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Invariant hypersurfaces for derivations in positive characteristic

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Abstract

Let A be an integral k-algebra of finite type over an algebraically closed field k of characteristic $p > 0$. Given a collection $\mathcal{D}$ of k-derivations on A, that we interpret as algebraic vector fields on $X = \mathrm{Spec}(A)$, we study the group spanned by the hypersurfaces $V(f)$ of X invariant under $\mathcal{D}$ modulo the rational first integrals of $\mathcal{D}$. We prove that this group is always a finite dimensional $\mathbb{F}_p$-vector space, and we give an estimate for its dimension. This is to be related to the results of Jouanolou and others on the number of hypersurfaces invariant under a foliation of codimension 1. As a application, given a k-algebra B between A^p and A, we show that the kernel of the pull-back morphism $\mathrm{Pic}(B) \to \mathrm{Pic}(A)$ is a finite $\mathbb{F}_p$-vector space. In particular, if A is a UFD, then the Picard group of B is finite.

MSC: 13N15; 14C20; 14R99

1. Introduction

Let A be an integral k-algebra of finite type over an algebraically closed field k of characteristic $p > 0$, and let $K(A)$ be its fraction field. Denote by X the spectrum of A, i.e. $X = \mathrm{Spec}(A)$. Let $\mathrm{Der}_k(A)$ be the set of k-derivations of A, or in other words the set of algebraic vector fields on X. Given a collection $\mathcal{D}$ of k-derivations on A, we would like to compare the set of hypersurfaces invariant under $\mathcal{D}$ with its set of rational first integrals, i.e. the rational functions on X annihilated by $\mathcal{D}$.

We begin with a few definitions and recall some results from the theory of foliations in positive characteristic (see [8] or [3]). The space $\mathrm{Der}_k(A)$ is provided with a Lie bracket defined by the rule $[D_1, D_2] = D_1D_2 - D_2D_1$. Moreover if D is a k-derivation on A, then $D^p = D \circ \cdots \circ D$ is again a k-derivation on A. This follows easily from the Leibniz formula with binomial coefficients for $D^p(fg)$, since $\mathrm{char}(k) = p$. *A foliation $\mathcal{F}$ on X* is a sub-A-module of $\mathrm{Der}_k(A)$, stable by Lie bracket and p-closed, that is:

- for any D, D' in $\mathcal{F}$, $[D, D'] = D \circ D' - D' \circ D$ belongs to $\mathcal{F}$,
- for any D in $\mathcal{F}$, $D^p = D \circ \cdots \circ D$ belongs to $\mathcal{F}$.

The definition of a foliation in characteristic zero is exactly the same without the p-closedness condition. The codimension of $\mathcal{F}$ is defined as $\mathrm{codim}(\mathcal{F}) = \dim(A) - \mathrm{rk}(\mathcal{F})$, where $\mathrm{rk}(\mathcal{F})$ is the rank of $\mathcal{F}$ as an A-module. For convenience, given a hypersurface H of X, denote by I_H the ideal of elements of A which vanish along H.

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Definition 1.1. Let $\mathcal{D}$ be a subset of $\mathrm{Der}_k(A)$. The hypersurface H is invariant by $\mathcal{D}$ if the ideal I_H is stable by all elements of $\mathcal{D}$, i.e. $D(I_H) \subseteq I_H$ for all $D \in \mathcal{D}$. The element f of $K(A)^*$ is a rational first integral if $D(f) = 0$ for all $D \in \mathcal{D}$.

Geometrically speaking, this means that all algebraic vector fields D in $\mathcal{D}$ are tangent to H. If the ground field is $\mathbb{C}$, Jouanolou proved that, under some conditions, a codimension 1 foliation has either finitely many invariant hypersurfaces, or has a nonconstant first integral (see [5]). This result was later improved by Brunella and Nicolau for complex codimension 1 foliations and for A-submodules $\mathcal{D}$ of $\mathrm{Der}_k(A)$ of corank 1 in positive characteristic (see [1]). More precisely, they proved that such an A-submodule with infinitely many irreducible invariant hypersurfaces must have a nontrivial first integral f, i.e. $f \notin K(A)^p$. The proof consists in showing first that $\mathcal{D}$ is a codimension 1 foliation, and then using the fact that foliations always have nontrivial first integrals in positive characteristic (see [8]).

However this result does not give any information about the relationship between invariant hypersurfaces and first integrals. Our purpose is to study this relationship when we restrict to hypersurfaces of the form $V(f)$, where f belongs to A. By analogy with the theory of foliations (see [6]), we give the following:

Definition 1.2. Let $\mathcal{D}$ be any collection of k-derivations on A. An element f of $K(A)^*$ is an algebraic solution of $\mathcal{D}$ if $D(f)/f$ belongs to A for any D in $\mathcal{D}$.

This definition extends the notion of invariant hypersurface in the following sense: if A is a UFD and $f \in A$ is irreducible, then f is an algebraic solution if and only if the ideal (f) is stable by $\mathcal{D}$, or in other words if $V(f)$ is invariant by $\mathcal{D}$. Note that any first integral is also an algebraic solution. Since

$$\frac{D(fg)}{fg} = \frac{D(f)}{f} + \frac{D(g)}{g}$$

for any $f, g \in K(A)^*$, the set of algebraic solutions (resp. first integrals) forms a multiplicative group. In order to compare the two notions, we introduce the quotient

$$\Pi(A, \mathcal{D}) = \frac{\{\text{Algebraic solutions of } \mathcal{D}\}}{\{\text{First integrals of } \mathcal{D}\}}.$$

Since $\Pi(A, \mathcal{D})$ is an abelian group, it is a $\mathbb{Z}$-module. But $\mathrm{char}(k) = p$, so $D(f^p) = 0$ for any $f \in K(A)$ and any $D \in \mathrm{Der}_k(A)$. Therefore $\Pi(A, \mathcal{D})$ is a $\mathbb{F}_p$-vector space, where $\mathbb{F}_p = \mathbb{Z}/p$. We are going to prove the following:

Theorem 1.3. *The $\mathbb{F}_p$-vector space $\Pi(A, \mathcal{D})$ is finite dimensional.*

In particular, the space of algebraic solutions differs from the first integrals by only a finite set. This theorem is to be related to a result of Pereira (see [9]) which asserts that, for $A = k[x_1, \ldots, x_n]$, every general algebraic vector field D has a nontrivial algebraic solution f, i.e. $f \in A - A^p$. In a sense, a general algebraic vector field in $k[x_1, \ldots, x_n]$ has nontrivial algebraic solutions, but not too many compared to its first integrals.

