JOURNAL OF COMBINATORIAL THEORY, Series A 68, 184–204 (1994)

m-Systems of Polar Spaces

E. E. SHULT

Kansas State University, Department of Mathematics, Cardwell Hall, Manhattan, Kansas 66506-2602

AND

J. A. THAS

University of Ghent, Department of Pure Mathematics, Krijgslaan 281, B-9000 Gent, Belgium

Communicated by Francis Buekenhout

Received March 15, 1993

Let P be a finite classical polar space of rank r, with $r \geq 2$. A partial m-system M of P, with $0 \leq m \leq r-1$, is any set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of k ($\neq 0$) totally singular m-spaces of P such that no maximal totally singular space containing π_i has a point in common with $(\pi_1 \cup \pi_2 \cup \cdots \cup \pi_k) - \pi_i$, $i = 1, 2, \ldots, k$. In each of the respective cases an upper bound δ for $|M|$ is obtained. If $|M| = \delta$, then M is called an m-system of P. For $m = 0$ the m-systems are the ovoids of P; for $m = r - 1$ the m-systems are the spreads of P. Surprisingly δ is independent of m, giving the explanation why an ovoid and a spread of a polar space P have the same size. In the paper many properties of m-systems are proved. We show that with m-systems of three types of polar spaces there correspond strongly regular graphs and two-weight codes. Also, we describe several ways to construct an m'-system from a given m-system. Finally, examples of m-systems are given.

1. FINITE CLASSICAL POLAR SPACES

Let P be a finite classical polar space of rank r, with $r \geq 2$ (see, e.g., Hirschfeld and Thas [9]). We use the following notation:

$W_n(q)$: the polar space arising from a symplectic polarity of $PG(n, q)$, n odd and $n \geq 3$: here $r = (n+1)/2$;

$Q(2n, q)$: the polar space arising from a non-singular quadric in $PG(2n, q)$, $n \geq 2$: here $r = n$;

$Q^+(2n+1, q)$: the polar space arising from a non-singular hyperbolic quadric in $PG(2n+1, q)$, $n \geq 1$: here $r = n + 1$;

0097-3165/94 $6.00

$Q^-(2n+1, q)$: the polar space arising from a non-singular elliptic quadric in $PG(2n+1, q)$, $n \geq 2$: here $r = n$;

$H(n, q^2)$: the polar space arising from a non-singular hermitian variety H in $PG(n, q^2)$, $n \geq 3$: for n odd $r = (n+1)/2$, for n even $r = n/2$.

Let $|P|$ denote the number of points of P, and let $\Sigma(P)$ be the set of all generators (or maximal totally singular subspaces) of P; all elements of $\Sigma(P)$ have dimension $r - 1$. For a proof of the following theorems we refer, e.g., to Hirschfeld and Thas [9].

THEOREM 1. *The numbers of points of the finite classical polar spaces are given by the formulae*

$$|W_n(q)| = (q^{n+1} - 1)/(q - 1),$$
$$|Q(2n, q)| = (q^{2n} - 1)/(q - 1),$$
$$|Q^+(2n+1, q)| = (q^n + 1)(q^{n+1} - 1)/(q - 1),$$
$$|Q^-(2n+1, q)| = (q^n - 1)(q^{n+1} + 1)/(q - 1),$$
$$|H(n, q^2)| = \left(q^{n+1} + (-1)^n\right)\left(q^n - (-1)^n\right)/(q^2 - 1).$$

THEOREM 2. *The numbers of generators of the finite classical polar spaces are given by*

$$|\Sigma(W_{2n+1}(q))| = (q+1)(q^2+1)\dots(q^{n+1}+1),$$
$$|\Sigma(Q(2n, q))| = (q+1)(q^2+1)\dots(q^n+1),$$
$$|\Sigma(Q^+(2n+1, q))| = 2(q+1)(q^2+1)\dots(q^n+1),$$
$$|\Sigma(Q^-(2n+1, q))| = (q^2+1)(q^3+1)\dots(q^{n+1}+1),$$
$$\left|\Sigma\left(H(2n, q^2)\right)\right| = (q^3+1)(q^5+1)\dots(q^{2n+1}+1),$$
$$\left|\Sigma\left(H(2n+1, q^2)\right)\right| = (q+1)(q^3+1)\dots(q^{2n+1}+1).$$

2. Ovoids and Spreads of Polar Spaces

Let P be a finite classical polar space of rank $r \geq 2$. An *ovoid* O of P is a pointset of P, which has exactly one point in common with each generator of P. A *spread* S of P is a set of generators, which constitutes a partition of the pointset. The following theorem is easily proved; cf., e.g., Thas [24].

THEOREM 3. *Let O be an ovoid and let S be a spread of the finite classical polar space P. Then*

$$\begin{array}{lll}
\textit{for} & P = W_{2n+1}(q), & |O| = |S| = q^{n+1} + 1, \\
\textit{for} & P = Q(2n,q), & |O| = |S| = q^{n} + 1, \\
\textit{for} & P = Q^{+}(2n+1,q), & |O| = |S| = q^{n} + 1, \\
\textit{for} & P = Q^{-}(2+1,q), & |O| = |S| = q^{n+1} + 1, \\
\textit{for} & P = H(2n,q^2), & |O| = |S| = q^{2n+2} + 1, \\
\textit{for} & P = H(2n+1,q^2), & |O| = |S| = q^{2n+1} + 1.
\end{array}$$

3. EXISTENCE AND NON-EXISTENCE OF SPREADS AND OVOIDS

3.1. *Spreads.* A spread of $W_n(q)$, $n = 2t + 1$, is also a t-spread of $PG(n,q)$, that is, a partition of $PG(n,q)$ by t-dimensional subspaces. For every $n = 2t + 1$ the polar space $W_n(q)$ has a spread which is also a regular t-spread of $PG(n,q)$; for details see, e.g., Thas [22]. Many other examples of spreads of $W_n(q)$ are known; see, e.g., Bader, Kantor and Lunardon [1], Dye [7], Kantor [10], Lüneburg [13], Thas [20], and Thas [28].

Proofs of the following results on spreads of quadrics can be found in Conway, Kleidman, and Wilson [6], Dye [7], Kantor [10–12], Moorhouse [14, 15], Payne and Thas [17], Shult [18], and Thas [27]. It is clear that $Q^{+}(4n+1,q)$ has no spread. For q even, $Q(2n,q)$, $Q^{-}(2n+1,q)$, and $Q^{+}(4n+3,q)$ always have a spread. For q odd, $Q^{+}(3,q)$ and $Q^{-}(5,q)$ have a spread; for $q = p$ an odd prime and for q odd with $q \equiv 0$ or 2 (mod 3), $Q^{+}(7,q)$ and $Q(6,q)$ have a spread; the polar space $Q(4n,q)$, with q odd, has no spread.

Concerning spreads of the polar spaces $H(n,q^2)$ the following results are known. They are respectively due to Thas [27] and Brouwer [4]: the polar spaces $H(2n+1,q^2)$ and $H(4,4)$ do not have a spread.

Open problems. The existence or non-existence of spreads in the following cases:

(a) $Q(6,q)$ for q odd, with $q \equiv 1 \pmod 3$ and q not a prime;

(b) $Q(4n+2,q)$ for $n > 1$ and q odd;

(c) $Q^{+}(7,q)$ for q odd, with $q \equiv 1 \pmod 3$ and q not a prime;

(d) $Q^{+}(4n+3,q)$ for $n > 1$ and q odd;

(e) $Q^{-}(2n+1,q)$ for $n > 2$ and q odd;

(f) $H(4,q^2)$ for $q > 2$;

(g) $H(2n,q^2)$ for $n > 2$.

