Some concepts of stability analysis in combinatorial optimization

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Abstract

This paper surveys the recent results in stability analysis for discrete optimization problems, such as a traveling salesman problem, an assignment problem, a shortest path problem, a Steiner problem, a scheduling problem and so on. The terms "stability", "sensitivity" or "postoptimal analysis" are generally used for the phase of an algorithm at which a solution (or solutions) of the problem has been already found, and additional calculations are also performed in order to investigate how this solution depends on changes in the problem data.

In this paper, the main attention is paid to the stability region and to the stability ball of optimal or approximate solutions. A short sketch of some other close results has been added to emphasize the differences in approach surveyed.

Keywords: Combinatorial optimization; Stability analysis; Stability radius; Schedule; Matroid; Graph

1. Introduction

A new direction in combinatorial optimization, connected with stability analysis of the solution, is reviewed. The major part of the paper deals with problems like the following one. Given solution \( t \) (or the whole solution set) of a discrete optimization problem \( Z \), stability analysis consists in finding an answer to the question: By how much can we perturb numerical parameters of the problem \( Z \) without loss of the property of \( t \) to be optimal (respectively, without extending the solution set)?
We formulate a discrete optimization problem as follows. Let \( E = \{e_1, e_2, \ldots, e_n\} \) be a given set and \( T_n = \{t_1, t_2, \ldots, t_m\}, m \geq 2, \) be a set of subsets of the set \( E \) (called trajectories). The weights \( w(e_1) = a_1, w(e_2) = a_2, \ldots, w(e_n) = a_n \) are ascribed to the elements of \( E \). Each weighting can be represented as a vector \( A = (a_1, a_2, \ldots, a_n) \) in the space \( R_n \) of all real vectors. Sometimes, if \( n = k^2 \), the components of the vector \( A \) will be considered as the elements of \((k \times k)\)-square matrix \( \tilde{A} \):

\[
\tilde{A} = \begin{bmatrix}
a_1 & a_2 & \cdots & a_k \\
 a_{k+1} & a_{k+2} & \cdots & a_{2k} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{(k-1)k+1} & a_{(k-1)k+2} & \cdots & a_{k^2}
\end{bmatrix}
\]

In this case, we shall write \( \tilde{A} \in R_{kk} \).

At every trajectory \( t_i \in T_n \), the value of functional \( t_i(A) \) (i.e., the length of a trajectory with weighting \( A \)) is defined. We shall consider the following objective functions:

\[
t_i(A) = \sum_{e_j \in t_i} a_j \quad \text{(for linear problem)} \tag{1}
\]

\[
t_i(A) = \max_{e_j \in t_i} |a_j| \quad \text{(for bottleneck problem).} \tag{2}
\]

Thus, for a given objective function, a discrete optimization problem \( Z \) may be determined by a triple \( \{E, T_n, A\} \). To solve this problem means to find a trajectory of the minimum (or conversely, maximum) length, i.e., optimal trajectory or optimal solution. Examples of such problems are furnished by a traveling salesman problem, a Steiner problem, a shortest path problem, an assignment problem, and a lot of other ones on graphs, matroids, and so forth.

Let \( Z_A \) be an instance of the problem \( \{E, T_n, A\} \). Denote by \( \varphi(A) \) the set of all subscripts of the trajectories which are optimal for the problem \( Z_A \); \( i \in \varphi(A) \) iff \( t_i \) is an optimal trajectory for the problem \( Z_A \) with weighting \( A \). In what follows we shall assume that \( \varphi(A) \) (or at least one element of \( \varphi(A) \)) is known in advance.

There exists a lot of approaches to analyze stability aspect in combinatorial optimization. In spite of the fact that in different publications the terms "postoptimal analysis", "sensitivity", "tolerance", "stability", and some others are used, most authors of such works attempt to answer to the following close questions.

How can one vary the elements of \( A \) in the problem \( Z_A \) such that a solution of the obtained problem \( Z_{A'} \) may be found by using the solution of the problem \( Z_A \)? How can one calculate the quantity of the above mentioned variations of the elements from \( A \)? How does the data inaccuracy influence the structure of the set \( \varphi(A) \)? How can one construct an efficient algorithm for solving a set of similar problems? Using the stability results, construct exact or, at least, approximate algorithm for a discrete optimization problem. How is the stability concept connected with the parametric optimization problems with one or more variable parameters?
This paper is devoted to a rather general approach to stability analysis and presents, mainly, the results available only in Russian. Some other approaches to stability analysis are only briefly reviewed in Section 2. In Section 3, we introduce the stability ball for so-called linear and bottleneck trajectory problems. Section 4 deals with the efficient algorithms for calculating a stability radius. Some results on the stability ball for the minimization of Boolean linear form are discussed in Section 5. The stability of an optimal schedule (or more exactly, the stability of an optimal digraph) is considered in Section 6. All necessary terms and definitions are introduced in the course of Sections 3-6, and all results are observed without any proof.

2. Some approaches to stability analysis

The stability notions have been introduced almost simultaneously with the appearance of the first methods for solving mathematical programming problems. However, direct transformation of such stability results to some discrete optimization problems has provided only very simple conclusions. The papers, presenting the specific stability concepts for combinatorial optimization, were published about twenty years ago. Nevertheless, by this time, there exist hundreds of such papers, and here we try to provide only a brief sketch of some typical results and approaches. Moreover, we shall concentrate on the publications in Russian which are almost inaccessible for the Western mathematicians.

Obviously, the most discrete optimization problems may be formulated as a particular case of integer linear programming. In turn, the latter is a special case of the general mathematical programming, and therefore, one can transform almost all numerous methods and results, that have been so far obtained for stability analysis in linear or nonlinear programming (for example, the results from [12, 64]), to the most discrete optimization problems. Unfortunately, such transformation does not exploit the specific combinatorial structure and other properties of the optimization problems on graphs, matroids, etc.