Using the arguments of the proof of Theorem 1.3, one can also derive an estimate for the dimension of $\Pi(A, \mathcal{D})$ if $\mathcal{D}$ is a foliation. Let deg be a degree function on $K(A)$ (see Section 3). Given a k-derivation D on A, we define the *degree* $\deg(D)$ of D as

$$\deg(D) = \sup_{f \in K(A)} \{\deg(D(f)) - \deg(f)\}.$$

For instance, if $A = k[x_1, \ldots, x_n]$ and deg stands for the standard homogeneous degree on A, then the degree of a k-derivation D is given by $\deg(D) = \sup\{\deg(D(x_i))\} - 1$. In this case, the degree $\deg(D)$ of any k-derivation D is finite. The boundedness of the degree for a k-derivation is analogous to the notion of continuity of a derivation with respect to a valuation (see [7]).

Definition 1.4. Let $\deg : K(A) \to \mathbb{Z} \cup \{-\infty\}$ be a degree such that $\deg(a) = 0$ for any element $a \in k^*$. Then deg is good if the following conditions hold:

- For any k-derivation D on A, the degree of D is finite.
- For any integer, the dimension $l(n)$ of $\{f \in A \mid \deg(f) \leq n\}$ over k is finite.

Corollary 1.5. *Let A be an integral k-algebra of finite type over an algebraically closed field k of characteristic p. Let* deg *be a good degree function on $K(A)$. Let $\mathcal{F}$ be a foliation on* $\mathrm{Spec}(A)$*, spanned by $D_1, \ldots, D_n$. Then we have*

$$\dim_{\mathbb{F}_p} \Pi(A, \mathcal{F}) \leq l(\deg(D_1)) + \cdots + l(\deg(D_n)).$$

We derive from Theorem 1.3 another result that has a priori nothing to do with foliations. Given two integral k-algebras A, B such that $A^p \subseteq B \subseteq A$, we would like to find a relationship between their Picard groups. Denote by $\mathcal{F}_B$ the maximal foliation on $\mathrm{Spec}(A)$ which vanishes on B, i.e. the set of k-derivations on A which vanish on B. If A and B are normal rings, then B is exactly the kernel of $\mathcal{F}_B$ (see [8]). Using this interpretation, we can prove the following:

Corollary 1.6. *Let A be a normal integral k-algebra of finite type over an algebraically closed field k of characteristic p. Let B be a normal subalgebra of A such that $A^p \subseteq B \subseteq A$. Then the kernel of the pull-back morphism $\pi : \mathrm{Pic}(B) \to \mathrm{Pic}(A)$, $M \mapsto M \otimes A$ is a finite $\mathbb{F}_p$-vector space, and we have*

$$\dim_{\mathbb{F}_p} \ker \pi \leq \dim_{\mathbb{F}_p} \Pi(A, \mathcal{F}_B).$$

In particular, if A is a UFD, then the Picard group of B is finite.

We end this paper with two examples. The first one illustrates the sharpness of the estimate given in Corollary 1.5, and the necessity of considering foliations in its formulation. We will use the second one to determine a Picard group.

2. Reduction to the case of foliations

Let A be an integral k-algebra of finite type, where k is algebraically closed of characteristic p. In this section, we are going to see how to restrict ourselves to finite collections of derivations satisfying some further conditions. For any collection $\mathcal{D}$ of $\mathrm{Der}_k(A)$, denote by $\mathcal{F}(\mathcal{D})$ *the smallest foliation containing* $\mathcal{D}$, i.e. the intersection of all foliations on X containing $\mathcal{D}$. By definition, it is well defined and unique.

Lemma 2.1. *Let A be an integral k-algebra of finite type, where* $\mathrm{char}(k) = p$*, and let $\mathcal{D}$ be a collection of k-derivations on A. Let $\{D_1, \ldots, D_n\}$ be a system of generators of $\mathcal{F}(\mathcal{D})$. Then $\mathcal{D}$ and $\{D_1, \ldots, D_n\}$ have exactly the same set of algebraic solutions (resp. first integrals). In particular $\Pi(A, \mathcal{D}) = \Pi(A, \{D_1, \ldots, D_n\})$.*

Proof. For any collection $\mathcal{D}_0$, denote by $\mathrm{AS}(\mathcal{D}_0)$ the group of algebraic solutions of $\mathcal{D}_0$, and by $\mathrm{FI}(\mathcal{D}_0)$ its group of first integrals. Our purpose is to show the equalities

$$\mathrm{AS}(\mathcal{D}) = \mathrm{AS}(\{D_1, \ldots, D_n\}) \quad \text{and} \quad \mathrm{FI}(\mathcal{D}) = \mathrm{FI}(\{D_1, \ldots, D_n\}).$$

We only need to prove the first one, since the proof for the second is entirely similar. So let f be an algebraic solution for $\{D_1, \ldots, D_n\}$. Then every element D of $\mathcal{D}$ is an A-linear combination of $D_1, \ldots, D_n$. Since $D_i(f)/f$ belongs to A for any i, $D(f)/f$ belongs to A. Since this holds for any $D \in \mathcal{D}$, f is an algebraic solution of $\mathcal{D}$ and we have

$$\mathrm{AS}(\{D_1, \ldots, D_n\}) \subseteq \mathrm{AS}(\mathcal{D}).$$

To prove the equality of the two sets, it suffices to show that every algebraic solution of $\mathcal{D}$ is also an algebraic solution of $\{D_1, \ldots, D_n\}$. So fix any algebraic solution f of $\mathcal{D}$. Consider the set $M(f)$ of k-derivations on A such that $D(f)/f$ belongs to A. By definition $M(f)$ is an A-submodule of $\mathrm{Der}_k(A)$ containing $\mathcal{D}$, and f is an algebraic solution of $M(f)$. We prove that $M(f)$ is stable by Lie bracket and p-closed. First, let D_1, D_2 be any elements of $M(f)$. Then there exist some elements R_1, R_2 of A such that $D_1(f) = R_1 f$ and $D_2(f) = R_2 f$, and we have

$$[D_1, D_2](f) = \{D_1(R_2) - D_2(R_1)\} f.$$

So $[D_1, D_2]$ belongs to $M(f)$. Second, let D be any element of $M(f)$. Then there exists an element R of A such that $D(f) = Rf$. Consider the sequence $\{R_n\}_{n>0}$ in A constructed by induction as follows:

$$R_1 = R \quad \text{and} \quad \forall n > 0, \quad R_{n+1} = D(R_n) + R R_n.$$

