JOURNAL OF ALGEBRA **181,** 934–961 (1996)
ARTICLE NO. 0155

Reflective Integral Lattices

Rudolf Scharlau and Britta Blaschke

Universität Dortmund, Fachbereich Mathematik, D 44221 Dortmund, Germany

Communicated by J. Tits

A lattice L with a positive definite quadratic form is called reflective if the unique largest subgroup generated by reflections of the orthogonal group $\mathrm{O}(L)$ has no fixed vector. Equivalently, the "root system" $\mathsf{R}(L)$ has maximal rank. The root system of a lattice is defined in Section 1; the roots are not necessarily of length 1 or 2. In Section 2, the structure of reflective lattices is worked out. They are described and classified by pairs $(\mathsf{R}, \mathscr{L})$, where R is a "scaled root system" and the "code" $\mathscr{L}$ is a subgroup of the "reduced discriminant group" $\overline{T}(\mathsf{R})$. The crucial point is that $\overline{T}(\mathsf{R})$ only depends on the combinatorial equivalence class of the root system R. In Section 3, we give a precise description of the full root system of a reflective lattice if one starts with a sub-root-system of combinatorial type $n\mathsf{A}_1$ or $m\mathsf{A}_2$. In Section 4, our techniques are applied to a complete and explicit description of all reflective lattices in dimensions ≤ 6.

Contents.

1. INTRODUCTION

We consider positive definite, integral quadratic lattices L. That is, L is a free $\mathbf{Z}$-module of finite rank, $L \cong \mathbf{Z}^n$, together with a positive definite symmetric bilinear form. The value of the form at vectors $x, y \in V := \mathbf{Q}L$ is denoted by $(x, y) \in \mathbf{Q}$. "Integral" means that $(x, y) \in \mathbf{Z}$ for all $x, y \in L$.

0021-8693/96 $18.00

For $v \in V$, the reflection

$$s_v: x \mapsto x - \frac{2(v,x)}{(v,v)} v$$

is an isometry of the quadratic vector space V.

Recall that a vector $v \in L$ is called *primitive* if $v/m \notin L$ for all integers $m > 1$.

DEFINITION 1.1. A vector $v \in L$ is called a *root* of L if it is primitive and if s_v maps L into itself. A lattice is called *reflective* if its roots generate a sublattice of full rank.

The following easy observation is the starting point of this paper.

PROPOSITION 1.2. *The set of all roots of L*

$$\mathsf{R}(L) := \{v \in L | v \text{ primitive}, s_v \in \mathsf{O}(L)\}$$

is a root system in the usual sense of Lie algebra theory.

Proof. We refer to [Bou68, Chap. VI] for the definition of a root system. If $v, v' \in \mathsf{R}(L)$, then $s_v v' \in \mathsf{R}(L)$ since $s_{s_v v'} = s_v s_{v'} s_v$. By the above reflection formula, a primitive vector $v \in L$ is a root if and only if $(v, L) \subseteq \mathbf{Z}(v,v)/2$. In particular, the "crystallographic condition," $2(v,v')/(v,v) \in \mathbf{Z}$ for any two $v, v' \in \mathsf{R}(L)$, holds. ∎

We shall consider $\mathsf{R}(L)$ as a root system R together with a specified quadratic form invariant under the Weyl group $\mathsf{W}(\mathsf{R}) = \langle s_v | v \in \mathsf{R}\rangle$, and the notion of isomorphism of root systems is that of an isometric bijection. Thus, $\mathsf{R}(L)$ is even a finer invariant than an ordinary root system; we shall sometimes speak of a *scaled root system*.

We observe that v is a root if and only if v is an element of $(v,v)/2$ times the *dual lattice* $L^\# = \{y \in V | (L, y) \subseteq \mathbf{Z}\}$. In particular, $(v,v)/2$ divides the exponent of the finite group $T(L) := L^\#/L$, the so-called *discriminant group* of L. If L is self-dual (unimodular), then the roots are precisely the vectors of norm $(v,v) = 1$ or 2. In general, all divisors of twice the exponent of $T(L)$ may occur as norms of roots (think of lattices with an orthogonal basis).

As examples, consider the two quadratic forms given by the following matrices (the indices at the vectors indicate the value (v,v)):

$$L_1 \cong \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \qquad \det L_1 = 8,$$

$$\begin{aligned} \mathsf{R}(L_1) &= \{\pm(1,0,0)_2, \pm(0,1,0)_2, (\pm 1, \pm 1, 0)_4, \pm(1,1,-2)_8\} \\ &\cong \mathsf{C}_2 \perp {}^4\mathsf{A}_1 \end{aligned}$$

$$L_2 \cong \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \qquad \det L_2 = 12$$

$$\mathsf{R}(L_2) = \{\pm(1,0,0)_2, \pm(1,-1,-1)_4, \pm(0,1,-1)_6\} \cong \mathsf{A}_1 \perp {}^2\mathsf{A}_1 \perp {}^3\mathsf{A}_1.$$

Here $\perp$ denotes the orthogonal sum of two root systems which is defined in the obvious way (as a set, it is the disjoint union). The notation A_l, C_l, etc., for irreducible root systems is the usual one; the additional left upper index denotes scaling of the quadratic form (cf. Section 2 below). The lattices L_1 and L_2 are reflective.

In this paper, we shall show that reflective lattices L are classified by pairs $(\mathsf{R}, \mathscr{L})$, where R is a scaled root system, and $\mathscr{L}$, the so-called *glue code* of L, is a subgroup of a group $\overline{T}(\mathsf{R})$, the *reduced discriminant group* of R. The reduced discriminant group $\overline{T}(\mathsf{R})$ is by definition a subgroup of the discriminant group $T(\mathsf{R})$ of the root lattice generated by R. The important fact is that $\overline{T}(\mathsf{R})$ only depends on the combinatorial equivalence class of R. We then prove a couple of results about decomposability, and about the full root system $\mathsf{R}(L)$ in a situation where one starts only with a sub-root-system $\mathsf{R} \subset \mathsf{R}(L)$. In the concluding Section 4, we shall see that these general results will lead without too much additional effort to the classification of all reflective lattices in dimensions ≤ 6.

The notion of reflective lattices had been introduced by Vinberg [Vin72] in the context of arithmetic groups acting on hyperbolic space, and generated by reflections in hyperplanes. In the course of investigating such groups, one is naturally led to define "reflective" lattices of signature $(m, 1)$, for some m. A basic lemma of Vinberg relates the reflectivity of such an indefinite lattice to the reflectivity of its positive definite sublattices. See [Vin72, Vin85, SW92] for details.

So far, positive definite reflective lattices have been investigated as objects in their own right only in the unimodular case [Ven80, Ker94], and recently in the 2-elementary case [SV94]. Kervaire in [Ker94] treats the dimension 32, where the determination of all admissible codes (for the various different root systems) is already very elaborate. The present paper shows, among other things, that unimodularity is inessential as far as the codes are concerned. Thus, concrete classifications along the lines of [Ker94] could be extended to other genera of not too large dimension and determinant.

The use of the scaled root system as a refined invariant of an arbitrary lattice is exemplified in the tables of [SH94].

2. GENERAL RESULTS ON THE STRUCTURE OF REFLECTIVE LATTICES

Let R be a scaled root system (in a rational vector space) and $R = \langle \mathsf{R} \rangle$ the $\mathbf{Z}$-lattice generated by R. Since R is supposed to carry an integral quadratic form (invariant under the Weyl group $W(L)$), we can consider its *discriminant group*

$$T(\mathsf{R}) := R^{\#}/R.$$

Let L be a reflective lattice with $\mathsf{R}(L) \supseteq \mathsf{R}$ and of the same rank as R. Since L is integral, it is contained in $R^{\#}$, and is obviously determined up to isomorphism by the pair $(\mathsf{R}, \mathscr{L})$, where $\mathscr{L} := L/R \subseteq T(\mathsf{R})$. Here an isomorphism of pairs $(\mathsf{R}, \mathscr{L})$ and $(\mathsf{R}', \mathscr{L}')$ is an isometry $\mathsf{R} \to \mathsf{R}'$ such that the induced map $T(\mathsf{R}) \to T(\mathsf{R}')$ maps $\mathscr{L}$ to $\mathscr{L}'$. To any reflective lattice there is naturally associated the pair $(\mathsf{R}(L), L/\langle \mathsf{R}(L) \rangle)$ with $\mathsf{R} = \mathsf{R}(L)$ the unique maximal choice. We call $L/\langle \mathsf{R}(L) \rangle$ the *glue code* or simply the *code* of L. More generally, if $\mathsf{R}_0 \subset \mathsf{R}(L)$ is a sub-root-system of full rank, then $L/\langle \mathsf{R}_0 \rangle \subseteq T(\mathsf{R}_0)$ is the *code of L over* R_0. Thus, the classification of reflective lattices is equivalent to the classification of certain pairs $(\mathsf{R}, \mathscr{L})$. Here R is a priori an arbitrary scaled root system. For instance, for the combinatorial type $\mathsf{A}_1 \perp \cdots \perp \mathsf{A}_1 = n\mathsf{A}_1$ (n-factors) we have arbitrary parameters $\alpha_1, \ldots, \alpha_n$ for the norms of the roots. Extending the standard terminology A, B, C, D, E, F, G for the usual root systems or Dynkin diagrams, we shall denote the irreducible scaled root systems as follows (α is any positive integer):

${}^{\alpha}\mathsf{A}_n, {}^{\alpha}\mathsf{D}_n, {}^{\alpha}\mathsf{E}_n$	roots of length 2α
${}^{\alpha}\mathsf{B}_n$	short roots of length α
${}^{\alpha}\mathsf{C}_n$	short roots of length 2α
${}^{\alpha}\mathsf{G}_2$	short roots of length 2α
${}^{\alpha}\mathsf{F}_4$	short roots of length 2α

By ${}^{\alpha}\mathsf{X}_n$ as above, we really mean the set of vectors of the corresponding root system; the lattice generated by these vectors will be denoted by $X_n = \langle {}^{\alpha}\mathsf{X}_n \rangle$. With the exception of $\mathsf{A}_1 = {}^2\mathsf{B}_1$ and $\mathsf{C}_2 = {}^2\mathsf{B}_2$, the normalization of the quadratic form is such that ${}^{\alpha}X_n$ is integral if and only if $\alpha \in \mathbf{N}$. A root system is called *normalized* if $\alpha = 1$ for all its irreducible components ${}^{\alpha}\mathsf{X}$.

The usual notion of isomorphism of two root systems R and R' is called *combinatorial equivalence* here: there exists a bijection $\varphi\colon \mathsf{R} \to \mathsf{R}'$ such that $(\varphi u, \varphi v)/(\varphi v, \varphi v) = (u, v)/(v, v)$ for all $u, v \in \mathsf{R}$. (This can be defined without referring to the scalar product.) Except for the ambiguity in rank 1 and 2 mentioned above, every scaled root system is combinatorially equivalent to a unique normalized root system; we can thus identify combinatorial types of root systems with normalized root systems.

The description of reflective lattices by pairs $(\mathsf{R}, \mathscr{L})$ can be considered as a special case (with additional structure, of course) of the following well known general principle.

Let M be an integral lattice. Then the integral over-lattices $L \supseteq M$ of the same rank are in one-to-one correspondence with the subgroups $\mathscr{L} \subseteq T(M)$ of the discriminant group of M which are totally isotropic, i.e., $\mathscr{L} \subseteq \mathscr{L}^{\perp}$, with respect to the *discriminant bilinear form* $T(M) \times T(M) \to \mathbf{Q}/\mathbf{Z}$. Since some of our results also apply to non-reflective lattices (in the sense that they deal with only a part of the root system), we occasionally use the above terminology in the following more general way: Let $\mathsf{R}_0 \subseteq \mathsf{R}(L)$ be a sub-root-system, possibly with rank $\mathsf{R}_0 < \operatorname{rank} \mathsf{R}(L)$ or $\operatorname{rank} \mathsf{R}(L) < n$. Then $\mathscr{L} := L/(\langle \mathsf{R}_0 \rangle \perp \langle \mathsf{R}_0^{\perp} \cap L \rangle) \subseteq T(\mathsf{R}_0) \oplus T(\mathsf{R}_0^{\perp} \cap L)$ is still called the (*glue*) *code of L over* R_0.

We recall that any positive definite lattice L is the orthogonal sum of uniquely determined indecomposable sublattices L_i. It is readily seen that each L_i is reflective if L is. See the end of this section for more details. Therefore, it is sufficient to classify indecomposable reflective lattices. We shall see that this assumption leads to great simplifications. In particular, large classes of root systems are eliminated.

