

$$f(N_t) = f(N_0) + \int_{(0,t]} \widetilde{\Delta}_N f(N_{s-})\, \mathrm{d}N_s \tag{25}$$

for $t \in [0, T]$. Since $\widetilde{\Delta}_N f(N_{s-})$ is a measurable function we can apply the formula (25) to $\widetilde{\Delta}_N f(N_{s-})$ in (25), which yields

$$\begin{aligned} f(N_t) &= f(N_0) + \int_{(0,t]} \widetilde{\Delta}_N f(N_0)\, \mathrm{d}N_s + \int_{(0,t]} \int_{(0,s_2)} (\widetilde{\Delta}_N)^2 f(N_{s_1-})\, \mathrm{d}N_{s_1}\, \mathrm{d}N_{s_2} \\ &= f(N_0) + \widetilde{\Delta}_N f(N_0) \int_{(0,t]} \mathrm{d}N_s + \int_{(0,t]} \int_{(0,s_2)} (\widetilde{\Delta}_N)^2 f(N_{s_1-})\, \mathrm{d}N_{s_1}\, \mathrm{d}N_{s_2} \end{aligned} \tag{26}$$

for $t \in [0, T]$. Here $(\widetilde{\Delta}_N)^q$ denotes for integer $q \in \{1, 2, \ldots\}$ the q times consecutive application of the function $\widetilde{\Delta}_N$ given in (24). Note that a double stochastic integral with respect to the counting process N naturally arises in (26). One can now continue in (26) to apply the formula (25) to the measurable function $(\widetilde{\Delta}_N)^2 f(N_{s_1-})$, which yields

$$f(N_t) = f(N_0) + \widetilde{\Delta}_N f(N_0) \int_{(0,t]} \mathrm{d}N_s + \left(\widetilde{\Delta}_N\right)^2 f(N_0) \int_{(0,t]} \int_{(0,s_2)} \mathrm{d}N_{s_1} \mathrm{d}N_{s_2} + \bar{R}_3(t), \tag{27}$$

with remainder term

$$\bar{R}_3(t) = \int_{(0,t]} \int_{(0,s_3)} \int_{(0,s_2)} (\widetilde{\Delta}_N)^3 f(N_{s_1-})\, \mathrm{d}N_{s_1}\, \mathrm{d}N_{s_2}\, \mathrm{d}N_{s_3}$$

for $t \in [0, T]$. In (27) we have obtained a double integral in the expansion part. Furthermore, we have a triple integral in the remainder term. We call (27) a stochastic Taylor expansion of the function $f(\cdot)$ with respect to the counting process N. Its expansion part only depends on multiple stochastic integrals with respect to the counting process N. These are weighted by some constant coefficient functions with values taken at the expansion point N_0. It is clear how to proceed to obtain higher order Taylor expansions by iterative application of formula (25).

Fortunately, the multiple stochastic integrals that arise can be easily computed. It is straightforward to prove by induction, see [5], that

$$\int_{(0,t]} \mathrm{d}N_s = N_t,$$

$$\int_{(0,t]} \int_{(0,s_2)} \mathrm{d}N_{s_1}\,\mathrm{d}N_{s_2} = \frac{1}{2!} N_t(N_t-1),$$

$$\int_{(0,t]} \int_{(0,s_3)} \int_{(0,s_2)} \mathrm{d}N_{s_1}\,\mathrm{d}N_{s_2}\,\mathrm{d}N_{s_3} = \frac{1}{3!} N_t(N_t-1)(N_t-2),$$

$$\int_{(0,t]} \int_{(0,s_l)} \cdots \int_{(0,s_2)} \mathrm{d}N_{s_1} \cdots \mathrm{d}N_{s_{l-1}} \mathrm{d}N_{s_l} = \begin{cases} \binom{N_t}{l} & \text{for } N_t \geqslant l, \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

for $t \in [0, T]$ and $l \in \{1, 2, \ldots\}$. Here we have used the common combinatorial abbreviation $\binom{i}{l}$ for $i \geqslant l$ with $0! = 1$.

