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Trivariate analysis of soil ranking-correlated characteristics and its application to probabilistic stability assessments in geotechnical engineering problems

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Abstract

An important issue in the probabilistic prediction modelling of multivariate soil properties (usually including cohesion, friction angle, and unit weight) is the measurement of dependence structure among these properties. The use of Pearson's correlation as a dependence measure has several pitfalls; therefore, it may not be appropriate to use probabilistic prediction models in geotechnical engineering problems based on this correlation. As an alternative, a copula-based methodology for prediction modelling and an algorithm to simulate multivariate soil data are proposed.

In this method, all different random variables are transformed to a rank/uniform domain in order to form a copula function by applying cumulative distribution function transformations. The technique of copulas, representing a promising alternative for solving multivariate problems to describe their dependence structure by a ranked correlation coefficient, is highlighted. Two existing observed soil data sets from river banks are used to fit a trivariate normal copula and a trivariate fully nested Frank copula. The ranking correlation coefficient Kendall's τ and the copula model parameters are estimated. The goodness-of-fit test to choose the best-fitting model is discussed.

A series of triplet samples (i.e., cohesion, friction angle, and unit weight) simulated from the trivariate normal copula with flexible marginal distributions are used as input parameters to evaluate the uncertainties of soil properties and to define their correlations. The influence of the cross-correlation of these soil properties on reliability-based geotechnical design is demonstrated with two simple geotechnical problems: (a) the bearing capacity of a shallow foundation resting on a clayey soil and (b) the stability of a cohesive-frictional soil in a planar slope. The sensitivity analysis of their correlations of random variables on the influence of the reliability index provides a better insight into the role of the dependence structure in the reliability assessment of geotechnical engineering problems.

Keywords: Copula; Correlation; Monte Carlo simulation; Stochastic dependence; Slope stability; Bearing capacity; Trivariate

1. Introduction

The measurement of geotechnical properties of clayey and sandy soils such as cohesion (c), inner friction angle (φ, called friction angle hereafter) and unit weight (or bulk density, usually distinguished with moist density γ_m, dry density γ_d, and saturated density γ_{sat}) by laboratory tests demonstrate large variations and uncertainties (Matsuo and Kuroda, 1974; Phoon

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Nomenclature

Symbol: description

AIC	Akaike Information Criterion
B	width of strip foundation
C	copula distribution function
C^n	empirical copula
c	cohesion
c_θ	density function of copula C_θ
CBSM	Copula-based sampling method
CoV	coefficient of variance
D_f	depth of footing
d_{area}	relative percentage area difference of confidence regions
F	marginal distribution
F_c	critical F_s
FNFC	fully nested Frank Copula
FORM	first order reliability method
F_s	factor of safety (defined with respect to shear strength)
H	distribution function
H_s	height of the slope
I_o	enclosed area of confidence region for observed data
I_p	enclosed area of confidence region for predicted data
L	log-pseudo likelihood function
l_θ	derivative of L
m	total number of failed cases
n	total number of simulations
N_c	factor of bearing capacity
N_q	factor of bearing capacity
N_γ	factor of bearing capacity
P	probability
P_f	probability of failure
p_c	level for the statistic S^n
q_a	allowable bearing capacity
q_u	ultimate bearing capacity
S	number of variables
S^n	Cramer-von Mises statistic
$\tan\phi$	tangent of inner friction angle
U_i	ith uniform random variable, $=F(Z_i)$
u_i	ith realisation of U_i
Z_i	ith random variable
z_i	ith realisation of Z_i
$\mathbf{Z}$	vector of random variables
s_c	shape factor of a foundation
s_q	shape factor of a foundation
s_γ	shape factor of a foundation
α	inclination of slope
β_{HL}	the Hasofer–Lind reliability index
β_{cb}	reliability index
γ	bulk unit weight of soil
γ_b	bulk unit weight of soil under the foundation base
γ_d	dry bulk unit weight of soil
γ_m	moist bulk unit weight of soil
γ_{sat}	saturated bulk unit weight of soil
γ_u	bulk unit weight of soil over the foundation base
μ	mean
ρ_p	Pearson's correlation coefficient
ρ_s	Spearman's correlation coefficient
σ	standard deviation
$\mathbf{\Sigma}$	symmetrical covariance matrix
τ	Kendall correlation coefficient
ϕ	inner friction angle
$\phi_\theta(\cdot)$	generator function of the Archimedean function
Φ	normal distribution
θ	inclination of a failure plane; Frank copula parameter
ψ	inclination of the front of a slope
Ω	failure domain

and Kulhawy, 1999; Nadim, 2007; Parker et al., 2008). The traditional deterministic analysis cannot fully explain the safety of geo-structures because the variations of these soil properties are not considered (Lumb, 1970; Matsuo and Kuroda, 1974; Whitman, 1984; Tamimi et al., 1989; Duncan, 2000). Probabilistic analyses provide an alternative approach for assessing the reliability of geotechnical engineering, whereby the variables included in the analyses are expressed in probabilistic terms to account for the inherent uncertainty of each input variable (Phoon and Kulhawy, 1999; Baecher and Christian, 2003). This becomes possible when our knowledge of the statistical properties of the soil is improved.

The values of these geotechnical parameters are influenced by the same geological and geotechnical processes in different ways and to different extents; therefore, there may be positive or negative correlation between any pair of variables. A positive correlation for a pair of variables implies that both variables tend to assume either large values or small values simultaneously. Conversely, a negative correlation means one variable may take on a large value while the other assumes a small one and vice versa. Examples of this have been previously presented, including a positive correlation between soil unit weight and soil strengths (including cohesion and friction angle) (Matsuo and Kuroda, 1974) and a negative correlation (Lumb, 1970; Wolff, 1985). The development of techniques to reproduce or establish the correlations while maintaining the desired accuracy is key for probabilistic assessment.

The correlation involved in a design equation is the most important issue encountered in the probabilistic analysis of geotechnical engineering. Some existing probabilistic models assume independence between the random variables by ignoring all possible correlations (Alonso, 1976; Tobutt, 1982; Nguyen and Chowdhury, 1984; Huang et al., 2010). Other models realise the importance of including the correlations in their numerical approximations (Nguyen and Chowdhury, 1985; Tamimi et al., 1989; Fenton and Griffiths, 2003; Ferson and Hajagos, 2006; Youssef Abdel Massih et al., 2008; Griffiths et al., 2009; Cho and Park, 2010; Lü and Low, 2011). However, Fenton and Griffiths (2003) found that the varying cross-correlation has only a minor influence on the stochastic behaviour of the bearing capacity. A more thorough investigation needs to be made to explore this controversial area.

The influence of the correlation between strength parameters on geotechnical stability analysis was investigated using various well-established numerical approximation techniques such as the First Order Second Moment (FOSM), First-Order Reliability Method (FORM) and Monte Carlo simulation. The reliability index estimated using the FOSM approach is not "invariant", which gives several expressions of performance function (Duncan, 2000; Nadim, 2007). Hasofer and Lind (1974) proposed an invariant definition for the reliability index β_{HL}, which is referred to as the FORM (Nguyen and Chowdhury, 1984) in standard normal space. Low and Tang (1997) proposed a more intuitive interpretation of the Hasofer and Lind reliability index for correlated variables in the original coordinate system. The non-normal distributions are replaced by an equivalent normal ellipsoid calculated using a Rackwitz–Fiessler transformation (Rackwitz and Fiessler, 1978). In addition, Zahn (1989) derived a transformation algorithm for the Hasofer–Lind index without requiring the user to leave the original space of correlated variables based on the existed formulas for independent variables. The commonly used Monte Carlo simulation technique for correlated variables as applicable to geotechnical calculations has been described in a reference (Nguyen and Chowdhury, 1985). A number of algorithms have been developed in the literature to generate correlated random numbers (Nawathe and Rao, 1979; Iman et al., 1980; Tamimi et al., 1989), and some of them integrate with highly optimised software programs such as @RISK (Palisade, 2000).

The above approximation approaches have the merit of simplicity, but they also possess shortcomings because the approximate theoretical formulation of the statistical distribution of soil parameters has to be known. Non-parametric methods are preferable when data sets are available because they impose no structure on the estimated distribution (Karmakar and Simonovic, 2009). Additionally, the application of traditional Pearson's correlation coefficient by the study efforts to consider the dependence of variables has been criticised by some researchers in regards to the statistical aspects (Kendall and Gibbons, 1990; Boyer et al., 1999; Embrechts et al., 2002). These problems can be avoided by resorting to very flexible joint distributions, copula functions, first mentioned by Sklar (1959), which permit the marginal parameters to be dependent on each other. Copulas are able to handle mixed marginal distributions families (including parametric or non-parametric) and account for the structure of dependence overlooking the margins (Nelsen, 2006; Clemen and Reilly, 1999; Lambert and Vandenhende, 2002). When using copulas, the dependence function can also be studied separately from the marginal distributions (Salvadori and De Michele, 2004).

In this study, the triplet parameters (c, φ, and γ) are fitted by a trivariate normal copula (also called Gaussian copula) and a trivariate fully nested Frank copula (FNFC) model. The dependence structures are examined by two available soil parameter data sets, and a goodness-of-fit for the different models is assessed with the Akaike Information Criterion (AIC, Akaike, 1974). Thus, a clear understanding of system reliability through multiple soil properties and their possible dependences, constructed by copula-based simulations, might pave the way for an objective description of the overall reliability of a practical geotechnical engineering problem.

The paper is organised as follows: the variability of geotechnical parameters and dependence structures among them are provided in Section 2. Section 3 contains some expressions of the multivariate copula theory. In Section 4, the fitted copulas of multi-geotechnical parameters are illustrated by existing laboratory observations, including the application of goodness-of-fit tests for the model selection. The overall results of the application to the bearing capacity of a footing and the stability of a planar slope are presented and discussed in Section 5, and the paper closes with a summary of the key findings and the conclusions in Section 6.