Another approach has been suggested in [11, 75, 90]. Let \( t = \{e_1, e_2, \ldots, e_l\} \) be an optimal trajectory of the problem \( Z_A \). It is required to find two numbers \( x_i \) and \( y_i \) for any element \( a_i \) such that varying only the weight \( a_i \) in the interval \( (a_i - x_i, a_i + y_i) \) keeps the trajectory \( t \) to be optimal. The complexity of calculating \( x_i \) and \( y_i \) is comparable with the complexity of the algorithms for solving the initial optimization problem (such as a shortest path problem, and a minimum spanning tree one).

Sometimes, it is required to solve two problems \( Z_{A^i} \) and \( Z_{A^i} \) such that the most components of \( A^i \) are equal to the corresponding components of \( A^2 \). In [8, 92] special methods have been proposed to solve the set of such close problems \( Z_{A^i}, Z_{A^i}, \ldots, Z_{A^r} \). A solution for each of the following problems was obtained using the solution of the previous one. If the number of different elements in each pairs \( A^i \) and \( A^{i+1} \), \( i = 1, 2, \ldots, r \), was sufficiently small, then the computational complexity of this method...
was proved to be less in comparison with the total complexity of solving the problems $Z_{A1}, Z_{A2}, \ldots, Z_{Ar}$ separately.

Various stability concepts have been considered for integer (or Boolean) linear programming:

Maximize \((c, x)\) \hspace{1cm} (3)

subject to \(\bar{A}x \leq b,\) \hspace{1cm} (4)

where \(c \in \mathbb{R}_+, b \in \mathbb{R}_+, \bar{A} \in \mathbb{R}_{m \times n}\) and vector \(x\) is integer (respectively, Boolean).

The case of fixed matrix \(\bar{A}\) and fixed vector \(c\) has been considered in [5]. Let \(\phi(b)\) be equal to the value of \((c, x)\) for the optimal solution \(x\) of an integer problem (3)–(4). It is proved, that any \(\phi(b)\) is a Gomory function, and on the other hand, any Gomory function corresponds to some function \(\phi(b)\) for an integer problem (3)–(4). A set of all integer linear programming problems with the same optimal basis has been described in [77].

The group of German mathematicians carry out the research of stability in polynomial, nonlinear and integer programming. You can find some of their results in [3, 4, 54]. In the case of integer programming their approach is close to [5].

The stability of a bottleneck linear programming problem has been studied in [27, 96]. For fixed \(\bar{A}\) and \(b\), it is possible to use a solution of the initial problem to solve a changed one. This method is based on a simplex algorithm.

A stability concept, concerning with the calculation errors and the data inaccuracy, was studied in [18] for a linear programming problem. Let \(B\) and \(B'\) be bases of two consecutive iterations in a simplex algorithm, and \(d\) is the maximal value of inaccuracy over all elements \(B\). The following question has been considered: What maximal inaccuracy \(d'\) in \(B'\) is induced by such change of basis? An expression for \(d'\) as a function of \(B, B'\) and \(d\) has been obtained in [18].

A Boolean programming problem and a maximum flow problem have been considered in [6], where so-called bounded function \(f(t)\) was introduced. In a maximum flow problem, the value of \(f(t)\) is equal to the value of a maximal flow, when the arc capacity is given in the form \(c_{ij} + b_{ij}t\). In [10] the arc capacity in a maximum flow problem was assumed to be changeable, and a solution of such changed problem was found by using a solution of the initial one.

A special case of a parametric traveling salesman problem has been considered in [60]. Instead of the fixed weight \(a_i\) of the arc \(e_i\), the whole set of the weights was ascribed to this arc, the actual weight of \(e_i\) depending on the arc position in the Hamiltonian cycle. A branch-and-bound method has been proposed to solve such problems. A problem with an initial data, depending on one or more parameters, has been studied in [25]. A parametric approach to stability analysis in integer programming has been considered in [35, 41, 71, 74]. The methods and problems presented in [3, 54, 71, 74] are close to those discussed in [70]. It should be noted that the list of references from [70] includes fourteen papers on stability in mathematical programming, and only four of them being related to integer programming.
There exists a lot of papers on stability analysis in a transportation problem, e.g., [31, 76, 77]. Usually, an algorithm for solving ordinary problems (general ones or their special cases) is modified for solving parametric transportation problems. Some algorithms for parametric problems have been presented in [31].

In [45, 67–69] the stability of a scheduling algorithm has been discussed. The network scheduling problem with operation duration $a_i$ belonging to the closed interval $[a_i^1, a_i^2]$ has been considered in [56]. The question under consideration was to find a set of optimal makespan schedules for each $a_i \in [a_i^1, a_i^2]$ and thus to describe a stability polyhedron.

Obviously, this section cannot be regarded as a full presentation of all known approaches to stability analysis in combinatorial optimization. It gives only a brief description of the stability research surroundings. The main purpose of this paper is to review an approach to stability analysis that, originally, has been proposed in [48, 49], and then has been developed in [19–29, 32, 48, 50–52] for linear and bottleneck trajectory problem, in [33, 39, 83, 85] for the minimization of Boolean linear form, and in [1, 2, 42, 43, 78–82, 84, 86, 87, 89] for optimal makespan scheduling and for some other sequencing and scheduling problems. In our opinion, this approach generalizes some of the above mentioned ones, and gives a new glance on the well-known discrete optimization problems.