3.2. *Ovoids.* In Thas [20] it is shown that $W_3(q)$ has an ovoid if and only if q is even. Moreover any ovoid of $W_3(q)$, q even, is an ovoid of $PG(3,q)$. Conversely, any ovoid of $PG(3,q)$, q even, is an ovoid of some $W_3(q)$ (see e.g., Hirschfeld [8]). Further, Thas [24] proves that $W_n(q)$, $n = 2t + 1$ with $t > 1$, has no ovoid.

Thas [24] also proves the non-existence of ovoids in $Q(2n,q)$, with q even and $n > 2$, and $Q^-(2n + 1, q)$, with $n > 1$. Kantor [10] shows that there is no ovoid in $Q^+(2n + 1, 2)$, $n \geq 4$, and Shult [19] proves that there is no ovoid in $Q^+(2n + 1, 3)$, $n \geq 4$. More generally, Blokhuis and Moorhouse [3] show that in $Q^+(2n + 1, q)$, with $q = p^h$, p prime, and

$$p^n > \binom{2n+p}{p-1},$$

there is no ovoid. For $n \geq 4$, this excludes $p = 2,3$; for $n \geq 5$ this excludes $p = 2,3,5,7$. The polar space $Q(4,q)$ always has an ovoid; see, e.g., Payne and Thas [17]. Clearly $Q^+(3,q)$ has an ovoid and for all q, $Q^+(5,q)$ admits an ovoid; see, e.g., Hirschfeld [8]. For $q = 3^h$ the polar space $Q(6,q)$ has an ovoid; see Kantor [10] and Thas [23, 27]. Applying triality (cf. Hirschfeld and Thas [9]) to the results on spreads of $Q^+(7,q)$ in 3.1, we find that $Q^+(7,q)$ has an ovoid in at least the following cases: q even, q an odd prime, and q odd with $q \equiv 0$ or $2 \pmod 3$.

Concerning ovoids of the polar spaces $H(n,q^2)$ the following results are known: it is easy to show that $H(3,q^2)$ admits ovoids (see, e.g., Payne and Thas [17] and Thas [25]) and in Thas [24] it is proved that $H(n,q^2)$, with n even, has no ovoid.

Open problems. The existence or non-existence of ovoids in the following cases:

(a) $Q(6,q)$ for q odd with $q \neq 3^h$;

(b) $Q(2n,q)$ for $n > 3$ and q odd;

(c) $Q^+(7,q)$ for q odd, with $q \equiv 1 \pmod 3$ and q not a prime;

(d) $Q^+(2n + 1, q)$ for $n > 3$, $q = p^h$, p prime, and

$$p^n \leq \binom{2n+p}{p-1};$$

(e) $H(n,q^2)$ for n odd and $n > 3$.

4. m-Systems and Partial m-Systems of Polar Spaces

4.1. Definition. Let P be a finite classical polar space of rank r, with $r \geq 2$. A *partial m-system* of P, with $0 \leq m \leq r - 1$, is any set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of k ($\neq 0$) totally singular m-spaces of P such that no generator containing π_i has a point in common with $(\pi_1 \cup \pi_2 \cup \cdots \cup \pi_k) - \pi_i$, $i = 1, 2, \ldots, k$. A partial 0-system of size k is also called a *partial ovoid*, or a *cap*, or a *k-cap*; a partial $(r-1)$-system is also called a *partial spread*.

4.2. THEOREM 4. *Let M be a partial m-system of the finite classical polar space P. Then*

$$\begin{array}{lll} \textit{for} & P = W_{2n+1}(q), & |M| \le q^{n+1} + 1, \\ \textit{for} & P = Q(2n, q), & |M| \le q^n + 1, \\ \textit{for} & P = Q^+(2n+1, q), & |M| \le q^n + 1, \\ \textit{for} & P = Q^-(2n+1, q), & |M| \le q^{n+1} + 1, \\ \textit{for} & P = H(2n, q^2), & |M| \le q^{2n+1} + 1, \\ \textit{for} & P = H(2n+1, q^2), & |M| \le q^{2n+1} + 1. \end{array}$$

Proof. By Theorem 3 we may assume that $m < r - 1$, with r the rank of the polar space.

If the polar space has ambient space $PG(s, q)$, then it will be denoted by P_s. Further, let $|P_s| = A_s$ and $|M| = \alpha$.

For each point $p_i \in P_s$ not in an element of M, let t_i be the number of totally singular $(m+1)$-spaces of P_s containing p_i and an element of M.

Now we count in different ways the number of ordered pairs (p_i, ξ), with $p_i \in P_s$ not in an element of M and with ξ a totally singular $(m+1)$-space of P_s containing p_i and an element of M.

We obtain

$$\sum t_i = \alpha q^{m+1} A_{s-2m-2}. \qquad (1)$$

In (1), $A_{s-2m-2} = |P_{s-2m2}$, with P_{s-2m-2} of the same type as P_s, and also polar spaces or rank 1 are admitted; for polar spaces of rank 1 we have $|W_1(q)| = |Q(2, q)| = |H(1, q^2)| = q + 1$, $|Q^+(1, q)| = 2$, $|Q^-(3, q)| = q^2 + 1$, $|H(2, q^2)| = q^3 + 1$.

Next we count in different ways the number of ordered triples (p_i, ξ, ξ'), with $p_i \in P_s$ not in an element of M and with ξ and ξ' distinct totally singular $(m+1)$-spaces of P_s containing p_i and an element of M.

We obtain

$$\sum t_i(t_i - 1) = \alpha(\alpha - 1)A_{s-2m-2}. \tag{2}$$

The remarks concerning (1) also hold for (2).

The number of points p_i equals

$$|I| = A_s - \alpha(q^{m+1} - 1)/(q - 1). \tag{3}$$

As $|I|\Sigma t_i^2 - (\Sigma t_i)^2 \geq 0$, we obtain from (1), (2), and (3)

$$\begin{aligned}-\alpha^2(q^{m+1} - 1) + \alpha\Big(-(q^{m+1} - 1)^2 + A_s(q - 1)& \\ -A_{s-2m-2}q^{2m+2}(q - 1)\Big)& \\ + A_s(q^{m+1} - 1)(q - 1) \geq 0.&\end{aligned} \tag{4}$$

Now an easy calculation gives us the bounds in the statement of the theorem. ▮

4.3. *m-Systems.* Let M be a partial m-system of the finite classical polar space P. If for $|M|$ the upper bound in the statement of Theorem 4 is reached, then M is called an *m-system* of P. So for an m-system M we have in the respective cases:

if	$P = W_{2n+1}(q)$,	then	$\lvert M\rvert = q^{n+1} + 1$,
if	$P = Q(2n, q)$,	then	$\lvert M\rvert = q^n + 1$,
if	$P = Q^+(2n + 1, q)$,	then	$\lvert M\rvert = q^n + 1$,
if	$P = Q^-(2n + 1, q)$,	then	$\lvert M\rvert = q^{n+1} + 1$,
if	$P = H(2n, q^2)$,	then	$\lvert M\rvert = q^{2n+1} + 1$,
if	$P = H(2n + 1, q^2)$,	then	$\lvert M\rvert = q^{2n+1} + 1$.

For $m = 0$, the m-system M is an ovoid of P; for $m = r - 1$, with r the rank of P, the m-system M is a spread of P. The fact that $|M|$ is independent of m gives us the explanation why an ovoid and a spread of a polar space P have the same size.