By induction, we have $D^n(f) = R_n f$ for any $n > 0$. In particular $D^p(f)/f$ belongs to A and D^p belongs to $M(f)$. Therefore $M(f)$ is a foliation containing $\mathcal{D}$. By definition of $\mathcal{F}(\mathcal{D})$, M contains $\mathcal{F}(\mathcal{D})$, and hence its generators $D_1, \ldots, D_n$. Since f is an algebraic solution of $M(f)$, f is also an algebraic solution of $\{D_1, \ldots, D_n\}$, and the result follows. ∎

3. A lemma on degree functions

In this section, we are going to establish a lemma that is crucial for the proof of Theorem 1.3. This lemma asserts that every integral k-algebra A of finite type carries a finite set of degree functions enjoying some nice properties. In fact, this result holds for any field k, but for convenience we will only prove it for when k is algebraically closed. In what follows, all degree functions that we consider are maps $\deg : K(A) \to \mathbb{Z} \cup \{-\infty\}$ satisfying the usual axioms, with the additional condition that $\deg(f) = 0$ for any $f \in k^*$.

Definition 3.1. Let A be an integral k-algebra, $K(A)$ its fraction field and deg a degree on $K(A)$. A k-derivation D on A is bounded for deg if $\deg(D)$ is finite.

Lemma 3.2. *Let A be an integral k-algebra of finite type over an algebraically closed field k. Then there exist a finite set $\{\deg_1, \ldots, \deg_r\}$ of degree functions on $K(A)$ satisfying the following conditions:*

- *for any $i = 1, \ldots, r$, every k-derivation D on A is bounded for $\deg_i$,*
- *for any integers $n_1, \ldots, n_r$, $\mathcal{L}(n_1, \ldots, n_r) = \{f \in A \mid \forall i = 1, \ldots, r,\ \deg_i(f) \le n_i\}$ is a k-vector space of finite dimension.*

Proof. Let X be the affine variety $\operatorname{Spec}(A)$. This variety is embedded in some k^n. Let X' be the projective closure of X in $\mathbb{P}^n(k)$, and denote by $\mathcal{X}$ its normalization. By construction, the variety $\mathcal{X}$ is projective, normal and birational to X. Let H' be the hyperplane at infinity in $\mathbb{P}^n(k)$ such that $X = X' - H'$. If $\varphi : \mathcal{X} \to X'$ denotes the normalization morphism, set $H = \varphi^{-1}(H')$. Then H is a finite union of prime Weyl divisors $Z_1, \ldots, Z_r$. Let ord_{Z_i} be the order along Z_i. Since $\mathcal{X}$ is birational to X, every element f of $K(A)$ can be considered as a rational function on $\mathcal{X}$. We set

$$\deg_i(f) = -\operatorname{ord}_{Z_i}(f).$$

Since ord_{Z_i} is a valuation, $\deg_i$ defines a degree function on $K(A)$. We are going to show that these degrees enjoy all the conditions of the lemma.

First step: Given any k-derivation D on A, we are going to prove that D is bounded for $\deg_i$. Consider an affine open set U_i in $\mathcal{X}$ such that $Z_i \cap U_i \neq \emptyset$. Let $x_1, \ldots, x_n$ be a set of generators of $\mathcal{O}_{U_i}$. Since the fraction field of $\mathcal{O}_{U_i}$ is equal to $K(A)$, there exist some elements $\alpha_1, \ldots, \alpha_n$ of $K(\mathcal{O}_{U_i})$ such that $D(x_j) = \alpha_j$. Write $\alpha_j = a_j/b_j$, where a_j, b_j belong to $\mathcal{O}_{U_i}$, and set $B = b_1 \cdots b_n$. By construction, the k-derivation BD maps $\mathcal{O}_{U_i}$ into itself, and in particular we have

$$BD(\mathcal{O}_{U_i, Z_i \cap U_i}) \subseteq \mathcal{O}_{U_i, Z_i \cap U_i}.$$

Since U_i is normal, $\mathcal{O}_{U_i, Z_i \cap U_i}$ is a discrete valuation ring (see for instance [4]). Let h_i be a generator of the unique maximal ideal of $\mathcal{O}_{U_i, Z_i \cap U_i}$, and set $\delta_i = \operatorname{ord}_{Z_i}(B)$. By construction, any rational map f on X with $\deg_i(f) = r$ can be written as

$$f = \frac{g}{h_i^r}$$

where g is invertible in $\mathcal{O}_{U_i, Z_i \cap U_i}$. By derivation and multiplication, we get

$$h_i^{r+1} BD(f) = h_i BD(g) - rBD(h_i)g.$$

So $h_i^{r+1} BD(f)$ belongs to $\mathcal{O}_{U_i, Z_i \cap U_i}$, its order along Z_i is nonnegative and we find

$$\deg_i(f) + 1 - \delta_i - \deg_i(D(f)) \ge 0.$$

Since this holds for any $f \in K(A)$, we obtain that $\deg_i(D) \le 1 - \delta_i$, and hence $\deg_i(D)$ is finite.

Second step: We are going to prove the second assertion of the lemma. Fix some integers $n_1, \ldots, n_r$. By definition, the hyperplane divisor H is a linear combination of the Z_i with positive coefficients. So there exists a positive integer n such that $nH \ge n_1 Z_1 + \cdots + n_r Z_r$. Let f be any element of $\mathcal{L}(n_1, \ldots, n_r)$, viewed as a regular function on $\mathcal{X} - H$. If $\operatorname{div}(f)$ denotes its Weyl divisor on $\mathcal{X}$, we have the relation

$$\operatorname{div}(f) + nH \ge \operatorname{div}(f) + n_1 Z_1 + \cdots + n_r Z_r \ge 0.$$

In particular, if $k(\mathcal{X})$ denotes the fraction field of $\mathcal{X}$, we have the inclusion

$$\mathcal{L}(n_1, \dots, n_r) \subseteq \{f \in k(\mathcal{X}) \mid \mathrm{div}(f) + nH \geq 0\}.$$

Since H is a hyperplane section, it is locally principal and it defines an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$. Moreover, the locally principal divisor $\mathrm{div}(f) + nH$ on $\mathcal{X}$ is effective if and only if $\mathrm{div}(f) + nH \geq 0$. Indeed since $\mathcal{X}$ is normal, a rational function is regular on an open set U of $\mathcal{X}$ if and only if it has no hypersurface of poles on U. So the right-hand side of the latter inclusion corresponds to the space $\Gamma(\mathcal{X}, \mathcal{L}^{-n})$ of global sections. Since $\mathcal{X}$ is projective, this space is finite dimensional by Serre's theorem (see [10]). ■