3. Stability radius for a trajectory problem (TP)

Consider a problem $(E, T_\alpha, \tilde{A})$ described in Section 1. Let $E$ and $T_\alpha$ be fixed. $\tilde{A}$ is $(k \times k)$-square matrix with real elements, $n = k^2$, and $R_{kk}$ is the space of all real $(k \times k)$-square matrices with the Chebyshev metric, i.e., the distance $r(\tilde{A}, \tilde{A}')$ between the matrices $\tilde{A} = ||a_{ij}||$ and $\tilde{A}' = ||a_{ij}'||$ from the set $R_{kk}$ is given by

$$\max\{|a_{ij} - a_{ij}': i = 1, 2, \ldots, k, j = 1, 2, \ldots, k\}.$$ 

The problem of finding an optimal trajectory $t \in T_\alpha$, in this case, will be called trajectory problem. For the case of reference, we shall use the abbreviation TP adjoined with an adjective linear or bottleneck depending on objective function (1) or (2) to be used. We assume that the set $T_\alpha$ in TP depends on $n$ only, and does not depend on the elements from $\tilde{A}$. In our consideration this assumption is necessary.

The following significant definitions have been introduced in [48, 49].

**Definition 3.1.** The open ball $O_{\rho}(\tilde{A}) \subset R_{kk}$ with the radius $\rho$ and the matrix $\tilde{A}$ as center is called a stability ball of $\tilde{A}$ if the inclusion $\varphi(\tilde{B}) \subseteq \varphi(\tilde{A})$ holds for each matrix $\tilde{B} \in O_{\rho}(\tilde{A})$.

**Definition 3.2.** Let $P$ be a set of all real $\rho$ that $O_{\rho}(\tilde{A})$ is a stability ball of $\tilde{A}$. The number $\rho(\tilde{A}) = \sup\{\rho: \rho \in P\}$ is called stability radius of $\tilde{A} \in R_{kk} \setminus R_0$, where $R_0 = \{\tilde{A}: \varphi(\tilde{A}) = \{1, 2, \ldots, m\}\}$. If $\tilde{A} \in R_0$, let us agree on $\rho(\tilde{A}) = 0$. 

The set $R_0$ includes all matrices $\tilde{A} \in R_{kk}$ such that all trajectories from $T_n$ have the same length with weighting $\tilde{A}$. In [50] it has been noted that $R_0$ is a subspace of $R_{kk}$.

The sense of Definitions 3.1 and 3.2 has been substantiated in [50, 52]. Here, it should be noted only that the knowledge of the stability radius $\rho(\tilde{A})$ and the set $\varphi(\tilde{A})$ for a given individual problem $Z_\tilde{A}$ makes possible to obtain a solution of all problems, whose each numerical parameter $a_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, k$, does not differ from the corresponding parameter of the given problem by a quantity greater than $\rho(\tilde{A})$. An invariant similar to $\rho(\tilde{A})$ turns out to be useful particularly for solving NP-complete problems, when a decision time increases exponentially with the problem dimensions, and when many information, being obtained, usually lost after decision process to be completed. As a rule, after solving an individual problem $Z_\tilde{A}$, we use only its solution. Calculating or, even though, estimating $\rho(\tilde{A})$ gives the possibility to use the obtained results for a new problem $Z_z$ with $r(\tilde{A}, \tilde{A}') \leq \rho(\tilde{A})$.

During consideration in Sections 3 and 4 a cardinality of the set $\varphi(\tilde{A})$ is very important, because all optimal trajectories are supposed to be known (see Definition 3.1). Fortunately, as it has been shown in [50, 52], the equality $|\varphi(\tilde{A})| = 1$ is valid "almost everywhere" (i.e., the Lebesgue measure of the set $\{\tilde{A} \in R_{kk}: |\varphi(\tilde{A})| > 1\}$ is equal to zero). More common situations, when $|\varphi(\tilde{A})| > 1$ is possible, are discussed in Sections 5 and 6.

The first analytical expression for the stability radius has been obtained in [23, 50]:

**Theorem 3.1.** The equality

$$\rho(\tilde{A}) = \min_{j \notin \varphi(\tilde{A})} \max_{i \in \varphi(\tilde{A})} \frac{|t_i(\tilde{A}) - t_j(\tilde{A})|}{|t_i| + |t_j| - 2|t_i \cap t_j|}$$

holds in the case of linear TP.

**Theorem 3.2.** The equality

$$\rho(\tilde{A}) = \min \{|t_i(\tilde{A}) - t_j(\tilde{A})|/2: i \in \varphi(\tilde{A}), j \notin \varphi(\tilde{A})\}$$

holds in the case of bottleneck TP.

The problems with both stable and variable elements of $\tilde{A}$ have been investigated in [27, 29].

**Definition 3.3.** The set $W \subseteq R_{kk}$ is called a covering of the set $V \subseteq R_{kk}$ by stability balls iff for any matrix $\tilde{B} \in V$ there exists a matrix $\tilde{A} \in W$ such that $\tilde{B}$ belongs to the stability ball of the matrix $\tilde{A}$.

In [26] it has been shown that there does not exist a finite and even a denumerable set $W$ which is a covering of the ball $O_1(\tilde{0})$. Nevertheless, for any given real $\varepsilon > 0$, the
finite \( \varepsilon \)-covering may be constructed: For \( \varepsilon \)-covering \( W \subseteq R_{kk} \) of the set \( V \subseteq R_{kk} \), the Lebesgue measure of the set

\[
\{ \tilde{B} \in V : \text{there exists a matrix } \tilde{A} \in W \text{ such that } \tilde{B} \in \overline{O}_\rho(\tilde{A}) \}
\]
is equal to the Lebesgue measure of \( O_1(0) \) minus \( \varepsilon \). Here and in what follows, \( \overline{O}_\rho(\tilde{A}) \) is a closure of \( O_\rho(\tilde{A}) \) and \( \tilde{O} \) is a \( k \times k \)-square matrix with all elements equal to zero.

The stability of a bottleneck TP has been studied in \([22, 23, 25-27]\). Other metrics in the space \( R_{kk} \) have been considered in \([21, 27, 52]\). For example, if \( \langle x \rangle \) is the Euclidean metric, we have the following assertion.