4.4. THEOREM 5. *Let M be an m-system of the finite classical polar space P of rank r, with $m < r - 1$. Then the number θ of totally singular $(m + 1)$-spaces of P containing an element of M and a given point $p \in P$ not in an element of M is independent of the choice of p. In the respective cases* *the number θ is given as follows:*

for	$P = W_{2n+1}(q)$,	$\theta = q^{n-m} + 1$,
for	$P = Q(2n, q)$,	$\theta = q^{n-m-1} + 1$,
for	$P = Q^+(2n + 1, q)$,	$\theta = q^{n-m-1} + 1$,
for	$P = Q^-(2n + 1, q)$,	$\theta = q^{n-m} + 1$,
for	$P = H(2n, q^2)$,	$\theta = q^{2n-2m-1} + 1$,
for	$P = H(2n + 1, q^2)$,	$\theta = q^{2n-2m-1} + 1$.

Proof. Let M be an m-system of the polar space P. As we then have equality in (4), it follows that in the proof of Theorem 4 the number t_i is the constant

$$\begin{aligned}\bar{t} &= \left(\sum t_i\right)/|I| \\ &= |M|q^{m+1}A_{s-2m-2}/\left(A_s - |M|\left(q^{m+1} - 1\right)/(q - 1)\right).\end{aligned}$$

Now an easy calculation gives us $\bar{t} = \theta$ in the respective cases. ∎

Remark. If p is a point of P not in an element of the m-system M, then for $m < r - 1$ and $P \neq W_{2n+1}(q)$, the tangent hyperplane of P at p contains exactly θ elements of M: for $m < r - 1$ and $P = W_{2n+1}(q)$, the hyperplane $p^\perp$ contains exactly θ elements of M.

5. Intersections with Hyperplanes

From now on let P be a finite classical polar space of rank r, and let M be an m-system of P.

5.1. Theorem 6. *For $P \neq W_{2n+1}(q)$, let ζ be the number of elements of M contained in a non-tangent hyperplane π of P; for $P = W_{2n+1}(q)$, let ζ be the number of elements of M contained in a hyperplane $p^\perp$, with p not in an element of M. Then*

for $P = W_{2n+1}(q)$, we have $\zeta = \theta = q^{n-m} + 1$,

for $P = Q(2n, q)$ and $\pi \cap Q(2n, q) = Q^+(2n - 1, q)$, we have $\zeta = q^{n-m-1} + 1$,

for $P = Q^-(2n + 1, q)$, we have $\zeta = q^{n-m} + 1$,

for $P = H(2n, q^2)$, we have $\zeta = q^{2n-2m-1} + 1$.

Proof. It is clear that for $P = W_{2n+1}(q)$ we have $\zeta = \theta$ (cf. the remark following Theorem 5).

Now let $P = Q(2n, q)$ and let π_i be an non-tangent hyperplane intersecting $Q(2n, q)$ in a polar space of type $Q^+(2n - 1, q)$. Further, let ζ_i be the number of elements of M in π_i.

Count in different ways the number of ordered pairs (γ, π_i), with $\gamma \in M$ in π_i. We obtain

$$\sum \zeta_i = (q^n + 1)q^n(q^{n-m-1} + 1)/2. \tag{5}$$

Now we count in different ways the number of ordered triples (γ, γ', π_i), with γ and γ' distinct elements of M in π_i. We obtain

$$\sum \zeta_i(\zeta_i - 1) = (q^n + 1)q^n q^{n-m-1}(q^{n-m-1} + 1)/2. \tag{6}$$

The number of hyperplanes π_i equals

$$|I| = q^n(q^n + 1)/2. \tag{7}$$

By (5), (6), and (7), we have $|I|\Sigma\zeta_i^2 - (\Sigma\zeta_i)^2 = 0$, and so

$$\zeta_i = \left(\sum \zeta_i\right)/|I| = q^{n-m-1} + 1$$

for all $i \in I$.

Next, let $P = Q^-(2n + 1, q)$ and let π_i be any non-tangent hyperplane of P. Further, let ζ_i be the number of elements of M in π_i. As in the previous section we obtain consecutively

$$\sum \zeta_i = (q^{n+1} + 1)q^n(q^{n-m} + 1), \tag{8}$$

$$\sum \zeta_i(\zeta_i - 1) = (q^{n+1} + 1)q^{2n-m}(q^{n-m} + 1), \tag{9}$$

$$|I| = q^n(q^{n+1} + 1). \tag{10}$$

By (8), (9), and (10) we have $|I|\Sigma\xi_i^2 - (\Sigma\zeta_i)^2 = 0$, and so

$$\zeta_i + \left(\sum \zeta_i\right)/|I| = q^{n-m} + 1$$

for all $i \in I$.

Now let $P = H(2n, q^2)$ and let π_i be any non-tangent hyperplane of P. Further, let ζ_i be the number of elements of M in π_i. As before we obtain consecutively

$$\sum \zeta_i = (q^{2n+1}+1)q^{2n}(q^{2n-2m-1}+1)/(q+1) \quad (11)$$

$$\sum \zeta_i(\zeta_i - 1) = (q^{2n+1}+1)q^{4n-2m-1}(q^{2n-2m-1}+1)/(q+1), \quad (12)$$

$$|I| = q^{2n}(q^{2n+1}+1)/(q+1). \quad (13)$$

By (11), (12), and (13) we have $|I|\Sigma\xi_i^2 - (\Sigma\zeta_i)^2 = 0$, and so

$$\zeta_i = \left(\sum \zeta_i\right)/|I| = q^{2n-2m-1}+1$$

for all $i \in I$. ∎

Remark. If $P = Q(2n,q)$ and $\pi \cap Q(2n,q) = Q^-(2n-1,q)$, then $\zeta = 0$ for $m = n-1$, and ζ depends on the choice of π for $m < n-1$.

If $P = Q^+(2n+1,q)$, then $\zeta = 0$ for $m = n$, and ζ depends on the choice of π for $m < n$.

If $P = H(2n+1,q^2)$, then $\zeta = 0$ for $n = m$, and ζ depends on the choice of π for $m < n$.

5.2. THEOREM 7. *For $P \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$ we have $\zeta = \theta$; that is, any hyperplane contains either one or θ elements of M. Hence the union $\tilde{M}$ of the elements of M has two intersection numbers β_1, β_2 with respect to hyperplanes.*

Proof. By Theorems 5 and 6, any hyperplane π which is not tangent at a point of P in $\tilde{M}$ for $P \neq W_{2n+1}(q)$, and which is not of the form $p^\perp$, with $p \in \tilde{M}$, for $P = W_{2n+1}(q)$, contains $\zeta = \theta$ elements of M. Any other hyperplane contains exactly one element of M. It is now clear that $\tilde{M}$ has two intersection numbers with respect to hyperplanes. ∎

Calculation of the intersection numbers β_1, β_2. In the respective cases we obtain

(a) $P = W_{2n+1}(q)$

$$\beta_1 = \frac{q^{m+1}-1}{q-1} + q^{n+1}\cdot\frac{q^m+1}{q-1} = \frac{(q^{m+1}-1)(q^n+1)}{q-1} - q^n,$$

$$\beta_2 = (q^{n-m}+1)\cdot\frac{q^{m+1}-1}{q-1} - (q^{n+1}-q^{n-m})\cdot\frac{q^m-1}{q-1}$$

$$= \frac{(q^{m+1}-1)(q^n+1)}{q-1}.$$

(b) $P = Q^-(2n+1, q)$

$$\beta_1 = \frac{q^{m+1}-1}{q-1} + q^{n+1}\cdot\frac{q^m-1}{q-1} = \frac{(q^{m+1}-1)(q^n+1)}{q-1} - q^n,$$

$$\beta_2 = (q^{n-m}+1)\cdot\frac{q^{m+1}-1}{q-1} + (q^{n+1}-q^{n-m})\cdot\frac{q^m-1}{q-1}$$

$$= \frac{(q^{m+1}-1)(q^n+1)}{q-1}.$$

(c) $P = H(2n, q^2)$

$$\beta_1 = \frac{q^{2m+2}-1}{q^2-1} + q^{2n+1}\cdot\frac{q^{2m}-1}{q^2-1}$$

$$= \frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1} - q^{2n-1},$$

$$\beta_2 = (q^{2n-2m-1}+1)\cdot\frac{q^{2m+2}-1}{q^2-1} + (q^{2n+1}-q^{2n-2m-1})\cdot\frac{q^{2m}-1}{q^2-1}$$

$$= \frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1}.$$

COROLLARY. *For $P \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$ any m-system defines a strongly regular graph and a two-weight code.*

Proof. By Theorem 7 this follows immediately from Calderbank and Kantor [5]. ∎

6. INTERSECTIONS WITH GENERATORS

6.1. THEOREM 8. *Let M be an m-system of the finite classical polar space P and let $\tilde{M}$ be the union of all elements of M. Then for any generator γ of P we have*

$$|\gamma \cap \tilde{M}| = (q^{m+1}-1)/(q-1).$$

Proof. Let γ_i be any generator of P_s, with s the dimension of the ambient space of P_s, and let $|\gamma_i| \cap \tilde{M}| = t_i$.