With (28) we can rewrite the stochastic Taylor expansion (27) in the form

$$f(N_t) = f(N_0) + \tilde{\Delta}_N f(N_0) \binom{N_t}{1} + (\tilde{\Delta}_N)^2 f(N_0) \binom{N_t}{2} + \bar{R}_3(t),$$

where

$$\tilde{\Delta}_N f(N_0) = \tilde{\Delta}_N f(0) = f(1) - f(0),$$

$$(\tilde{\Delta}_N)^2 f(N_0) = f(2) - 2f(1) + f(0).$$

In the given case this leads to the expansion

$$f(N_t) = f(0) + (f(1) - f(0))N_t + (f(2) - 2f(1) + f(0))\tfrac{1}{2} N_t(N_t-1) + \bar{R}_3(t)$$

for $t \in [0, T]$. More generally, by induction it follows the stochastic Taylor expansion

$$f(N_t) = \sum_{k=0}^{l} (\tilde{\Delta}_N)^k f(N_0) \binom{N_t}{k} + \bar{R}_{l+1}(t), \tag{29}$$

with

$$\bar{R}_{l+1}(t) = \int_{(0,t]} \cdots \int_{(0,s_2)} (\tilde{\Delta}_N)^{l+1} f(N_{s_1-})\,\mathrm{d}N_{s_1} \cdots \mathrm{d}N_{s_{l+1}}$$

for $t \in [0, T]$ and $l \in \{0, 1, \ldots\}$, where $(\tilde{\Delta}_N)^0 f(N_0) = f(N_0)$. By neglecting the remainder term in (29) one does not consider the occurrence of a higher number of jumps and obtains a useful truncated Taylor approximation of a measurable function f with respect to a counting process N. Note that in (29) the truncated expansion is exact if no more than l jumps occur until time t in the realization of N. Consequently, if there is a small probability that more than l jumps occur over the given time period, then the truncated stochastic Taylor expansion can be expected to be quite accurate under any reasonable criterion.

Similar to (29) let us now derive a stochastic Taylor expansion for functions of solutions of the pure jump SDE (17). We define similarly as above the measurable function $\tilde{\Delta}_N f(\cdot)$ such that

$$\tilde{\Delta}_N f(X_{t-}) = \Delta f(X_t) = f(X_t) - f(X_{t-}) \tag{30}$$

for all $t \in [0, T]$. In the same manner as previously shown this leads to the expansion

$$\begin{aligned} f(X_t) &= f(X_0) + \int_{(0,t]} \widetilde{\Delta}_N f(X_{s-})\,\mathrm{d}N_s \\ &= f(X_0) + \int_{(0,t]} \left(\widetilde{\Delta}_N f(X_0) + \int_{(0,s_2)} (\widetilde{\Delta}_N)^2 f(X_{s_1-})\,\mathrm{d}N_{s_1} \right) \mathrm{d}N_{s_2} \\ &= f(X_0) + \sum_{k=1}^{l} (\widetilde{\Delta}_N)^k f(X_0) \int_{(0,t]} \cdots \int_{(0,s_2)} \mathrm{d}N_{s_1} \cdots \mathrm{d}N_{s_k} + \tilde{R}^{l+1}_{f,t} \\ &= f(X_0) + \sum_{k=1}^{l} (\widetilde{\Delta}_N)^k f(X_0) \binom{N_t}{k} + \tilde{R}^{l+1}_{f,t}, \end{aligned} \tag{31}$$

with

$$\tilde{R}^{l+1}_{f,t} = \int_{(0,t]} \cdots \int_{(0,s_2)} (\widetilde{\Delta}_N)^{l+1} f(X_{s_1-})\,\mathrm{d}N_{s_1} \cdots \mathrm{d}N_{s_{l+1}}$$

for $t \in [0, T]$ and $l \in \{1, 2, \ldots\}$. One notes that (31) generalizes (29) in a simple fashion.