**Theorem 3.3.** If \( |\varphi(\tilde{A})| = 1 \) and \( k \in \varphi(\tilde{A}) \), then the equality

\[
\rho(\tilde{A}) = \min_{i \notin \varphi(\tilde{A})} \frac{|t_i(\tilde{A}) - t_k(\tilde{A})|}{\langle t_i - t_k \rangle}
\]
holds in the case of linear TP.

Theorems 3.4–3.6 have been proved in \([27, 28, 52]\) for linear TP.

**Theorem 3.4.** \( \rho(\tilde{A}) \leq \max \{|a_{ij}| : i = 1, 2, \ldots, k, j = 1, 2, \ldots, k \} \).

**Theorem 3.5.** If \( |t| = k \) for each \( t \in T_n \), matrix \( \tilde{A} = ||a_{ij||} \) belongs to \( R_{kk} \setminus R_0 \), all \( a_{ij} \) are rational numbers: \( a_{ij} = p_{ij}/q_{ij} \), and \( q \) is the least common multiple of \( q_{ij} \), \( i = 1, 2, \ldots, k, j = 1, 2, \ldots, k \), then we have \( \rho(\tilde{A}) \geq 1/(2kq) \).

**Theorem 3.6.** If \( \varphi(\tilde{A}) = \varphi(\tilde{B}) \) and \( \tilde{A} + \tilde{B} \) means a sum of matrices \( \tilde{A} \) and \( \tilde{B} \), then \( \rho(\tilde{A} + \tilde{B}) \geq \rho(\tilde{A}) + \rho(\tilde{B}) \).

The following notions have been introduced in \([52]\).

**Definition 3.4.** The matrix \( \tilde{A} \) is called stable with respect to \( t_s \), \( s \in \varphi(\tilde{A}) \), if there exists an open ball \( O_\rho(\tilde{A}) \subset R_{kk}, \rho > 0 \), such that \( s \in \varphi(\tilde{B}) \) for each \( \tilde{B} \in O_\rho(\tilde{A}) \). If \( P \) is a set of the all such radius \( \rho \), then \( \rho_s(\tilde{A}) = \sup \{ \rho : \rho \in P \} \).

**Definition 3.5.** The matrix \( \tilde{A} \) is called stable if

\[
\rho_0(\tilde{A}) = \max_{s \in \varphi(\tilde{A})} \rho_s(\tilde{A}) > 0.
\]

The following claim has been obtained in \([52]\).

**Theorem 3.7.** The matrix \( \tilde{A} \) is stable iff \( |\varphi(\tilde{A})| = 1 \).

Obviously, in the case \( |\varphi(\tilde{A})| = 1 \), it follows \( \rho_0(\tilde{A}) = \rho(\tilde{A}) \). Thus, \( \rho_s(\tilde{A}) = 0 \) for \( s \in \varphi(\tilde{A}) \) if there exists an optimal trajectory different from \( t_s \).
Let set $\Omega_i$ consist of all matrices $\tilde{A} \in \mathbb{R}^{kk}$ with $i \in \varphi(\tilde{A})$, i.e., $\Omega_i$ is a stability region of the trajectory $t_i$. In [48, 52] the following assertions about a stability region have been proved:

1. $\Omega_i$ is a convex closed cone for each $i = 1, 2, \ldots, m$;
2. $R_{kk} = \bigcup_{i=1}^{m} \Omega_i$;
3. $R_0 = \bigcap_{i=1}^{m} \Omega_i$ is a subspace of $R_{kk}$;
4. The cone $O_i$ is finitely generated.

4. The algorithms for computing a stability radius

Because of generality, formulas (5) and (6) require the complicated search for computing a stability radius in TP. For practical use, it is necessary to find more simple formulas, taking into account the specific features of an individual problem. Here, we present some effective algorithms for computing a stability radius for some kinds of TP.

As it was mentioned in Section 3, we are forced to assume that $|\varphi(A)| - 1$ (or at least, $|\varphi(A)|$ is bounded by a polynomial in $n$). It must be done even for the most polynomially solvable combinatorial optimization problems, since generally, the number of their solutions may depend on $n$ exponentially. This fact plays a significant role for computing $\rho(A)$. The probability of $|\varphi(A)|$ to be equal to 1 for TP with $a_t \in \{0, 1, \ldots, N - 1\}$ has been studied in [28].

Let us consider two theorems from [21] to compare the complexity of an initial problem $Z_4$ and a problem of calculating $\rho(A)$. Assume $A = \{a_1, a_2, \ldots, a_n\} \in \mathbb{R}_a$, where each $a_i$ is a rational number, i.e., $a_i = p_i/q_i$ with integer $p_i$ and integer $q_i$. Let $q$ mean the least common multiple of $q_i$, $i = 1, 2, \ldots, n$, and $n_0 = \max \{|t_i|: t = 1, 2, \ldots, n\}$. Let $p(n)$ be a polynomial in $n$ and $g(n, q)$ be a polynomial in $n$ and $q$.

**Theorem 4.1.** If $O(p(n_o))$ is the complexity of an algorithm for the linear TP and $A \in \mathbb{R}_a \setminus R_0$, then there exists an algorithm for calculating $\rho(A)$ in time $O(|\varphi(A)|g(n_0, aq))$, where $a = \max \{|a_i|: i = 1, 2, \ldots, n\}$.

**Theorem 4.2.** If $O(p(n_o))$ is the complexity of an algorithm for the bottleneck TP and $A \in \mathbb{R}_a \setminus R_0$, then there exists an algorithm for calculating $\rho(A)$ in time $O(|\varphi(A)|p(n_o))$.

To emphasize the significance of these claims we call your attention to [22], where it has been shown that the problem of finding $\rho(A)$ for a shortest path problem is NP-complete (see Theorem 4.10 in this section).

In [24] a specific transformation of a branch-and-bound algorithm for a traveling salesman problem to a procedure for calculating its stability radius was suggested. A little change in a branch-and-bound based program allows to calculate $\rho(\tilde{A})$.