Count in two different ways the number of ordered pairs (p, γ_i), with $p \in \tilde{M}$ contained in the generator γ_i. We obtain

$$\sum_i t_i = \tilde{\alpha}|\Sigma(P_{s-2})|, \tag{14}$$

with $\tilde{\alpha} = |\tilde{M}|$, with $\Sigma(P_{s-2})$ the set of all generators of P_{s-2}, and with P_{s-2} of the same type as P_s.

Next, count in two different ways the number of ordered triples (p, p', γ_i), with p and p' different points of $\tilde{M}$ in the generator γ_i.

We obtain

$$\sum_i t_i(t_i - 1) = \tilde{\alpha}\left(q\frac{q^m - 1}{q - 1} + (\alpha - 1)\frac{q^m - 1}{q - 1}\right)|\Sigma(P_{s-4})|, \tag{15}$$

with $\alpha = |M|$ and with P_{s-4} of the same type as P_s.

The number of generators γ_i equals

$$|I| = |\Sigma(P_s)|. \tag{16}$$

By (14), (15), and (16), and relying on

$$\alpha|\Sigma(P_{s-2})| = |\Sigma(P_s)|$$

and

$$\alpha(q + \alpha - 1)|\Sigma(P_{s-4})| = q|\Sigma(P_s)|,$$

we obtain

$$\sum_i t_i = (q^{m+1} - 1)|\Sigma(P_s)|/(q - 1)$$

and

$$\sum_i t_i(t_i - 1) = (q^{m+1} - 1)(q^m - 1)q|\Sigma(P_s)|/(q - 1)^2.$$

Hence $|I|\Sigma_i t_i^2 - (\Sigma_i t_i)^2 = 0$, and so

$$t_i = \left(\sum_i t_i\right)/|I| = (q^{m+1} - 1)/(q - 1)$$

for all $i \in I$. ∎

6.2. *k-Ovoids of polar spaces.* Let P be a finite polar space of rank r, $r \geq 2$. A pointset K of P is called a *k-ovoid* of P if each generator of P

contains exactly k points of K. A k-ovoid with $k = 1$ is an ovoid. For $r = 2$, k-ovoids were already introduced by Thas [26].

By Theorems 8, the union of all elements of an m-system M of P is a k-ovoid with $k = (q^{m+1} - 1)/(q - 1)$.

7. $m' =$ Systems Arising from a Given m-System

Here we describe some constructions of m'-systems starting from a given m-system. As the cases "ovoids arising from a given ovoid" and "spreads arising from a given spread" were already considered in the papers on ovoids and spreads mentioned in Section 3, these constructions are not repeated here.

7.1. Theorem 9. *If $Q^-(2n + 1, q)$ has an m-system, then also $Q(2n + 1, q)$ and $Q^+(2n + 3, q)$ have m-systems.*

Proof. Let M be an m-system of $Q^-(2n + 1, q)$. Embed $Q^-(2n + 1, q)$ in a $Q(2n + 2, q)$ and embed $Q(2n + 2, q)$ in a $Q^+(2n + 3, q)$. Any generator π of $Q^+(2n + 3, q)$ containing $\pi_i \in M$, intersects $PG(2n + 1, q) \supset Q^-(2n + 1, q)$ in a generator of $Q^-(2n + 1, q)$; any generator π' of $Q(2n + 2, q)$ containing $\pi_i \in M$, intersects $PG(2n + 1, q)$ in a generator of $Q^-(2n + 1, q)$. Hence π and π' are skew to each element of $M - \{\pi_i\}$. As $|M|$ is the number of elements of an m-system of $Q(2n + 2, q)$, respectively $Q^+(2n + 3, q)$, the set M is also an m-system of $Q(2n + 2, q)$, respectively $Q^+(2n + 3, q)$. ∎

7.2. Theorem 10. *The polar space $Q(2n, q)$, q even, has an m-system if and only if the polar space $W_{2n-1}(q)$ has an m-system.*

Proof. Consider the polar space $Q(2n, q)$, q even, in $PG(2n, q)$. The nucleus of the quadric Q defining $Q(2n, q)$ is denoted by z. Let $PG(2n - 1)$, be a hyperplane of $PG(2n, q)$ not containing z. The projections from z onto $PG(2n - 1, q)$ of the totally singular s-dimensional subspaces of $Q(2n, q)$ are the totally singular s-dimensional subspaces of a polar space $W_{2n-1}(q)$ in $PG(2n - 1, q)$. Hence an m-system of $Q(2n, q)$ is projected onto an m-system of $W_{2n-1}(q)$, and, conversely, any m-system of $W_{2n-1}(q)$) is the projection of an m-system of $Q(2n, q)$. ∎

7.3. Theorem 11. *Let S_1 and S_2 be spreads of $Q^+(7, q)$, where the generators of S_1 and the generators of S_2 belong to different families. Then for each $\xi_i \in S_1$ there is exactly one $\eta_j \in S_2$ with $\xi_i \cap \eta_j = \pi_{ij}$ a plane. Also, the $q^3 + 1$ planes π_{ij} form a 2-system of $Q^+(7, q)$.*

Proof. Let $S_1 = \{\xi_1, \xi_2, \dots\}$ and let $S_2 = \{\eta_1, \eta_2, \dots\}$. Fix a generator ξ_i. If $|\xi_i \cap \eta_j| = t_{ij}$, then

$$\sum_j t_{ij} = q^3 + q^2 + q + 1. \tag{17}$$

As ξ_i and η_j are generators of different families, we have $t_{ij} \in \{1, q^2 + q + 1\}$. Also, the number of indices j is exactly $|S_2| = q^3 + 1$. Now it is clear that for exactly one index j we have $t_{ij} = q^2 + q + 1$, while for any other index j' we have $t_{ij'} = 1$.

Since any two elements of S_1 are skew, any two distinct elements ξ_{i_1} and ξ_{i_2} of S_1 define planes $\pi_{i_1 j_1}$ and $\pi_{i_2 j_2}$ in distinct generators η_{j_1} and η_{j_2}. Now indices can be chosen in such a way that $\xi_i \cap \eta_i = \pi_{ii}$ is a plane.

As ξ_i and η_i are the only generators containing π_{ii}, it is clear that every generator containing π_{ii} is skew to π_{jj}, for all $j \neq i$.

We conclude that the $q^3 + 1$ planes π_{ii} form a 2-system of $Q^+(7, q)$. ∎

Remark. If $Q^+(4n + 3, q)$, $n \geq 1$, admits a $2n$-system, then it admits a spread. This spread is obtained by considering all generators of a given family of generators of $Q^+(4n + 3, q)$, containing an element of the $2n$-system.

7.4. THEOREM 12. (a) *If* $H(2n, q^2)$ *admits an m-system* M, *then* $Q^-(4n + 1, q)$ *admits a* $(2m + 1)$*-system* M'.