2. Variability and correlation of geotechnical parameters

2.1. Probability distribution and coefficient of variation

Among soil properties, c, ϕ, and γ are the most frequently studied variables in regards to the reliability analysis of geotechnical engineering practice (Phoon and Kulhawy, 1999; Duncan, 2000). In the vast body of literature on statistical and probabilistic applications in geotechnics, the normal, log-normal, beta, and truncated normal distributions have been customarily used to model the variability and uncertainty of these geotechnical parameters (Lumb, 1970; Wolff, 1985). The applicability of the normal distribution to soil properties has gained popularity (Lumb, 1970; Tobutt, 1982; Duncan, 2000; Baecher and Christian, 2003). To avoid negative values, the log-normal (Brejda et al., 2000; Fenton and Griffiths, 2003), gamma distribution (Baecher and Christian, 2003; Forrest and Orr, 2010), and a truncated normal distribution (Duncan, 2000) can be used. Beta distributions (Lumb, 1970; Harrop-Williams, 1986; Harr, 1987; Fenton and Griffiths, 2003) are very versatile distributions that can be used to replace almost any of the common distributions and that do not suffer from extreme value problems because the domain (range) is bounded by specified values.

The variation of a parameter is commonly described with a coefficient of variation (CoV) because of the advantages of being dimensionless as well as providing a meaningful measure of the relative dispersion of the data around the sample mean. The typical CoV values for soil properties were summarised by Alonso (1976), Phoon et al. (1995), Becker (1996), and Alawneh et al. (2006) in their publications. The typical distribution types and values of CoV for soil properties published in some leading journals are revisited and summarised in Table 1. The CoV of c (9–145%), as estimated from

Table 1
CoV of geotechnical parameters

Type	No.	CoV (%)	Distribution	Reference
CoV of c	C1	16.2–31.6	/	Alonso (1976)
	C2	12–85	/	Becker (1996)
	C3	10–70	/	Cherubini (2000a)
	C4	12–145	/	Cherubini (2000b)
	C5	20–40	Normal	Forrest and Orr (2010)
	C6	14.5–106.8	/	Hata et al. (2011)
	C7	9–23	/	Lumb (1970)
	C8	20–40	/	Matsuo and Kuroda (1974)
	C9	18–49	/	Nguyen and Chowdhury (1984)
CoV of ϕ	F1	13.8–22.9 ($\tan\phi$)	/	Alonso (1976)
	F2	5–25	/	Becker (1996)
	F3	5–50	/	Cherubini (2000a)
	F4	1–87.2	/	Cherubini (2000b)
	F5	5–15	Normal	Forrest and Orr (2010)
	F6	5.7–76	/	Hata et al. (2011)
	F7	12–56	/	Lee et al. (1983)
	F8	0.8–5.9 ($\tan\phi$)	/	Lumb (1970)
	F9	5.8–46.6	/	Matsuo and Kuroda (1974)
	F10	2–5	/	Nadim (2007)
	F11	14–33	/	Nguyen and Chowdhury (1984)
	F12	5–15	/	Phoon and Kulhawy (1999)
CoV of γ	G1	2.5–6.8	/	Alonso (1976)
	G2	4–16	/	Becker (1996)
	G3	3–10	/	Cherubini (2000a)
	G4	1–27.9	/	Cherubini (2000b)
	G5	3–7	/	Duncan (2000)
	G6	1–10	Normal	Forrest and Orr (2010)
	G7	1–8	/	Matsuo and Kuroda (1974)
	G8	3–20	/	Phoon and Kulhawy (1999)

Note: / denotes for the author does not assign any distribution type.

the minimum and maximum values, respectively, given in the table and the CoV for ϕ (or $\tan\phi$) are between approximately 0.8% and 87.2%. Unit weight varies in a relatively limited range with a CoV between 1% and 27.9%.

2.2. Dependence of soil properties

The traditional concept of correlation coefficient ρ_p (i.e., the Pearson's product–moment correlation coefficient) is a measure of linear dependence between two random variables Z_1 and Z_2, and can be written as

$$\rho_p(Z_1, Z_2) = \frac{\mathrm{cov}(Z_1, Z_2)}{\sqrt{\sigma^2(Z_1)\sigma^2(Z_2)}} \quad (1)$$

where $\mathrm{cov}(Z_1, Z_2)$ is the covariance between Z_1 and Z_2, $\mathrm{cov}(Z_1, Z_2) = \mu(Z_1, Z_2) - \mu(Z_1)\mu(Z_2)$. $\mu(Z_1)$ and $\sigma(Z_1)$ denotes the mean and standard deviation of Z_1, respectively. There are some limitations, as $\rho_p = 0$ does not necessarily imply independence (Embrechts et al., 2002); it is not informative for asymmetric distributions (Boyer et al., 1999); and the attainable correlation coefficient values within the $[-1, 1]$ range depend upon the marginal distributions $F_1(\cdot)$ and $F_2(\cdot)$ (Embrechts et al., 2002).

This has led statisticians to the use of concordance measures to characterise dependence (Capéraà and Genest, 1993) such as Kendall's τ and Spearman's ρ_s. As Kendall and Gibbons (1990) argue, the confidence intervals for Spearman's ρ_s are less reliable and less interpretable than confidence intervals for Kendall's τ. Kendall's τ measure of dependence between two random variables (Z_1, Z_2) is defined as the probability of concordance minus the probability of discordance (Conover, 1998). This can be written as

$$\tau(Z_1, Z_2) = P[(Z_1 - \tilde{Z}_1)(Z_2 - \tilde{Z}_2) > 0] - P[(Z_1 - \tilde{Z}_1)(Z_2 - \tilde{Z}_2) < 0] \quad (2)$$

where $(\tilde{Z}_1, \tilde{Z}_2)$ is an independent copy of the vector (Z_1, Z_2). The first expression on the right side is the probability of concordance of (Z_1, Z_2) and $(\tilde{Z}_1, \tilde{Z}_2)$, and the second expression on the right side is the probability of discordance of the same two vectors. In common with other measures of correlation, Kendall's τ will produce values between -1 and 1, with a positive correlation indicating that the ranks of both variables increase together whilst a negative correlation indicates that as the rank of one variable increases, that of the other one decreases.

Soil strengths (usually described by c and φ) based on the Mohr–Coulomb criterion are associated with a single observation, so they are not independent. The dependence between the random variables is mostly determined by the Pearson's coefficient ρ_p of linear correlation, as reported by Lumb (1970), Cherubini (1997), Forrest and Orr (2010). Negative correlation is mainly reported against their laboratory measurements (Lumb, 1970; Cherubini, 1997; Forrest and Orr, 2010; Hata et al., 2011). However, this is not always the case, as some results of positive correlation have been reported by Lumb (1970) and Wolff (1985). More extensive discussion of this important subject requires more data that are realistic. There is a scarcity in the availability of raw data to examine the correlation between c and γ, and the correlation between ϕ and γ. Available data supported with experimental tests are only given by Matsuo and Kuroda (1974) and Parker et al. (2008). A positive correlation coefficient is more often just assumed in the leading literature articles (Chowdhury and Xu, 1992; Low and Tang, 1997).

A summary of the correlation coefficients between these pairs is provided in Table 2.

3. The copula approach for multivariate distributions

With the identification of marginal distributions and dependence measures among a set of random variables, the construction of a joint description of these cross-correlated geotechnical parameters through copula approaches is introduced below.

3.1. Theory of copulas

Let 'observed' pairs $(z_1^1, \ldots, z_S^1), \ldots, (z_1^n, \ldots, z_S^n)$ be drawn from a multivariate population of $Z_1, \ldots, Z_S$, where n is the number of observations and S is the number of variables. As stated by Sklar's Theorem (Sklar 1959), the joint multivariate distribution, denoted as $H_{Z_1,\ldots,Z_S}(z_1, \ldots, z_S)$ or $P(Z_1 \le z_1, \ldots, Z_S \le z_S)$ of these populations are connected with their one-dimensional marginal

Table 2
Correlation coefficient between soil properties

Property	ρ_p	Notes	Reference
c, tanϕ or c, ϕ	−0.61		Cherubini (2000a)
	−0.47	Triaxial test (clay)	Forrest and Orr (2010)
	0.25	Consolidated undrained triaxial test	Harr (1987)
	−0.1	Consolidated drained triaxial test	Hata et al. (2011)
	−0.81		
	−0.87		
	−0.572		
	−0.554		
	−0.49		
	−0.359		
	−0.557		
	−0.7 to −0.37		Lumb (1970)
	−0.412 (Soil 1)	Direct shear test (clay)	Matsuo and Kuroda (1974)
	0.316 (Soil 2)		
	0.369 (Soil 3)		
	−0.474 (Soil 3)		
	−0.748 (Soil 3)		
	−0.47	Drained triaxial test	Wolff (1985)
γ, c	0.25	/	Babu and Srivastava (2007)
	0.5		
	0.75		
	0.4	/	Chowdhury and Xu (1992)
	0.5	/	Low and Tang (1997)
	0.44	Direct shear test (clay)	Matsuo and Kuroda (1974)
γ, tan ϕ or γ, ϕ	0.25	/	Babu and Srivastava (2007)
	0.5		
	0.75		
	0.7	/	Chowdhury and Xu (1992)
	0.5	/	Low and Tang (1997)
	0.713 (soil 1)	Direct shear test (clay, silt, silt sand)	Matsuo and Kuroda (1974)
	0.656 (soil 2)		
	0.926 (soil 3)		
	0.859 (soil 3)		
	−0.943 (soil 3)		

Note: / denotes for non-experimental data.

probability distributions $F_i(z_i), i=1,\ldots,S$ through copula (Nelsen, 2006). Seeing as the continuous marginal probability distribution $F_i(z_i)$ is defined on the range [0, 1], a copula function, written as $C(F_1(z_1),\ldots,F_S(z_S))$, is the S-dimensional probability distribution on a unit hyper-cube $[0,1]^S$, i.e.,

$$H(z_1,\ldots,z_S)=C(F_1(z_1),\ldots,F_S(z_S)) \tag{3}$$

Thus, the relationship between $H(z_1,\ldots,z_S)$ and $F_i(z_i)(i=1,\ldots,S)$ is well established by unique copula mapping. Letting $u_i=F_i(z_i)$ represent a sample of a standard uniform random variable, a copula function can be recast as the joint cumulative distribution function of $(U_1,\ldots,U_S)$

$$C(u_1,\ldots,u_S)=P(U_1\le u_1,\ldots,U_S\le u_S) \tag{4}$$

where $P(\cdot)$ represents the joint probability. The significance of the cumulative distribution function transformation lies in the property that its application does not affect the dependence structure of the original observations. This implies that the multivariate model of original observations can be constructed in the transformed unit hyper-cube using a copula as a basis. The procedure is flexible because no restrictions are placed on the marginal distributions (Clemen and Reilly, 1999; McNeil et al., 2005), which is the main reason for the popularity of copula theory in many areas of research (Embrechts et al., 2002; Lambert and Vandenhende, 2002; Zhang and Singh, 2007). Most importantly, this approach can handle arbitrarily complicated dependence between the input variables. This makes the approach significantly more general than the methods implemented in common risk analysis software packages, which model correlations but not dependencies in general (Ferson and Hajagos, 2006).