4. A lemma on polynomials

In this section, we are going to establish a lemma on the shape of some polynomials of low degree having roots in $\mathbb{F}_p^s$. This result can be stated as follows:

Lemma 4.1. *Let A be an integral k-algebra where k is a field of characteristic p. Let $P(t_1, \dots, t_s)$ be an element of $A[t_1, \dots, t_s]$, of degree $\leq p$ in $t_1, \dots, t_s$, such that $P(z_1, \dots, z_s) = 0$ for any $(z_1, \dots, z_s) \in \mathbb{F}_p^s$. Then there exist some unique elements $a_1, \dots, a_s$ of A such that*

$$P(t_1, \dots, t_s) = a_1(t_1^p - t_1) + \cdots + a_s(t_s^p - t_s).$$

Proof. By induction on $s \geq 1$, the case $s = 1$ being clear. Indeed if $P(t)$ vanishes at any element z of $\mathbb{F}_p$, then $P(t)$ is divisible by $(t^p - t)$. Since P has degree $\leq p$ in t, $P(t) = a(t^p - t)$ for some element $a \in A$, and this element is unique. So assume the property holds to the order $s-1$, and let $P(t_1, \dots, t_s)$ be an element of $A[t_1, \dots, t_s]$, of degree $\leq p$ in $t_1, \dots, t_s$, such that $P(z_1, \dots, z_s) = 0$ for any $(z_1, \dots, z_s) \in \mathbb{F}_p^s$. Denote by a_s the coefficient of t_s^p in the expression of P, and set

$$Q(t_1, \dots, t_s) = P(t_1, \dots, t_s) - a_s(t_s^p - t_s).$$

By definition, the polynomial Q has degree $\leq p$ in $t_1, \dots, t_s$ and vanishes at any point of $\mathbb{F}_p^s$. Moreover its degree with respect to the variable t_s is $< p$. Therefore it has an expansion of the form

$$Q(t_1, \dots, t_s) = Q_0(t_1, \dots, t_{s-1}) + \cdots + Q_{p-1}(t_1, \dots, t_{s-1})t_s^{p-1}$$

where each polynomial Q_i has degree $\leq p - i$ in $t_1, \dots, t_{s-1}$. For any fixed element $z = (z_1, \dots, z_{s-1})$ of $\mathbb{F}_p^{s-1}$, consider the polynomial

$$Q_z(t_s) = Q(z_1, \dots, z_{s-1}, t_s).$$

By construction Q_z vanishes at any point z_s of $\mathbb{F}_p$, and has degree $< p$ in t_s. So it is of the form $a(t_s^p - t_s)$, where a belongs to A. Since Q_z has degree $< p$, it is the zero polynomial. In particular, for any $z = (z_1, \dots, z_{s-1})$ of $\mathbb{F}_p^{s-1}$, we have

$$Q_0(z_1, \dots, z_{s-1}) = Q_1(z_1, \dots, z_{s-1}) = \cdots = Q_{p-1}(z_1, \dots, z_{s-1}) = 0.$$

For any index $i \geq 0$, the polynomial Q_i vanishes on $\mathbb{F}_p^{s-1}$. Since its degree in $t_1, \dots, t_{s-1}$ is $\leq (p-i)$, the polynomial Q_i is by induction an A-linear combination of $(t_1^p - t_1), \dots, (t_{s-1}^p - t_{s-1})$. But for any $i > 0$, Q_i has degree $\leq p - i < p$, so it is the zero polynomial. In particular, we find

$$Q(t_1, \dots, t_s) = Q_0(t_1, \dots, t_{s-1}) = a_1(t_1^p - t_1) + \cdots + a_{s-1}(t_{s-1}^p - t_{s-1})$$

for some elements $a_1, \dots, a_{s-1}$ of A. By construction of Q, this yields

$$P(t_1, \dots, t_s) = a_1(t_1^p - t_1) + \cdots + a_s(t_s^p - t_s).$$

The uniqueness of $a_1, \dots, a_s$ is obvious. ■

5. Proof of Theorem 1.3 and Corollary 1.5

Let A be an integral k-algebra of finite type, where k is algebraically closed of characteristic p. Let $\mathcal{D}$ be a collection of k-derivations on A. We fix a finite set $\{D_1, \ldots, D_n\}$ of generators for the foliation $\mathcal{F}(\mathcal{D})$. By Lemma 2.1, we know that $\Pi(A, \mathcal{D}) = \Pi(A, \{D_1, \ldots, D_n\})$, and that the following group morphism:

$$L : \Pi(A, \mathcal{D}) \longrightarrow A^n, \qquad f \longmapsto \left(\frac{D_1(f)}{f}, \ldots, \frac{D_n(f)}{f}\right)$$

is well defined and injective. In particular, its image I is isomorphic to $\Pi(A, \mathcal{D})$. In order to establish Theorem 1.3 and Corollary 1.5, we only need to show that I is a finite dimensional $\mathbb{F}_p$-vector space, and estimate its dimension. This is exactly what we will do in the following subsections.

5.1. *Dimension of* $\mathrm{Vect}_k(I)$

Let $\mathrm{Vect}_k(I)$ be the vector space over k spanned by I in A^n. We prove that $\mathrm{Vect}_k(I)$ *is finite dimensional*, as follows. Let $\{\deg_1, \ldots, \deg_r\}$ be a collection of degrees on $K(A)$ satisfying the conditions of Lemma 3.2. Then there exists some constant $\{d_{i,j}\}$ such that, for any $f \in K(A)$,

$$\deg_i(D_j(f)) \leq d_{i,j} + \deg_i(f).$$

Let f be any algebraic solution of $\mathcal{F}$, and set $L_j(f) = D_j(f)/f$ for any j. Then $L_j(f)$ belongs to A for any j, and we have $\deg_i(L_j(f)) \leq d_{i,j}$ for any i, j. In particular, we obtain the inclusion

$$I \subseteq \bigoplus_{j=1}^{n} \mathcal{L}(d_{1,j}, \ldots, d_{r,j}).$$

By Lemma 3.2, all the components of this sum are finite dimensional over k. Therefore I spans a finite dimensional k-vector space.