Let us give an illustration. For the particular example of the linear SDE (18) we obtain for any measurable function f the function

$$\widetilde{\Delta}_N f(X_{\tau-}) = f(X_{\tau-}(1+\psi)) - f(X_{\tau-})$$

for the jump times $\tau \in [0, T]$ with $\Delta N_\tau = 1$. Therefore, in the case $l = 2$, we get from (31) and (28) the expression

$$\begin{aligned} f(X_t) = {} & f(X_0) + (f(X_0(1+\psi)) - f(X_0))(N_t - N_0) + (f(X_0(1+\psi)^2) - 2f(X_0(1+\psi)) + f(X_0)) \\ & \times \tfrac{1}{2}(N_t - N_0)((N_t - N_0) - 1) + \tilde{R}^3_{f,t} \end{aligned}$$

for $t \in [0, T]$. By neglecting the remainder term $\tilde{R}^3_{f,t}$ we obtain, for this simple example, a truncated Taylor expansion of $f(X_t)$ at X_0. Let us emphasize that in the derivation of the stochastic Taylor expansion (31) only measurability of the function f and the coefficients $(\widetilde{\Delta}_N)^k f(\cdot)$, for $k \in \{1, \ldots, l\}$ is required. This contrasts with the case of diffusion and jump-diffusion SDEs where differentiability conditions are needed to obtain a stochastic Taylor expansion.

3.5. *Order* 1.0 *strong Taylor approximation*

The Euler scheme (21) can be interpreted as being derived from the expansion (31) applied to each time step by setting $f(x) = x$, choosing $l = 1$ and neglecting the remainder term. By choosing $l = 2$ in the corresponding truncated Taylor expansion, when applied to each time discretization interval $[t_n, t_{n+1}]$ with $f(x) = x$, we obtain the *order* 1.0 *strong Taylor approximation*

$$Y_{n+1} = Y_n + c\Delta N_n + (c(t_n, Y_n + c) - c)\tfrac{1}{2}(\Delta N_n)(\Delta N_n - 1)$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with $Y_0 = X_0$ and $\Delta N_n = N_{t_{n+1}} - N_{t_n}$.

In the special case of our linear example (18), the order 1.0 strong Taylor approximation turns out to be of the form

$$Y_{n+1} = Y_n \left\{ 1 + \psi \Delta N_n + \frac{\psi^2}{2} \Delta N_n (\Delta N_n - 1) \right\} \tag{32}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with $Y_0 = X_0$.

For the linear SDE (18) and a given sample path of the Poisson process, we plot in Fig. 1 the exact solution (19), the Euler approximation (22) and the order 1.0 strong Taylor approximation (32). We selected a time step size $\Delta = 0.25$ and the following parameters: $X_0 = 1$, $T = 1$, $\psi = -0.15$ and $\lambda = 20$. Note in Fig. 1 that the order 1.0 strong Taylor approximation is at the terminal time $t = 1$ rather close to the exact solution. It appears visually better than the Euler

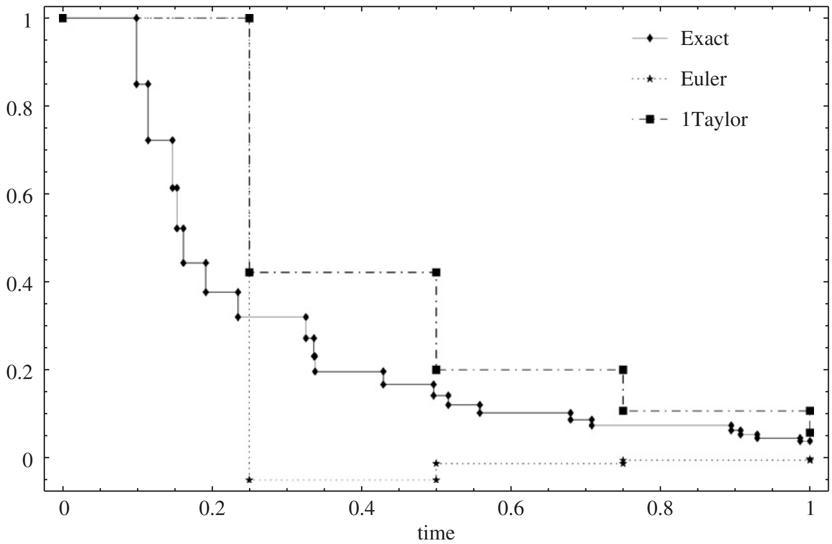

Fig. 1. Exact solution, Euler and order 1.0 Taylor approximations.

approximation, which is even negative. The convergence Theorem 1, presented in Section 2.4, and the convergence Theorem 5, to be presented in Section 3.7, provide a firm basis for judging the performance of higher order schemes.