The classification of (indecomposable) reflective lattices in a fixed dimension n can roughly be subdivided into the following steps.

Step I. Determine an appropriate set of combinatorial root systems R_0 of rank n such that each root system of an indecomposable reflective lattice of dimension n contains one of the R_0's.

One could consider all minimal root systems in dimension n (see Proposition 2.6). We shall see later in Propositions 2.8 and 2.10 how indecomposability reduces the list. On the other hand, it is not adequate to really restrict oneself to minimal root systems, as will become clear later.

Step II. For each root system R_0 from step I, now considered as a scaled root system ${}^{\alpha}\mathsf{X} \perp {}^{\beta}\mathsf{Y} \ldots$ for arbitrary $\alpha, \beta, \ldots,$ determine all glue codes $\mathscr{L}_0 \subseteq T(\mathsf{R}_0)$ such that the following holds:

(i) $\mathscr{L}_0 \subseteq T(\mathsf{R}_0)$ is invariant under $W(\mathsf{R})$.

(ii) The vectors $v \in \mathsf{R}_0$ remain primitive in the inverse image $L = L(\mathsf{R}_0, \mathscr{L}_0)$ of $\mathscr{L}_0$ in $\mathbf{Q}R_0$.

(iii) L is integral.

(iv) L is indecomposable.

Conditions (i) and (ii) are equivalent to the desired inclusion $\mathsf{R}_0 \subseteq \mathsf{R}(L)$. If one does not want to mention L explicitly, one can reformulate (iii) by requiring that $\mathscr{L}$ should be totally isotropic with respect to the discriminant bilinear form on $T(\mathsf{R}_0)$. This amounts to certain congruence conditions on the scales $\alpha, \beta \ldots$.

Step III. For each pair $(\mathsf{R}_0, \mathscr{L}_0)$ coming from Step II, calculate the full root system $\mathsf{R}(L)$ of the lattice $L = L(\mathsf{R}_0, \mathscr{L}_0)$.

Clearly, the outcome of Step III will essentially depend on the scales $\alpha, \beta \ldots$. For instance, if these are all distinct, then there will be only few possibilities for additional reflections, because these still have to permute the roots of some fixed length. Recall that we eventually only want to list pairs $(\mathsf{R}, \mathscr{L})$ where R is the full root system of $L(\mathsf{R}, \mathscr{L})$; Step III gives the necessary restrictions on the scales.

Step IV. Select one representative in each isomorphism class of pairs $(\mathsf{R}, \mathscr{L})$ (or lattices L) obtained in Step III.

For a fixed $\mathsf{R} = {}^{\alpha}\mathsf{X} \perp {}^{\beta}\mathsf{Y} \ldots$, Step IV amounts to choosing one representative in each orbit of the $\mathscr{L}$'s under the orthogonal group $\mathrm{O}(\mathsf{R})$. It is easily achieved in an ad hoc way in all dimensions for which the procedure will be fully carried out. In practice, most of the work will already be done in Step III where we shall of course suppress codes which are equivalent to others with respect to automorphisms of R. It depends on the choice of root systems we start from in Step I to what extent the same lattice might be obtained from different sub-root-systems.

We now prove a series of general propositions which allow us to carry out explicitly Steps I, II, and III of the above procedure in small dimensions.

Before formulating the main general result—related to Step II—which we prove about the reflective lattices, we have to introduce the reduced discriminant group of R. If M is any lattice and $\alpha \in \mathbf{Q}$, then $({}^{\alpha}M)^{\#} = (1/\alpha)M^{\#}$ (as abelian groups). Thus, if $\alpha \in \mathbf{N}$, then $M^{\#}/M$ is a subgroup of $({}^{\alpha}M)^{\#}/{}^{\alpha}M$. In particular, we see that $T(\mathsf{X}) \subseteq T({}^{\alpha}\mathsf{X})$ for any root system X and any $\alpha \in \mathbf{N}$. Now let R be an arbitrary scaled root system. Write $\mathsf{R} = {}^{\alpha}\mathsf{X} \perp {}^{\beta}\mathsf{Y} \perp \dots$, where each of $\mathsf{X}, \mathsf{Y}, \dots$ is irreducible, normalized, and $\neq \mathsf{B}_1, \mathsf{B}_2$ if $\alpha, \beta, \dots$ is even. Thus R is uniquely determined. The desired subgroup $\overline{T}(\mathsf{R}) \subseteq T(\mathsf{R})$ will be defined component-wise: $\overline{T}(\mathsf{R}) := \overline{T}({}^{\alpha}\mathsf{X}) \oplus \overline{T}({}^{\beta}\mathsf{Y}) \oplus \cdots$. If X is not of type B or C, we simply set $\overline{T}({}^{\alpha}\mathsf{X}) := T(\mathsf{X})$. For a component ${}^{\beta}\mathsf{B}_l$ or ${}^{\gamma}\mathsf{C}_l$, we shall single out a still smaller subgroup as follows. We use the following explicit description of the root systems ${}^{\alpha}\mathsf{B}_l$ and ${}^{\alpha}\mathsf{C}_l$ and their lattices:

$$ {}^{\alpha}B_l = \oplus \mathbf{Z} e_i, \ (e_i, e_j) = \alpha \delta_{i_j} $$

$$ \text{roots: } \pm e_i \text{ of norm } \alpha, \ \pm e_i \pm e_j \text{ of norm } 2\alpha; $$

$$ {}^{\alpha}C_l = \left\{ \Sigma x_i e_i \in {}^{\alpha}B_l | \Sigma x_i \equiv 0(2) \right\} $$

$$ \text{roots: } \pm e_i \pm e_j \text{ of norm } 2\alpha, \ \pm 2 e_i \text{ of norm } 4\alpha. $$

Now we set $\overline{T}({}^{\beta}\mathsf{B}_l) = 0$ if β is odd, and

$$ \overline{T}(\mathsf{R}) = \mathbf{F}_2 \bar{g} \cong \mathbf{F}_2, $$

where
$$ g = \tfrac{1}{2}(e_1 + \cdots + e_l) \qquad \text{if } \mathsf{R} = {}^{2\alpha}\mathsf{B}_l $$
$$ g = e_1 \qquad \text{if } \mathsf{R} = {}^{\alpha}\mathsf{C}_l. $$

Notice that $\overline{T}$ is independent of the choice of coordinates.

DEFINITION 2.1. Let R be an arbitrary scaled root system. The subgroup $\overline{T}(\mathsf{R})$ of $T(\mathsf{R})$ is called the *reduced discriminant group* of R.

THEOREM 2.2. *Let L be any lattice and R a root system contained in L. Let $\mathscr{L} = L/(\langle \mathsf{R} \rangle \perp (\mathsf{R}^{\perp} \cap L))$ be the glue code of L over R, and $\mathrm{pr}\,\mathscr{L} \subset T(\mathsf{R})$ its projection onto the factor R. Then the elements of R are actually roots of L if and only if $\mathscr{L}$ is contained in the reduced discriminant group $\overline{T}(\mathsf{R})$, and furthermore $\mathscr{L}$ contains no elements of the form $e/2$, where e is a basis vector of an A_1-component.*

The last condition says that, if $\mathsf{R}_0 = n_0 \mathsf{A}_1$ is the product of all copies A_1 in $\mathsf{X} \perp \mathsf{Y} \perp \dots$, then $\mathscr{L} \cap T(\mathsf{R}_0) \subseteq \mathbf{F}_2^{n_0}$ has minimal weight ≥ 2.

COROLLARY 2.3. *Reflective lattices L are classified by pairs $(\mathsf{R}, \mathscr{L})$, where R is a scaled root system and $\mathscr{L} \subseteq \overline{T}(\mathsf{R})$ a certain subgroup of the reduced discriminant group.*

COROLLARY 2.4. *To a reflective lattice whose components are of type B or C we can canonically associate a binary code of length m, where m is the number of components of $\mathsf{R}(L)$.*

Proof of Theorem 2.2. We can treat the components of R separately and may thus assume that $\mathsf{R} = {}^{\alpha}\mathsf{X}$ is irreducible. First consider the case where X is of type A, D, or E. If $\mathsf{R} \subseteq \mathsf{R}(L)$, then for any root $a \in \mathsf{X}$ we have

$$2(a, L)/(a, a) = (a, L)/\alpha = \big(a, \mathrm{pr}_X(L)\big)/\alpha \in \mathbf{Z}.$$

That is, $\mathrm{pr}_X(L) \subseteq \alpha\langle {}^{\alpha}\mathsf{X}\rangle^{\#} = \langle \mathsf{X}\rangle^{\#}$, as desired.

Conversely, assume that $\mathscr{L} \subseteq \overline{T}(\mathsf{R})$. Then the first part of the proof shows that even $s_a(u) \equiv u \bmod \langle \mathsf{R}\rangle$ for all $u \in L$, i.e., s_a fixes $\mathscr{L}$ pointwise. (This is well known: the Weyl group acts trivially on the weight lattice modulo the root lattice, and for types A, D, E, the weight lattice is the dual of the root lattice.) In particular, $s_a \in \mathrm{O}(L)$. For all irreducible diagrams A_l, D_l, E_l, except A_1, it is immediate from the explicit bases that $a/m \notin \langle \mathsf{R}\rangle^{\#}$ for all $a \in \mathsf{R}$, $m \in \mathbf{N}$, $m > 1$. That is, the roots remain automatically primitive in any L given by $\mathscr{L} \subseteq T(\mathsf{R})$.

The case $\mathsf{R} = {}^{\alpha}\mathsf{B}_l$, α odd is settled by Proposition 2.7 below.

Now we treat the case where $\mathsf{R} = {}^{2\alpha}\mathsf{B}_l$ or ${}^{\alpha}\mathsf{C}_l$, and we first assume that $\mathsf{R} \subseteq \mathsf{R}(L)$. By the previous case, applied to $l\,{}^{\alpha}\mathsf{A}_1 \subset {}^{2\alpha}\mathsf{B}_l$, respectively ${}^{\alpha}\mathsf{D}_l \subset {}^{\alpha}\mathsf{C}_l$, we know that the corresponding projection of $\mathscr{L}$ is contained in $T(l\mathsf{A}_1) = \oplus \mathbf{F}_2\overline{e_i/2}$, respectively $T(\mathsf{D}_l) = T(\mathsf{C}_l) = \langle \bar{e}_1, \bar{g}\rangle \cong \mathbf{F}_2 \times \mathbf{F}_2$, where $g = \frac{1}{2}(e_1 + \cdots + e_l)$.

Consider first the case $\mathsf{R} = {}^{2\alpha}\mathsf{B}_l$, and assume that L contains a vector of the form

$$x = \frac{e_1}{2} + x',\; x' \in \langle e_1, e_2\rangle^{\perp}.$$

Let σ be the reflection in $(e_1 - e_2)^{\perp}$, then $\sigma e_1 = e_2$ and $\sigma x' = x'$, and thus $x - \sigma x = \frac{1}{2}(e_1 - e_2) \in L$. But $e_1 - e_2 \in {}^{2\alpha}\mathsf{B}_l \subset \mathsf{R}(L)$ is primitive, a contradiction.

Now consider the case $\mathsf{R} = {}^{\alpha}\mathsf{C}_l$, and assume that L contains a vector of the form

$$x = \tfrac{1}{2}(\pm e_1 + e_2 + \cdots + e_l) + x',\; x' \in \mathsf{R}^{\perp}.$$

Using the reflection τ in $e_1^{\perp}$, we see that $x - \tau x = e_1 \in L$. But then $2e_1 \in \mathsf{R} \subset \mathsf{R}(L)$ is not primitive, a contradiction.

Conversely, it is easily seen that all elements of ${}^{2\alpha}\mathsf{B}_l$ respectively ${}^{\alpha}\mathsf{C}_l$ really are roots of L if $\mathrm{pr}\,\mathscr{L} \subseteq \overline{T}(\mathsf{R})$. ∎

One important consequence of 2.2 is that it naturally leads to a notion of combinatorial equivalence of reflective lattices which refers to both the root system and the code, but it is considerably weaker than isometry. To that end, observe that any combinatorial equivalence $\varphi\colon \mathsf{R} \to \mathsf{R}'$ between two scaled root systems induces an isomorphism, defined component-wise, of the associated reduced discriminant groups $\overline{T}(\mathsf{R})$ and $\overline{T}(\mathsf{R}')$.

DEFINITION 2.5. Two reflective lattices L and L', given by root systems $\mathsf{R} = \mathsf{R}(L)$ and $\mathsf{R}' = \mathsf{R}(L')$ and glue codes $\mathscr{L} \subset \overline{T}(\mathsf{R})$ and $\mathscr{L} \subset \overline{T}(\mathsf{R}')$ are called *combinatorially equivalent* if there exists a combinatorial equivalence from R onto R' such that the associated isomorphism from $\overline{T}(\mathsf{R})$ onto $\overline{T}(\mathsf{R}')$ maps $\mathscr{L}$ onto $\mathscr{L}'$.