5.2. *Associated polynomials*

By the previous subsection, there exist some algebraic solutions $f_1, \ldots, f_s$ such that the elements $L(f_1), \ldots, L(f_s)$ form a basis of $\mathrm{Vect}_k(I)$. *We fix these algebraic solutions from now on.* By definition, given an algebraic solution f, the element $L(f)$ is a k-linear combination of $L(f_1), \ldots, L(f_s)$. The problem is, we do not know a priori what are the possible coefficients of this linear combination. In this subsection, we are going to construct some polynomials in s variables with the following remarkable property: *these polynomials vanish at every point* $(z_1, \ldots, z_s)$ *such that* $z_1 L(f_1) + \cdots + z_s L(f_s)$ *is equal to some* $L(f)$, *where* f *is an algebraic solution.* The construction proceeds as follows. Since $D_1, \ldots, D_n$ span a foliation, there exist some elements $a_{i,j}$ of A such that, for any $i = 1, \ldots, n$,

$$D_i^p = \sum_{j=1}^{n} a_{i,j} D_j.$$

We fix these elements $a_{i,j}$, and extend the action of D_i to the ring $A[t_1, \ldots, t_s]$ by setting $D_i(t_\alpha) = 0$ for any $\alpha \in \{1, \ldots, s\}$. For any $g \in A[t_1, \ldots, t_s]$ and any $i = 1, \ldots, n$, we introduce the k-linear operator:

$$T_{i,g} : A[t_1, \ldots, t_s] \longrightarrow A[t_1, \ldots, t_s], \qquad f \longmapsto D_i(f) + gf.$$

For convenience, we set $L_i(f) = D_i(f)/f$. Note that $L(f) = (L_1(f), \ldots, L_n(f))$. For any $i = 1, \ldots, n$, consider the following element P_i of $A[t_1, \ldots, t_s]$:

$$P_i(t_1, \ldots, t_s) = \left(T_{i, \sum_{\alpha=1}^{s} t_\alpha L_i(f_\alpha)}\right)^{p-1} \left(\sum_{\alpha=1}^{s} t_\alpha L_i(f_\alpha)\right) - \sum_{j=1}^{n} a_{i,j} \left(\sum_{\alpha=1}^{s} t_\alpha L_j(f_\alpha)\right).$$

Definition 5.1. $P_1, \ldots, P_n$ are the polynomials associated with $\{D_1, \ldots, D_n; f_1, \ldots, f_s\}$.

Lemma 5.2. *Let f be a nonzero element of $K(A)$. If we set $g = D_i(f)/f$, then we have $D_i^p(f) = (T_{i,g})^{p-1}(g)f$.*

Proof. It suffices to prove that $D_i^N(f) = (T_{i,g})^{N-1}(g)f$ for any $N > 0$. We do it by induction on $N > 0$, the case $N = 1$ being clear by construction. Assume the assertion holds to the order $N - 1$. Then we have to the order N

$$D_i^N(f) = D_i(D_i^{N-1}(f)) = D_i((T_{i,g})^{N-2}(g)f) = D_i((T_{i,g})^{N-2}(g))f + (T_{i,g})^{N-2}(g)D_i(f).$$

By definition of g, we have $D_i(f) = gf$ and this implies

$$D_i^N(f) = D_i((T_{i,g})^{N-2}(g))f + (T_{i,g})^{N-2}(g)gf = (T_{i,g} \circ (T_{i,g})^{N-2}(g))f = (T_{i,g})^{N-1}(g)f. \quad \blacksquare$$

Lemma 5.3. *Let f be any algebraic solution of $\{D_1, \ldots, D_n\}$. Then for any $i = 1, \ldots, n$, $L_1(f), \ldots, L_n(f)$ satisfy the relation $(T_{i,L_i(f)})^{p-1}(L_i(f)) = \sum_{j=1}^n a_{i,j} L_j(f)$.*

Proof. Let f be an algebraic solution of $D_1, \ldots, D_n$. If $g = D_i(f)/f = L_i(f)$, then we find by applying Lemma 5.2 to f

$$D_i^p(f) = (T_{i,L_i(f)})^{p-1}(L_i(f))f.$$

Since $D_i^p = \sum_j a_{i,j} D_j$ for any $i = 1, \ldots, n$ by construction, this implies

$$\frac{D_i^p(f)}{f} = (T_{i,L_i(f)})^{p-1}(L_i(f)) = \sum_{j=1}^n a_{i,j} \frac{D_j(f)}{f} = \sum_{j=1}^n a_{i,j} L_j(f). \quad \blacksquare$$

Lemma 5.4. *Let f be any algebraic solution of $\{D_1, \ldots, D_n\}$. Let $z_1, \ldots, z_s$ be some elements of k such that $L(f) = z_1 L(f_1) + \cdots + z_s L(f_s)$. Then for any $i = 1, \ldots, n$, we have $P_i(z_1, \ldots, z_s) = 0$. Moreover $P_1, \ldots, P_n$ vanish at any point $(z_1, \ldots, z_s)$ of $\mathbb{F}_p^s$.*

Proof. Let f be an algebraic solution of $\{D_1, \ldots, D_n\}$. If $L(f) = z_1 L(f_1) + \cdots + z_s L(f_s)$, then we have $L_i(f) = z_1 L_i(f_1) + \cdots + z_s L_i(f_s)$ for any index i. By definition of the associated polynomials and by Lemma 5.3, we get

$$\begin{aligned} P_i(z_1, \ldots, z_s) &= \left(T_{i, \sum_{\alpha=1}^s z_\alpha L_i(f_\alpha)}\right)^{p-1} \left(\sum_{\alpha=1}^s z_\alpha L_i(f_\alpha)\right) - \sum_{j=1}^n a_{i,j} \left(\sum_{\alpha=1}^s z_\alpha L_j(f_\alpha)\right) \\ &= (T_{i,L_i(f)})^{p-1}(L_i(f)) - \sum_{j=1}^n a_{i,j} L_j(f) = 0 \end{aligned}$$

and the first assertion follows. For the second, let $z_1, \ldots, z_s$ be some elements of $\mathbb{F}_p$. Let $y_1, \ldots, y_s$ be some integers whose classes modulo p are equal to $z_1, \ldots, z_s$ respectively. Then the element $f = f_1^{y_1} \cdots f_s^{y_s}$ is also an algebraic solution of $\{D_1, \ldots, D_n\}$, and $L(f) = z_1 L(f_1) + \cdots + z_s L(f_s)$. Therefore $P_i(z_1, \ldots, z_s) = 0$ for any $i = 1, \ldots, n$ by the first assertion. $\blacksquare$