3.6. *Order* 1.5 *and order* 2.0 *strong Taylor approximations*

If we use the truncated stochastic Taylor expansion (31) with $l=3$, when applied to each time interval $[t_n, t_{n+1}]$ with $f(x)=x$, we obtain the *order* 1.5 *strong Taylor approximation*

$$Y_{n+1} = Y_n + c\Delta N_n + \{c(t_n, Y_n + c) - c\} \binom{\Delta N_n}{2}$$
$$+ \{c(t_n, Y_n + c + c(t_n, Y_n + c)) - 2c(t_n, Y_n + c) + c\} \binom{\Delta N_n}{3}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with $Y_0 = X_0$.

In the case of our particular example (18), the order 1.5 strong Taylor approximation is of the form

$$Y_{n+1} = Y_n \left\{1 + \psi\Delta N_n + \psi^2 \binom{\Delta N_n}{2} + \psi^3 \binom{\Delta N_n}{3}\right\}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with $Y_0 = X_0$.

To construct an approximation with second order of strong convergence we need to choose $l = 4$ in the truncated expansion (31) with $f(x)=x$. Then we obtain the *order* 2.0 *strong Taylor approximation*

$$Y_{n+1} = Y_n + c\Delta N_n + \{c(Y_n + c(Y_n)) - c(Y_n)\} \binom{\Delta N_n}{2}$$
$$+ \{c(t_n, Y_n + c + c(t_n, Y_n + c)) - 2c(t_n, Y_n + c) + c\} \binom{\Delta N_n}{3}$$
$$+ \{c(t_n, Y_n + c + c(t_n, Y_n + c) + c(t_n, Y_n + c + c(t_n, Y_n + c)))$$
$$- 3c(t_n, Y_n + c + c(t_n, Y_n + c)) + 3c(t_n, Y_n + c) - c\} \binom{\Delta N_n}{4}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with $Y_0 = X_0$.

For the linear SDE (18) the order 2.0 strong Taylor approximation is of the form

$$Y_{n+1} = Y_n \left\{ 1 + \psi \Delta N_n + \psi^2 \binom{\Delta N_n}{2} + \psi^3 \binom{\Delta N_n}{3} + \psi^4 \binom{\Delta N_n}{4} \right\}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ with $Y_0 = X_0$.

3.7. Convergence theorem for pure jump dynamics

It is desirable to be able to construct systematically highly accurate discrete time approximations for solutions of pure jump SDEs. For this purpose we use the stochastic Taylor expansion (31) to construct the order γ strong Taylor scheme for pure jump processes, for $\gamma \in \{0.5, 1, 1.5, \ldots\}$.

In this section we consider a pure jump process described by a more general SDE than the SDE (17) considered so far. For the pure jump SDE (17) driven by one Poisson process it is possible, as shown above, to derive higher order strong schemes that involve only one Poisson random variable in each time step. However, it is important to study also more general multi-dimensional pure jump processes, which allow the modeling of more complex features as state-dependent intensities, for instance. For this reason, we consider here the d-dimensional pure jump SDE

$$\mathrm{d}X_t = \int_{\mathscr{E}} c(t, X_{t-}, v) p_\phi(\mathrm{d}v \times \mathrm{d}t), \tag{33}$$

for $t \in [0, T]$, with $X_0 \in \mathbb{R}^d$. Here the jump coefficient c and the Poisson random measure are defined as in (2). Note that the mark space $\mathscr{E}$ of the Poisson random measure is, in general, multi-dimensional and, thus, generates several sources of jumps. The case of a multi-dimensional SDE driven by several Poisson processes is a specific case of the SDE (33). The SDE (33) is equivalent to the jump-diffusion SDE (2) when the drift coefficient a and the diffusion coefficient b equal zero.

Theorem 1, presented in Section 2.4, establishes the strong order of convergence of strong Taylor approximations for jump-diffusion SDEs. When specifying the mentioned theorem to the case of SDEs driven by pure jump processes, it turns out that it is possible to weaken the assumptions on the coefficients of the stochastic Taylor expansions. As we will see below, the Lipschitz and growth conditions on the jump coefficient are already sufficient to establish the convergence of strong Taylor schemes of any given strong order of convergence $\gamma \in \{0.5, 1, 1.5, 2, \ldots\}$. Differentiability of the jump coefficient is not required. This is due to the structure of the increment operator $L^{(-1)}$, see (14), appearing in the coefficient of the stochastic Taylor expansion for pure jump processes.