A rich set of copula types has been generated using inversion and other methods (Nelsen, 2006). Considering that the inference for copula models is still under development (Genest and Favre, 2007), the normal copula (Elliptical class) and the Frank copula (Archimedean class) will be discussed here, as they permit both positive and negative dependence. These popular copulas are given in the context of trivariate expressions.

3.2. Normal copula

For $S=3$, the trivariate normal copula can be written as

$$C_\rho^G(u_1,u_2,u_3;\rho_p)=\Phi_\rho(\Phi^{-1}(u_1),\Phi^{-1}(u_2),\Phi^{-1}(u_3);\rho_p)$$

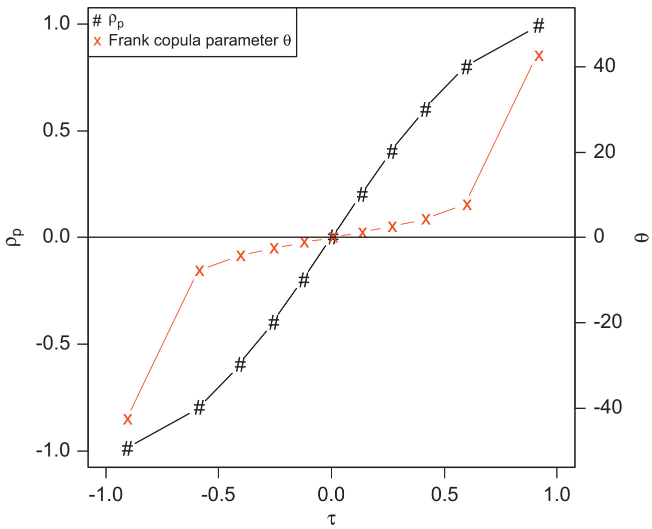

Fig. 1. Relationships of Pearson's ρ_p or Frank copula parameter θ with Kendall's τ.

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \int_{-\infty}^{\Phi^{-1}(u_3)} \frac{1}{(2\pi)^{3/2}|\mathbf{\Sigma}_3|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{W}^T(\mathbf{\Sigma}_3)^{-1}\mathbf{W}\right)\mathrm{d}\mathbf{W} \quad (5)$$

where $\Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \Phi^{-1}(u_3); \rho_p)$ denotes the joint cumulative distribution function of a multivariate normal distribution with mean vector zero and covariance matrix equal to the correlation matrix $\mathbf{\Sigma}_3$, $\Phi_\rho(t) = \int_{-\infty}^{z} (1/\sqrt{2\pi})e^{-t^2/2}\mathrm{d}t$ denotes the standard normal distribution and $\Phi^{-1}(\cdot)$ is the quantile function of the univariate standard normal distribution. $u_1 = F_1(z_1)$ is the marginal cumulative distribution function of Z_1. The integral variable $W = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}$, and $\mathbf{\Sigma}_3 = \begin{bmatrix} 1 & \rho_p^{12} & \rho_p^{13} \\ \rho_p^{12} & 1 & \rho_p^{23} \\ \rho_p^{13} & \rho_p^{23} & 1 \end{bmatrix}$ is the symmetrical covariance matrix with the linear Pearson's correlation coefficient ρ_p, which is correlation parameter and restricted to the interval from −1 to 1 (see Embrechts et al., 2002). The necessary parameters within marginal normal distributions of these normal copulas have further extended the concept of linear dependence to the modelling of correlated non-normal random variables (Clemen and Reilly, 1999; Lambert and Vandenhende, 2002), which has obviously gained more popularity.

3.3. Frank copula

Another widely used family of copula functions is the Archimedean family (Nelsen, 2006), which is constructed in a completely different way from the normal copula by using a generator function ϕ_θ. The generator function of the Frank copula is $\phi_\theta = -\ln(e^{-\theta t}-1)/(e^{-\theta}-1)$, where θ is copula dependence parameter. For $S=3$, the trivariate Frank copula is given by

$$C_\varphi(u_1, u_2, u_3; \theta) = \varphi^{-1}(\varphi(u_1), \varphi(u_2), \varphi(u_3); \theta)$$
$$= -\frac{1}{\theta}\ln\left[1 + (e^{-\theta u_1}-1)(e^{-\theta u_2}-1)(\frac{e^{-\theta u_3}-1)}{(e^{-\theta}-1)^2}\right] \quad (6)$$

where the Frank copula $C_\varphi(u_i, u_j)$ and φ_θ is a convex decreasing function with $\varphi_\theta(1) = 0$, and $\varphi_\theta{}^{-1}(\cdot)$ is the pseudo-inverse of $\varphi_\theta(\cdot)$.

For the above trivariate Frank copula, the generator function solely characterises the dependence structure of random variables and is often described by a univariate function with the model parameter θ. It usually is designated as an exchangeable (or symmetric) multivariate Archimedean copula (Mcneil, 2008). If one correlation parameter is extended, such as the node u_1 and u_2 are coupled through copula C_{11}, and node u_3 is coupled with $C_{11}(u_1, u_2)$ through copula C_{21}, an asymmetric Frank copula will be formed, which requires two bivariate copulas C_{11} and C_{21}, with corresponding the generators ϕ_{11} and ϕ_{21}

$$C(u_1, u_2, u_3; \theta_{11}, \theta_{21}) = C_{21}(u_3, C_{11}(u_1, u_2; \theta_{11}); \theta_{21})$$
$$= \phi_{21}\left(\phi_{11}^{-1}\{\phi_{11}(u_1) + \phi_{11}(u_2)\}\right) \quad (7)$$

In this structure, which Berg and Aas (2007) refer to as fully nested, all bivariate margins are themselves Archimedean copulas. Hence, the asymmetric 3-dimensional model can be properly applied when two variables, e.g., u_1, u_2, are likely correlated with the third one u_3, and the degree of dependence between u_1, u_2, is stronger than that of u_1, u_2 with u_3.

More specifically, the two pairs (u_1,u_3) and (u_2,u_3) both have copula C_{21} with the dependence parameter θ_{21}. To generate a proper copula, the level of nesting must decrease with the level of nesting, i.e., $\theta_{11} > \theta_{21}$. For more related studies on trivariate analyses, the reader is encouraged to examine the literature (McNeil et al., 2005; Berg and Aas, 2007; Zhang and Singh, 2007; Serinaldi and Grimaldi, 2007; Mcneil, 2008; Wong et al., 2010).

3.4. Relationship between Kendall's τ and copula's parameter

For the bivariate normal copula, there is a relationship between the linear correlation ρ_p and the rank correlation τ (Frees and Valdez, 1998):

$$\tau(u_1, u_2) = \frac{2}{\pi}\arcsin(\rho_{12}) \quad (8)$$

where $\arcsin(t)$ is an inverse trigonometric function such that $\sin(\arcsin(t)) = t$. This expression prompts an alternative estimation of ρ_p. The use of Eq. (8) may be more advantageous as τ is rank-dependent and invariant with respect to strictly monotonic nonlinear transformations.

For the bivariate Frank copula C_φ, Genest and MacKay (1986) have shown that τ depends on the generator $\varphi_\theta(\cdot)$ and its derivative in the simple following form:

$$\tau(u_1, u_2) = 4\int_0^1 \int_0^1 C(u_1, u_2; \theta)\mathrm{d}C(u_1, u_2; \theta) - 1$$

$$= 1 + 4\int_0^1 \frac{\varphi_\theta(t)}{\varphi_\theta^{'}(t)}\mathrm{d}t \tag{9}$$

If the Kendall's τ is known, the correlation parameter of the copula, θ, can be estimated using this expression. An illustration of these expressions between ρ_p or θ, and τ is shown in Fig. 1.

If appropriate marginal distributions are already available, then the likelihood function for the data can be derived. The resulting estimate of θ or τ would then be marginal dependent. For the semi-parametric estimation method, let a random sample $\{z_1^k, z_2^k, k = 1, \ldots, n\}$ be drawn from the distribution $H(z_1, z_2) = C_\theta(F_1(z_1), F_2(z_2))$. The log-pseudo likelihood function $L(\theta)$ of the copula $C_\theta()$ can be expressed as

$$L(\theta) = \sum_{k=1}^{n} \log[c_\theta(F_1^n(z_1^k), F_2^n(z_2^k))] \tag{10}$$

where $c_\theta()$ denotes the density function of the copula, which has the same meaning as the density function of univariate random variables, $F_1^n(z_1^k)$ is marginal empirical distribution function of Z_1 multiplied by n/n+1 for avoiding the difficulty of the potential unboundedness of $\log(c_\theta(F_1(z_1), F_2(z_2)))$, as discussed by Zhang and Singh (2007). The same applies to the distribution function of Z_2. Analogous to the maximum likelihood estimation, Eq. (10) is maximised by taking the partial derivative of $L(\theta)$ and equating it to zero as

$$\frac{1}{n}\frac{\partial}{\partial\theta}L(\theta) = \frac{1}{n}\sum_{k=1}^{n} l_\theta[\theta, (F_1^n(z_1^k), F_1^n(z_2^k))] = 0 \tag{11}$$

where $L(\theta)$ denotes the log-likelihood function and l_θ denotes the derivative of L with respect to parameter θ. The copula parameter θ can be obtained by solving this equation numerically.