Lemma 5.5. *For any $i = 1, \ldots, n$, we have $P_i(t_1, \ldots, t_s) = \sum_{\alpha=1}^s L_i(f_\alpha)^p (t_\alpha^p - t_\alpha)$.*

Proof. For any index i, it is easy to check by induction on $N > 0$ that the expression

$$\left(T_{i, \sum_{\alpha=1}^s t_\alpha L_i(f_\alpha)}\right)^{N-1} \left(\sum_{\alpha=1}^s t_\alpha L_i(f_\alpha)\right)$$

is a polynomial in $A[t_1, \ldots, t_s]$ of degree N in $t_1, \ldots, t_s$, whose leading term is equal to

$$\left(\sum_{\alpha=1}^s t_\alpha L_i(f_\alpha)\right)^N.$$

In particular, the polynomial P_i has degree p in $t_1, \ldots, t_s$, and its leading term is given by

$$\left(\sum_{\alpha=1}^{s} t_\alpha L_i(f_\alpha)\right)^p = \sum_{\alpha=1}^{s} t_\alpha^p L_i(f_\alpha)^p.$$

Moreover P_i vanishes at all points of $\mathbb{F}_p^s$ by Lemma 5.4. By Lemma 4.1, there exist some elements $a_{i,1}, \ldots, a_{i,s}$ of A such that

$$P_i(t_1, \ldots, t_s) = a_{i,1}(t_1^p - t_1) + \cdots + a_{i,s}(t_s^p - t_s).$$

Since the leading term of P is equal to $\sum_\alpha L_i(f_\alpha)^p t_\alpha^p$, we have $L_i(f_\alpha)^p = a_{i,\alpha}$ for any $\alpha = 1, \ldots, s$ and the result follows. ■

5.3. *Proof of Theorem 1.3*

Let $f_1, \ldots, f_s$ be some algebraic solutions of $\{D_1, \ldots, D_n\}$ such that $L(f_1), \ldots, L(f_s)$ form a basis of $\mathrm{Vect}_k(I)$. We are going to show that $L(f_1), \ldots, L(f_s)$ form a basis of I over $\mathbb{F}_p$. Since $\Pi(A, \mathcal{D})$ is isomorphic to I, Theorem 1.3 will follow. By construction, these elements are linearly independent over k (and hence over $\mathbb{F}_p$). So it suffices to prove that they span I over $\mathbb{F}_p$. Let x be any element of I. By definition of I, there exists an algebraic solution f of $\{D_1, \ldots, D_n\}$ such that $x = L(f)$. Since the $L(f_j)$ form a basis of the k-vector space spanned by I, there exist some elements $z_1, \ldots, z_s$ of k such that $L(f) = z_1 L(f_1) + \cdots + z_s L(f_s)$. We only need to check that $z_1, \ldots, z_s$ belong to $\mathbb{F}_p$. By Lemmas 5.4 and 5.5, we have for any $i = 1, \ldots, n$

$$P_i(z_1, \ldots, z_s) = \sum_{\alpha=1}^{s} L_i(f_\alpha)^p (z_\alpha^p - z_\alpha) = 0.$$

But k is algebraically closed. So for any $\alpha = 1, \ldots, s$, there exists an element y_α of k such that $y_\alpha^p = z_\alpha^p - z_\alpha$. Since $\mathrm{char}(k) = p$, we get for any $i = 1, \ldots, n$

$$\sum_{\alpha=1}^{s} y_\alpha^p L_i(f_\alpha)^p = \left(\sum_{\alpha=1}^{s} y_\alpha L_i(f_\alpha)\right)^p = 0.$$

Since A is an integral k-algebra, it has no nilpotent elements and $\sum_{\alpha=1}^{s} y_\alpha L_i(f_\alpha) = 0$ for any index i. But this is equivalent to writing

$$\sum_{\alpha=1}^{s} y_\alpha L(f_\alpha) = 0.$$

Since $L(f_1), \ldots, L(f_s)$ are linearly independent over k, we get $y_1 = \cdots = y_s = 0$. Therefore $z_\alpha^p - z_\alpha = 0$ for any $\alpha = 1, \ldots, s$, every z_α belongs to $\mathbb{F}_p$ and Theorem 1.3 is proved.

5.4. *Proof of Corollary 1.5*

Let A be a k-algebra of finite type over an algebraically closed field k of characteristic p. Let $\mathcal{F}$ be a foliation on $\mathrm{Spec}(A)$, and denote by $D_1, \ldots, D_n$ a set of generators of $\mathcal{F}$ as an A-module. We follow the notation of the previous subsection with $\mathcal{D} = \mathcal{F}$. Let deg be a good degree function on A (see the introduction). For any integer n, let $\mathcal{L}(n)$ be the k-vector space of the elements f of A such that $\deg(f) \leq n$. Let L stand for the morphism defined at the beginning of this section for $D_1, \ldots, D_n$. Then we have the inclusion

$$I = L(\Pi(A, \mathcal{F})) \subseteq \mathcal{L}(\deg(D_1)) \times \cdots \times \mathcal{L}(\deg(D_n))$$

by definition of the degree of a k-derivation. In particular, we find

$$\dim_k \mathrm{Vect}_k(I) \leq l(\deg(D_1)) + \cdots + l(\deg(D_n)).$$

Let $f_1, \ldots, f_s$ be some algebraic solutions of $\mathcal{F}$ such that $L(f_1), \ldots, L(f_s)$ form a basis over k of $\mathrm{Vect}_k(I)$. By the arguments of the previous subsection, $L(f_1), \ldots, L(f_s)$ also form a basis of I over $\mathbb{F}_p$. In particular we have

$$\dim_{\mathbb{F}_p} I = \dim_k \mathrm{Vect}_k(I).$$

Since L is an isomorphism between $\Pi(A, \mathcal{F})$ and I, we obtain the inequality

$$\dim_{\mathbb{F}_p} \Pi(A, \mathcal{F}) \leq l(\deg(D_1)) + \cdots + l(\deg(D_n))$$

which is exactly the assertion of Corollary 1.5.