For an equidistant time discretization with maximum step size $\Delta \in (0, 1)$ we define, the *order γ strong Taylor scheme* for pure jump SDEs by

$$Y_{n+1}^{\Delta} = Y_n^{\Delta} + \sum_{k=0}^{2\gamma-1} \int_{t_n}^{t_{n+1}} \int_{\mathscr{E}} \cdots \int_{t_n}^{s_1} \int_{\mathscr{E}} (L^{(-1)})^k c(t_n, Y_n^{\Delta}, v^0) p_\phi(\mathrm{d}v^0 \times \mathrm{d}s^0) \ldots p_\phi(\mathrm{d}v^k \times \mathrm{d}s^k), \tag{34}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ and $\gamma \in \{0.5, 1, 1.5, \ldots\}$. Recall that the increment operator $L^{(-1)}$ is defined in (14) and we denote by $(L^{(-1)})^0$ the identity operator.

The following three lemmas show that for SDEs driven by pure jump processes Lipschitz and growth conditions imply the sufficient conditions required by Theorem 1 to guarantee the corresponding orders of strong convergence.

Lemma 2. *Assume that the jump coefficient satisfies the Lipschitz condition*

$$|c(t, x, u) - c(t, y, u)| \leqslant K|x - y|, \tag{35}$$

for $t \in [0, T]$, $x, y \in \mathbb{R}^d$ and $u \in \mathscr{E}$, with some constant $K \in (0, \infty)$. Then for any $\gamma \in \{0.5, 1, 1.5, \ldots\}$ and $k \in \{0, 1, 2, \ldots, 2\gamma - 1\}$ the kth coefficient $(L^{(-1)})^k c(t, x, u)$ of the order γ strong Taylor scheme, satisfies the Lipschitz condition

$$|(L^{(-1)})^k c(t, x, u) - (L^{(-1)})^k c(t, y, u)| \leqslant C_k |x - y|, \tag{36}$$

for $t \in [0, T]$, $x, y \in \mathbb{R}^d$, $u \in \mathscr{E}^k$ and some constant $C_k \in (0, \infty)$ which depends only on k.

Proof. We prove the assertion (36) by induction on k. For $k=0$, by the Lipschitz condition (35) we obtain

$$|(L^{(-1)})^0 c(t,x,u)-(L^{(-1)})^0 c(t,y,u)| = |c(t,x,u)-c(t,y,u)| \leqslant K|x-y|.$$

For $k=l+1$, by the induction hypothesis, Jensen's inequality and the Lipschitz condition (35) we obtain

$$\begin{aligned}
&|(L^{(-1)})^{l+1}c(t,x,u)-(L^{(-1)})^{l+1}c(t,y,u)|\\
&\quad = |(L^{(-1)})^{l}c(t,x+c(t,x,v),u)-(L^{(-1)})^{l}c(t,x,u)\\
&\qquad -(L^{(-1)})^{l}c(t,y+c(t,y,v),u)+(L^{(-1)})^{l}c(t,y,u)|\\
&\quad \leqslant C_l|x-y+(c(t,x,v)-c(t,y,v))|+C_l|x-y|\\
&\quad \leqslant 2C_l|x-y|+C_lK|x-y|\\
&\quad = C_{l+1}|x-y|,
\end{aligned}$$

which completes the proof of Lemma 2. □

Lemma 3. *Assume that the jump coefficient satisfies the growth condition*

$$|c(t,x,u)|^2 \leqslant \widetilde{K}(1+|x|^2) \tag{37}$$

for $t\in[0,T]$ and $x\in\mathbb{R}^d$ and $u\in\mathscr{E}$, with some constant $\widetilde{K}\in(0,\infty)$. Then for any $\gamma\in\{0.5,1,1.5,\ldots\}$ and $k\in\{0,1,2,\ldots,2\gamma-1\}$ the kth coefficient $(L^{(-1)})^k c(t,x,u)$ of the order γ strong Taylor scheme, satisfies the growth condition

$$|(L^{(-1)})^k c(t,x,u)|^2 \leqslant \widetilde{C}_k(1+|x|^2) \tag{38}$$

for $t\in[0,T]$, $x,y\in\mathbb{R}^d$, $u\in\mathscr{E}^k$ and some constant $\widetilde{C}_k\in(0,\infty)$ which depends only on k.