(b) *If* $H(2n + 1, q^2)$ *admits an m-system* M, *then* $Q^+(4n + 3, q)$ *admits a* $(2m + 1)$*-system* M'.

Proof. (a) Consider the polar space $Q^-(4n + 1, q)$, $n \geq 1$. In the extension $PG(4n + 1, q^2)$ of $PG(4n + 1, q)$, the polar space $Q^+(4n + 1, q)$ extends to the polar space $Q^+(4n + 1, q^2)$. On the polar space $Q^-(4n + 1, q^2)$ it is possible to choose a projective $2n$-space π with $\pi \cap \bar{\pi} = \varnothing$, where $\bar{\pi}$ is conjugate to π with respect to the quadratic extension $GF(q^2)$ of $GF(q)$. The lines of $Q^-(4n + 1, q)$ whose extensions intersect π and $\bar{\pi}$ form a partition T of the pointset of $Q^-(4n + 1, q)$. It can be shown that the points common to π and the extensions of the lines of T form a hermitian variety H of π. Hence for $n \geq 2$ there arises a polar space $H(2n, q^2)$.

Let M be an m-system of $H(2n, q^2)$, $n \geq 2$; then $|M| = q^{2n+1} + 1$. If $\pi_i \in M$ and if $\bar{\pi}_i$ is conjugate to π_i, then the $(2m + 1)$-dimensional totally singular subspace of $Q^-(4n + 1, q)$ defined by π_i and $\bar{\pi}_i$ will be denoted by π'_i. These $q^{2n+1} + 1$ spaces π'_i are mutually skew. Assume by way of contradiction that there is a $(2m + 2)$-dimensional totally singular subspace ζ of $Q^-(4n + 1, q)$ which contains π'_i and has a point x in common with π'_j, $i \neq j$. If π_{ij} is the $(2m + 1)$-dimensional subspace of π generated by π_i and π_j, then, as π_i and π_j belong to an m-system of $H(2n, q^2)$, $\pi_{ij} \cap H(2n, q^2)$ is a polar space $H(2m + 1, q^2)$ if $m > 0$ and a Baer-subline $H(1, q^2)$ of π_{ij} if $m = 0$. Hence the lines of T whose extensions intersect π in the points of $H(2m + 1, q^2)$ have as union the pointset of a polar space $Q^+(4m + 3, q)$. Clearly π'_i, π'_j, ζ are totally singular subspaces of$Q^+(4m + 3, q)$. As ζ has dimension $2m + 2$, we have a contradiction. Consequently, no generator of $Q^-(4n + 1, q)$ containing π'_i has a point in common with $\pi'_j, i \neq j$.

As a $(2m + 1)$-system of $Q^-(4n + 1, q)$ has size $q^{2n+1} + 1$, that is, the number of spaces π'_i, we conclude that the set $M' = \{\pi'_1, \pi'_2, \ldots\}$ is a $(2m + 1)$-system of $Q^-(4n + 1, q)$.

(b) Consider the polar space $Q^+(4n + 3, q)$, $n \geq 0$. In the extension $PG(4n + 3, q^2)$ of $PG(4n + 3, q)$, the polar space $Q^+(4n + 3, q)$ extends to the polar space $Q^+(4n + 3, q^2)$. On the polar space $Q^+(4n + 3, q^2)$ it is possible to choose a projective $(2n + 1)$-space π with $\pi \cap \bar{\pi} = \varnothing$, where $\bar{\pi}$ is conjugate to π with respect to the quadratic extension $GF(q^2)$ of $GF(q)$. The lines of $Q^+(4n + 3, q)$ whose extensions intersect π and $\bar{\pi}$ form a partition T of the pointset of $Q^+(4n + 3, q)$. It can be shown that the points common to π and the extensions of the lines of T form a hermitian variety H of π. Hence for $n \geq 1$ there arises a polar space $H(2n + 1, q^2)$.

Let M be an m-system of $H(2n + 1, q^2)$, $n \geq 1$; then $|M| = q^{2n+1} + 1$. If $\pi_i \in M$ and if $\bar{\pi}_i$ is conjugate to π_i, then the $(2m + 1)$-dimensional totally singular subspaces of $Q^+(4n + 3, q)$ defined by π_i and $\bar{\pi}_i$ will be denoted by π'_i. These $q^{2n+1} + 1$ spaces π'_i are mutually skew. Assume by way of contradiction that there is a $(2m + 2)$-dimensional totally singular subspace ζ of $Q^+(4n + 3, q)$ which contains π'_i and has a point x in common with $\pi'_j, i \neq j$. If π_{ij} is the $(2m + 1)$-dimensional subspace of π generated by π_i and π_j, then, as π_i and π_j belong to an m-system of $H(2n + 1, q^2)$, $\pi_{ij} \cap H(2n + 1, q^2)$ is a polar space $H(2m + 1, q^2)$ if $m > 0$ and a Baer-subline $H(1, q^2)$ of π_{ij} if $m = 0$. Hence the lines of T whose extensions intersect π in the points of $H(2m + 1, q^2)$ have as union the pointset of a polar space $Q^+(4m + 3, q)$. Clearly π'_i, π'_j, ζ are totally singular subspaces of $Q^+(4m + 3, q)$. As ζ has dimension $2m + 2$, we have a contradiction. Consequently, no generator of $Q^+(4n + 3, q)$ containing π'_i has a point in common with $\pi'_j, i \neq j$.

As a $(2m + 1)$-system of $Q^+(4n + 3, q)$ has size $q^{2n+1} + 1$, that is, the number of spaces π'_i, we conclude that the set $M' = \{\pi'_1, \pi'_2, \ldots\}$ is a $(2m + 1)$-system of $Q^+(4n + 3, q)$. ∎

For an irreducible conic of $PG(2, q)$ we also use the notation $Q(2, q)$. A 0-system of the conic $Q(2, q)$ is defined to be the set of all points of the conic. For an elliptic quadric of $PG(3, q)$ we also use the notation $Q^-(3, q)$. A 0-system of the elliptic quadric $Q^-(3, q)$ is defined to be the set of all points of the quadric.

7.5. THEOREM 13. (a) *If $Q(2n, q^2)$, with $n \geq 1$ and q odd, admits an m-system M, then $Q^+(4n+1, q)$ admits a $(2m+1)$-system M'. If $Q(2n, q^2)$, with $n \geq 1$ and q even, admits an m-system M, then $Q(4n, q)$, and hence also $Q^+(4n+1, q)$, admits a $(2m+1)$-system M'.*

(b) *If $Q^-(2n+1, q^2)$, with $n \geq 1$, admits an m-system M, then $Q^-(4n+3, q)$ admits a $(2m+1)$-system M'.*

Proof. (a) In the extension $PG(4n+1, q^2)$ of $PG(4n+1, q)$ we consider two $2n$-dimensional subspaces π and $\bar{\pi}$ which are conjugate with respect to the extension $GF(q^2)$ of $GF(q)$, which are skew for q odd, and which have just one point p in common for q even. Clearly p belongs to $PG(4n+1, q)$. In $PG(4n+1, q)$ we now consider a polar space $Q^+(4n+1, q)$ such that π and $\bar{\pi}$ are polar with respect to the polarity θ defined by the extension $Q^+(4n+1, q^2)$ of $Q^+(4n+1, q)$; for q even, we assume that p is not a point of $Q^+(4n+1, q)$. Then $\pi \cap Q^+(4n+1, q^2)$ is a polar space $Q(2n, q^2)$ for $n > 1$, and an irreducible conic $Q(2, q^2)$ for $n = 1$. For q even, the $4n$-dimensional space $PG(4n, q^2)$ defined by π and $\bar{\pi}$ extends a space $PG(4n, q)$ which intersects $Q^+(4n+1, q)$ in a polar space $Q(4n, q)$; the kernel of the polar spaces $Q(2n, q^2)$ and $Q(4n, q)$ is the point p.