6. Proof of Corollary 1.6

Let A be an integral k-algebra of finite type over an algebraically closed field k of characteristic $p > 0$. Let B be a subalgebra of A such that $A^p \subseteq B \subseteq A$. We assume that A and B are normal. Consider the pull-back morphism π defined by

$$\pi : \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(A), \qquad M \longmapsto M \otimes_B A.$$

Since A, B are normal, there exists a foliation $\mathcal{F}_B$ on $\mathrm{Spec}(A)$ such that $B = \bigcap_{D \in \mathcal{F}_B} \ker D$ (see [8]). Denote by G *the group of algebraic solutions of $\mathcal{F}_B$ modulo the subgroup spanned by the invertible elements of A and the first integrals of $\mathcal{F}_B$*. Note that every invertible element of A is an algebraic solution. By construction, we have a surjective morphism

$$F : \Pi(A, \mathcal{F}_B) \longrightarrow G.$$

Since $\Pi(A, \mathcal{F}_B)$ is finite dimensional over $\mathbb{F}_p$ by Theorem 1.3, G is also finite dimensional. We are going to construct an injective group morphism:

$$\theta : \ker \pi \longrightarrow G.$$

This will imply that $\ker \pi$ is finite dimensional over $\mathbb{F}_p$, and give the estimate

$$\dim_{\mathbb{F}_p} \ker \pi \leq \dim_{\mathbb{F}_p} G \leq \dim_{\mathbb{F}_p} \Pi(A, \mathcal{F}_B)$$

which is exactly the result given by Corollary 1.6.

The construction proceeds as follows. Let M' be a finitely generated, locally free B-submodule of $K(B)$. If $M' \otimes A \simeq A$, then $A.M' \simeq A$ and there exists an element f' of $K(A)$ such that $A.M' = A\{f'\}$. Since $K(B)$ is annihilated by $\mathcal{F}_B$, $D(M') = 0$ for any $D \in \mathcal{F}_B$ and the A-module $A.M'$ is stable by $\mathcal{F}_B$. In particular, for any element D of $\mathcal{F}_B$, $D(f')$ belongs to $A\{f'\}$ and f' is an algebraic solution of $\mathcal{F}_B$. Note that f' is uniquely determined up to multiplication by an element of A^*. If $\mathcal{E}$ denotes the set of finitely generated, locally free B-submodules M' of $K(B)$ such that $M' \otimes A \simeq A$, then we have a well-defined correspondence:

$$\Theta : \mathcal{E} \longrightarrow G, \qquad M' \longmapsto [f'].$$

Let M'' be another finitely generated, locally free B-submodule of $K(B)$. If $M'' \simeq M'$, then the isomorphism from M' to M'' is induced by the multiplication by an element g of $K(B)^*$, i.e. $M'' = g.M'$. Since $A.M'' = A\{f''\}$, we have

$$A.M'' = A\{f''\} = g.A.M' = A\{gf'\}$$

and f''/gf' is an invertible element of A. So the class $[f']$ of f' in G only depends on the isomorphism class of M'. Since every invertible sheaf M on $\mathrm{Spec}(B)$ can be represented by a finitely generated, locally free B-submodule M' of $K(B)$, Θ induces a map:

$$\theta : \ker \pi \longrightarrow G, \qquad [M'] \longmapsto [f'].$$

Lemma 6.1. *The map θ is a group morphism.*

Proof. Let M', M'' be two finitely generated, locally free B-submodules of $K(B)$ such that $M' \otimes A \simeq A$ and $M'' \otimes A \simeq A$. Let $M'M''$ be the B-submodule of $K(B)$ spanned by all products $x'x''$, where $x' \in M'$ and $x'' \in M''$. Since $M' \otimes A$ and $M'' \otimes A$ are trivial, there exist some elements f', f'' of $K(A)$ such that $A.M' = A\{f'\}$ and $A.M'' = A\{f''\}$. But then we have

$$A.M'M'' = (A.M')(A.M'') = A\{f'\}.A\{f''\} = A\{f'f''\}$$

and the A-module $A.M'M''$ is spanned by $f'f''$. Since the module $M'M''$ represents the invertible sheaf $M' \otimes M''$ in $\mathrm{Pic}(B)$, we obtain

$$\theta(M' \otimes M'') = \theta(M'M'') = [f'f''] = [f'][f''] = \theta(M')\theta(M''). \quad \blacksquare$$

Lemma 6.2. *The morphism θ is injective.*

Proof. Let M' be a finitely generated, locally free B-submodule of $K(B)$. Assume that $M' \otimes A \simeq A$ and that $[f'] = 0$. Then there exists a rational first integral g of $\mathcal{F}_B$, and an invertible element h of A, such that $A.M' = A\{gh\} = A\{g\}$. Since $D(g) = 0$ for any $D \in \mathcal{F}_B$, g belongs to $K(B)$. First we show that $M' \subseteq B\{g\}$. Let x be any element of M'. Since x belongs to $A.M'$, there exists an element a of A such that $x = a.g$. Since $D(x) = D(g) = 0$ for any $D \in \mathcal{F}_B$, $D(a) = 0$ for any $D \in \mathcal{F}_B$ and a belongs to B. In particular, x belongs to $B\{g\}$ and we have

$$M' \subseteq B\{g\}.$$

Second we show the equality $M' = B\{g\}$. Since M' and $B\{g\}$ are finite modules over a noetherian ring, it suffices to prove that $M'_{\mathcal{M}} = B_{\mathcal{M}}\{g\}$ for any maximal ideal $\mathcal{M}$ of B. Let $\mathcal{M}$ be a maximal ideal of B. Since $A^p \subseteq B \subseteq A$, A is integral over B. By the Cohen going-up theorem (see [2]), there exists a maximal ideal $\mathcal{M}_A$ of A such that $\mathcal{M}_A \cap B = \mathcal{M}$. Since M' is locally free of rank 1, there exists an element x of M' such that $M'_{\mathcal{M}} = B_{\mathcal{M}}\{x\}$. But then we find

$$A_{\mathcal{M}_A} M' = A_{\mathcal{M}_A} M'_{\mathcal{M}} = A_{\mathcal{M}_A} B_{\mathcal{M}}\{x\} = A_{\mathcal{M}_A}\{x\}.$$