Proof. We prove the assertion of Lemma 3 again by induction on k. For $k=0$, by applying the growth condition (37) we obtain

$$|(L^{(-1)})^0 c(t,x,u)|^2 = |c(t,x,u)|^2 \leqslant \widetilde{K}(1+|x|^2).$$

For $k=l+1$, by the induction hypotheses, Jensen's inequality and the growth condition (37) we obtain

$$\begin{aligned}
|(L^{(-1)})^{l+1}c(t,x,u)|^2 &= |(L^{(-1)})^{l}c(t,x+c(t,x,v),u)-(L^{(-1)})^{l}c(t,x,u)|^2\\
&\leqslant 2(\widetilde{C}_l(1+|x+c(t,x,v)|^2)+\widetilde{C}_l(1+|x|^2))\\
&\leqslant 2(\widetilde{C}_l(1+2(|x|^2+|c(t,x,v)|^2))+\widetilde{C}_l(1+|x|^2))\\
&\leqslant \widetilde{C}_{l+1}(1+|x|^2),
\end{aligned}$$

which completes the proof of Lemma 3. □

Lemma 4. *Let us assume that*

$$E(|X_0|^2)<\infty \tag{39}$$

and the jump coefficient satisfies the Lipschitz condition

$$|c(t,x,u)-c(t,y,u)| \leqslant K_1|x-y| \tag{40}$$

and the growth condition

$$|c(t,x,u)|^2 \leqslant K_2(1+|x|^2) \tag{41}$$

for $t \in [0,T]$, $x, y \in \mathbb{R}^d$, *and* $u \in \mathscr{E}$, *with constants* $K_1, K_2 \in (0,\infty)$. *Then for any* $\gamma \in \{0.5, 1, 1.5, \ldots\}$ *and* $k \in \{0, 1, 2, \ldots, 2\gamma - 1\}$ *the kth coefficient* $(L^{(-1)})^k c(t,x,u)$ *of the order* γ *strong Taylor scheme satisfies the integrability condition*

$$(L^{(-1)})^k c(\cdot, x, \cdot) \in \mathscr{H}_k,$$

for $x \in \mathbb{R}^d$, *where* $\mathscr{H}_k$ *is the set of predictable stochastic process* $g = \{g(t), t \in [0,T]\}$ *such that*

$$E\left(\int_0^T \int_{\mathscr{E}} \int_0^{s_k} \int_{\mathscr{E}} \cdots \int_0^{s_2} |g(s, v^1, \ldots, v^k, \omega)|^2 \phi(\mathrm{d}v^1)\, \mathrm{d}s_1 \ldots \phi(\mathrm{d}v^k)\, \mathrm{d}s_k\right) < \infty.$$

Proof. By Lemma 3 for any $\gamma \in \{0.5, 1, 1.5, \ldots\}$ and $k \in \{0, 1, 2, \ldots, 2\gamma - 1\}$ the kth coefficient $(L^{(-1)})^k c(t,x,u)$ of the order γ strong Taylor scheme satisfies the growth condition

$$|(L^{(-1)})^k c(t,x,u)|^2 \leqslant \widetilde{C}_k (1 + |x|^2) \tag{42}$$

for $t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $u \in \mathscr{E}^k$, with the constant $\widetilde{C}_k \in (0,\infty)$. Therefore, for any $\gamma \in \{0.5, 1, 1.5, \ldots\}$ and $k \in \{0, 1, 2, \ldots, 2\gamma - 1\}$, by condition (42) and Fubini's theorem we obtain