Let M be an m-system of $Q(2n, q^2)$, $n \geq 1$; then $|M| = q^{2n} + 1$. If $\pi_i \in M$ and if $\bar{\pi}_i$ is conjugate to π_i with respect to the extension $GF(q^2)$ of $GF(q)$, then the $(2m+1)$-dimensional totally singular subspace of $Q^+(4n+1, q)$ defined by π_i and $\bar{\pi}_i$ will be denoted by π'_i. These $q^{2n}+1$ spaces π'_i are mutually skew. Assume by way of contradiction that there is a $(2m+2)$-dimensional totally singular subspace ζ of $Q^+(4n+1, q)$ which contains π'_i and has a point x in common with π'_j, $i \neq j$. If π_{ij} is the $(2m+1)$-dimensional subspace of π generated by π_i and π_j, then, as π_i and π_j belong to an m-system of $Q(2n, q^2)$, $\pi_{ij} \cap Q(2n, q^2)$ is a polar space $Q^+(2m+1, q^2)$ if $m > 0$ and a point-pair $Q^+(1, q^2)$ of π_{ij} if $m = 0$. Now it is easy to show that the $(4m+3)$-dimensional space $PG(4m+3, q^2)$ generated by π'_i and π'_j intersects $Q^+(4n+1, q^2)$ in a polar space $Q^+(4m+3, q^2)$; the space π_{ij} and its conjugate $\bar{\pi}_{ij}$ with respect to the extension $GF(q^2)$ of $GF(q)$ are polar with respect to the polarity θ' induced by θ in $PG(4m+3, q^2)$. The space $PG(4m+3, q^2)$ extends a space $PG(4m+3, q)$, and $PG(4m+3, q) \cap Q^+(4n+1, q^2) = PG(4m+3, q) \cap Q^+(4m+3, q^2)$ is a polar space $Q^+(4m+3, q)$. Clearly

π_i', π_j', ζ are totally singular subspaces of $Q^+(4m+3, q)$. As ζ has dimension $2m+2$, we have a contradiction. Consequently, no generator of $Q^+(4n+1, q)$ containing π_i' has a point in common with $\pi_j', i \neq j$.

As a $(2m+1)$-system of $Q^+(4n+1, q)$ has size $q^{2n}+1$, that is, the number of spaces π_i', we conclude that the set $M' = \{\pi'_1, \pi'_2, \ldots\}$ *is* a $(2m+1)$-system of $Q^+(4n+1, q)$.

In the even case all spaces of M' are totally singular spaces of the polar space $Q(4n, q)$. As a $(2m+1)$-system of $Q(4n, q)$ has size $q^{2n}+1 = |M'|$, it is clear that M' is also a $(2m+1)$-system of $Q(4n, q)$.

(b) In the extension $PG(4n+3, q^2)$ of $PG(4n+3, q)$ we consider two $(2n+1)$-dimensional subspaces π and $\bar{\pi}$ which are conjugate with respect to the extension $GF(q^2)$ of $GF(q)$ and for which $\pi \cap \bar{\pi} = \emptyset$. In $PG(4n+3, q)$ we now consider a polar space $Q^-(4n+3, q)$ such that π and $\bar{\pi}$ are polar with respect to the polarity θ defined by the extension $Q^+(4n+3, q^2)$ of $Q^-(4n+3, q)$. Assume by way of contradiction that $\pi \cap Q^+(4n+3, q^2)$ is a polar space $Q^+(2n+1, q^2)$. If ξ is a generator of $Q^+(2n+1, q^2)$ and $\bar{\xi}$ is conjugate to ξ with respect to the extension $GF(q^2)$ of $GF(q)$, then ξ and $\bar{\xi}$ define a $(2n+1)$-dimensional totally singular subspace of $Q^-(4n+3, q)$, a contradiction. Hence $\pi \cap Q^-(4n+3, q^2)$ is a polar space $Q^-(2n+1, q^2)$ for $n > 1$, and an elliptic quadric $Q^-(3, q^2)$ for $n = 1$.

Let M be an m-system of $Q^-(2n+1, q^2)$, $n \geq 1$; then $|M| = q^{2n+2} + 1$. Then, as in case (a), one shows that M defines a $(2m+1)$-system of the polar space $Q^-(4n+3, q)$. ∎

Remark. Recall that by 5.2 every m-system of $Q^-(2n+1, q^2)$, respectively $Q^-(4n+3, q)$, defines a strongly regular graph and a two-weight code.

For a non-singular hermitian curve of $PG(2, q^2)$ we also use the notation $H(2, q^2)$. A 0-system of the hermitian curve $H(2, q^2)$ is defined to be the set of all points of the curve.

7.6. THEOREM 14. *If $H(2n, q^2)$, $n \geq 1$, admits an m-system M, then $W_{4n+1}(q)$ admits a $(2m+1)$-system M'.*

Proof. In the extension $PG(4n+1, q^2)$ of $PG(4n+1, q)$ we consider two mutually skew $2n$-dimensional subspaces π and $\bar{\pi}$ which are conjugate with respect to the extension $GF(q^2)$ of $GF(q)$. We now consider a polar space $W_{4n+1}(q)$ in $PG(4n+1, q)$, such that π, and then also $\bar{\pi}$, is self-polar with respect to the symplectic polarity θ defined by the extension $W_{4n+1}(q^2)$ of $W_{4n+1}(q)$. Let $x \in \pi$, let $\bar{x}$ be the point of $\bar{\pi}$ conjugate to x with respect to the extension $GF(q^2)$ of $GF(q)$, and let $\bar{x}^\theta \cap \pi = \pi_x$. It is clear that the mapping θ_π: $x \to \pi_x$ is a (non-singular) polarity of the projective space π. The absolute points of θ_π are exactly the points x of π for which the line $x\bar{x}$ is totally singular for θ; in such a case $x\bar{x} \cap PG(4n+1, q)$ is a totally singular line of $W_{4n+1}(q)$. We now show that θ_π is a unitary polarity of π.

Let L be any line of π and let $\bar{L}$ be the corresponding line of $\bar{\pi}$. The lines L and $\bar{L}$ generate a threespace $PG(3, q^2)$, which is an extension of $PG(3, q)$. Now we determine the number α of absolute points of θ_π on L; that is, we determine the number of totally singular lines of $W_{4n+1}(q)$ in $PG(3, q)$ whose extensions contain a point of L and $\bar{L}$. First, assume that the polarity induced by θ in $PG(3, q^2)$ is singular with radical $PG(3, q^2)$. Then all lines of $PG(3, q^2)$ are totally singular, and so $\alpha = q^2 + 1$. Next, assume that the polarity induced by θ in $PG(3, q^2)$ is singular with radical $PG(1, q^2)$. In such a case the totally singular lines of θ in $PG(3, q^2)$ are all lines of $PG(3, q^2)$ having a non-empty intersection with $PG(1, q^2)$. Since L and $\bar{L}$ are totally singular, the lines L and $\bar{L}$ have respective points x and $\bar{x}$ in common with $PG(1, q^2)$. It is now clear that $x\bar{x}$ is the only totally singular line of θ in $PG(3, q^2)$, intersecting L and $\bar{L}$ in points conjugate with respect to the extension $GF(q^2)$ of $GF(q)$; hence, $\alpha = 1$. Finally, assume that the polarity induced by θ in $PG(3, q^2)$ is non-singular. Then there are exactly $\alpha = q + 1$ totally singular lines of θ in $PG(3, q^2)$, interesting L and $\bar{L}$ in conjugate points. Consequently $\alpha \in \{1, q+1, q^2 + 1\}$. Since the dimension of π is even θ_π, either is a unitary polarity or a pseudopolarity. Assume, by way of contradiction, that θ_π is a pseudopolarity. Then θ_π has a $(2n-1)$-dimensional space ζ of absolute points. As for the line L joining any two of these absolute points it holds that $\alpha = q^2 + 1$, and we necessarily have $PG(3, q^2) = L\bar{L} \subset PG(3, q^2)^\theta$. Let ζ' be the $(4n-1)$-dimensional space generated by ζ and $\bar{\zeta}$. Further, let N be a line over $GF(q)$ contained in ζ'. If the extension N' of N contains a point of ζ and $\bar{\zeta}$, then clearly N' is totally singular for θ. So assume that N' has an empty intersection with ζ and $\bar{\zeta}$. The lines containing a point of N', ζ and $\bar{\zeta}$, intersect ζ and $\bar{\zeta}$ in the points of lines L and $\bar{L}$ which are conjugate with respect to the extension $GF(q^2)$ of $GF(q)$. Since every line of $L\bar{L}$ is totally singular for θ, it follows that N' is totally singular for θ. Hence ζ' is totally singular for θ, a contradiction as ζ' has dimension $4n-1$. We conclude that θ_π is a unitary polarity.