On the other hand, we have

$$A_{\mathcal{M}_A} M' = (A.M')_{\mathcal{M}_A} = A_{\mathcal{M}_A}\{g\}.$$

In particular, the fraction $a = x/g$ is an invertible element of $A_{\mathcal{M}_A}$. Since x and g belong to $K(B)$, a and a^{-1} belong to $K(B)$. But $A^p \subseteq B \subseteq A$, so $A_{\mathcal{M}_A}$ is a finite $B_{\mathcal{M}}$-module. In particular, a and a^{-1} are integral over $B_{\mathcal{M}}$. Since B is normal, $B_{\mathcal{M}}$ is also normal and a, a^{-1} belong to $B_{\mathcal{M}}$. Therefore a is an invertible element of $B_{\mathcal{M}}$. Since $M'_{\mathcal{M}} \subseteq B_{\mathcal{M}}\{g\}$, we obtain the equality

$$M'_{\mathcal{M}} = B_{\mathcal{M}}\{x\} = B_{\mathcal{M}}\{g\}.$$

Since this holds for any maximal ideal $\mathcal{M}$, we have $M' = B\{g\}$. In particular, the invertible sheaf M represented by M' is trivial, and injectivity follows. $\blacksquare$

7. Two examples

7.1. Computation of $\Pi(A, \mathcal{F})$

Let A be the k-algebra $k[x, y]$, where k is algebraically closed of characteristic p. Note that A is normal and $A^* = k^*$. Let deg be the standard homogeneous degree on A. Given an element t of k, consider the following k-derivation D_t on A:

$$D_t = x\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}.$$

By an easy computation, we find $(D_t)^p = D_{t^p}$. Then two cases may occur.
First case: t belongs to $\mathbb{F}_p$.

Then the foliation $\mathcal{F}(D_t)$ is equal to $A\{D_t\}$. Since D_t has degree zero, the dimension of $\Pi(A, \mathcal{F}(D_t))$ is bounded by $l(0) = 1$ by Corollary 1.5. Since $D_t(x) = x$, x is an algebraic solution whose class in $\Pi(A, \mathcal{F}(D_t))$ is nonzero. Therefore,

$$\dim_{\mathbb{F}_p} \Pi(A, \{D_t\}) = \dim_{\mathbb{F}_p} \Pi(A, \mathcal{F}(D_t)) = 1.$$

Second case: t does not belong to $\mathbb{F}_p$.

Then $(D_t)^p$ and D_t are A-linearly independent, and the foliation $\mathcal{F}(D_t)$ contains the A-module $A\{x\partial/\partial x, y\partial/\partial y\}$. Since this latter is p-closed, stable by Lie bracket and contains D_t, we have $\mathcal{F}(D_t) = A\{x\partial/\partial x, y\partial/\partial y\}$. So $\mathcal{F}(D_t)$ is spanned by two derivations of degree 0. By Corollary 1.5, the dimension of $\Pi(A, \mathcal{F}(D_t))$ is bounded by $l(0) + l(0) = 2$. Consider now the polynomials x and y. Since $D_t(x)/x = 1$, $D_t(y)/y = t$ and $t \notin \mathbb{F}_p$, x and y are $\mathbb{F}_p$-linearly independent in $\Pi(A, \mathcal{F}(D_t))$. Therefore,

$$\dim_{\mathbb{F}_p} \Pi(A, \{D_t\}) = \dim_{\mathbb{F}_p} \Pi(A, \mathcal{F}(D_t)) = 2.$$

Note that for all t, the Picard group of the kernel B of $\mathcal{F}(D_t)$ is reduced to zero. Therefore the estimate given in Corollary 1.6 is not the best one in the case $t \in \mathbb{F}_p^*$. In either case, we can notice that the dimension of $\Pi(A, \mathcal{D})$ depends not only on the degrees of the elements of $\mathcal{D}$, but also on those of the whole foliation $\mathcal{F}(\mathcal{D})$.

7.2. *Computation of a Picard group*

Let A be the k-algebra $k[x, y]$, where k is algebraically closed of characteristic 2. Let B be the subalgebra $k[x^2, y^2, x(1 + xy)]$. Then B is isomorphic to $k[u, v, w]/(w^2 - u(1 + uv))$. This latter ring is normal because the variety $\mathrm{Spec}(B)$ is a complete intersection which is nonsingular in codimension 1 (see [4]). So B is a normal subring of A, which is itself normal. We would like to compute the Picard group of B by means of Corollary 1.6. To that purpose, consider the following k-derivation D:

$$D = x^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Since $D^2 = 0$, the foliation $\mathcal{F}(D)$ is equal to $A.\{D\}$. Moreover D vanishes on B, and hence $B \subseteq B' = \ker D$. Since $K(B) \subseteq K(B') \subseteq K(A)$ and $K(A)/K(B)$ is an extension of degree 2, $K(B')$ is equal either to $K(B)$ or to $K(A)$. The latter case is impossible because D is a nonzero derivation, and so $K(B') = K(B)$. Since B and B' contain A^2, B' is integral over A^2, and hence over B. As $B' \subseteq K(B)$ and B is normal, we deduce that $B' = B$. In particular, we have

$$\mathcal{F}_B = A.\{D\}.$$

Now A is a UFD, $A^* = k^*$ and B is normal. By Lemma 6.2, the morphism θ injects $\mathrm{Pic}(B)$ in $\Pi(A, \{D\})$. First we determine $\Pi(A, \{D\})$. Let deg be the standard homogeneous degree on $k[x, y]$. Consider the injective morphism

$$L : \Pi(A, \{D\}) \longrightarrow A, \qquad f \longmapsto \frac{D(f)}{f}.$$

Since D has degree 1, $L(f)$ is of the form $a + bx + cy$, where a, b, c belong to k. By Lemma 5.3, this polynomial satisfies the equation $D(L(f)) = L(f)^2$, which implies

$$D(bx + cy) = bx^2 + c = (a + bx + cy)^2.$$

So $a = 0$, b belongs to $\mathbb{F}_2$ and $c = 0$. In particular, $\dim_{\mathbb{F}_2} \Pi(A, \{D\}) \leq 1$. Since $D(x) = x^2$, $\Pi(A, \{D\})$ is nonzero of dimension 1, spanned by x. Consider now the ideal M' of B defined by $M' = (x^2, x(1 + xy))$. The open sets $D(x^2)$ and $D(1 + x^2y^2)$ build a covering of $\mathrm{Spec}(B)$, and we have

$$M'_{(1/x^2)} = B_{(1/x^2)} \quad \text{and} \quad M'_{(1/1+x^2y^2)} = B_{(1/1+x^2y^2)}\{x\}.$$

Therefore M' defines an invertible sheaf M on $\mathrm{Spec}(B)$. Since $A.M' = A\{x\}$ and $D(x) \neq 0$, we find $\theta(M) = [x] \neq 0$. In particular θ is an isomorphism from $\mathrm{Pic}(B)$ to $\Pi(A, \{D\})$, and $\mathrm{Pic}(B) = \mathbb{F}_2$.

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