$$\begin{aligned} &E\left(\int_0^T \int_{\mathscr{E}} \int_0^{s_k} \int_{\mathscr{E}} \cdots \int_0^{s_1} \int_{\mathscr{E}} |(L^{(-1)})^k c(t, X_{s_0}, u^0)|^2 \phi(\mathrm{d}u^0)\, \mathrm{d}s_0 \ldots \phi(\mathrm{d}u^k)\, \mathrm{d}s_k\right) \\ &\leqslant E\left(\int_0^T \int_{\mathscr{E}} \int_0^{s_k} \int_{\mathscr{E}} \cdots \int_0^{s_1} \int_{\mathscr{E}} \widetilde{C}_k (1 + |X_{s_0}|^2) \phi(\mathrm{d}u^0)\, \mathrm{d}s_0 \ldots \phi(\mathrm{d}u^k)\, \mathrm{d}s_k\right) \\ &= \widetilde{C}_k \frac{(T\lambda)^k}{k!} + \widetilde{C}_k \int_0^T \int_0^{s_k} \cdots \int_0^{s_1} E(|X_{s_0}|^2)\, \mathrm{d}s_0 \ldots \mathrm{d}s_k < \infty. \end{aligned}$$

The last passage holds, since conditions (39), (40) and (41) ensure that

$$\sup_{t \in [0,T]} E(|X_t|^2) < \infty,$$

see [33], and this completes the proof of Lemma 4. □

Theorem 5. *For given* $\gamma \in \{0.5, 1, 1.5, 2, \ldots\}$, *let* $Y^\Delta = \{Y^\Delta(t), t \in [0,T]\}$ *be the order* γ *strong Taylor scheme* (34) *for the SDE* (33) *corresponding to a time discretization* $(t)_\Delta$ *with* $\Delta \in (0,1)$. *We assume for the jump coefficient* $c(t,x,v)$ *the Lipschitz condition* (35) *and the growth condition* (37). *Moreover, suppose that*

$$E(|X_0|^2) < \infty \quad \textit{and} \quad \sqrt{E(|X_0 - Y_0^\Delta|^2)} \leqslant K_1 \Delta^\gamma.$$

Then the estimate

$$\sqrt{E\left(\max_{0 \leqslant n \leqslant n_T} |X_n - Y_n^\Delta|^2\right)} \leqslant K \Delta^\gamma$$

holds, where the constant K does not depend on Δ.

The proof of Theorem 5 is a direct consequence of the convergence Theorem 1 for jump diffusions presented in Section 2.4. This is the case because by the Lemmas 2–4, the coefficients of the order γ strong Taylor scheme (34) satisfy the conditions required by the convergence Theorem 1. Note that the differentiability condition in Theorem 1, that is $f_{-\alpha} \in \mathscr{C}^{1,2}$ for all $\alpha \in \mathscr{A}_\gamma \cup B(\mathscr{A}_\gamma)$, is not required for pure jump SDEs. This condition is used in the convergence proof, see [2], for the derivation of the Wagner–Platen expansion. In the pure jump case, as shown in Section 3.4, one needs only measurability of the jump coefficient c to derive the stochastic expansion. We emphasize that in the case of pure jump SDEs, unlike the more general case of jump diffusions, no extra differentiability conditions on the jump coefficient c are required when deriving higher order approximations.

Theorem 5 states that the order γ strong Taylor scheme for pure jump SDEs achieves a strong order of convergence equal to γ. In fact Theorem 5 states that the strong convergence of order γ is not just at the endpoint T but it is also uniform over all time discretization points. Thus, by including enough terms from the stochastic Taylor expansion (31) we are able to construct schemes of any given strong order of convergence $\gamma \in \{0.5, 1, 1.5, \ldots\}$.

For the mark-independent pure jump SDE (17) driven by one Poisson process, the order γ strong Taylor scheme (34) reduces to

$$Y_{n+1}^{\Delta} = Y_n^{\Delta} + \sum_{k=1}^{2\gamma} (\widetilde{\Delta}_N)^k f(Y_n^{\Delta}) \binom{\Delta N_n}{k} \tag{43}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$, with $f(x) = x$, where the operator $\widetilde{\Delta}_N$ is defined in (30). In this case the generation of the multiple stochastic integrals involved is straightforward, since only one Poisson distributed random variable at each time step is required, as we have seen in (28). This allows the above schemes to be easily constructed. Such a construction is more complex in the case of genuine jump-diffusion SDEs and it is worth to know the advantages that one has when deriving higher order strong Taylor schemes for pure jump SDEs. We also note that for other pure jump SDEs driven by a counting process, one obtains similar schemes as those in (43). This is not restricted to Poisson processes.

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