The considered approach has been applied to the Steiner problem on a graph (briefly: SP). Here, we shall consider the stability of SP more thoroughly than stability
of other problems. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a set of edges in a graph $G = (V, E)$; $A \in \mathbb{R}_+$ be a vector of edge weights; $T_n$ be a set of Steiner trees in the graph $G$; and $V_1$ be a set of terminals, $V_1 \subseteq V$, $|V_1| = k$. Obviously, Steiner problem $Z_k$ is a special case of linear TP and formula (5) is valid for its $\rho(A)$. If the components of the vector $A$ are rational numbers, one can apply Theorem 4.1 for calculating $\rho(A)$ with complexity $O(|\varphi(A)|g(n_0, a_q))$. The constructive proof of this claim [21] gives an algorithm for such calculation. By the way, the special transformation of the branch-and-bound method [14] gives another algorithm for calculating $\rho(A)$ for SP. Note that the most known algorithms for SP are topological ones, and Steiner trees may be constructed in running time of such algorithm.

**Theorem 4.3.** If there exists an $O(g(n, k))$ algorithm for constructing all Steiner trees, then one can find $\rho(A)$ for SP in time $O(|\varphi(A)|g(n, k))$.

The last theorem gives the opportunity to apply any topological algorithm for calculating $\rho(A)$ for SP. Usually, the complexity of such calculation is essentially less than $g(n, k)$, and it is comparable, in asymptotic behavior, with the complexity of the initial topological algorithm. As inequality $p(n) < g(n, k)$ holds in many cases (e.g., either $a_1, a_2, \ldots, a_n$ are integers or $|\varphi(A)| = 1$ or $n$ is enough small), an algorithm based on Theorem 4.1 is more efficient than a topological one.

Since SP with many hundreds vertices are considered in practice, there exist numerous heuristic algorithms for SP in networks, see [94]. Most of them are based on the algorithms for a special case of SP with $V = V_1$, namely: minimum spanning tree problem (MSTP). For example, some very efficient algorithms for MSTP have been presented in [90, 94]. In [21] it has been shown that if an optimal trajectory is unique, the running time of an algorithm for calculating $\rho(A)$ for MSTP is equal to (in asymptotic behavior) the running time of the method from [38]: We can find $\rho(A)$ in time $O(na(n, |V|))$, where

$$a(x, y) = \min \{i: \log^{(i)} y \leq x/y\}$$

and

$$\log^{(i)} y = \underbrace{\log\log\cdots\log}_{i \text{ times}} y.$$

This algorithm is based on a simple formula for $\rho(A)$ that, in turn, follows from the next Theorem 4.4. Let $T_n$ be a set of the spanning trees such that for any $t_j \in T_n$ we have $|t_i \cap t_j| = |V| - 2$, where $t_i$ is a fixed optimal spanning tree in MSTP and $t_j(A) \neq t_i(A)$.

**Theorem 4.4.** If $t_i$ is an optimal spanning tree and $A \in \mathbb{R}_n \setminus R_0$, then we have

$$\rho(A) = \min \{t_j(A) - t_i(A): t_j \subseteq T_n\}/2.$$
Emphasize two essential properties of formula (8): Absence of search in the set $q(A)$ and distinction $t_i$ and $t_j$ only in one element. Let us consider an optimal trajectory (i.e., a minimum spanning tree) $t_s = (e_{s_1}, e_{s_2}, \ldots, e_{s_p})$. For each element $e_{s_i}$, let $c(e_{s_i})$ denote $\min\{w(e_j) - w(e_{s_i})\}$, where the minimum is taken over all elements $e_j \in E \setminus t_s$ such that $w(e_j) \neq w(e_{s_i})$ and $\{t_s \setminus \{e_{s_i}\}\} \cup \{e_j\}$ is a spanning tree. If a set of such elements $e_j$ is empty, let us agree that $c(e_{s_i})$ is not defined (thus, either $c(e_{s_i}) > 0$ or it is not defined). Assume

$$d_s(A) = \min\{c(e_{s_i}) : c(e_{s_i}) \text{ is defined}, i = 1, 2, \ldots, p\}.$$  \hspace{1cm} (9)

Due to Theorem 4.4, the value $d_s(A)$ is the same for all optimal trajectories. Thus, the subscript $s$ may be dropped in the notation $d_s(A)$ and we obtain

$$\rho(A) = d(A)/2.$$ \hspace{1cm} (10)

As a result of (10), it is not difficult to describe a very simple algorithm [21]. Nevertheless, one can use just the more effective (in some cases) procedure, namely, an algorithm from [90]. Let us use Tarjan's algorithm in the following way. Let $t_s$ be a solution of MSTP. Note that in [90] only one element $a_i, i = 1, 2, \ldots, n$, is perturbed and two numbers $x_i$ and $y_i$ are calculated such that for any variation of the weight of $e_i$ within the closed interval $[x_i, y_i]$, the trajectory $t_s$ remains optimal. If the optimal trajectory $t_s$ is unique, due to Theorem 4.4 and equality (10), we obtain

$$\rho(A) = \min\{y_i - a_i : e_i \in t_s\}/2.$$ 

Further, due to the fact that for the minimum spanning tree $t_s$ we have $x_i = -\infty$ for all $i$ such that $e_i \in t_s$, we can simplify the last formula:

$$\rho(A) = \min\{y_i - a_i : e_i \in t_s\}/2.$$

After having all such numbers $y_i, i = 1, 2, \ldots, n$, been defined, we can calculate $\rho(A)$ in time $O(n)$. In [90], it has been proved that these numbers may be calculated in time $O(na(n, |V|))$. So noting (7), we conclude

**Theorem 4.5.** If $A \in R_n \setminus R_0$ and $|\varphi(A)| = 1$, then $\rho(A)$ in the MSTP can be calculated in time $O(na(n, |V|))$.