For Step I of the above general procedure, the following result is useful.

PROPOSITION 2.6. *Every root system contains a sub-root-system of the same dimension having only components* A_l, $l \geq 1$ (*up to scaling*). *More precisely, the irreducible root systems* R *contain the following sub-root-systems* R_0:

R	R_0
B_n	$n\mathsf{A}_1$
C_n	$n\mathsf{A}_1$
D_n, *n even*	$n\mathsf{A}_1$
D_n, *n odd*	$(n-3)\mathsf{A}_1\mathsf{A}_3$
F_4	$4\mathsf{A}_1, 2\mathsf{A}_2$
E_6	$\mathsf{A}_1\mathsf{A}_5, 3\mathsf{A}_2$
E_7	$7\mathsf{A}_1, \mathsf{A}_1 2\mathsf{A}_3, \mathsf{A}_2\mathsf{A}_5, \mathsf{A}_7$
E_8	$8\mathsf{A}_1, 2\mathsf{A}_1 2\mathsf{A}_3, \mathsf{A}_1\mathsf{A}_2\mathsf{A}_5, \mathsf{A}_1\mathsf{A}_7, 4\mathsf{A}_2, 2\mathsf{A}_4, \mathsf{A}_8$

These sub-root-systems exhaust all minimal sub-root-systems of full rank.

This result is certainly well known, or can be considered as a standard exercise. The main step is contained in [Bou68, Chap. VI, Sect. 4, Exer. 4], where the maximal sub-root-systems are determined. The minimal ones are then found simply by descending as far as possible without decreasing the rank.

In Proposition 2.6, only the combinatorial types of the R_0 are listed. For the root systems having roots of different lengths, the scaling of the R_0's is as follows:

R	R_0
B_n	$k\mathsf{B}_1(n-k)\mathsf{A}_1$, $0 \leq k \leq n$, $n-k \equiv 0(2)$
C_n	$k\mathsf{A}_1(n-k)\,{}^2\mathsf{A}_1$, $0 \leq k \leq n$, $k \equiv 0(2)$
F_4	$2\mathsf{A}_1 2\,{}^2\mathsf{A}_1, \mathsf{A}_2\,{}^2\mathsf{A}_2$
G_2	$\mathsf{A}_1\,{}^3\mathsf{A}_1, \mathsf{A}_2$

We now turn to condition (iv) of Step II, that is, to decomposability of lattices. First of all, we notice that indecomposable lattices contain only roots of even norm:

PROPOSITION 2.7. *Let v be a root in a lattice L such that (v, v) is odd. Then v splits off*: $L = \mathbf{Z}v \perp (v^{\perp} \cap L)$.

Proof. Since v is a root, $2(v, x)/(v, v) \in \mathbf{Z}$ for all $x \in L$, necessarily $(v, x)/(v, v) \in \mathbf{Z}$. Thus

$$x = \left(x - \frac{(v, x)}{(v, v)}v\right) + \frac{(v, x)}{(v, v)}v \in (v^{\perp} \cap L) + \mathbf{Z}v,$$

as desired. ▮

The next two propositions which are immediate corollaries of Theorem 2.2 reduce very efficiently the number of (minimal) root systems which remain to be considered under assumption (iv).

PROPOSITION 2.8. *Let* R *be any root system, assume that* $\mathsf{R} = \mathsf{R}_1 \perp \mathsf{R}_2$ *with* $|\overline{T}(\mathsf{R}_1)|$ *and* $|\overline{T}(\mathsf{R}_2)|$ *co-prime. Then any reflective lattice L with* $\mathsf{R} \subseteq \mathsf{R}(L)$ *of full rank is decomposable as* $L = L \cap \mathbf{Q}\mathsf{R}_1 \perp L \cap \mathbf{Q}\mathsf{R}_2$.

Proof. Obviously $\mathscr{L} = \mathscr{L} \cap \overline{T}(\mathsf{R}_1) \perp \mathscr{L} \cap \overline{T}(\mathsf{R}_2)$, where as usual $\mathscr{L} = L/\langle \mathsf{R} \rangle \subseteq \overline{T}(\mathsf{R})$ is the code of L over R. ▮

The next proposition is a little technical to state but equally easy to prove; it allows to exclude quite frequently factors $\mathsf{A}_3 = \mathsf{D}_3$ from the list of root systems to be considered.

PROPOSITION 2.9. *Let* $L, \mathsf{R}, \mathsf{R}_1, \mathsf{R}_2$ *be as in Proposition* 2.8, *assume that* $\mathsf{R}_1 = {}^{\alpha}\mathsf{D}_m$ *with odd m, and that the exponent of* $\overline{T}(\mathsf{R}_2)$ *is not divisible by* 4. *Then* $\mathsf{R}(L) \supseteq {}^{2\alpha}\mathsf{B}_m \perp \mathsf{R}_2$ *or* ${}^{\alpha}\mathsf{C}_m \perp \mathsf{R}_2$.

Proof. We want to show that $\mathsf{R}(L) \cap \mathsf{R}_1$ is strictly larger than D_m. Denote by $\mathrm{pr}_1, \mathrm{pr}_2$ the projections of $\overline{T}(\mathsf{R})$ onto $\overline{T}(\mathsf{R}_1) \cong \mathbf{Z}/4\mathbf{Z}$, resp. $\overline{T}(\mathsf{R}_2)$. Take D_m in its standard form $\{(z_1, z_2, \dots, z_m) \in \mathbf{Z}^m | \sum x_i \equiv 0(2)\}$. Then $T(D_m)$ is generated by (the class of) $u := (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. First of all, notice that, if $2u \in L$, then $e_1, e_2, \dots, e_m \in L$, and these vectors are additional roots:

$$\frac{2(e_1, x)}{(e_1, e_1)} = 2(e_1, x) \in \mathbf{Z} \text{ for all } x \in L.$$

Now assume that $2u \notin L$. Since $T(\mathsf{D}_m) = \langle u \rangle$ is cyclic, this means that $\mathscr{L} \cap T(\mathsf{D}_m) = \{0\}$, that is, the restriction of pr_2 to $\mathscr{L}$ is injective. By the assumption on $\overline{T}(\mathsf{R}_2)$, this implies that the exponent of $\mathscr{L}$ is not divisible by 4, and thus $\mathrm{pr}_1(\mathscr{L}) \subseteq \langle 2u \rangle$. This means that the projection of L to $\mathbf{Q}\mathsf{D}_m$ is contained in $\mathbf{Z}^m$. If follows that $2e_1, 2e_2, \dots, 2e_m$ (which are primitive by our assumption $2u \notin L$) are roots:

$$\frac{2(2e_i, x)}{(2e_i, 2e_i)} = (e_i, x) \in \mathbf{Z} \text{ for all } x \in L. \quad \blacksquare$$

Observe that Proposition 2.9 is applicable whenever $2T(\mathsf{R}_2) = 0$, in particular in the case $\mathsf{R}_2 = \emptyset$, $m = \dim L$. This means that D_m with m odd never occurs as the root system of a reflective lattice.

PROPOSITION 2.10. *Let L be an arbitrary lattice*, X *a component of its root system*, $\mathsf{X} \cong {}^{\alpha}\mathsf{G}_2$, ${}^{\alpha}\mathsf{F}_4$ *or* ${}^{\alpha}\mathsf{E}_8$. *Then L is decomposable as* $L = (L \cap \mathbf{Q}\mathsf{X}) \perp (L \cap \mathbf{Q}\mathsf{X}^{\perp})$. *Furthermore,* $L \cap \mathbf{Q}\mathsf{X} \cong {}^{\alpha}A_2, {}^{\alpha}D_4, {}^{\alpha}\mathsf{E}_8$, *respectively.*

Proof. Let us consider the case F_4. We use the above standard coordinates for $\mathsf{R}_1 = {}^{\alpha}\mathsf{F}_4 \subseteq \mathbf{Q}^4$. Write $V = \mathbf{Q}L = \mathbf{Q}^4 \perp V'$ and consider an arbitrary lattice vector $v = (x_1, x_2, x_3, x_4, v')$. By assumption, $2e_i$, $i = 1, \ldots, 4$ and $u := e_1 + e_2 + e_3 + e_4$ are roots of L. Thus $2(v, 2e_i)/4\alpha = x_i \in \mathbf{Z}$ and $2(v, u)/4\alpha = (x_1 + x_2 + x_3 + x_4)/2 \in \mathbf{Z}$. This shows that $(x_1, x_2, x_3, x_4) \in D_4$, as claimed.

The case G_2 is similar. The case E_8 is trivial, since ${}^{\alpha}E_8$ is α-modular. $\blacksquare$

We close this section by giving some more details on Step II, condition (iv). We state two propositions which make the reduction to the indecomposable case explicit and precise.

PROPOSITION 2.11. *Let L be any lattice, and $L = L_1 \perp \cdots \perp L_s$ its decomposition into indecomposables, $L_i = \mathbf{Z}e_i$ for $i = 1, \ldots, r$ and* $\dim L_i \geq 2$ *for $i = r + 1, \ldots, s$. Then*

$$\mathsf{R}(L) = \left\{ \pm e_i \pm e_j | 1 \leq i < j \leq r \text{ such that } (e_i, e_i) = (e_j, e_j) \right\} \cup \bigcup_{i=r+1}^{s} \mathsf{R}(L_i).$$

Proof. If v is any root, the reflection s_v permutes the L_i. Since it fixes a hyperplane, it must leave invariant all L_i of dimension at least two, and can interchange at most two, necessarily isometric, one-dimensional components. Furthermore, if M is any s-invariant sublattice, then it is readily seen that $v \in M$ or $v \in M^{\perp}$. The result follows immediately. $\blacksquare$

PROPOSITION 2.12. *Let a reflective lattice L be given by a root system* R *and a code* $\mathscr{L} \subseteq \overline{T}(\mathsf{R})$. *Exclude the trivial case that L is an orthogonal sum of one-dimensional lattices* (*and thus* $\mathscr{L} = 0$). *If L is decomposable, then the code $\mathscr{L}$ is decomposable along an appropriate decomposition of* R. *That is,* *one can decompose* $\mathsf{R} = \mathsf{R}_1 \perp \mathsf{R}_2$ *non-trivially such that* $\mathscr{L} = (\mathscr{L} \cap \overline{T}(\mathsf{R}_1)) \perp (\mathscr{L} \cap \overline{T}(\mathsf{R}_2))$.

Proof. Decompose L as in 2.11, set $L_0 = \sum_{i=1}^{r} \mathbf{Z} e_i$, $V_0 = \mathbf{Q} L_0$. Using this proposition, write $\mathsf{R} = \mathsf{R}_0 \perp \mathsf{R}_{r+1} \perp \cdots \perp \mathsf{R}_s$, where $\mathsf{R}_0 = \mathsf{R} \cap V_0$ and $\mathsf{R}_i = \mathsf{R}(L_i)$, $i = r+1, \dots, s$. Then it is clear that

$$\begin{aligned} \mathscr{L} = L/R &= L_0/R_0 \perp L_{r+1}/R_{r+1} \perp \cdots \perp L_s/R_s \\ &= \{0\} \perp \left(\mathscr{L} \cap T(\mathsf{R}_{r+1})\right) \perp \cdots \perp \left(\mathscr{L} \cap T(\mathsf{R}_s)\right). \quad \blacksquare \end{aligned}$$

Notice that the case $\mathscr{L} \cap \overline{T}(\mathsf{R}_i) = 0$ is allowed in Proposition 2.12 (and obviously must be allowed).

3. REFLECTIVE LATTICES WITH COMPONENTS $\mathsf{B}_l, \mathsf{C}_l$, OR A_2

In this section, we prove some results concerning Step III of the general classification procedure outlined in Section 2. In certain cases, we determine the full root system $\mathsf{R}(L) \supseteq \mathsf{R}_0$ of a reflective lattice L given by some root system R_0 and a code $\mathscr{L}_0 \subseteq T(\mathsf{R}_0)$. The particularly important cases (for a full classification in small dimensions) $\mathsf{R}_0 = n\mathsf{A}_1$ and $m\mathsf{A}_2$ are treated in detail in Theorems 3.1 and 3.2; the last result, Proposition 3.3, deals with components of type B or C. For the explicit enumeration of all reflective lattices of some specified dimension, the resulting conditions on $\mathscr{L}_0$ and on the scales $\alpha, \beta, \dots$ for the equality $\mathsf{R}_0 = \mathsf{R}(L)$ are the most important consequences of this section.