3.5. Identification of the best-fitting copula

The goodness-of-fit for the alternative copulas is assessed with Cramer-von Mises statistics (Genest et al., 2009). The Cramer-von Mises statistic is based on the empirical process comparing the empirical copula with a parametric estimate of the copula derived under the null hypothesis, H_0. The Cramer-von Mises function, defined by Genest et al. (2009), represents the type of distance between the true and observed copula:

$$S^n = \sum_{i=1}^{n} \{C^n(U_1^{i,n}, U_2^{i,n}) - C^{\theta_n}(U_1^{i,n}, U_2^{i,n})\}^2 \tag{12}$$

where C^n is the empirical copula used as the most objective benchmark and C^{θ_n} is an estimator of C under the hypothesis that H_0: $C \in \{C^\theta\}$ holds. Here, θ^n is an estimator of θ computed from the ranked pseudo-observations $(U_1^{1,n}, U_2^{1,n}), \ldots, (U_1^{n,n}, U_2^{n,n})$ and can be estimated via the inversion of Kendall's τ. Large values of S^n lead to the rejection of H_0. Approximate p_c-values for the test statistic S^n are obtained by means of a parametric bootstrapping approach (Kojadinovic and Yan, 2010). The p_c-value represents the level at which the copula is not rejected, meaning that models with higher p_c-values are better in terms of them not being rejected.

The best-fitting copula from among the candidate copulas can be identified by AIC (Akaike, 1974), which is defined as

$$\begin{aligned}\mathrm{AIC} = &-2 \times \ln(\text{maximized likelihood for the model}) \\ &+2 \times \text{number of fitted parameters}\end{aligned} \tag{13}$$

A copula associated with the smallest AIC value is considered to be the best-fitting copula.

3.6. Numerical implementation

R was used (R Development Core Team, 2008) as a convenient working environment for carrying out simulations in the next sections, and the programming language of this software environment integrates with a large number of statistical functions, such as AIC and logLik (computes the logarithm of the likelihood). Two new computing resources (R packages) specifically developed to help risk assessors in their projects were utilised. The first package, 'fitdistrplus', gathers tools for choosing and fitting a parametric univariate distribution to a given dataset (Pouillot and Delignette-Muller, 2010). The second package, 'copula', helps to build and study multivariate modelling for fitting copulas (Yan, 2007; Yan and Kojadinovic, 2010). After a 'mvdc' class designed to construct multivariate distributions with given margins and their dependence using copulas is imposed, the package easily allows the generation of random variables through 'rmvdc' function or 'rcopula'. The density of multivariate distributions is defined by 'dmvdc'. The command 'gofCopula', where in default the approximate p_c-values for the test statistics are obtained using the parametric bootstrap, makes the goodness-of-test procedure easier to compute. These R packages are freely available at the Comprehensive R Archive Network (cran.r-project.org).

4. Modelling observed soil properties through copula functions

The following two data sets, including the Watarase River (Matsuo and Kuroda, 1974) and Goodwin Creek Watershed (Parker et al., 2008), were used to examine the joint behaviours assembled by the above multivariate copulas among three quantities characterising soil properties, in particular c, φ, and γ. The first step of the procedure to define the trivariate joint distribution is to identify the margins of the three variables. The second step is to characterise the correlation structure of available samples, such as Kendall's τ. The third step is to construct the trivariate normal copula and to estimate the θ_{11} and θ_{21} parameters of the FNFC.

4.1. Unsaturated soil from the Watarase River in Japan

A series of soil strength parameters of an unsaturated homogeneous silty soil layer near the Watarase River in Japan were investigated by Matsuo and Kuroda (1974), using triaxial compression tests under the unconsolidated-undrained condition. The physical properties, such as moist density of those soil layers, were also tested in the laboratory. The properties of

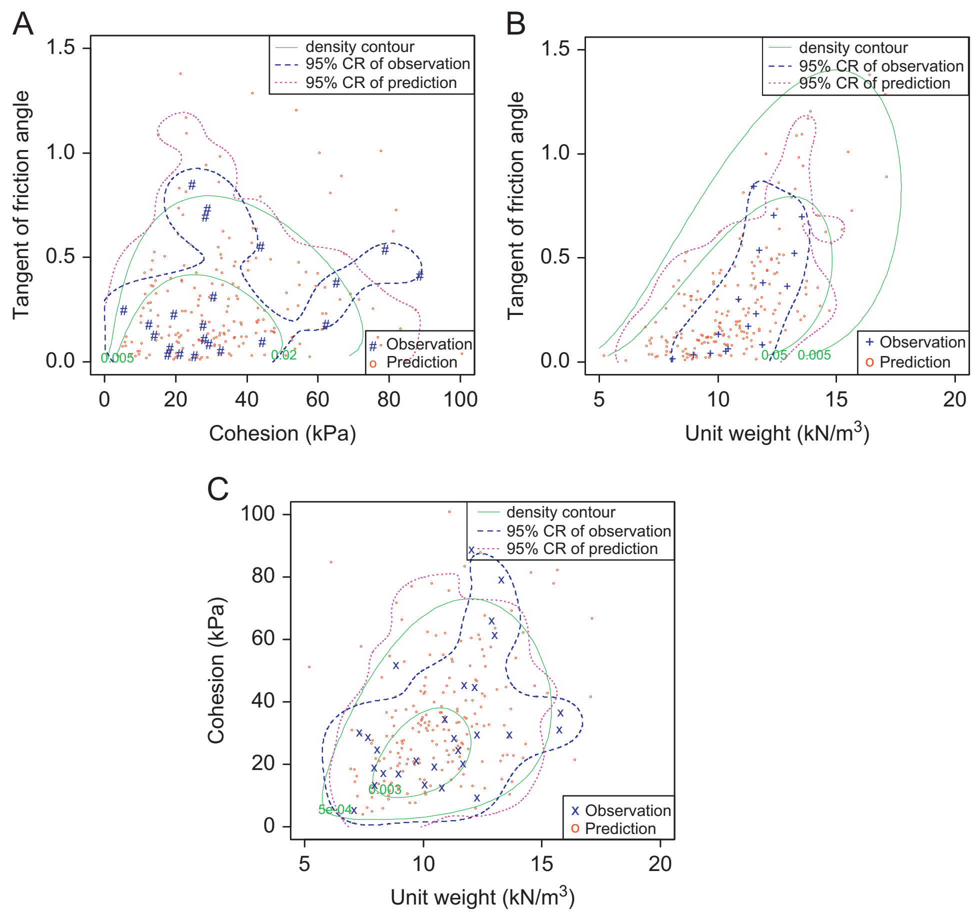

Fig. 2. Watarase River data set: scatter plot of observed pairs: (A) $(c, \tan\phi)$, (B) $(\gamma_d, \tan\phi)$ and (C) (γ_d, c) of unsaturated soil 1 (after Matsuo and Kuroda, 1974) overlapped onto the synthetic ones (200 samples) through the trivariate normal copula.

several unsaturated soils as fill materials of an embankment were investigated in their publication, with soil 1 being used as an example in this study. Their observed pairs $(c, \tan\phi)$, $(\gamma_d, \tan\phi)$, and (γ_d, c) are illustrated in Fig. 2.

Some statistical characteristics, including the mean, standard deviation, and CoV, of the observations are provided in Table 3. The values of the CoV for cohesion and friction angle are 64.5% and 89.83%, respectively. The CoV of the physical property, i.e., dried unit weight, is 22.41%.

Among the various possible candidate marginal distributions, the following functions were tested in the fit of these parameters: normal, log-normal, Gumbel, Weibull, and gamma distributions. No detailed explanation of these distributions will be given here, as they are readily available in many standard textbooks (Montgomery and Runger, 1999). The suitability of distributions for these experimental data is quantified using the Anderson–Darling (AD) test (Anderson and Darling, 1954), and the AIC, as listed in Table 4. The values of the AD statistic and AIC provide an objective way of determining which model among a set of models is the most parsimonious. The results of the AD statistic in Table 4 indicate that most distributions can pass the 5% Anderson–Darling test, i.e., 0.752 (Montgomery and Runger, 1999), with the best distribution achieving the lower probability level (best-fitting). The problems with this method have been well documented (Akaike, 1974); thus, the decisions concerning best-fitting were evaluated using the information criteria. The AIC is determined by log-likelihood, number of parameters, and effective sample size as formulated in Eq. (13). The best-fitting distribution is highlighted by its value in Table 4. The gamma distribution was selected as the best-fitting marginal distributions for c and γ_d, and the $\tan\phi$ was confirmed to follow the Weibull distribution, as shown in Fig. 3a, b and c, respectively. The normal distribution and the kernel density estimation (kde, see Venables and Ripley, 2002) are superimposed to assist with the interpretation of these density curves. The estimated parameters of the best-fitting margins are listed in Table 5.

When the parameters c, ϕ, and γ_d were jointly analysed, they were all cross-correlated with each other, as their dependence measurements demonstrate in Table 6. The friction angle of soils

Table 3
Summary statistics of the Watarase River and Goodwin Creek data

Case	Variable	Sample size	Mean	Standard deviation	CoV (%)
Watarase	c (kPa)	22	33.67	21.72	64.5
	$\tan\phi$	17	0.28	0.25	89.83
	γ_d (kN/m^3)	28	10.78	2.42	22.41
Goodwin	c (kPa)	17	4.63	3.94	85.08
	$\tan\varphi$	17	0.63	0.14	22.09
	γ_m (kN/m^3)	17	18.59	0.43	2.29

Table 4
AD statistic and AIC of marginal distributions

Case	Type	AD statistic					AIC				
		Normal	Lognormal	Gumbel	Weibull	Gamma	Normal	Lognormal	Gumbel	Weibull	Gamma
Watarase	c (kPa)	1.32	0.38	0.56	0.65	0.5	200.85	193.07	193.54	194.23	**192.94**
	$\tan\varphi$	0.72	0.45	0.62	0.37	0.36	6.64	−1.00	3.56	**−2.33**	−2.29
	γ_d (kN/m^3)	0.39	0.52	0.61	0.38	0.46	131.84	131.55	132.45	132.55	**131.33**
Goodwin	c (kPa)	0.46	1.61	0.48	1.17	1.02	97.79	95.09	95.85	89.33	**88.42**
	$\tan\varphi$	0.46	0.45	0.47	0.5	0.44	−15.85	−15.81	−15.15	−15.63	**−16.01**
	γ_m (kN/m^3)	0.42	0.41	0.41	0.49	0.42	22.22	22.17	22.37	23.66	**22.16**