In [20] the stability approach under consideration have been applied to some problems on a weighted matroid and on an intersection of the two weighted matroids. It should be noted again that the former is considered without a restriction on $|\varphi(A)|$. The following statement generalizes the previous MSTP consideration and uses the above notations, in particular (9).

Let $w(A)$ be the least nonzero weight of the elements from an optimal trajectory (or a trajectory is a base of matroid) and $c(n)$ restricts time for checking the independence of any subset of $E$. 
Theorem 4.6. If \((E, T_n)\) is a matroid with weighting \(A \in R^+_n \setminus R_0\), then for the optimal trajectories \(t_n\) and \(t_{n'}\), the equalities
\[
d_n(A) = d_{n'}(A) = d(A) \quad \text{and} \quad \rho(A) = \min(w(A), d(A)/2)
\]
hold, and \(\rho(A)\) can be computed in \(O(n^2c(n))\) time.

The case with intersection \(T_n = F_1 \cap F_2\) of the two matroids \(M_1 = (E, F_1)\) and \(M_2 = (E, F_2)\) includes many problems on a bipartite matching, on a maximal matching under symmetry condition, on a maximal directed spanning tree, an assignment problem, and some others.

Let \(r\) be the smallest rank of matroids \(M_1\) and \(M_2\), and \(C(n) = \max\{c_1(n), c_2(n)\}\), where \(c_1(n)\) and \(c_2(n)\) mean the same quantities for \(M_1\) and \(M_2\) as \(c(n)\) does for matroid \((E, T_n)\).

Theorem 4.7. If \(A \in R^+_n \setminus R_0\), \(|\phi(A)| = 1\), and \(T_n = F_1 \cap F_2\), while \(M_1 = (E, F_1)\) and \(M_2 = (E, F_2)\) are matroids, then \(\rho(A)\) can be computed in time \(O(n^r + nrC(n))\).

Stability in a shortest path problem has been investigated in [22]. Consider a digraph \(G = (V, E)\) with a set of vertices \(V = \{v_1, v_2, \ldots, v_k\}\) and a set of arcs \(E = \{e_1, e_2, \ldots, e_n\}\). Let \(v\) and \(w\) be two fixed vertices from the set \(V\). Denote the set of all simple paths from \(v\) to \(w\) by \(T_n = \{t_1, t_2, \ldots, t_m\}\) and suppose \(m > 1\). Usually, \((k \times k)\)-square matrix \(A = ||a_{ij}||\) of distances is used instead of the weight vector. Assume

\[
a_{ij} = \begin{cases} a_p & \text{if } (i, j) = e_p \in E, \\ \infty & \text{otherwise.}
\end{cases}
\]

Thus, a matrix \(\bar{A}\) consists of elements \(a_1, a_2, \ldots, a_n\) and \(k^2 - n\) its elements are equal to \(\infty\). Let space \(R_{sk}\) have the Chebyshev metric and the set \(t_i \oplus t_j = (t_i \cup t_j) \setminus (t_i \cap t_j)\) be ordered: The arcs of the set \(t_i \setminus t_j\) follow after the arcs of the set \(t_j \setminus t_i\) and the arcs from \(t_i\) and \(t_j\), respectively, being ordered in the same way as in \(t_i\) and \(t_j\). Let us present \(T_n\) as a union of two sets \(T\) and \(T'\), where
\[
T = \{t_j: j \notin \phi(\bar{A})\}, \text{ there exists an optimal path } t_i \text{ such that } t_i \oplus t_j \text{ is a circuit}\}
\]
and \(T' = T_n \setminus T\). The proof of the following statement has been given in [22],

Theorem 4.8. If \(a_p \geq 0\) for all \(p = 1, 2, \ldots, n\) and \(\phi(\bar{A}) = \{i\}\), then
\[
\rho(\bar{A}) = \min_{t_i \in T} \frac{t_j(\bar{A}) - t_i(\bar{A})}{|t_i| + |t_j| - 2|t_i \cap t_j|}.
\]

Let \(\{\mu_1, \mu_2, \ldots, \mu_u\}\) be the set of all circuits in digraph \(G\) and \(l^A(\mu_i)\) mean the length of the circuit \(\mu_i\) with weighting \(\bar{A}\). The circuit \(\mu_j\) is called shortest ratio circuit if
\[
l^A(\mu_j)/|\mu_j| = \min\{l^A(\mu_i)/|\mu_i|: i = 1, 2, \ldots, u\}.
\]
Theorem 4.8 provides a simple algorithm for calculating $\rho(\bar{A})$ when $|\varphi(\bar{A})| = 1$. It uses a procedure of finding a shortest ratio circuit, including a given path in digraph $G$. At first, such path is an arc from $t_i$, then it is a pair of adjacent arcs from $t_i$, then it is a triple of adjacent arcs, and etc. Since the number of such paths is equal to $O(n^2)$, we obtain

**Theorem 4.9.** If there exists an $O(g(n))$ algorithm for finding a shortest ratio circuit, including a fixed path, then one can find $\rho(\bar{A})$ in time $O(n^2g(n))$ provided that $|\varphi(\bar{A})| = 1$.

Consider a recognition version of the problem of calculating $\rho(\bar{A})$.

**Input:** A graph $G$, a matrix $\bar{A}$, a fixed number $r$.

**Output:** Do there exist path $t_j$ and path $t_i$, $j \notin \varphi(\bar{A})$, $i \in \varphi(\bar{A})$, such that

$$\frac{t_j(\bar{A}) - t_i(\bar{A})}{|t_i| + |t_j| - 2|t_i \cap t_j|} < r?$$

Obviously, the above question is equivalent to the following one: Does the inequality $\rho(\bar{A}) < r$ hold? Let us agree to denote this problem by $\text{RADIUS}$. Call to mind, that a shortest path problem in a digraph without negative circuits is polynomially solvable. Nevertheless, the following claim takes place.

**Theorem 4.10.** The problem $\text{RADIUS}$ is NP-complete even in the case of absence of negative cycles in digraph $G$.