So for $n = 1, \theta_\pi$ defines a non-singular hermitian curve $H(2, q^2)$, and for $n > 1$ θ_π defines a polar space $H(2n, q^2)$.

Let M be an m-system of $H(2n, q^2)$, $n \geq 1$; then $|M| = q^{2n+1} + 1$. If $\pi_i \in M$ and if $\bar{\pi}_i$ is conjugate to π_i with respect to the extension $GF(q^2)$ of $GF(q)$, then, by a reasoning analogous to the one used in the last part of the previous section, the $(2m+1)$-dimensional subspace π_i' of $PG(4m+1, q)$ defined by π_i and $\bar{\pi}_i$ is a totally singular subspace of $W_{4n+1}(q)$. These $q^{2n+1} = 1$ spaces π_i' are mutually skew. Assume by way of contradiction that there is a $(2m + 2)$-dimensional totally singular subspace η of $W_{4n+1}(q)$ which contains π_i' and has a point x in common with π_j', $i \neq j$. Let $z\bar{z}$, with $z \in \pi_j$ and $\bar{z} \in \bar{\pi}_j$, contain x. Then for any point y of π_i the space $\bar{y}^\theta$ contains z. Hence the line yz is a line of $H(2n, q^2)$. Consequently $\pi_i z$ is an $(m + 1)$-dimensional totally singular subspace of $H(2n, q^2)$ containing a point of π_j, a contradiction as M is an m-system of $H(2n, q^2)$. So no generator of $W_{4n+1}(q)$ containing π_i' has a point in common with π_j', $i \neq j$.

As a $(2m + 1)$-system of $W_{4n+1}(q)$ has size $q^{2n+1} + 1$, that is, the number of spaces π_i', we conclude that $M' = \{\pi_1', \pi_2', \dots\}$ is a $(2m + 1)$-system of $W_{4n+1}(q)$. ∎

Remark. Recall that by 5.2 every m-system of $H(2n, q^2)$, respectively $W_{4n+1}(q)$, defines a strongly regular graph and a two-weight code.

8. Examples of m-Systems

8.1. *m-Systems of $Q^-(2n + 1, q)$, $n \geq 1$.* We apply Theorem 7.5(b).

(a) *Let $q = p^{2^h u}$, with p any prime and u odd.* As $Q^-(3, q)$ has a 0-system, the polar space $Q^-(2^{s+2} - 1, p^{2^{h-s}u})$ has a $(2^s - 1)$-system for all $0 \leq s \leq h$.

As $Q^-(5, q)$ has a spread, the polar space $Q^-(3.2^{s+1} - 1, p^{2^{h-s}u})$ has a $(2^{s+1} - 1)$-system for all $0 \leq s \leq h$.

(b) *Let $q = 2^{2^h u}$, with u odd.* As $Q^-(2n + 1, q), n \geq 1$, has an $(n - 1)$-system, the polar space $Q^-((n + 1)2^{s+1} - 1, 2^{2^{h-s}u})$ has an $(n2^s - 1)$-system for all $0 \leq s \leq h$.

8.2. *m-Systems of $Q(2n, q)$, $n \geq 2$.* 1. We apply Theorem 9 and rely on 8.1.

(a) *Let $q = p^{2^h u}$, with p any prime and u odd.* The polar space $Q(2^{s+2}, p^{2^{h-s}u})$ has a $(2^s - 1)$-system for all $0 \leq s \leq h$.

The polar space $Q(3.2^{s+1}, p^{2^{h-s}u})$ has a $(2^{s+1} - 1)$-system for all $0 \leq s \leq h$.

(b) *Let $q = 2^{2^h u}$, with u odd.* The polar space $Q((n + 1)2^{s+1}, 2^{2^{h-s}u})$ has an $(n2^s - 1)$-system for all $0 \leq s \leq h$.

2. Consider the classical generalized hexagon $H(q)$ of order q embedded in the non-singular quadric Q of $PG(6, q)$ (cf. Thas [23]). A spread of $H(q)$ is a set S of lines of $H(q)$, any two of which are at distance 6 in the incidence graph of $H(q)$, such that each line of $H(q)$ not in S is concurrent with a unique line of S. Clearly $|S| = q^3 + 1$. In [23] it is

shown that $H(q)$ always has a spread and that for $q = 3^{2h+1}$, $h \geq 0$, $H(q)$ admits at least two projectively inequivalent spreads.

Let S be a spread of $H(q)$. We will show that S is a 1-system of the polar space $Q(6, q)$ arising from the quadric Q. Assume by way of contradiction that the generator π of $Q(6, q)$ containing $L \in S$ has a point x in common with $M \in S - \{L\}$. Then $d(x, L) = 3$, so $d(L, M) = 4$, a contradiction.

We conclude that every spread S of $H(q)$ is also a 1-system of $Q(6, q)$.

Problems. (1) Does there exist a 1-system of $Q(6, q)$, $q \neq 3^{2h+1}$, which is not a spread of a $Q^-(5, q) \subset Q(6, q)$?

(2) Does there exist a spread of $H(q)$, $q \neq 3^{2h+1}$, which is not a spread of a $Q^-(5, q) \subset Q(6, q)$, with $Q(6, q)$ the polar space defined by the quadric Q in which $H(q)$ is embedded?

8.3. *m-systems of* $Q^+(2n + 1, q)$, $n \geq 2$. 1. We apply Theorem 9 and rely on 8.1.

(a) *Let* $q = p^{2^h u}$, *with* p *any prime and* u *odd.* The polar space $Q^+(2^{s+2} + 1, p^{2^{h-s}u})$ has a $(2^s - 1)$-system for all $0 \leq s \leq h$.

The polar space $Q^+(3.2^{s+1} + 1, p^{2^{h-s}u})$ has a $(2^{s+1} - 1)$-system for all $0 \leq s \leq h$.

(b) *Let* $q = 2^{2^h u}$, *with* u *odd.* The polar space $Q^+((n + 1)2^{s+1} + 1, 2^{2^{h-s}u})$ has an $(n2^s - 1)$-system for all $0 \leq s \leq h$.

2. (a) Let O be any ovoid of $H(3, q^2)$. Then, by Theorem 12(b), with O there corresponds a 1-system of $Q^+(7, q)$.

(b) Let Σ_1 and Σ_2 be the families of generators of $Q^+(7, q)$. For q even, for $p = q$ an odd prime, and for q odd with $q \equiv 0$ or $2 \pmod 3$, $Q^+(7, q)$ admits a spread S. Assume, e.g., that the elements of S belong to Σ_1. Then Σ_2 has a spread S' projectively equivalent to S. By Theorem 11 the spreads S and S' define a 2-system of $Q^+(7, q)$.