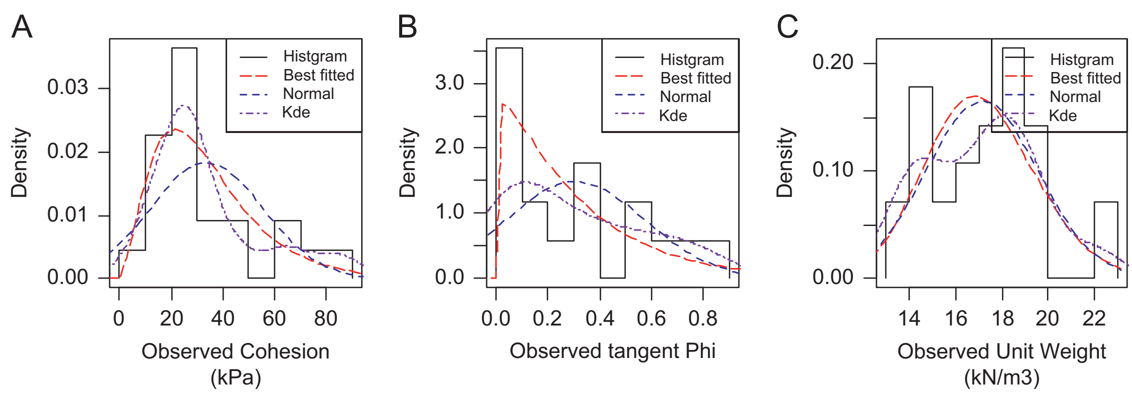

Fig. 3. Watarase River data set: Best-fitted marginal density distributions of observations: (A) c, (B) $\tan\phi$, (C) γ_d, overlapped with their histograms and corresponding normal distributions and kernel density estimations (kde)

Table 5
Estimated parameters of marginal distributions of the Watarase River and Goodwin Creek data

Case	Variable	Distribution	Estimated parameters
Watarase	c	Gamma	Shape = 2.77, rate = 0.08
	$\tan\phi$	Weibull	Shape = 1.07, scale = 0.31
	γ_d	Gamma	Shape = 52.38, rate = 3.05
Goodwin	c	Gamma	Shape = 0.7, rate = 0.15
	$\tan\phi$	Gamma	Shape = 20.98, rate = 33.55
	γ_m	Gamma	Shape = 2025.29, rate = 108.93

Table 6
Correlations of the Watarase River and Goodwin Creek data

Case		Sample size	ρ_p	τ
Watarase	c, $\tan\phi$	22	0.29	0.19
	γ_d, $\tan\phi$	17	0.72	0.65
	γ_d, c	28	0.44	0.35
Goodwin	c, $\tan\phi$	17	−0.49	−0.29
	γ_m, $\tan\phi$	17	−0.14	−0.06
	γ_m, c	17	−0.13	−0.02

has a strong positive correlation with their dry density γ_d. The value for Kendall's τ for the parameters $\tan\phi$ and γ_d is higher ($\tau=0.65$). The cohesion c of soils has a positive correlation ($\tau=0.35$) with the dry density γ_d (no correlation was reported by Matsuo and Kuroda's investigation). A positive correlation between the cohesion and the friction angle was revisited with a calculation with $\tau=0.19$ (no correlation based on initial Matsuo and Kuroda's analysis). As expected, the absolute values for Kendall's τ are lower than the ones of ρ_p, evidently by the expression between them (Lindskog et al., 2003).

The pairs from this observed triplets can be drawn in a ranked pseudo space, i.e., distributed uniformly over a unit square, as shown in Fig. 4. They further provide graphic evidence of dependence structures.

Based on the above analyses and graphic observations, both parameter pairs $(c, \tan\phi)$ and (γ_d, c) have similar values of dependence, and besides pairs $(\tan\phi, \gamma_d)$ have a strong dependence. There is no physical reason for the two parameter pairs to have the same (or at least similar) rank correlations, but it is required for the FNFC model (if the rank correlations are different, a more complex Pairwise Frank copula approach can be used instead of the FNFC model, refer to Berg and Aas (2007)). Here, both criteria for the application of the FNFC model are fulfilled: the pairs $(\tan\phi, \gamma_d)$ and (c, γ_d) have the same copula (i.e., the same structure of dependence), and the existent dependency is weaker than that of the pair $(c, \tan\phi)$.

Visual scatter plots of realisations from the candidate trivariate normal copula are shown in Fig. 2. Only 200 random samples were selected for legibility. The confidence region (CR) is defined in the original physical space of two random

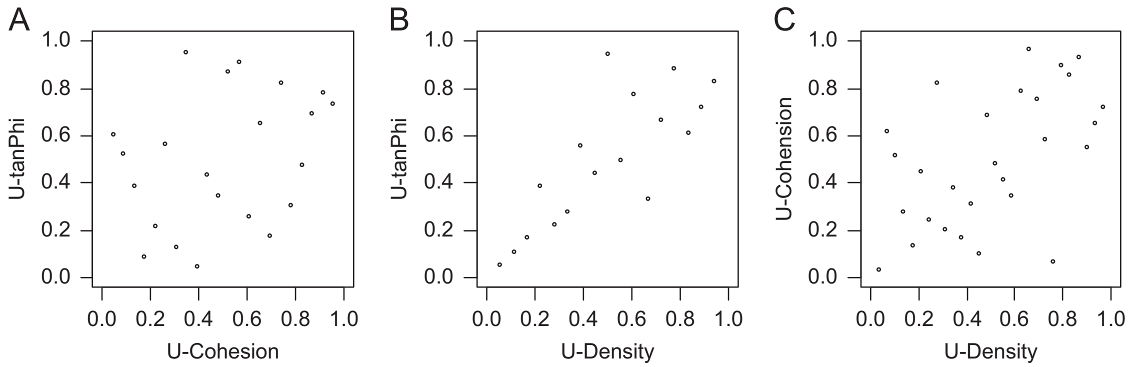

Fig. 4. Watarase River data set: scatter plot of pairs: (A) $(c, \tan\phi)$, (B) $(\gamma_d, \tan\phi)$ and (C) (γ_d, c) of pseudo-observations.

variables to characterise the different spread of the sampled data in all directions around a point (or mean), which can help the reader visualise the differences of their coverage. At a 95% confidence level, the confidence curves for both the observed (enclosed area I_O) and predicted data (enclosed area I_P), determined by a 2D kernel density estimator ('kde2d' of MASS package in R, see Venables and Ripley, 2002) using 300 grid points in each direction, are illustrated in this graph. To quantify the differences of these confidence regions, the percentage form of relative change d_{area} between the simulated and measured regions can be expressed by the ratio of the absolute change and divided by the measured region, i.e., $d_{area} = (\mathrm{abs}(I_P - I_O)/I_O) \times 100$. Here, the I_O associated with the measured region is taken as a reference value. If the relative percentage difference d_{area} is large, the predictions are less valuable than the observations. However, the relative performance of the model can be related with some other geometrical characteristics of a confidence region, such as axial inclination, shape, and consistency with geometric mean, although this technique is beyond the scope of the study. The relative percentage difference d_{area} of predictions from the candidate trivariate normal copula are listed in Table 7 and ranged between 21.57% and 84.55%. This graphical technique can provide an alternative tool to understand the performance of a simulation and to make a pre-selection of appropriate copulas. The density contours of the joint theoretical distribution with the best-fitting margins given in Table 5 are overlapped (the same below), as produced by the function of 'dmvdc' as described earlier. When the sample size of prediction is large enough, the confidence regions will evolve and converge to shapes similar to these theoretical density contours according to the central limit theorem.

The scatter diagrams of the candidate FNFC were fitted, as shown in Fig. 5, and the superimposed confidence and percentages of their relative area differences are listed in Table 7. The differences are not pronounced, and both copula models can provide a good description of the given experimental data. The estimated θ_{11} and θ_{21} parameters of the Frank asymmetric Archimedean copula are shown in Table 7.

It is difficult to compare the fit of the two copulas directly because they are non-nested models. However, the AIC results are −87.94 and −42.06 for the normal and FNFC models, respectively.

Table 7
Estimated copula parameters, p_c-value and AIC of various copulas.

Parameter	Normal		Frank	
Normal/Frank	Watarase	Goodwin	Watarase	Goodwin
ρ_p^{12}/θ_{11}	0.186	−0.296	1.216	1.036
ρ_p^{13}/θ_{21}	0.647	−0.059	2.521	1.417
ρ_p^{23}	0.35	−0.015	/	/
d_{area} of 95% CR for $(c, \tan\phi)$	70.15	71.97	75.54	123.57
d_{area} of 95% CR for $(\gamma_d, \tan\phi)$	84.55	48.39	76.25	53.83
d_{area} of 95% CR for (γ_d, c)	21.57	32.05	23.92	101.5
p_c-value	0.66	0.768	0.579	0.508
AIC	−87.94	−27.645	−42.06	−13.152

The smaller AIC value for the normal model indicates that this model is preferred. The approximate p_c-values given in Table 7 indicate that there is very little evidence against the two copula models. The p_c-value achieved by the normal copula is slightly larger than that generated by the FNFC.

Fig. 6 illustrates a further comparison of the density contours for the bivariate pair of $(c, \tan\phi)$ through different models. As is seen from this figure, the level curves of the empirical density for a bivariate normal model (parameters of mean and standard deviation are taken from Table 3) are elliptic, whereas the level curves of the density through copulas with the best-fitting marginal distributions (listed in Table 5) have a quite different shape. The observed data were superimposed on the contour plots. The normal copula with the best-fitting margins provides a distribution quite similar, although not identical, to the one by the bivariate Frank copula, and they are a more reasonable visual representation of the observed data than the traditional bivariate normal model.