In the following theorem, vectors $c \in \mathbf{F}_2^n$ (“codewords”) are written $110\dots0$ etc. The corresponding coset in $\frac{1}{2}L_0/L_0$, where $L_0 = \mathbf{Z}^n$, is denoted by $L_0(c) = \{\frac{1}{2}(x_1, \dots, x_n) | x_i \in \mathbf{Z}, (x_1, \dots, x_n) \equiv c \pmod 2\}$.

THEOREM 3.1. *Let* $\mathsf{R}_0 = {}^{\alpha_1}\mathsf{A}_1 {}^{\alpha_2}\mathsf{A}_1 \cdots {}^{\alpha_n}\mathsf{A}_1$, $n \geq 3$, *and* $\mathscr{L}_0 \subseteq \mathbf{F}_2^n = \overline{T}(\mathsf{R}_0)$. *The additional roots of the lattice* $L := L(\mathsf{R}_0, \mathscr{L}_0)$ *are as follows, up to permutation of the coordinates.*

I. *A vector* $v \in L_0 \setminus \mathsf{R}_0 = \mathbf{Z}^n \setminus \{\pm e_i\}$ *is a root if and only if*

$$v = (\pm 1, \pm 1, 0, \dots, 0) \qquad \alpha_1 = \alpha_2$$
$$110 \cdots 0 \in \mathscr{L}_0^{\perp} \setminus \mathscr{L}_0$$

Let $c \in \mathscr{L}_0 \setminus \{0\}$. *A vector* v *in the coset* $L_0(c)$ *is a root if and only if one of the following cases II to V holds*:

II.

$c = 110 \cdots 0$ $\qquad \alpha_1 = \alpha_2$

$v = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, 0, \ldots, 0\right)_{\alpha_1}$ $\qquad 110 \cdots 0 \in \mathscr{L}_0$

III.

$c = 110 \cdots 0$ $\qquad \alpha_1 = \alpha_2 = 2\alpha_3$

$v = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm 1, 0, \ldots, 0\right)_{2\alpha_1}$ $\qquad 110 \cdots 0 \in \mathscr{L}_0, 1110 \cdots 0 \in \mathscr{L}_0^{\perp}$

IV.

$c = 1110 \cdots 0$ $\qquad \alpha_1 = 2\alpha_2 = 2\alpha_3$

$v = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, 0, \ldots, 0\right)_{\alpha_1}$ $\qquad 1110 \cdots 0 \in \mathscr{L}_0, 0110 \cdots 0 \in \mathscr{L}_0^{\perp}$,

or $n = 3$

V.

$c = 11110 \cdots 0$ $\qquad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$

$v = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, 0, \ldots, 0\right)_{2\alpha_1}$ $\qquad 11110 \cdots 0 \in \mathscr{L}_0 \cap \mathscr{L}_0^{\perp}$

Proof. Let $v = (x_1, x_2, \ldots, x_n)$ be an additional root, assume that $x_1, \ldots, x_k \neq 0$, $x_{k+1} = \cdots = x_n = 0$. then $e_1, \ldots, e_k$ together with v generate an irreducible root system. In particular, there are at most two different values for the lengths of these roots. Assume that

$$\alpha_1 = \cdots = \alpha_l = 2\beta, \ \alpha_{l+1} = \cdots = \alpha_k = \beta$$

for some β and an index $l \leq k$ (possibly $l = 0$).

First Case. $(v, v) = 2\beta$, i.e., v is a "short root." Since $(v, v)/2\beta + 2(x_1^2 + \cdots + x_l^2) + x_{l+1}^2 + \cdots + x_k^2 = 1$ and $x_i \in \mathbf{Z} \cdot \frac{1}{2}$ for all i, one of the following must hold:

(a) $\qquad k = l = 2 \qquad v = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm 1, 0, \ldots, 0\right)$

(b) $\qquad k = 3, l = 1 \qquad v = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, 0, \ldots, 0\right)$

(c) $\qquad k = 4, l = 0 \qquad v = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, 0, \ldots, 0\right)$.

The condition that this v actually be a root is that $2(v, w)/(v, v) \in \mathbf{Z}$ for all $w \in L$, i.e., $4(v_1 w_1 + \ldots v_l w_l) + 2(v_{l+1} w_{l+1} + \cdots + v_k w_k) \in \mathbf{Z}$. Of course this just means $w(v_{l+1} w_{l+1} + \cdots + v_k w_k) \in \mathbf{Z}$. This condition depends only on $v_i \bmod \mathbf{Z}$, i.e., on the image of v in $\mathscr{L}$. It amounts to the condition as given in Case II (no condition), Case IV, or Case V of the theorem, respectively.

Second Case. $(v,v) = 4\beta$, i.e., v is a "long root." Then $2(x_1^2 + \cdots + x_l^2) + x_{l+1}^2 + \cdots + x_k^2 = 2$, and thus

$$\text{(d)} \qquad k = 2, l = 0 \qquad v = (\pm 1, \pm 1, 0, \dots, 0)$$

$$\text{(e)} \qquad k = 3, l = 2 \qquad v = \left(\pm \tfrac{1}{2}, \pm \tfrac{1}{2}, \pm 1, 0, \dots, 0\right).$$

The condition for v being a root now is $2(v_1 w_1 + \cdots v_l w_l) + (v_{l+1} w_{l+1} + \cdots + v_k w_k) \in \mathbf{Z}$. This again depends only on $v \bmod L$ and is what we required in Case I, resp. Case III. The additional condition $110 \cdots 0 \notin \mathscr{L}_0$ in Case I expresses that $(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0) \notin L$, i.e., that v as in (d) is actually primitive. ∎

The new roots of type III or IV in Theorem 3.1 never arise separately, but always in connection with new roots of type II or I, respectively. More precisely, we can say the following.

Remark. Suppose that the minimal weight of $\mathscr{L}_0$ is at least 2.

(a) Under the assumption of Theorem 3.1, III, the root system $\mathsf{R}(L)$ contains ${}^{\alpha_3}\mathsf{B}_3$ as a sub-system.

(b) Under the assumption of Theorem 3.1, IV, the root system $\mathsf{R}(L)$ contains ${}^{\alpha_2}\mathsf{C}_3$ as a sub-system.

To see this, we simply observe that $L \cap (\mathbf{Q}e_1 + \mathbf{Q}e_2 + \mathbf{Q}e_3)$ contains the following roots:

Case III:

2 old short roots $\pm e_3$, 4 new short roots of type II,

4 old long roots $\pm e_1, \pm e_2$, 8 new long roots of type III.

Case IV:

4 old short roots $\pm e_2, \pm e_3$, 8 new short roots of type IV,

2 old long roots $\pm e_1$, 4 new long roots of type I (indices 2, 3).

The condition $0110 \cdots 0 \notin \mathscr{L}_0$ of I is fulfilled, since $10 \cdots 0 \notin \mathscr{L}_0$, by assumption.

We now come to the root system $m\mathsf{A}_2$ (with arbitrary scaling). By $\{e_i, f_i\}$ we denote a root basis for the ith factor; the ith factor of the reduced discriminant group $\overline{T}$ is generated by the residue class $\bar{g}_i$ of $g_i := (e_i - f_i)/3 \in A_2^\#$; the $\bar{g}_i$ define an isomorphism $\overline{T} \cong \mathbf{F}_3^m$. For $c \in \mathbf{F}_3^m$, the notation $L_0(c) \subset L_0^{\#}$, where $L_0 = mA_2$, is analogous to the previous case $L_0 = nA_1$.

THEOREM 3.2. *Let* $\mathsf{R}_0 = {}^{\alpha_1}\mathsf{A}_2{}^{\alpha_2}\mathsf{A}_2 \cdots {}^{\alpha_m}\mathsf{A}_2$, $m \geq 2$, *and* $\mathscr{L}_0 \subseteq \mathbf{F}_3^m = \overline{T}(\mathsf{R}_0)$. *The additional roots of the lattice* $L := L(\mathsf{R}_0, \mathscr{L}_0)$ *are as follows, up to permutation of the factors* A_2.

I. *A vector* $v \in L_0 \setminus \mathsf{R}_0$ *is a root if and only if* $v \in \{\pm(e_1 - f_1), \pm(2e_1 + f_1), \pm(e_1 + 2f_2)\}$ (*i.e., v is a long root in one of the* A_2*-factors*), *and* $\mathscr{L}_0 \subseteq \{0\} \times \mathbf{F}_3^{m-1}$. *In this case, L is decomposable as* $L = L \cap (\mathbf{Q}e_1 + \mathbf{Q}f_1) \perp L \cap (\mathbf{Q}e_2 + \cdots + \mathbf{Q}f_m)$.

Let $c \in \mathscr{L}_0 \setminus \{0\}$. *The coset* $L_0(c)$ *contains new roots if and only if one of the following cases* II *to* IV *holds, up to sign factors*:

II. $c = 10\ldots0$, $\alpha_1 \equiv 0 \pmod 3$, *new roots* $g_1, -e_1 + g_1, g_2 + f_1$. *In this case, L is decomposable as in Case* I.

III. $c = 110\ldots0$, $\alpha_1 = 2\alpha_2$, $9 + 9$ *new roots of norms* α_1 *and* $2\alpha_2$. *In this case,* $\mathsf{R}(L) \cap (\mathbf{Q}e_1 + \mathbf{Q}f_1 + \mathbf{Q}e_2 + \mathbf{Q}f_2) = {}^{\alpha_2}\mathsf{F}_4$ *and L is decomposable as* $L = L \cap (\mathbf{Q}e_1 + \mathbf{Q}f_1 + \mathbf{Q}e_2 + \mathbf{Q}f_2) \perp L \cap (\mathbf{Q}e_3 + \cdots + \mathbf{Q}f_m)$.

IV. $c = 1110\ldots0$, $\alpha_1 = \alpha_2 = \alpha_3$, $u_1 + u_2 + u_3 \equiv 0 \pmod 3$ *for all* $u \in \mathscr{L}_0$, 27 *new roots of norm* $2\alpha_1$. *In this case,* $\mathsf{R}(L) \cap (\mathbf{Q}e_1 + \cdots + \mathbf{Q}f_3) \cong {}^{\alpha_1}\mathsf{E}_6$.

We do not give a detailed proof here, which is similar to the proof of Theorem 3.1, but simpler because of the fact that G_2, F_4, E_6, and E_8 are the only irreducible over-root-system with the same rank of a root system consisting only of scaled copies of A_2.

The decomposability in Cases I to III of the last theorem follows from Proposition 2.10.

The next proposition refers to the description of a reflective lattice with m components R_i of type B_l or C_l by a binary code of length m, as given in Corollary 2.4. In that corollary, it was allowed that $\mathsf{R}_1 \perp \mathsf{R}_2 \perp \cdots \perp \mathsf{R}_m$ is only a sub-root-system of $\mathsf{R}(L)$. In analogy with Theorem 3.2 we study now the question of the full root system as depending on the scales of the R_i and $\mathscr{L}$. An over-root-system of ${}^{\alpha}\mathsf{B}_l$, $l \geq 3$, can only be a ${}^{\beta}\mathsf{B}_m$, $m > l$, with $\beta = \alpha$; similarly for ${}^{\alpha}\mathsf{C}_l$. The only over-root-system of the same rank could be the appropriate ${}^{\beta}\mathsf{F}_4$ ($\beta = \alpha/2$ or $\beta = \alpha$) which was treated in Proposition 2.10. Therefore, the following proposition, if necessary applied several times, covers all cases of enlarging components of type B or C. As in Theorem 2.2, we allow non-reflective lattices and further components of arbitrary type in the root system.

We wish to emphasize that the important case of one component being one-dimensional is included in the following proposition. We look at this situation more closely before formulating the general result. First of all, we extend the definition of the root system C_l, $l \geq 2$ to the case $l = 1$ by setting $\mathsf{C}_1 = {}^2\mathsf{A}_1 = {}^4\mathsf{B}_1$ (one pair of roots of norm 4). This definition is suggested by the standard representation of C_l (see the discussion preceding Definition 2.1), and we shall now see that it fits perfectly well to describe all sub-root-systems of C_n, $n \geq 3$, of combinatorial type $\mathsf{A}_1\mathsf{C}_{n-1}$. Indeed, one sees immediately that, in standard coordinates, the only choice up to automorphism for such a sub-root-system is $\{\pm 2e_1\} \perp \{\underline{x} = (x_1, \ldots, x_n) \in \mathsf{C}_n | x_1 = 0\} \cong {}^2\mathsf{A}_1\mathsf{C}_{n-1} = \mathsf{C}_1\mathsf{C}_{n-1}$. Similarly, for B_n, $n \geq 3$, the only choice is $\mathsf{B}_1\mathsf{B}_{n-1}$. Therefore, the following proposition really covers all cases of gluing some ${}^\alpha\mathsf{B}_1$ with some ${}^\beta\mathsf{B}_l$ or ${}^\gamma\mathsf{C}_l$, $l \geq 2$.