8.4. *m-Systems of* $W_{2n+1}(q)$, $n \geq 1$. 1. Applying Theorem 10 and relying on 8.2 we see that the polar space $W_{(n+1)2^{s+1}-1}(2^{2^{h-s}u})$, with u odd, has an $(n2^s - 1)$-system for all $n \geq 1$ and $0 \leq s \leq h$.

2. Applying Theorem 14 we see that $W_5(q)$ admits a 1-system for each prime power q.

8.5. *m-Systems of* $H(3n - 2, q^2)$, *n odd.* Under the trace map: $GF(q^{2n}) \to GF(q^2)$ when n is odd, a non-degenerate Hermitian form on $GF(q^{2n})^{(m)}$ becomes a non-degenerate Hermitian form on $GF(q^2)^{(nm)}$. Upon applying this when $m = 3$, a unital $H(2, q^{2n})$ becomes a $(n - 1)$-system of $H(3n - 1, q^2)$. Thus $W(6n - 1, q)$ and $Q^-(6n - 1, q)$ admit a $(2n - 1)$-system (resp. Theorem 14 and Theorem 12), so that in turn, $Q(6n, q)$ and $Q^+(6n + 1, q)$ admit a $(2n - 1)$-system (Theorem 9).

9. Classification of All 1-Systems of $Q^+(5,q)$

THEOREM 15. *Up to a projectivity $Q^+(5,q)$, with q odd, has a unique 1-system. For q even each 1-system of $Q^+(5,q)$ is a spread of a $Q(4,q) \subset Q^+(5,q)$.*

Proof. Let M be a 1-system of $Q^+(5,q)$. Consider Q^+ as the Klein quadric of the lines of $PG(3,q)$; see Hirschfeld [8]. With the q^2+1 lines of M there correspond q^2+1 pencils of lines, with respective vertices $p_0, p_1, \ldots$ and contained in the respective plans $\pi_0, \pi_1, \ldots$. As M is a 1-system we have $p_i \notin \pi_j$ for $i \neq j$. By Theorem 8 each plane π of $PG(3,q)$ contains either one or $q+1$ points of $0 = \{p_0, p_1, \ldots\}$. Now by Thas [21] the set O is an ovoid of $PG(3,q)$.

Let q be odd. Then by the theorem of Barlotti [2] and Panella [16] the ovoid O is an elliptic quadric. Hence, up to a projectivity, O and M are uniquely defined.

Let q be even. Then the tangents of O are the lines of a polar space $W_3(q)$ in $PG(3,q)$; see Hirschfeld [8]. Hence M is a spread of the image $Q(4,q)$ of $W_3(q)$ onto $Q^+(5,q)$; see also Payne and Thas [17]. ∎

Note added in proof. Let $\Sigma = \{\pi_i\}$ be an m-system of the classical polar space Δ. In the appropiate Grassmannian space G, Σ may be regarded as sets of points which are as far apart as possible. Also for each π in Σ, its perp-space in Δ defines a geometric hyperplane $H(\pi)$ of G which contains the Grassmann point $G(\pi)$ representing π. In turn this geometric hyperplane is known to arise from a projective hyperplane of the projective space of the $(m+1)$-exterior product into which the Grassmannian embedds. Since, for distinct π and π' in Σ, $G(\pi)$ is never contained in $H(\pi')$ recently discovered inequalities of Blokhuis and Moorehouse bounding p-ranks of point-hyperplane incidence matrices can be applied to give a bound on Σ. Here are two consequences: In characteristic 2, there can be no 1-systems of $Q^+(2s+1,q)$ for s at least 7, nor of $H(2s,q^2)$ with s at least 6.

References

1. L. Bader, W. M. Kantor, and G. Lunardon, Symplectic spreads from twisted fields, to appear.
2. A. Barlotti, Un 'estensione del teorema di Segre–Kustaanheimo, *Boll. Un. Mat. Ital.* **10** (1955), 96–98.
3. A. Blokhuis and E. Moorhouse, private communication, 1992.
4. A. E. Brouwer, private communication, 1981.
5. A. R. Calderbank and W. M. Kantor, The geometry of two-weight codes, *Bull. London Math. Soc.* **18** (1986), 97–122.
6. J. H. Conway, P. B. Kleidman, and R. A. Wilson, New families of ovoids in O_8^+, *Geom. Dedicata* **26** (1988), 157–170.
7. R. H. Dye, Partitions and their stabilizers for line complexes and quadrics, *Ann. Mat. Pura Appl.* **114** (1977), 173–194.
8. J. W. P. Hirschfeld, "Finite Projective Spaces of Three Dimensions," Oxford Univ. Press, Oxford, 1985.
9. J. W. P. Hirschfeld and J. A. Thas, "General Galois Geometries," Oxford Univ. Press, Oxford, 1991.

10. W. M. KANTOR, Ovoids and translation planes, *Canad. J. Math.* **34** (1982), 1195–1207.
11. W. M. KANTOR, Spreads, translation planes and Kerdock sets, I, *SIAM J. Algebra Discuss. Math.* **3** (1982), 151–165.
12. W. M. KANTOR, Spreads, translation planes and Kerdock sets, II, *SIAM J. Algebra Discuss. Math.* **3** (1982), 308–318.
13. H. LÜNEBURG, "Die Suzukigruppen und ihre Geometrien," Lecture Notes in Mathematics, Vol. 10, Springer-Verlag, Berlin, 1965.
14. G. E. MOORHOUSE, Root lattice constructions of ovoids, *in* "Finite Geometry and Combinatorics," Cambridge Univ. Press, Cambridge, 1993.
15. G. E. MOORHOUSE, Ovoids from the E_8 root lattice, *Geom. Dedicata*, to appear.
16. G. PANELLA, Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito, *Boll. Un. Mat. Ital.* **10** (1955), 507–513.
17. S. E. PAYNE AND J. A. THAS, "Finite Generalized Quadrangles," Research Notes in Mathematics, Vol. 110, Pitman, Boston/London/Melbourne, 1984.
18. E. E. SHULT, A sporadic ovoid in $\Omega^+(8,7)$, *Algebras Groups Geom.* **2** (1985), 495–513.
19. E. E. SHULT, Nonexistence of ovoids in $\Omega^+(10,3)$, *J. Combin. Theory Ser. A* **51** (1989), 250–257.
20. J. A. THAS, Ovoidal translation planes, *Arch. Math. (Basel)* **23** (1972), 110–112.
21. J. A. THAS, A combinatorial problem, *Geom. Dedicata* **1** (1973), 236–240.
22. J. A. THAS, Two infinite classes of perfect codes in metrically regular graphs, *J. Combin. Theory Ser. B* **23** (1977), 236–238.
23. J. A. THAS, Polar spaces, generalized hexagons and perfect codes, *J. Combin. Theory Ser. A* **29** (1980), 87–93.
24. J. A. THAS, Ovoids and spreads of finite classical polar spaces, *Geom. Dedicata* **10** (1981). 135–144.
25. J. A. THAS, Semi-partial geometries and spreads of classical polar spaces, *J. Combin. Theory Ser. A* **35** (1983), 58–66.
26. J. A. THAS, Interesting pointsets in generalized quadrangles and partial geometries, *Linear Algebra Appl.* **114/115** (1989), 103–131.
27. J. A. THAS, Old and new results on spreads and ovoids of finite classical polar spaces, *Ann. Discrete Math.* **52** (1992), 529–544.
28. J. A. THAS, Projective geometry over a finite field, *in* "Handbook of Incidence Geometry" (F. Buekenhout, Ed.), Chap. 7, North-Holland, Amsterdam, to appear.