4.2. Bank material properties along the Goodwin Creek Experimental Watershed in Mississippi

Parker et al. (2008) demonstrated the variation of soil properties and discussed the impacts of uncertainty parameters

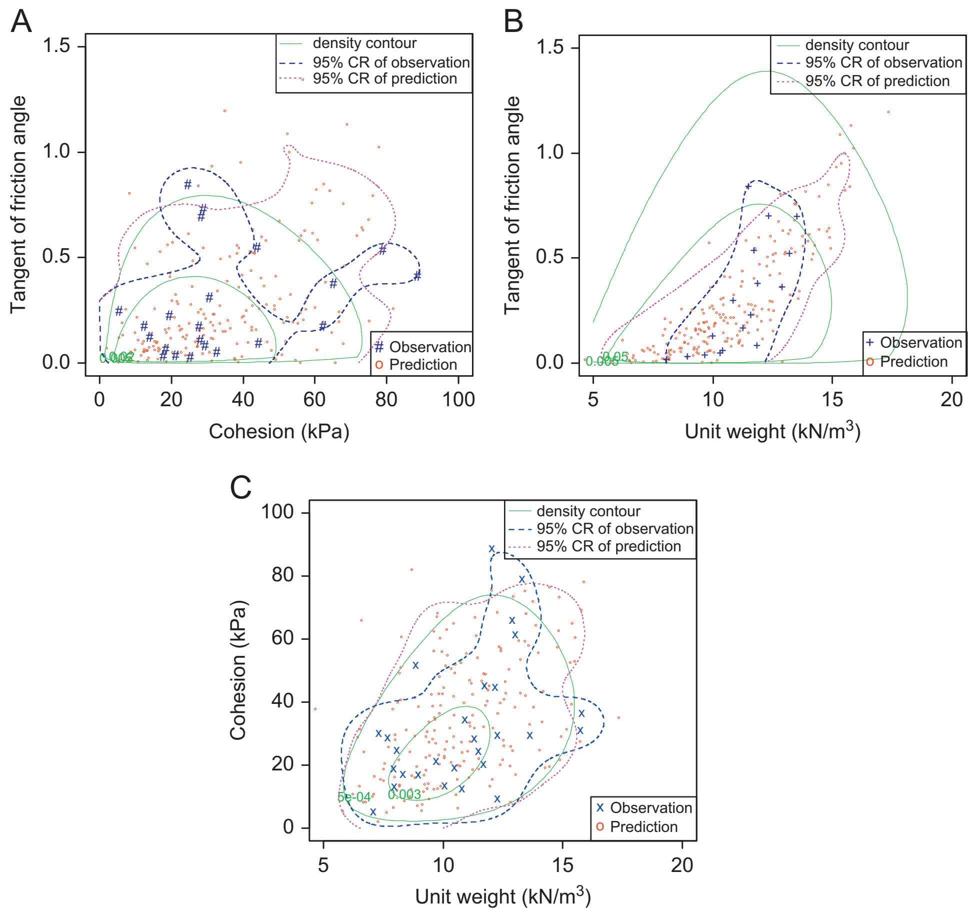

Fig. 5. Watarase River data set: scatter plot of pairs: (A) $(c, \tan\phi)$, (B) $(\gamma_d, \tan\phi)$ and (C) (γ_d, c) of predictions through the FNFC.

on bank stability. Bank materials along a stretch of the Goodwin Creek Experimental Watershed (in the Yazoo Basin, Mississippi, USA) consist of 1–2 m of moderately cohesive brown clayey-silt of Late Holocene age overlying approximately 1.5 m of Early Holocene grey, blocky silt of low cohesion and permeability. A series of in situ shear strength measurements were taken using an Iowa Borehole Shear Tester (BST; Lutteneggerr and Hallberg, 1981) to obtain the values for apparent cohesion and friction angle. The bulk unit weight was tested in the laboratory. The Late Holocene layer (approximately 1.00 m depth, upper layer) and Early Holocene layer (approximately 2.00 m depth, lower layer) were collected from cross-sections A through G (total 17 test samples). The upper layer as an example is adopted here, and the observed pairs $(c, \tan\phi)$, $(\gamma_d, \tan\phi)$, and (γ_d, c) are shown in Fig. 7.

As summarised in Table 3, the mean and standard deviation of effective cohesion are 4.63 kPa and 3.94 kPa, respectively. The mean and standard deviation of the tangent of friction angle are 0.63 and 0.14, respectively. The mean and standard deviation of saturated unit weight γ_s are 18.6 kN/m^3 and 0.43 kN/m^3. The best-fitting distributions of these properties are preferred to the generalised gamma distribution, as shown in Fig. 8a–c, according to the lowest AIC values in Table 4. The normal distribution is superimposed to assist with the interpretation of these density curves. The estimated parameters of these best-fitting distributions are given in Table 5.

The correlations among the soil properties including Pearson's ρ_p and Kendall's τ are summarised in Table 6. These dependence structures can be seen clearly via the scatter plot of pairs in ranked pseudo space, as illustrated in Fig. 9. The cohesion and the friction angle are more negative correlated ($\tau = -0.29$). The friction angle of soils has a weak negative correlation with the saturated density γ_s ($\tau = -0.06$). The cohesion c of soils has a very weak correlation with the density γ_s ($\tau = -0.02$).

The scatter plot for points from the trivariate normal copula is provided in Fig. 7. The relative percentages of area difference of confidence regions quantifying the observed and predicted data range from 32.05% to 71.97%. The diagrams for the predictions by the candidate FNFC are not shown due to space limitations, but the values of d_{area} are slightly larger than with the normal case. The models are compared using the p_c-value and AIC. The p_c-values higher than 0.05 (Kojadinovic and Yan, 2010) indicate that the null hypothesis cannot be rejected by these two models. AICs differ as listed in Table 7, which indicates the normal copula is better suited to this observation. The estimated parameters of these copulas by the method of maximum likelihood are given in Table 7.

Fig. 10 shows a comparison of density contours for the pair of $(c, \tan\varphi)$ through different models. As is seen from this figure, the level curves of the empirical density for the bivariate normal model (parameters are taken from Table 3)

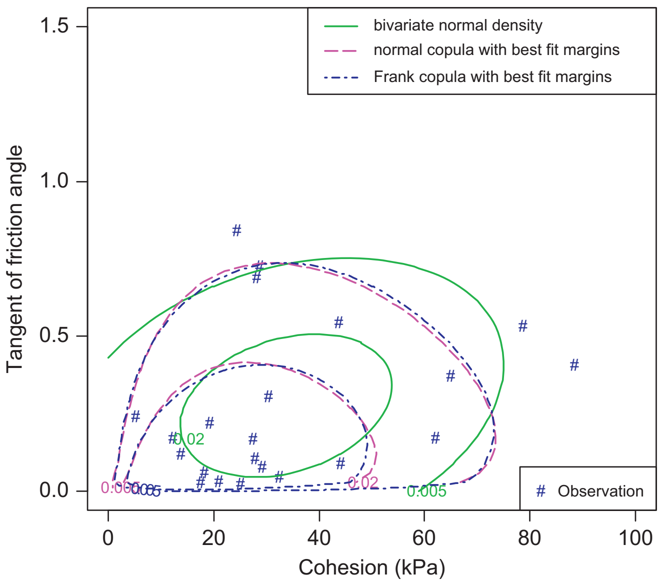

Fig. 6. Density contours of bivariate models for $(c, \tan\phi)$ of Watarase River data set with (1) normal margins, (2) gamma c marginal and Weibull $\tan\phi$ marginal using normal copula, (3) gamma c marginal and Weibull $\tan\phi$ marginal using the FNFC.

are elliptic, which indicates that the traditional multivariate linear model has difficulty providing a reasonable fit to the observed data (superimposed in this graph). The level curves of the density through copulas with the best-fitting marginal distributions have a quite different shape. The normal copula with the best-fitting margins (as given in Table 5) provides a distribution similar to the bivariate Frank copula. Both copula models appear to do a good job of mimicking the true distributions of these observations when a sufficient sample size is reached whose confidence regions will follow the shape of their theoretical density distributions.

It should be noted that boundaries, which 95% of measurements falling along the regression lines of the correlation amongst assumed normally distributed geotechnical variables, were imposed by Parker et al. (2008) to maintain the dependences when random sampling. Unfortunately, no further evidence was provided in their publications that would help confirm the influence of parameter correlations on the bank stability results.

5. Practical application to the bearing capacity of clay and in slope stability problems

Potentially, the increasing availability of multivariate data for complex systems will lead to an interest in the joint density distribution of soil properties, but it becomes numerically infeasible to evaluate such a high-dimensional integral, especially when no explicit expression is associated with most geotechnical stability problems. Based on the former investigations, a series of simulations can be drawn from each input variable (defined probability distribution or even non-parametric distribution as presented by Karmakar and Simonovic (2009)) while maintaining the correlation relationships between variables (defined in a manner similar to those experimental observations) through a copula. After a number of simulations, the probability of failure can be defined by

$$P_f = \iiint_{\Omega} f(c, \phi, \gamma) \mathrm{d}c\, \mathrm{d}\phi\, \mathrm{d}\gamma \tag{14}$$

where Ω is the region of failure ($F_s(c, \phi, \gamma) \leq F_c$), F_s is the factor of safety, F_c is the critical value of F_s. This leads to the following computational procedure:

[1] Establish the number of simulation points to be used.
[2] For each point i, generate triplet values c,ϕ,γ from the distribution $f(c,\phi,\gamma)$, including the dependence.
[3] Calculate the value of the factor of safety $F_s(c,\phi,\gamma)$ and count the number to be added to a running sum m if $F_s(c, \phi, \gamma) \leq F_c$ and go to the next;
[4] After all realisations have been evaluated, the estimate of P_f is the running sum n, $P_f = m/n$. The reliability index can be calculated by the ratio $\beta_{cb} = (\mu(F_s) - 1)/\sigma(F_s)$ under the assumption of $F_s(c,\phi,\gamma)$ is normally distributed.

The procedure described herein is similar to a Monte Carlo simulation, except for the samples generated by the copula model, called the Copula-based sampling method (CBSM). When the simulation number is sufficiently large, the standard deviation of estimation values F_s will be obtained by simulating sample inverses with the square root of simulation number. Thus, the accuracy increases with the increase of the simulation number. In general, when the simulation number is more than $n \geq 100/P_f$ the accuracy may be satisfactory (Tobutt, 1982; Husein Malkawi et al., 2000), and the probability of failure P_f can represent a deterministic solution. In the following examples, the reliability index β_{cb} is evaluated by the CBSM for comparison against results β_{HL} obtained by modified Hasofer and Lind's approach. Due to space limitations, for further details of this FORM, readers are asked to refer to Zahn (1989), Nguyen (1985), and Low and Tang (1997).