PROPOSITION 3.3 (Fusion of Components B or C). *Let L be a lattice with $\mathsf{R}(L) \supseteq \mathsf{R}_1 \perp \mathsf{R}_2$ where R_1 and R_2 are of type B or C. Let $\mathscr{L} = L/\langle \mathsf{R}_1 \perp \mathsf{R}_2\rangle \perp (L \cap \langle \mathsf{R}_1 \perp \mathsf{R}_2\rangle^\perp)$ its glue code over $\mathsf{R}_1 \perp \mathsf{R}_2$. Denote by $\bar{g}_i$, $i = 1, 2$, the generator of $\overline{T}(\mathsf{R}_i) \cong \mathbf{F}_2$.*

(a) *Let $\mathsf{R}_1 = {}^\alpha\mathsf{B}_l$, $\mathsf{R}_2 = {}^\alpha\mathsf{B}_m$, $m \geq 2$. Then $\mathsf{R}(L) \supseteq {}^\alpha\mathsf{B}_{l+m}$ if and only if $\bar{g}_1 + \bar{g}_2 \in \mathscr{L}^\perp$.*

(b) *Let $\mathsf{R}_1 = {}^\alpha\mathsf{C}_l$, $\mathsf{R}_2 = {}^\alpha\mathsf{C}_m$, $m \geq 2$. Then $\mathsf{R}(L) \supseteq {}^\alpha\mathsf{C}_{l+m}$ if and only if $\bar{g}_1 + \bar{g}_2 \in \mathscr{L}$.*

Proof. (a) We use the standard basis vector $e_1, \ldots, e_l$ for $\mathsf{R}_1 \cong \mathsf{B}_l$ and $e_{l+1}, \ldots, e_{l+m}$ for R_2, so that

$$g_1 = \frac{1}{2}\sum_{i=1}^{l} e_i, \qquad g_2 = \frac{1}{2}\sum_{i=l+1}^{l+m} e_i.$$

First observe that ${}^\alpha\mathsf{B}_{l+m} \subseteq \mathsf{R}(L)$ if and only if $e_1 + e_{l+1} \in \mathsf{R}(L)$. Let $U = (\mathsf{R}_1 \perp \mathsf{R}_2)^\perp$, and denote by σ the reflection in $(e_1 - e_{l+1})^\perp$. Let $v = x_1 g_1 + x_2 g_2 + u$, $x_i \in \{0, 1\}$, $u \in U$ be a typical glue vector of L over $\mathsf{R}_1 \perp \mathsf{R}_2$. If $g_1 + g_2 \in \mathscr{L}^\perp$, then x_1 and x_2 are simultaneously 0 or 1, and clearly $\sigma v = v$. Thus $\sigma \in \mathsf{O}(L)$. Since $\frac{1}{2}(e_1 - e_{l+1})$ is not in L (this uses $m \geq 2$ and $L \subseteq \langle \mathsf{R}_1 \perp \mathsf{R}_2, g_1, g_2\rangle + U$), the vector $e_1 - e_{l+1}$ is actually primitive and thus a root. Now assume that $g_1 + g_2 \notin \mathscr{L}^\perp$, but still $e_1 - e_{l+1} \in \mathsf{R}(L)$ and thus $\sigma L = L$. There exists a v as above with $x_1 = 1$, $x_2 = 0$. Then $v - \sigma v = \frac{1}{2}(e_1 - e_{l+1}) \in L$, but we already observed that this is not the case.

(b) We use the same notation $e_1, \ldots, e_{l+m}, U, \sigma$ as in part a) and recall from the proof of Theorem 2.2 that $L \subseteq \mathbf{Z}e_1 + \cdots + \mathbf{Z}e_{l+m} + U$. Observe that ${}^\alpha\mathsf{C}_{l+m} \subseteq \mathsf{R}(L)$ if and only if $e_1 - e_{l+1} \in \mathsf{R}(L)$. If $\bar{g}_1 + \bar{g}_2 \in \mathscr{L}$, then $e_1 - e_{l+1} \in L$. Furthermore, this vector is clearly a root, since it is contained in $\alpha L^\#$. Conversely, if $e_1 - e_{l+1} \in \mathsf{R}(L)$, then $\bar{g}_1 + \bar{g}_2 = \bar{e}_1 - \bar{e}_2 \in \mathscr{L}$ by definition. ∎

4. THE CLASSIFICATION OF ALL REFLECTIVE LATTICES IN DIMENSIONS ≤ 6

In this concluding section, we apply the previous general techniques to obtain the complete classification of all reflective lattices in dimensions ≤ 6. In dimensions 5 and 6, we list all combinatorial types, but we shall refrain from writing down the restrictions on the scales of the irreducible components. This is no essential restriction, in view of Theorems 3.1 and 3.2 and Proposition 3.3. By Propositions 2.11 and 2.12, we can and shall assume that the lattices to be considered are indecomposable. Under this assumption, the next main tool is Proposition 2.8, which drastically reduces the list of root systems to be considered. The precise result concerning this is the following:

LEMMA 4.1. *Let L be an indecomposable reflective lattice of dimension* ≤ 6. *Then the combinatorial type of its root system is one of the following*:

$\mathsf{A}_1; 2\mathsf{A}_1, \mathsf{C}_2, \mathsf{G}_2; 3\mathsf{A}_1, \mathsf{A}_1\mathsf{C}_2, \mathsf{B}_3, \mathsf{C}_3;$

$4\mathsf{A}_1, 2\mathsf{A}_1\mathsf{C}_2, \mathsf{A}_1\mathsf{B}_3, \mathsf{A}_1\mathsf{C}_3, 2\mathsf{A}_2, 2\mathsf{C}_2, \mathsf{A}_4, \mathsf{F}_4;$

$5\mathsf{A}_1, 3\mathsf{A}_1\mathsf{C}_2, 2\mathsf{A}_1\mathsf{B}_3, 2\mathsf{A}_1\mathsf{C}_3, \mathsf{A}_1 2\mathsf{C}_2, \mathsf{A}_1\mathsf{B}_4, \mathsf{A}_1\mathsf{C}_4, \mathsf{C}_2\mathsf{B}_3, \mathsf{C}_2\mathsf{C}_3, \mathsf{A}_5, \mathsf{B}_5, \mathsf{C}_5;$

$6\mathsf{A}_1, 4\mathsf{A}_1\mathsf{C}_2, 3\mathsf{A}_1\mathsf{B}_3, 3\mathsf{A}_1\mathsf{C}_3, 2\mathsf{A}_1 2\mathsf{C}_2, 2\mathsf{A}_1\mathsf{B}_4, 2\mathsf{A}_1\mathsf{C}_4, 2\mathsf{A}_1\mathsf{D}_4, \mathsf{A}_1\mathsf{C}_2\mathsf{B}_3,$

$\mathsf{A}_1\mathsf{C}_2\mathsf{C}_3,$

$\mathsf{A}_1\mathsf{A}_5, \mathsf{A}_1\mathsf{B}_5, \mathsf{A}_1\mathsf{C}_5, 3\mathsf{A}_2, 3\mathsf{C}_2, \mathsf{C}_2\mathsf{B}_4, \mathsf{C}_2\mathsf{C}_4, \mathsf{C}_2\mathsf{D}_4, 2\mathsf{B}_3, \mathsf{B}_3\mathsf{C}_3, 2\mathsf{C}_3, 2\mathsf{D}_3,$

$\mathsf{A}_6, \mathsf{B}_6, \mathsf{C}_6, \mathsf{D}_6, \mathsf{E}_6.$

Proof. One starts by systematically enumerating, for a fixed dimension n, all decompositions $n = n_1 + n_2 + \cdots + n_r$ into dimensions of r components, with $n_1 \leq n_2 \leq \cdots \leq n_r$, in lexicographic order. For instance, for $n = 5$ we have (writing 3.1 instead of $1 + 1 + 1$ etc.) $5.1, 3.1 + 2, 1 + 2.2, 1 + 4, 2 + 3, 5$. For each such "dimension vector", one writes down all possible combinatorial root systems $\mathsf{X}_1\mathsf{X}_2 \cdots \mathsf{X}_r$, where $\mathsf{X}_i \in \{\mathsf{A}_{n_i}, \mathsf{B}_{n_i}, \dots\}$, e.g., $5\mathsf{A}_1, 3\mathsf{A}_1\mathsf{A}_2, 3\mathsf{A}_1\mathsf{C}_2, 3\mathsf{A}_1\mathsf{G}_2, \mathsf{A}_1 2\mathsf{A}_2, \dots$. In this list, all root systems satisfying the assumptions of 2.4 or 2.6 (and thus giving rise to decomposable lattices) are suppressed. In particular, A_2 must not be combined with $\mathsf{A}_1, \mathsf{B}_l, \mathsf{C}_l$, and G_2 and F_4 must stand alone. All remaining occurrences of $\mathsf{A}_3 = \mathsf{D}_3$ and D_5 except for $2\mathsf{D}_3$ are ruled out by using Proposition 2.8. (To exclude, for instance, $\mathsf{D}_3\mathsf{C}_3$, take $\overline{\mathsf{R}}_3 = 3\mathsf{A}_1 \subset \mathsf{C}_3$ in that proposition.)

The preceding arguments already lead to the list given in the lemma except that the following root systems have to be ruled out still:

$\mathsf{B}_4, \mathsf{C}_4, \mathsf{D}_4$: Let L be a four-dimensional lattice with $\mathsf{R}(L) \supseteq^{\alpha} \mathsf{D}_l$ for some α. By 2.2, $L \subseteq^{\alpha} (D_4^{\#})$ (dual of the unscaled D_4-lattice). If $L =^{\alpha} D_4$, then $\mathsf{R}(L) = {}^{\alpha}\mathsf{F}_4$. If $L = {}^{\alpha}(D_4^{\#})$, then $L \cong {}^{\alpha/2}D_4$ and thus $\mathsf{R}(L) \cong {}^{\alpha/2}\mathsf{F}_4$. If L is of index 2 over D_4, then for each of the three cases, L is isometric to ${}^{\alpha}I_4$ (unit lattice) and thus decomposable. (This is of course a special feature of D_n, $n = 4$.)

$\mathsf{C}_2\mathsf{D}_4$, more generally $\mathsf{B}_l\mathsf{D}_4, \mathsf{C}_l\mathsf{D}_4$ for any l: In view of Theorem 2.2, we have $[L: \langle \mathsf{R}(L)\rangle] = 2$. Again using the special symmetry of D_4, we may assume that the D_4-component of the unique glue vector of L equals $e_1/2$ (where as usual $\mathsf{D}_4 \subset \mathbf{Z}e_1 + \cdots + \mathbf{Z}e_4$, $(e_i, e_i) = 1$). But then clearly s_{e_2} is an additional reflection (in fact, all e_i are in L and are additional roots). ∎

For the explicit construction of reflective lattices, we have to know what the appropriate codes look like. In dimensions less than 8, we only have to deal with binary codes plus some other, trivial cases. Every code can be decomposed uniquely into indecomposable ones, as introduced in Proposition 2.12. Therefore, we list only those. Observe that an indecomposable code of length at least 2 satisfies $\min \mathscr{L} \geq 2$ and $\min \mathscr{L}^{\perp} \geq 2$. In the next lemma, we list them up to length 6 and in particular introduce the notation to be referred to later. $\mathscr{L}_{n,k}$ denotes a k-dimensional subspace of $\mathbf{F}_2^n$. For $k = 1$ or $k = n-1$, the only possibilities are

$$\mathscr{L}_{n,1} := \langle 11 \cdots 1 \rangle$$

$$\mathscr{L}_{n,n-1} := \mathscr{L}_1^{\perp} = \langle 110 \cdots 0, 0110 \cdots 0, \ldots, 0 \cdots 011 \rangle$$

Later, when determining the types of lattices, (i.e., codes and combinatorial root systems) we have to take into account the symmetries of the codes. Therefore, we also list the orbits of the permutation group of each code on the positions 1 to n.