The proof of Theorem 4.10 has been obtained in [22] by the reduction of a Hamiltonian cycle problem to $\text{RADIUS}$.

5. The minimization problem of Boolean linear form (MBLF)

As it was outlined in Section 4, the complexity of $\rho(A)$ calculation depends on $|\varphi(A)|$. Therefore, it is advisable to develop the approach described in Sections 3 and 4 for the cases in which there exist many solutions of the considered problem. In this section, we shall demonstrate such development for slightly general linear TP using an $\varepsilon$-approximate solution instead of exact one. It must be noted that $\varepsilon$-approximate solution $t$ of the individual problem $Z_\varepsilon$ may be stable also when there exist other $\varepsilon$-approximate solutions of $Z_\varepsilon$ (it is not like this for an exact solution because of Theorem 3.7). Another development of the approach described in Sections 3 and 4 will be presented in Section 6 for more complicated problems from the scheduling theory.

Consider the next problem. Let $N$ be a set $\{1, 2, \ldots, n\}$ of positive integers; $X^n$ be the set of all $(0, 1)$-vectors $x = (x_1, x_2, \ldots, x_n)$; $R^n_+$ be the space of all nonnegative real vectors $A = (a_1, a_2, \ldots, a_n)$ with the Chebyshev metric:

$$r(A, A') = \max \{|a_i - a'_i|: i \in N\}$$
is a distance between vectors \( A \) and \( A' = (a'_1, a'_2, \ldots, a'_n) \). The set \( X \subseteq X^n \) of all feasible vectors is known, \( X \neq \emptyset \). The problem MBLF is to find a vector \( x^A = (x^A_1, x^A_2, \ldots, x^A_n) \in X \) such that

\[
F(A, x^A) = \min \{F(A, x) : x \in X\},
\]

where \( F(A, x) = \sum_{i \in N} a_i x_i \) is the objective function. The vector \( x^A \) is called a solution of MBLF and we assume (similar to TP) that the set \( X \) of feasible vectors depends on \( n \) only (does not depend on \( a_1, a_2, \ldots, a_n \)). Obviously, we can represent any linear TP with nonnegative \( a_i, i = 1, 2, \ldots, n \), in terms of MBLF in the following way. Let \( E = N \) and each trajectory \( t_i \in T_n \) correspond to the vectors \( x = (x_1, x_2, \ldots, x_n) \in X \) such that

\[
x_j = \begin{cases} 1 & \text{if } j \in t_i, \\ 0 & \text{otherwise}, \end{cases}
\]

and vice versa. Thus, we have one-to-one correspondence between the elements of \( T_n \) and the elements of \( X \).

Let \( x \in X \) be \( \varepsilon \)-approximate solution of BLFM, namely

\[
F(A, x) \leq (1 + \varepsilon) F(A, x^A).
\]  

The set of the vectors \( A \in R^+_n \), satisfying inequality (12), will be called a stability region of \( \varepsilon \)-approximate solution \( x \) and will be denoted by \( \Omega(x, \varepsilon) \). Let us consider the following definitions [39].

**Definition 5.1.** The closed ball \( \bar{O}_\rho(A) \) with the radius \( \rho \) and the center \( A \in \Omega(x, a) \), \( \varepsilon \geq 0 \), will be called a stability ball of an \( \varepsilon \)-approximate solution \( x \) of MBLF if \( \bar{O}_\rho(A) \cap R^+_n \subseteq \Omega(x, \varepsilon) \).

**Definition 5.2.** The largest value of the radius \( \rho \) of a stability ball \( \bar{O}_\rho(A) \) is called stability radius of the \( \varepsilon \)-approximate solution \( x \) and it is denoted by \( \rho_\varepsilon(x, A) \).

By \( U \) we denote the reflexive binary relation \( U \subseteq X \times X \) with maximal cardinality such that \( (x, x') \in U \) holds iff the equation \( x_i = 1 \) implies \( x'_i = 1 \) for all \( i \in N \). Here, \( x = (x_1, x_2, \ldots, x_n) \) and \( x' = (x'_1, x'_2, \ldots, x'_n) \). The following results have been proved in [39].

**Theorem 5.1.** We have \( \rho_\varepsilon(x, A) = \infty \) iff inclusion \( (x, x') \in U \) holds for each vector \( x' \in X \).

**Theorem 5.2.** We have \( \rho_\varepsilon(x, A) = 0 \) iff condition (12) is an equality and there exists a solution \( x^A \) of MBLF that inclusion \( (x, x^A) \in U \) does not hold.

**Theorem 5.3.** Let MBLF be a traveling salesman problem (or assignment problem) then necessary and sufficient conditions for \( \rho_\varepsilon(x, A) = 0 \) are the following: Relation (12) is an equality and there exists a solution \( x^A \) different from \( x \).
Moreover, the formulas for calculating $\rho_e(x, A)$, and lower and upper bounds of $\rho_e(x, A)$ have been obtained in [39]. The next claim has been proved in [85].

**Theorem 5.4.** The closed ball $\overline{B}_p(A)$, $p > 0$, is a stability ball of $x$ iff the inclusion $(A + \Delta) \in \Omega(x, \varepsilon)$ holds for each vector $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n) \in R^n_+$ with components $\Delta_i \in \{p, \max\{-a_i, -p\}\}$, $i = 1, 2, \ldots, n$.

Theorem 5.4 gives the possibility to calculate value $\rho_e(x, A)$ with the given accuracy $\delta > 0$ by the $O(2^n \log_2(a/\delta))$ algorithm [83, 85]. Here $a = \max\{a_i : i \in N\}$. It has been shown also that there exists a finite covering (see Definition 3.3) of any bounded region in the space $R^n_+$ by stability balls of $\varepsilon$-approximate solutions of MBLF for any given $\varepsilon > 0$.