5.1. Shallow strip foundation resting on varying clayey soil

In the conventional design of shallow foundations resting on a cohesive frictional soil, the allowable pressure can be calculated based on shear failure criterion (ultimate limit state). Here, a shallow strip foundation of width B resting on a horizontal ground at a depth D_f and loaded with a concentric vertical loading is considered. It is assumed this foundation rests on the soil, as in the case of Watarase River data set sampled by the trivariate normal copula, as given in Tables 5 and 6 for its properties. The influence of these dependent random variables on the computed reliability index is studied here.

The ultimate bearing capacity (q_u) is usually expressed by the well-known Terzaghi equation (Terzaghi, 1943)

$$q_u = cN_c s_c + \gamma_b D_f N_q s_q + 0.5B\gamma_u N_\gamma s_\gamma \tag{15}$$

where N_c, N_q, and N_γ is bearing capacity factors, D_f denotes depth of the footing and B is width of the foundation, γ_b is the

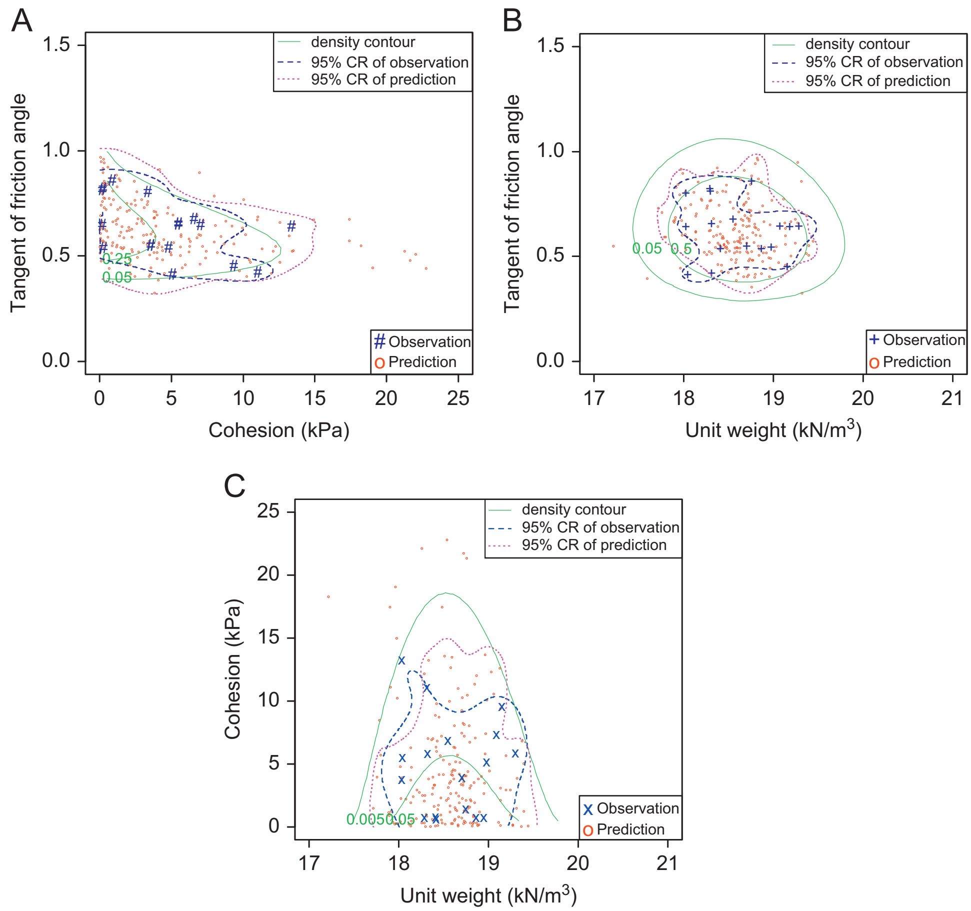

Fig. 7. Goodwin Creek data set: scatter plot of observed pairs: (A) ($c, \tan\phi$), (B) ($\gamma_m, \tan\phi$) and (C) (γ_m,c) of Bank material properties (after Parker et al., 2008) overlapped onto synthetic ones (200 samples) through the trivariate normal copula.

unit weight of the soil under the foundation base, γ_u is the unit weight of the soil over the foundation base, the mean of these unit weights are set with 17.5 kN/m^3 (adjusted from the mean of γ_d), and s_c, s_q, and s_γ are shape factors depending on the shape of foundations, setting their values to 1 for a strip-type foundation (Cherubini, 2000a). The bearing capacity factors are given below (Vesic, 1973):

$$N_c = (N_q - 1)\cot\phi \tag{16}$$

$$N_q = e^{\pi\tan\phi}\tan^2(45 + \phi/2) \tag{17}$$

$$N_\gamma = 2(N_q + 1)\tan\phi \tag{18}$$

The allowable bearing capacity (q_a) is obtained after applying a factor of safety to the ultimate bearing capacity (q_u), and a value between 2.5 and 3 is usually considered appropriate in the case of a shallow foundation. Using a conventional deterministic approach for mean values of input soil properties, the mean value of ultimate bearing capacity from shear failure criteria is obtained as 492.83 kPa. Using a factor of safety of 3.0, the allowable bearing capacity of the foundation soil is 164.27 kPa. From serviceability requirements, an applied pressure is set as 80 kPa.

The reliability index of 1.65 is estimated by the CBSM as indicated with a horizontal line in Fig. 11 where the observed correlations given in Table 6 are imposed. To study the influence of the cross-correlations of soil properties on the reliability index, one pair of ranked correlations with ranges varying from −0.91 to 0.91, and the other correlations were set with independent parameters. The Pearson's ρ_p, Kendall's τ, and the bivariate Frank's copula model parameter θ are listed as labels of the horizontal axial. Fig. 11 shows the variation of reliability indexes with varying correlation coefficients, and the reliability indexes increase with the decreasing of τ between cohesion and friction angle. Similar conclusions were reported by Cherubini (2000a), Lü and Low (2011) and Soubra and Mao (2012), although their numerical schemes are different, and the correlation coefficients are limited to negative values. Very different β_{cb} values are obtained with very high negative $\tau(c, \tan\phi)$ values, especially for $\tau(c, \tan\phi) = -0.91$.

The impact of the correlation coefficient of the pairs (γ, $\tan\phi$) and (γ,c) on the β_{cb} is not significant, and β_{cb} slightly increases as correlation decreases.

The values of reliability index β_{cb} obtained by the CBSM were compared with those obtained by Hasofer and Lind's method β_{HL} (indicating with FORM) with varying of the correlation coefficients between the cohesion and friction angle, as shown in

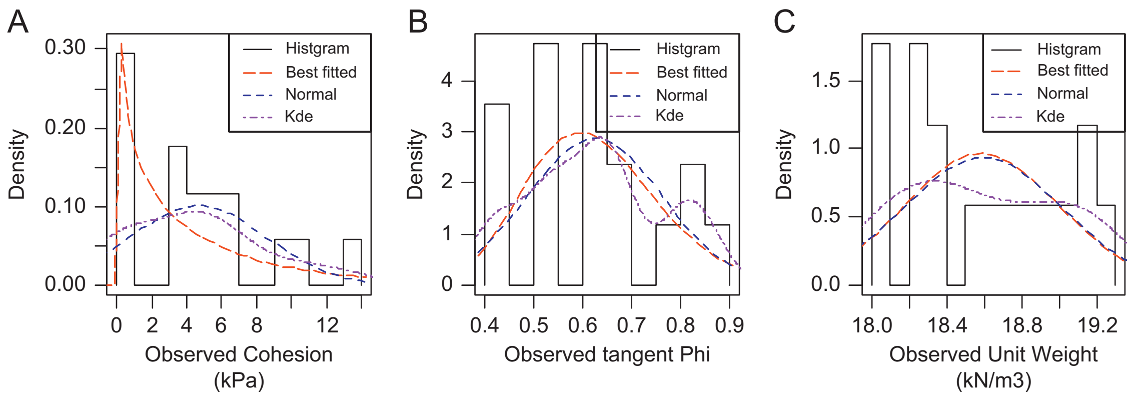

Fig. 8. Goodwin Creek data set: best-fitted marginal density distributions of observations: (A) c, (B) $\tan\phi$, (C) γ_m, overlapped with their histograms and corresponding normal distributions and kernel density estimations (kde).

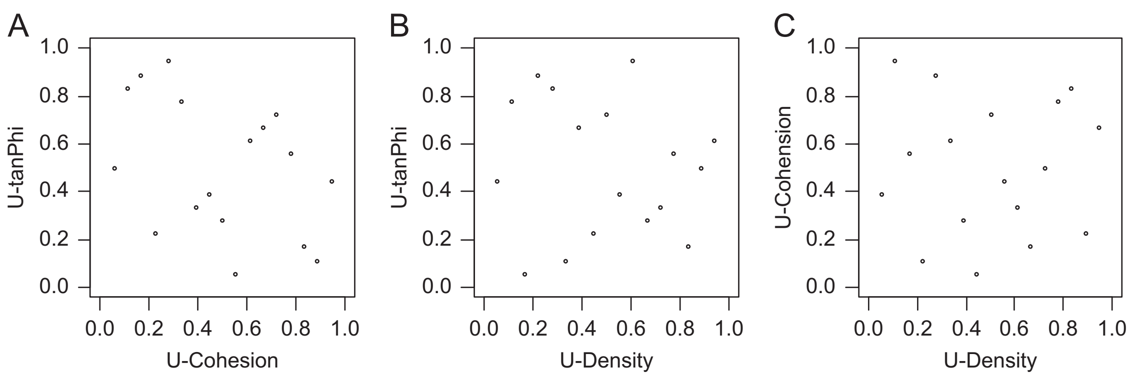

Fig. 9. Goodwin Creek data set: scatter plot of pairs: (A) $(c, \tan\phi)$, (B) $(\gamma_m, \tan\phi)$ and (C) (γ_m, c) of pseudo-observations.

Fig. 10. The proposed approach appears to give slightly larger values, although most of them are matched.