LEMMA 4.2. *The indecomposable binary codes $\mathscr{L}$ of length n with $2 \leq n \leq 6$ are, up to isomorphism, the following*:

$\mathscr{L}_{2,1}$; $\quad \mathscr{L}_{3,1}, \mathscr{L}_{3,2}$;

$\mathscr{L}_{4,1}$

$\mathscr{L}_{4,2} = \langle 1110, 1101 \rangle$ orbits $\{1,2\}, \{3,4\}$

$\mathscr{L}_{4,3}$;

$\mathscr{L}_{5,1}$,

$\mathscr{L}_{5,2a} = \langle 11000, 01111 \rangle$ orbits $\{1,2\}, \{3,4,5\}$

$\mathscr{L}_{5,2b} = \langle 11100, 10011 \rangle$ orbits $\{1\}, \{2,3,4,5\}$

$\mathscr{L}_{5,3a} = \mathscr{L}_{5,2a}^{\perp} = \langle 00011, 00110, 11100 \rangle$

$\mathscr{L}_{5,3b} = \mathscr{L}_{5,2b}^{\perp} = \langle 00011, 01100, 11010 \rangle$

$\mathscr{L}_{5,4}$;

$\mathscr{L}_{6,1}$

$\mathscr{L}_{6,2a} = \langle 110000, 011111 \rangle$ orbits $\{1,2\}\{3,4,5,6\}$

$\mathscr{L}_{6,2b} = \langle 111000, 100111 \rangle$ orbits $\{1\}\{2,3\}\{4,5,6\}$

$\mathscr{L}_{6,2c} = \langle 111100, 110011 \rangle$ transitive,

$\mathscr{L}_{6,3a} = \langle 001100, 000011, 110101 \rangle$ orbits $\{1,2\}\{3,4,5,6\}$

$\mathscr{L}_{6,3b} = \langle 110000, 011000, 001111 \rangle$ orbits $\{1,2,3\}\{4,5,6\}$

$\mathscr{L}_{6,3c} = \langle 001100, 111000, 010011 \rangle$ orbits $\{1\}\{2\}\{3,4\}\{5,6\}$

$\mathscr{L}_{6,3d} = \langle 110000, 011100, 010011 \rangle$ orbits $\{1,2\}\{3,4,5,6\}$

$\mathscr{L}_{6,3e} = \langle 100011, 010101, 001110 \rangle$ transitive

$\mathscr{L}_{6,4a} = \mathscr{L}_{6,2a}^{\perp}$

$\mathscr{L}_{6,4b} = \mathscr{L}_{6,2b}^{\perp}$

$\mathscr{L}_{6,4c} = \mathscr{L}_{6,2c}^{\perp}$

$\mathscr{L}_{6,4d} = \mathscr{L}_{6,2d}^{\perp}$

$\mathscr{L}_{6,4e} = \mathscr{L}_{6,2e}^{\perp}$

$\mathscr{L}_{6,5}$.

Proof. One first proves the general formula

$$\sum_{x \in \mathscr{L}} \|x\| = |\mathscr{L}| \cdot \frac{n}{2},$$

where $\|x\|$ denotes the Hamming weight of $x \in \mathbf{F}_2^n$. This formula readily follows from the Mac−Williams identity and the fact that the coefficient of X^1 in the weight distribution of $\mathscr{L}^{\perp}$ is zero, by assumption. The equivalent fact that for each position i between 1 and n, there exists at least one codeword with a 1 in that position, is used without explicit mentioning in the following discussion. We give a full proof only in the case $n = 6$. The cases where $n \leq 5$ are comparatively trivial.

First consider the case $\dim \mathscr{L} = 2$. We use the notation $0^1 1^0 2^{m_2} 3^{m_3} \ldots$ for the weight distribution, where m_i is the number of codewords of weight i. If $m_2 = m_3 = 0$, then clearly $\mathscr{L} \cong \mathscr{L}_{6,2c}$. If $m_2 = 0$ and $m_3 > 0$, then $m_3 = 1$, for otherwise $\mathscr{L}$ would be generated without loss of generality by 111000 and 000111 and thus decomposable. Clearly, $\mathscr{L} \cong \mathscr{L}_{6,2b}$. Similarly, if $m_2 > 0$, then $m_2 = 1$, and $\mathscr{L} \cong \mathscr{L}_{6,2a}$.

Now consider the case $\dim \mathscr{L} = 3$. The minimum weight of $\mathscr{L}$ cannot be 4, since in that case the sum of the weights would be at least $7 \cdot 4 = 28$. Let the minimum weight be equal to 3. Then $\mathscr{L}$ contains four odd and three non-zero even codewords, and the weight distribution necessarily is

$0^1 3^4 4^3$. The sum of any two codewords of weight 3 is of weight 4. This easily leads to the code $\mathscr{L}_{6.\,3e}$.

Now assume that the minimum weight is 2, and first consider the case $m_2 = 1$. The code cannot be even, since it would then have a total weight of at least $2 + 6 \cdot 4 = 26$. Now it follows that $m_3 \geq 3$, for otherwise the total weight would again be at least $2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 = 26$. We see that $\mathscr{L}$ is generated, up to permutation, by 110000 and two codewords y_1, y_2 of weight 3. Because of indecomposability, one of them has one 1 in common with the first generator; without loss of generality $y_1 = 011100$. It is now obvious that there are essentially two choices for y_2, namely 010011 which gives $\mathscr{L}_{6,\,3d}$, and 000111 which leads to $\mathscr{L}_{6,\,3c}$ after permuting the positions (13) and (24). We finally proceed to the case where $m_2 > 1$. Then the subspace $\mathscr{L}'$ of $\mathscr{L}$ generated by the words of weight 2 must be 2-dimensional, for otherwise $\mathscr{L} \cong \langle 110000, 001100, 000011 \rangle$ would be decomposable. If $\mathscr{L}' = \langle 110000, 001100 \rangle$, then the third generator must be equal to 101011 modulo $\mathscr{L}'$, for any other possibility would lead to a decomposable code. So we have $\mathscr{L}_{6,\,3a}$, up to permutation. If $\mathscr{L}' = \langle 110000, 011000 \rangle$, the same argument leads to the generator 001111 and the code $\mathscr{L}_{6,\,3b}$. ∎

As a last preparatory step before the explicit listing of all reflective lattices in dimensions ≤ 6, we now record what the reflective lattices with irreducible root system look like.

LEMMA 4.3. (a) *For $l \in \mathbf{N}$, $l \geq 3$, $l \neq 4$, and $\alpha \in 2\mathbf{N}$ such that $l\alpha \in 4\mathbf{N}$, there is a unique indecomposable reflective lattice with root system ${}^{\alpha}\mathsf{B}_l$. This is the lattice $B_l + \mathbf{Z}g = ({}^{\alpha}D_n)^{\#}$, where the glue vector g is as in Definition* 2.1.

(b) *For $l \in \mathbf{N}$, $l \geq 2$, $l \neq 4$ and $\alpha \in \mathbf{N}$, there is a unique indecomposable reflective lattice with root system ${}^{\alpha}\mathsf{C}_l$, namely ${}^{\alpha}C_l = {}^{\alpha}D_l$.*

For other pairs l, α, with $l \geq 3$ resp. $l \geq 2$, such a lattice does not exist.

Proof. (a) If L is indecomposable with root system ${}^{\alpha}\mathsf{B}_l$, then α must be even, by Proposition 2.7, and the glue code must be non-trivial, since ${}^{\alpha}B_l$ itself is an orthogonal sum of 1-dimensional lattices. By Theorem 2.2, $L = {}^{\alpha}B_l + \mathbf{Z}g$ as claimed. Integrality is equivalent to $(g, g) = l\alpha/4 \in \mathbf{Z}$. For $l \neq 4$, the root system ${}^{\alpha}\mathsf{B}_l$ is maximal in its dimension, and therefore the full root system $\mathsf{R}(L)$ must be equal to ${}^{\alpha}\mathsf{B}_l$. For $l = 4$, we have $L \cong {}^{\alpha/2}D_4$ and $\mathsf{R}(L) \cong {}^{\alpha/2}\mathsf{F}_4$.

(b) If L has root system ${}^{\alpha}\mathsf{C}_l$, then necessarily $\mathscr{L} = 0$, by Theorem 2.2. (For otherwise, $e_i \in L$, $i = 1, \ldots, l$ in standard coordinates, and the long roots would not be primitive.) The case $l = 4$ is excluded like in part a). ∎

We briefly look at the other irreducible root systems where the result is too obvious to be formulated as a separate lemma. We already observed, after Proposition 2.9, that ${}^{\alpha}\mathsf{D}_n$ with odd n never occurs as a root system of a reflective lattice. On the other hand, if n is even and $n \geq 4$, $n \neq 8$ there is a unique (up to automorphism) such lattice, namely $\widetilde{{}^{\alpha}D_n} = {}^{\alpha}D_n + \mathbf{Z}g$, where g is one of the long glue vectors, for instance $g = (e_1 + e_2 + \cdots + e_n)/2$ in standard coordinates. Here, α must be even if $n \equiv 2(4)$.

The root systems A_2 and $\mathsf{A}_3 \cong \mathsf{D}_3$ do not occur (A_2 enlarges to G_2). The discriminant group of A_n is cyclic, of order $n+1$, thus for any divisor k of $n+1$, the root lattice A_n has a unique over-lattice $A_n[k]$, contained in $A_n^{\#}$ and of index k over A_n. These lattices are well known from the literature, e.g., on perfect forms. Because of its uniqueness, $A_n[k]$ is preserved by all automorphisms of A_n, in particular by the reflections. Furthermore, it is readily seen that the roots of A_n remain primitive in $A_n[k]$. Thus we have $\mathsf{A}_n \subseteq \mathsf{R}(A_n[k])$. For $n \geq 4$, we have equality except for the cases $A_7[2] \cong E_7$, $A_8[3] \cong E_8$, since A_n, $n \geq 4$, $n \neq 7, 8$ is a maximal root system in its dimension. We do not reproduce the well known explicit shape of the glue vectors which shows that ${}^{\alpha}A_n[k]$ is integral if and only if $\alpha k(n+1-k)/(n+1) \in \mathbf{N}$. Summing up, we see that the combinatorial types of reflective lattices with root system A_n, $n \geq 4$ are exactly represented by the $A_n[k]$, $(n,k) \neq (7,2), \neq (8,3)$.

For $\mathsf{E}_6, \mathsf{E}_7$, the discriminant group is Z_3, Z_2 respectively, and the combinatorial types are represented by E_6, $E_6^{\#}$, E_7, $E_7^{\#}$. Finally, for G_2, F_4, or E_8, the glue code of a lattice with this root system must be trivial (otherwise, the roots of G_2 and F_4 would not stay primitive in the larger lattice; cf. also the proof of Proposition 2.10).

Since 2-dimensional lattices have a unique reduced Gram matrix, it seems appropriate to list also the reflective ones in this form. This is done in part (ii) of the following proposition, whereas part (i) uses the setup of this paper. The result is anyway obvious and only stated for the sake of completeness.

PROPOSITION 4.4. (i) *The* 2*-dimensional reflective lattices, given in terms of their root system* R *and code* $\mathscr{L}$, *are precisely the following*:

	R	$\mathscr{L}$	restrictions	determinant
(a)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1$	$\mathscr{L}_{2,1}$	$\alpha < \beta$ $\alpha + \beta \equiv 0(2)$	$\alpha\beta$
(b)	${}^{\alpha}\mathsf{G}_2$	0	no	$3\alpha^2$
(c)	${}^{\alpha}\mathsf{B}_1{}^{\beta}\mathsf{B}_1$	0	$\alpha < \beta$	$\alpha\beta$
(d)	${}^{\alpha}\mathsf{B}_2$	0	no	

(ii) *Let L be given by a reduced Gram matrix* $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, *that is* $0 \leq 2b \leq a \leq c$. *Then L is reflective if and only if* $\mathsf{O}(L) \neq \{\pm 1\}$ *if and only if* $2b = a$ *or*

$a = c$ or $b = 0$. The root system of L is

(a)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1$	*with* $\alpha = a/2$, $\beta = 2c - a/2$, *if* $2b = a < c$
		with $\alpha = a - b$, $\beta = a + b$, *if* $0 < 2b < a = c$
(b)	${}^{\beta}\mathsf{G}_2$	*if* $2b = a = c$
(c)	${}^{a}\mathsf{B}_1{}^{c}\mathsf{B}_1$	*if* $b = 0$, $a < c$
(d)	${}^{a}\mathsf{B}_2$	*if* $b = 0$, $a = c$

The lattice is decomposable precisely in the cases (a) *and* (b), *the code is non-zero in case* (a) *and zero otherwise.*

Proof. Part (i) is obvious after the previous general discussion. For part (ii), one verifies that the following vectors are roots, where e_1, e_2 are basis vectors with the above Gram matrix:

$e_1 - e_2, e_1 + e_2$	of norms $2(a - b)$ resp. $2(a + b)$
	if $b = 0$
$e_1, -e_1 + 2e_2$	of norms a resp. $4c - a$
	if $2b = a$ ∎