5.2. Slope of varying cohesive-frictional soil

Another simple example is a slope resistance against sliding on a single plane, known as the Culmann analysis (Taylor, 1948). The margin of safety is

$$M_s = c + \left[0.5\frac{H_s}{\sin\psi}\sin(\psi-\alpha)\cos\alpha\right][\tan\phi - \tan\alpha]\gamma_m \quad (19)$$

where α is the inclination of the failure plane, ψ is the inclination of the front of the slope, H_s is the vertical distance from the crest to the toe of the slope. Slope sliding occurs when $M_s \le 0$. Suppose that the Watarase River data set as sampled by the trivariate normal copula is used here. Then, the mutually dependent uncertain variables are the cohesion, the tangent of the friction angle, and the unit weight, i.e., (c, $\tan\phi$, and γ_m). The other parameters have fixed values (Baecher and Christian, 2003), i.e., $H_s = 10$ m, $\theta = 20^\circ$, and $\psi = 26^\circ$. In the slope stability analysis, the evaluation of probability of failure is normally required to account for variability and uncertainty in geotechnical parameters. The slope analysis used for this example concerns the stability assessment of an embankment having a potential planar failure mode and considering the correlation characteristic of these uncertain variables.

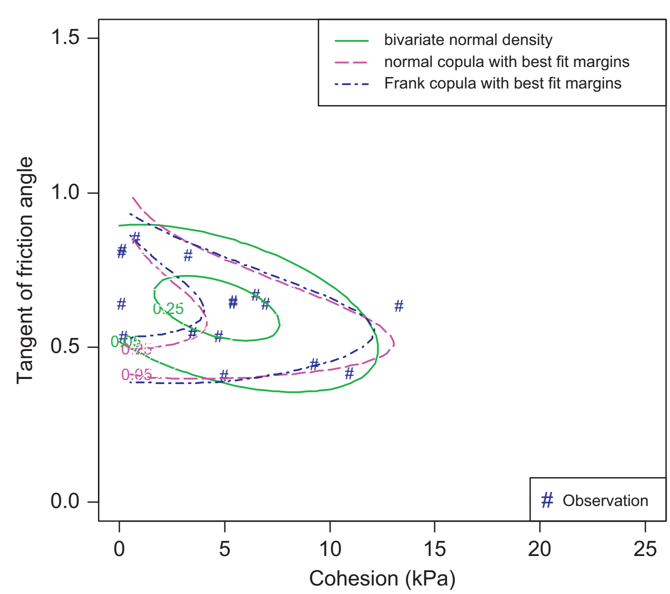

Fig. 10. Density contours of bivariate models for $(c, \tan\phi)$ of Goodwin Creek data set with (1) normal margin, (2) gamma c marginal and gamma $\tan\phi$ marginal using normal copula, (3) gamma c marginal and gamma $\tan\phi$ marginal using the FNFC.

The values of the reliability index as evaluated by the CBSM and the FORM for the range of ranked correlation coefficient from −0.91 to 0.91 between the soil shear strengths are presented in Fig. 12. The reliability index as determined by the CBSM is

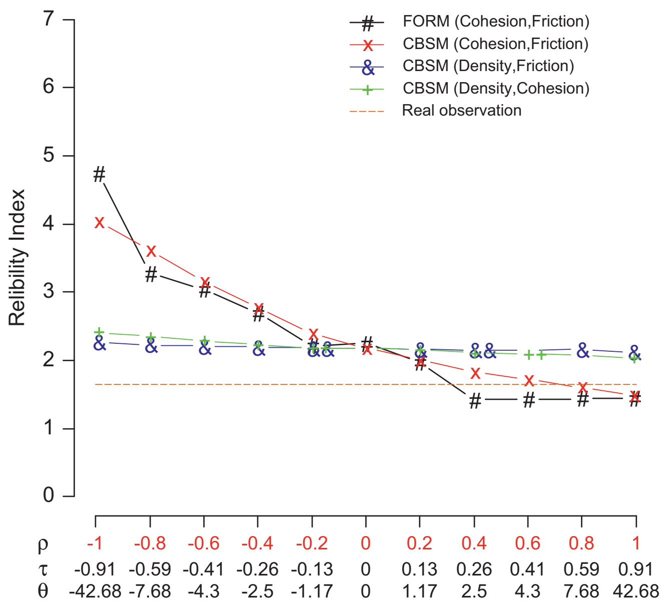

Fig. 11. Reliability index versus correlations of soil properties for the strip example.

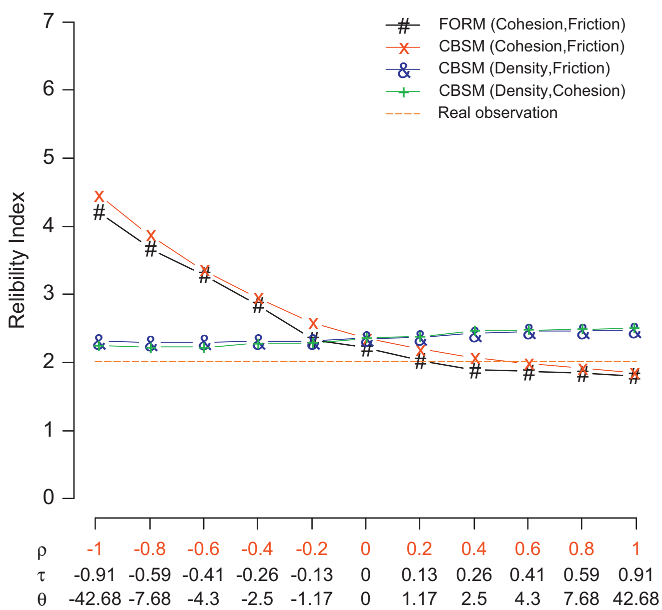

Fig. 12. Reliability index versus correlations of soil properties for the slope example.

consistently validated against the results obtained by the FORM. The analysis of the results illustrates the importance with respect to the value of β_{cb} of the ranked correlation coefficient τ between cohesion and friction angle. The evaluation of effect of cross-correlation on reliability index by this proposed procedure is in accordance with the results of some other investigators, such as Nguyen (1985) using Hasofer and Lind's method (in Fig. 5 of his publication) and Cho and Park (2010) using random fields procedures (in Fig. 11 of their publication).

The impact of the correlation coefficient of the pairs $(\gamma_m, \tan\phi)$ and (γ_m, c) on the β_{cb} is not significant, as illustrated in Fig. 12. The reliability indices based on both definitions are increased slightly when the correlation coefficients are changed from negative to positive. Soil strength parameters, including cohesion c and friction angle ϕ, contribute significantly to the resisting component of the slope against failure. A small variation of the dependent of cohesion c and friction angle ϕ has enormous influence on the reliability index of slope stability, which necessitates additional extensive research to gather more reliable experimental data to better explore their dependence structures. A reliability index of 2.01 was obtained for the case with correlated soil properties given in Table 6, indicated by a horizontal line in Fig. 12.

5.3. Discussion

The results of the CBSM shown in Figs. 11 and 12 , where the correlation coefficients are zero, can be considered as Monte Carlo sampling results without consideration of soil property interdependence. However, the evaluation of the applicability of the probabilistic assessment method is incomplete if the correlation between the geotechnical parameters is not taken into account.

The random values are generated from the cumulative density function, which are fed into the performance functions to achieve a probabilistic prediction of outcomes. The traditional multivariate probability distribution is mostly limited to the multivariate normal one or mixtures of it, but arbitrary margins are not allowed. The proposed copula-based simulation has more flexibility to express the correlation structures of variables. It proved to provide a practical and efficient method to facilitate a probabilistic geotechnical stability analysis through the CBSM and to measure the dependence of those random variables if a detailed observation is available. The method does not make any assumptions relating to the geometry of the failure surface (or performance function) and can be applied to any complex clay engineering geometry, layering, or pore pressure conditions. It is can be extended to the other geotechnical problems, especially cases involving correlated soil properties.

The results indicate that there is a real need for further studies, field or laboratory, on the correlation aspect of geotechnical parameters, as the reliability index, and hence the probability of failure, are highly dependent on the value of the correlation coefficient between variables. Unfortunately, this type of analysis requires a large amount of data that is often not available in practice. Nevertheless, this preliminary evidence encourages further investigation of soil-correlated characteristics.

The limitation of the present trivariate model is that it neglects the spatial autocorrelation of soils. The inherent spatial soil variability due to different depositional conditions and different loading histories should be described within the framework of random fields (Rungbanaphan et al., 2012; Kim and Sitar, 2013). Considering this, both cross-correlation and autocorrelation need to be taken into consideration in further studies.

When faced with a large number of uncertain variables, the analyst would do well to ask whether all of them are needed for the analysis. A reduction in the number of active variables not only makes the computations more tractable but also increases the chances that the results can be interpreted practically. There is an interest in considering only the correlation between shear strengths in the case of the Goodwin Creek data set because of the weakness of the other correlations, especially as there is no significant influence of reliability indices associated with these correlations.

6. Conclusion

A procedure to characterise stochastic dependence in a multivariate context was implemented with the aid of the copula theory. The methodology proposes the application of the cumulative distribution function transformation to each random variable, leading to the transition to a common uniform/rank domain. The modelling of stochastic dependence takes place in this common domain, using specific distributions with uniform marginals, the copula functions. Copulas are very flexible and can handle dependent parameters with mixed marginal distributions. Moreover, it is possible to construct higher dimensional copula models to incorporate further important soil parameters. The trivariate fully nested Frank copula and the trivariate normal copula were applied to construct a multivariate model incorporating the correlation of the soil parameters into the numerical modelling.

The presence of correlation has considerable influence on the performance of the geotechnical systems, as demonstrated by the bearing capacity of shallow foundation resting on clayey soil and the slope stability assessment, especially for the correlation between cohesion and friction angle. This correlation should be well recognised in the calculation of reliability. By comparing the results obtained from the CBSM with those calculated by the FORM for the bearing capacity and slope stability analysis, the proposed method for the correlated variability modelling of multivariate soil properties has been demonstrated as effective and helpful in performing a site-specific probabilistic stability analysis. This contribution sets the foundations for the modelling of stochastic dependence in geotechnical system studies. Further work will focus on a variety of applications and extend the modelling algorithms for the treatment of problems where the rank correlation matrix is non-positive definite of multivariate normal copula and the investigation of the applicability of different copula families to geotechnical system uncertainty analysis.

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