THEOREM 4.5. *The* 3*-dimensional, indecomposable, reflective lattices, given in terms of their root system* R *and glue code* $\mathscr{L}$ *are precisely the following*:

	R	$\mathscr{L}$	restrictions	determinant
(a)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{A}_1$	$\mathscr{L}_{3,1}$	$\alpha < \beta < \gamma$ $\alpha + \beta + \gamma \equiv 0(2)$	$2\alpha\beta\gamma$
(b)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{A}_1$	$\mathscr{L}_{3,2}$	$\alpha < \beta < \gamma$ $\alpha \equiv \beta \equiv \gamma \equiv 0(2)$	$\alpha\beta\gamma/2$
(c)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_2$	$\mathscr{L}_{2,1}$	$\alpha \neq \beta, 2\beta$ $\alpha \equiv 0(2)$	$2\alpha\beta^2$
(d)	${}^{\alpha}\mathsf{B}_3$	$\neq 0$	$\alpha \equiv 0(4)$	$\alpha^3/4$
(e)	${}^{\alpha}\mathsf{C}_3$	0	no	$4\alpha^3$

Proof. By Lemma 4.1, the root system $\mathsf{R}(L)$ contains a sub-root-system $\mathsf{R}_0 = {}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{A}_1$. The two possible codes $\mathscr{L} \subset \overline{T}(\mathsf{R}_0) = \mathbf{F}_2^3$ are $\mathscr{L}_{3,1} = \langle 111 \rangle$ and $\mathscr{L}_{3,2} = \langle 110, 011 \rangle$. We now discuss the full root system of $L = L(\mathsf{R}_0, \mathscr{L})$, using Theorem 3.1. If α, β, γ are pairwise distinct, then none of the Cases I to V of that theorem applies, and thus $\mathsf{R}(L) = \mathsf{R}_0$. This gives the cases (a) and (b) of Theorem 4.5. The stated congruence conditions express the integrality of the glue vectors and thus of the whole lattice. If at least two of α, β, γ are equal, then there are additional roots (at least) of type I or type II of Theorem 3.1 in the cases $\mathscr{L} = \mathscr{L}_{3,1}, \mathscr{L}_{3,2}$, respectively. Thus, the full root system is one of ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_2$, ${}^{\alpha}\mathsf{B}_3$, or ${}^{\alpha}\mathsf{C}_3$. In the last two cases, there exists a unique reflective lattice according to 4.3. In the case ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_2$, the uniqueness comes from Theorem 2.2 (the glue vector necessarily equals $e_1/2 + g = e_1/2 + e_2$ according to Definition 2.1). We finally have to check that the stated congruence conditions on α and β are the correct ones. After the discussion preceding Proposition 3.3, the only possibilities of enlarging ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_2$ are ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_2 = {}^{\beta}\mathsf{C}_1{}^{\beta}\mathsf{C}_2$, i.e., $\alpha = 2\beta$, or ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_2 = {}^{2\beta}\mathsf{B}_1{}^{2\beta}\mathsf{B}_2$, i.e., $\alpha = \beta$. In either of these cases, the root system does enlarge, since the condition of Proposition 3.3 is clearly satisfied. The glue vector has norm $\alpha/2 + \beta$, and thus $\alpha \equiv 0(2)$. ∎

We remark that the conditions for ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_2$ could also have been obtained from the remark following Theorem 3.1, using $2\,{}^{\beta}\mathsf{A}_1 \subset {}^{\beta}\mathsf{C}_2$.

COROLLARY 4.6. *A* 3-*dimensional reflective lattice of determinant not divisible by* 4 *is decomposable*.

This is obtained from Theorem 4.5 simply by inspection of the congruence conditions and determinant, in each case (a) to (e).

THEOREM 4.7. *The* 4-*dimensional, indecomposable, reflective lattices, given in terms of their root system* R *and glue code* $\mathscr{L}$ *are precisely the following*:

	R	$\mathscr{L}$	restrictions	determinant
(a)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{A}_1{}^{\delta}\mathsf{A}_1$	$\mathscr{L}_{4,1}$	$\alpha < \beta < \gamma < \delta$ $\alpha + \beta + \gamma + \delta \equiv 0(2)$	$4\alpha\beta\gamma\delta$
(b)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{A}_1{}^{\delta}\mathsf{A}_1$	$\mathscr{L}_{4,2}$	$\alpha < \beta, \gamma < \delta$ $\alpha + \beta \equiv 0(2), \gamma \equiv \delta \equiv 0(2)$	$\alpha\beta\gamma\delta$
(c)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{A}_1{}^{\delta}\mathsf{A}_1$	$\mathscr{L}_{4,3}$	$\alpha < \beta < \gamma < \delta$ $\alpha \equiv \beta \equiv \gamma \equiv \delta \equiv 0(2)$	$\alpha\beta\gamma\delta/4$
(d)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{C}_2$	$\mathscr{L}_{3,1}$	$\alpha < \beta, \alpha \neq \gamma \neq \beta$	$4\alpha\beta\gamma^2$
(e)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{C}_2$	$\mathscr{L}_{3,2}$	$\alpha < \beta, \alpha \neq 2\gamma \neq \beta$ $\alpha \equiv \beta \equiv 0(2)$	$\alpha\beta\gamma^2$
(f)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{B}_3$	$\mathscr{L}_{2,1}$	$2\alpha \neq \beta, \beta \equiv 0(2)$	$\alpha\beta^3/2$
(g)	${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_3$	$\mathscr{L}_{2,1}$	$\alpha \neq 2\beta, \alpha \equiv 0(2)$	$4\alpha\beta^3$
(h)	${}^{\alpha}\mathsf{A}_2{}^{\beta}\mathsf{A}_2$	$\mathscr{L}_{2,1}$	$\alpha < \beta, 2\alpha \neq \beta$ $\alpha + \beta \equiv 0(3)$	$\alpha^2\beta^2$
(i)	${}^{\alpha}\mathsf{C}_2{}^{\beta}\mathsf{C}_2$	$\mathscr{L}_{2,1}$	$\alpha < \beta$	$4\alpha^2\beta^2$
(j)	${}^{\alpha}\mathsf{A}_4$	0	no	$5\alpha^4$
(k)	${}^{\alpha}\mathsf{A}_4$	$\neq 0$	$\alpha \equiv 0(5)$	$\alpha^4/5$
(l)	${}^{\alpha}\mathsf{F}_4$	0	$\alpha \equiv 0(2)$.	$4\alpha^4$

Proof. Like in the previous case of dimension 3, most of the work has already been done before, essentially in the main Theorem 2.2 and in

Lemma 4.1, where the occurring root systems are listed. We now work out the congruence conditions on the scales $\alpha, \beta, \ldots$. We omit the trivial discussion of those conditions which refer to the integrality of L (e.g., $\alpha, \beta, \gamma, \delta$ all must be even in case (b)). We begin with the root system ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{A}_1{}^{\delta}\mathsf{A}_1$. If the code $\mathscr{L}$ is $\mathscr{L}_{4,1}$ or $\mathscr{L}_{4,3}$, then any equality between two of the scales leads to additional roots of type I or II of Theorem 3.1. In view of the symmetry properties of the codes, we may normalize the scales by the condition $\alpha < \beta < \gamma < \delta$, as stated in the theorem. For $\mathscr{L} = \mathscr{L}_{4,2}$, the codewords of weight 2 are $0011 \in \mathscr{L}$ and $1100 \in \mathscr{L}^{\perp}$. Therefore, we have to require $\alpha \neq \beta$, $\gamma \neq \delta$, without loss $\alpha < \beta$, $\gamma < \delta$, to avoid additional roots of type I, respectively II.

For the root system ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{C}_2$, we have to consider two codes $\mathscr{L}_{3,1}$ and $\mathscr{L}_{3,2}$. The condition $\alpha \neq \beta$, without loss $\alpha < \beta$ is necessary and sufficient to avoid additional roots of type I respectively type II. The second condition is that ${}^{\alpha}\mathsf{A}_1{}^{\gamma}\mathsf{C}_2 = {}^{2\alpha}\mathsf{B}_1{}^{2\gamma}\mathsf{B}_2 = {}^{\alpha/2}\mathsf{C}_1{}^{\gamma}\mathsf{C}_2$ (or analogously ${}^{\beta}\mathsf{A}_1{}^{\gamma}\mathsf{C}_2$) must enlarge to neither ${}^{2\gamma}\mathsf{B}_3$ nor ${}^{\gamma}\mathsf{C}_3$. According to 3.3, the first case is possible only for $\mathscr{L}_{3,1}$ and is excluded by requiring $\alpha \neq \gamma$, and the second is possible only for $\mathscr{L}_{3,2}$ and is excluded by the condition $\alpha \neq 2\gamma$. The root systems ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{B}_3 = {}^{2\alpha}\mathsf{B}_1{}^{\beta}\mathsf{B}_3$, ${}^{\alpha}\mathsf{A}_1{}^{\beta}\mathsf{C}_3 = {}^{\alpha/2}\mathsf{C}_1{}^{\beta}\mathsf{C}_3$ and ${}^{\alpha}\mathsf{C}_2{}^{\beta}\mathsf{C}_2$ are treated by exactly the same argument.

For the root system ${}^{\alpha}\mathsf{A}_2{}^{\beta}\mathsf{A}_2$, the reduced discriminant group equals $\overline{T} = \mathbf{Z}_3 \times \mathbf{Z}_3$, and in view of indecomposability we have $0 \neq \mathscr{L} \neq \overline{T}$. Thus, $|\mathscr{L}| = 3$, and $\mathscr{L}$ is diagonally embedded and in fact unique up to automorphisms of the root system. (By slight abuse of notation, we have denoted it by $\mathscr{L}_{2,1}$ again.) According to Theorem 3.2, the only possible new roots are of type III of that theorem, and are excluded by $\beta \neq 2\alpha$ (already assuming $\beta > \alpha$). The cases ${}^{\alpha}\mathsf{A}_4$ and ${}^{\alpha}\mathsf{F}_4$ are obvious. ∎

Remark. Originally, in [Bla90], the classification in dimension 4 had been obtained by considering only the minimal root systems $4\mathsf{A}_1, 2\mathsf{A}_2, \mathsf{A}_4$, making extensive use of Theorem 3.1. This approach is more straightforward, since it does not use the reduced discriminant group and Theorem 2.2 for components of type B or C. The disadvantage is that root systems like $2\mathsf{A}_1\mathsf{C}_2$ or $\mathsf{A}_1\mathsf{B}_3$ are then obtained several times, starting from differently scaled subsystems $4\mathsf{A}_1$ (cf. Proposition 2.6). Bringing these lattices into some normal form and suppressing isometric ones amounts to almost proving the "only if"-part of Theorem 2.2 in the case B or C.

LEMMA 4.8. *Let k, l be odd, ≥ 3 and $\alpha, \beta \in \mathbf{N}$ such that $\alpha \neq \beta$ and $\alpha k + \beta l \equiv 0(4)$. Then there is a unique reflective lattice with root system ${}^{\alpha}\mathsf{D}_k{}^{\beta}\mathsf{D}_l$.*

Proof. The proof is similar to the proof of Proposition 2.9. If u is the canonical glue vector of D_k used in that proof, and u' the corresponding vector for D_l, then our desired code $\mathscr{L}$ necessarily equals $\langle \overline{u + u'} \rangle$, up to automorphisms of $\mathsf{D}_k\mathsf{D}_l$. It is readily checked that the corresponding lattice is indeed not preserved by the reflections in the basis vectors. Using the assumption $\alpha \neq \beta$, it is now clear that the root system ${}^{\alpha}\mathsf{D}_k{}^{\beta}\mathsf{D}_l$ does not enlarge. The condition $\alpha k + \beta l \equiv 0(4)$ expresses that $(u + u', u + u') \in \mathbf{Z}$. ∎

LEMMA 4.9. *Consider a root system of the form* ${}^{\alpha}\mathsf{D}_4{}^{\beta}\mathsf{R}_1{}^{\gamma}\mathsf{R}_2$, *where* $\mathsf{R}_1, \mathsf{R}_2 \in \{{}^2\mathsf{B}_l,\ l \geq 1\} \cup \{\mathsf{C}_l,\ l \geq 2\}$. *Assume that* α *is even, and that the coefficient* δ *of every component* ${}^{\delta}\mathsf{B}_l$ *with odd l is divisible by* 4. *Exclude the case* $\mathsf{R}_1 = \mathsf{R}_2 = \mathsf{A}_1$, $\alpha = \beta = \gamma$. *Up to isomorphism, there is a unique reflective lattice with this root system.*

Proof. Let ${}^{\alpha}\mathsf{D}_4 \subseteq \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$, where $(e_i, e_j) = \alpha\delta_{ij}$ as usual, and $u = \frac{1}{2}(e_1 + \cdots + e_4)$. Denote by g_1, g_2 the glue vectors of $\mathsf{R}_1, \mathsf{R}_2$ according to Proposition 3.3. We claim that

$$\mathscr{L} = \langle \bar{e}_1 + \bar{g}_1, \bar{u} + \bar{g}_2 \rangle \cong Z_2 \times Z_2$$

is the essentially unique solution for the desired code $\mathscr{L} \subseteq T(D_4) \perp \overline{T}(\mathsf{R}_1) \perp \overline{T}(\mathsf{R}_2)$. The projection $\operatorname{pr}\mathscr{L}$ of $\mathscr{L}$ onto $T(D_4)$ is all of $T(D_4)$, for otherwise it would without loss of generality be contained in $\langle \bar{e}_1 \rangle$, and D_4 would enlarge to B_4 or C_4 (see earlier proofs). On the other hand, the intersection of $\mathscr{L}$ with $T(D_4)$ must be zero (otherwise, without loss $\bar{e}_1 \in \mathscr{L}$, and $\mathsf{R}(L)$ enlarges again). So the projection of L onto $\overline{T}(\mathsf{R}_1) \perp \overline{T}(\mathsf{R}_2) \cong Z_2 \times Z_2$ is injective as well, and $\mathscr{L}$ must consist of 4 elements, projecting isomorphically onto both $T(D_4)$ and $\overline{T}(\mathsf{R}_1) \perp \overline{T}(\mathsf{R}_2)$. Now recall that the glue vectors e_1, u and $u - e_1$ can be permutated by automorphisms of D_4. The uniqueness follows immediately from this symmetry property. In the excluded case, the root system enlarges to ${}^{\alpha}\mathsf{D}_6$. One verifies that it does not enlarge in all other cases. ∎

For simplicity, the following two theorems deal with combinatorial types only, and not with the scaling of the irreducible components. Therefore, the proof will immediately follow from the list of root systems in Lemma 4.1, the list of codes in Lemma 4.2, of course using Theorem 2.2 and Corollary 2.4.

THEOREM 4.10. *There exist precisely* 26 *combinatorial equivalence classes of indecomposable reflective lattices in dimension* 5. *They are given by the following root systems and codes*:

$5\mathsf{A}_1$: $\mathscr{L}_{5,1}, \mathscr{L}_{5,2a}, \mathscr{L}_{5,2b}, \mathscr{L}_{5,3a}, \mathscr{L}_{5,3b}, \mathscr{L}_{5,4}$;

$3\mathsf{A}_1\mathsf{C}_2$: $\mathscr{L}_{4,1}, \mathscr{L}_{4,2}, \mathscr{L}_{4,3}$, $\mathsf{C}_2 3\mathsf{A}_1$: $\mathscr{L}_{4,2}$;

$\mathsf{A}_1 2\mathsf{C}_2$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$, $2\mathsf{A}_1\mathsf{B}_3$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$, $2\mathsf{A}_1\mathsf{C}_3$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$;

$\mathsf{A}_1\mathsf{B}_4$; $\mathsf{A}_1\mathsf{C}_4$; $\mathsf{C}_2\mathsf{B}_3$; $\mathsf{C}_2\mathsf{C}_3$;

B_5; C_5; A_5: four codes.

THEOREM 4.11. *There exist precisely* 67 *combinatorial equivalence classes of indecomposable reflective lattices in dimension* 6. *They are given by the following root systems and codes*:

$6A_1$: 15 codes $\mathscr{L}_{6,1}, \dots, \mathscr{L}_{6,5}$ (see 4.2);

$4A_1C_2$: 6 codes $\mathscr{L}_{5,1}, \dots, \mathscr{L}_{5,4}$, $C_2 4A_1$: $\mathscr{L}_{5,2a}, \mathscr{L}_{5,2b}, \mathscr{L}_{5,3a}, \mathscr{L}_{5,3b}$;

$3A_1B_3$: $\mathscr{L}_{4,1}, \mathscr{L}_{4,2}, \mathscr{L}_{4,3}$, $B_3 3A_1$: $\mathscr{L}_{4,2}$;

$3A_1C_3$: $\mathscr{L}_{4,1}, \mathscr{L}_{4,2}, \mathscr{L}_{4,3}$, $C_3 3A_1$: $\mathscr{L}_{4,2}$;

$2A_1 2C_2$: $\mathscr{L}_{4,1}, \mathscr{L}_{4,2}, \mathscr{L}_{4,3}$, $A_1C_2A_1C_2$: $\mathscr{L}_{4,2}$, $2C_2 2A_1$: $\mathscr{L}_{4,2}$;

$2A_1B_4$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$, $2A_1C_4$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$; $2A_1D_4$;

$A_1C_2B_3$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$, $A_1C_2C_3$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$; $3C_2$: $\mathscr{L}_{3,1}, \mathscr{L}_{3,2}$;

$3A_2$: 2 codes;

A_1A_5: 2 codes; A_1B_5; A_1C_5; C_2B_4; C_2C_4;

$2B_3$; B_3C_3; $2C_3$; $2D_3$;

A_6: 2 codes; B_6; C_6; D_6; E_6: 2 codes.

Proof of Theorems 4.10 *and* 4.11. For a root system with only one type of irreducible components, the combinatorial equivalence classes of reflective lattices with this root system are clearly in one-to-one correspondence with the (isomorphism classes of) the relevant codes. This remark, Corollary 2.4, and Lemma 4.2 give the result for $5A_1$ and $6A_1$. For a root system of the shape $(m-1)XY$, where X and Y are distinct and of type B or C (including A_1), two reflective lattices given by binary codes $\mathscr{L}$ and $\mathscr{L}'$ of length m, are combinatorially equivalent if and only if there exists an isomorphism of $\mathscr{L}$ onto $\mathscr{L}'$ preserving the mth coordinate (where the component Y occurs). In other words, if we fix $\mathscr{L}$ and allow root systems equivalent to R, i.e., obtained by permutation of the irreducible components, then the combinatorial equivalence class is in one-to-one correspondence with the orbit under the automorphism group of $\mathscr{L}$ of the position $i \in \{1, \dots, m\}$ where the component Y is located. This remark settles the cases with root systems $3A_1C_2$, $3A_1B_3$, $3A_1C_3$ ($m = 4$) and $4A_1C_2$ ($m = 5$), in view of the last column of 4.2. For the root system $2A_1 2C_2$, an analogous argument applies: the automorphism group of $\mathscr{L}_{4,2}$ has 3 orbits on 2-element subsets of $\{1, 2, 3, 4\}$, represented by $\{1, 2\}, \{1, 3\}, \{3, 4\}$. For the codes $\mathscr{L}_{m,1}$ and $\mathscr{L}_{m,m-1}$, in particular for $m \leq 3$, the combinatorial equivalence class of the lattice only depends on the class of its root system, since the automorphism group of $\mathscr{L}$ is the full symmetric group.

The possible irreducible root system $\mathsf{A}_5, \mathsf{B}_5, \mathsf{C}_5, \mathsf{A}_6, \mathsf{B}_6, \mathsf{C}_6, \mathsf{D}_6, \mathsf{E}_6$ have been treated in Lemma 4.3 and the remarks following that lemma. For the case $2\mathsf{D}_3$, we refer to Lemma 4.8, and for $2\mathsf{A}_1\mathsf{D}_4$ to Lemma 4.9. We have now treated all root systems of Lemma 4.1 except for the following two:

$\mathsf{A}_1\mathsf{A}_5$: Here one necessarily has $\mathscr{L} = \langle g_1 + 3g_2 \rangle$ or $\mathscr{L} = \langle g_1 + 3g_2, 2g_2 \rangle$, where g_1, g_2 are generators of $T(\mathsf{A}_1), T(\mathsf{A}_5) \cong Z_6$ respectively. To see this, observe that, for any glue code $\mathscr{L} \subseteq \prod_{i=1}^{r} \overline{T}(\mathsf{R}_i)$ of an indecomposable reflective lattice, where the R_i are irreducible root systems, one has $\mathscr{L} \cap \overline{T}(\mathsf{R}_i) \neq 0, \ \neq \overline{T}(\mathsf{R}_i)$. (Otherwise, $\mathscr{L}$ and thus the lattice is decomposable.)

$3\mathsf{A}_2$: Here $\mathscr{L} \subseteq T(3\mathsf{A}_2) \cong Z_3^3$ must be indecomposable as before, and may be modified by automorphisms of the root system, i.e., permutations and multiplication by ± 1 independently in each component. (This is the usual notion of equivalence of ternary codes, and applies to root systems $m\mathsf{A}_2$ for arbitrary m.) It is readily seen that $\mathscr{L} = \langle 111 \rangle$ and $\mathscr{L} = \langle 110, 011 \rangle$ are essentially the only possibilities. ∎

Computational and geometric aspects of reflective lattices will be pursued in subsequent work. In this paper, we only include one result about the determination of a root basis, and thus the isomorphism type of the root system, of an arbitrary lattice L. Our method is a variation of well known ideas. We assume that the set R of all roots has already been determined, and we choose a half-space $\{x \in V \mid f(x) > 0\}$, where $f\colon V \to \mathbf{Q}$ is a linear form with $f(v) \neq 0$ for all roots $v \in \mathsf{R}$. As usual, we consider the set $\mathsf{R}^+ = \{v \in \mathsf{R} \mid f(v) > 0\}$ of positive roots. We (re)order the elements of $\mathsf{R}^+ = \{v_1, v_2, v_3, \dots, v_i, \dots\}$ in such a way that $\varphi(v_1) \leq \varphi(v_2) \leq \cdots \leq \varphi(v_i) \leq \varphi(v_{i+1}) \leq \dots$, where $\varphi(v) := f(v)^2/(v,v)$. Now, the indices $i_2, i_2, \dots, i_r$ are determined as follows: $i_1 = 1$, and i_{m+1} is the smallest i greater than i_m such that $(v_{i_j}, v_i) \leq 0$ for all $j = 1, \dots, m$. If no such i exists any more, then $m = r$, and the algorithm stops. The vectors $v_{i_1}, \dots, v_{i_r}$ are clearly linearly independent, since they lie in a common half-space and have pairwise nonpositive scalar product. More is true:

PROPOSITION 4.12. *The vectors $v_{i_1}, v_{i_2}, \dots, v_{i_r}$ determined by the above procedure form a root basis of* R (*namely, the unique root basis contained in* R^+).

For the proof, one considers a hypothetical vector $v \in \mathsf{R}^+$ which is not a non-negative linear combination of $v_{i_1}, \dots, v_{i_r}$, and with minimal φ-value. Let j be the smallest index with $(v, v_{i_j}) > 0$; this exists. One verifies that the vector $v' = s_j v = v - c v_{i_j}$, where s_j is the reflection with respect to v_{i_j} and $c > 0$, is still in R^+. Clearly $\varphi(v') < \varphi(v)$, and therefore v' and consequently also v is a non-negative linear combination of the v_{i_j}.

REFERENCES

[Bla90] B. Blaschke, ''Klassifikation reflektiver Gitter in kleinen Dimensionen,'' Diplomarbeit, Bielefeld, 1990.

[Bou68] N. Bourbaki, ''Groupes et Algèbre de Lie,'' Chap. IV, V, and VI, Hermann, Paris, (1968).

[Ker94] M. Kervaire, Unimodular lattices with a complete root system, *Enseign. Math.* **40** (1994), 59–104.

[SH94] R. Scharlau and B. Hemkemeier, Classification of integral lattices with large class number, Universität Bielefeld, SFB-preprint 94-102.

[SV94] R. Scharlau and B. Venkov, The genus of the Barnes–Wall Lattice, *Comment. Math. Helv.* **69** (1994), 322–333.

[SW92] R. Scharlau and C. Walhorn, Integral lattices and hyperbolic reflection groups, *Astérisque* **209** (1992), 279–291.

[Ven80] B. B. Venkov, On the classification of integral even unimodular 24-dimensional quadratic forms, *Proc. Steklov Inst. Math.* **4** (1980), 63–74; also reprinted in J. H. Conway and N. J. A. Sloane, ''Sphere Packings, Lattices and Groups,'' Chap. 18, Springer, New York, 1988.

[Vin72] E. B. Vinberg, On groups of unit elements of certain quadratic forms, *Math. USSR-Sb.* **16** (1972), 17–35.

[Vin85] E. B. Vinberg, The absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension, *Trans. Moscow Math. Soc.* **47** (1985), 75–112.