Consider some results for the case MBLF with $a_1, a_2, \ldots, a_m$ being variables and the components $a_{m+1}, a_{m+2}, \ldots, a_n$ being stable. If $A \in R^n_+$, let $A'$ be a vector $(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m) = (a_1, a_2, \ldots, a_m) \in R^n_+$ with variable components. The stability region and the stability radius are defined in this case by the following equalities:

$$
\Omega(x, A, m) = \{A' \in R^n_+ : F(A, x) \leq (1 + \varepsilon)F(A, x^A)\},
$$

$$
\rho_e(x, A, m) = \max\{\rho : \overline{B}_p(A') \cap R^n_+ \subseteq \Omega(x, \varepsilon, m)\}.
$$

The results similar to Theorem 5.1, Theorem 5.2 and Theorem 5.3 have been proved in [83, 85] for $\rho_e(x, A, m)$. Theorem 5.3 analogy and the next estimation

$$
\rho_e(x, A, m) < a_e = \max \left\{ \max\{a_i : i = 1, 2, \ldots, n\}, (1 + \varepsilon) \sum_{i=m+1}^{n} a_i \right\}
$$

have as a consequence the following theorem.

**Theorem 5.5.** If there exists an $O(p(n))$ algorithm for calculating $F(A, x^A)$, then there exists $O(p(n)2^m \log_2(a_\varepsilon/\delta))$ algorithm for calculating $\rho_e(x, A, m)$ with the given accuracy $\delta > 0$.

Thus we conclude "somewhat opposite" to Theorem 4.10.

**Corollary 5.1.** If MBLF is a polynomially solvable problem and the number $m$ of the variable components increases as $O(\log_2 n)$, then the problem of calculating $\rho_e(x, A, m)$ with the given accuracy $\delta$ is a polynomially solvable problem too.

In [32] the stability of an $\varepsilon$-approximate solution of BLFM with the Euclidean metric has been considered. The formula for calculating a stability radius and some properties of a stability region have been obtained. The properties of $\Omega(x, \varepsilon)$, similar to (i)-(iv) from Section 3, have been proved in [39].
6. Optimal makespan scheduling (OMS)

As in Section 5, the stability of the concrete solution will be considered here also. The condition \(|\rho(A)| = 1\) that was very important for TP (see Sections 3 and 4), is not essential for the schedule problem, because analogy of Theorem 3.7 is not valid for a makespan scheduling problem and for some other problems that are more complicated than TP. It must be noted also that stability of the concrete solution seems to be more useful in practice than the invariant \(\rho(A)\), analyzed in Sections 3 and 4.

Let us consider the following scheduling problem. There is a set \(Q = \{1, 2, \ldots, n\}\) of \(n\) operations that have to be processed on the machines of set \(M = \{M_1, M_2, \ldots, M_m\}\). \(Q_k\) denotes the set of operations that are to be processed on machine \(M_k \in M\). Let vector \(A = (a_1, a_2, \ldots, a_n)\), \(A \in \mathbb{R}^*_+,\) represent the duration \(a_i\) of each operation \(i \in Q\). The preemptions of operations are not allowed. The set of operations \(Q\) is supposed to be partially ordered by the precedence constraints:

\[
\text{if } i \rightarrow j, \text{ then } c_i \leq c_j - a_j,
\]

where \(c_i\) is the completion time of the operation \(i \in Q\). Since at any time each machine can process one operation at most, we can conclude that one of the inequalities

\[
c_p \leq c_q - a_q \quad \text{or} \quad c_q \leq c_p - a_p
\]

must hold for each pair of operations \(p\) and \(q\) from the same set \(Q_k\). The optimal makespan scheduling problem (OMS) is to find a feasible schedule \((c_1, c_2, \ldots, c_n)\) in order to minimize the value of the objective function \(\Phi(c_1, c_2, \ldots, c_n) = \max\{c_i : i \in Q\}\). This problem data can be conveniently represented by means of a disjunctive graph \(G = (Q, C \cup D)\), where:
- \(Q\) is the set of vertices (operations), nonnegative weight \(a_i\) being attached to each vertex \(i \in Q\);
- \(C\) is the set of directed (conjunctive) arcs, representing the given conditions (13):
  \[C = \{(i, j) : i \rightarrow j, i \in Q, j \in Q\};\]
- \(D\) is the set of directed (disjunctive) arcs, representing the conditions (14):
  \[D = \{(p, q), (q, p) : p \in Q_k, q \in Q_k, k = 1, 2, \ldots, m\}.
\]

A pair of disjunctive arcs \(\{(p, q), (q, p)\}\) must be settled (i.e., one of the two arcs must be added to a subset \(D' \subset D\) of chosen arcs and the other one must be rejected \([63, 66, 88]\)). The choice of arc \((p, q)\) (respectively \((q, p)\)) defines the precedence of operation \(p\) (of operation \(q\)) over operation \(q\) (of operation \(p\)) on their common machine \(M_k \in M\). A feasible schedule is defined by a subset \(D^s \subset D\) such that

(i) \((p, q) \in D^s\) iff \((q, p) \in D \setminus D^s\) and

(ii) digraph \(G^s = (Q, C \cup D^s)\) has no circuits.

Since the criterion is regular (i.e., an objective function is a nondecreasing one of \(n\) completion times of operations \(Q\)), we may consider only semiactive schedules \([7, 63, 89]\). Let \(P(G) = \{G^1, G^2, \ldots, G^k\}\) be the set of all digraphs \(G^s\) that satisfy the conditions (i) and (ii). Each digraph \(G^s \in P(G)\) defines a unique semiactive schedule \(s = (c_1(s), c_2(s), \ldots, c_n(s))\), where \(c_i(s)\) is the earliest possible completion time of operation
with respect to \( G^s \). On the other hand, each schedule (and each semiactive schedule in particular) defines unique digraph \( G^s \in P(G) \). The graph \( G^s \in P(G) \) is called optimal if \( s \) is an optimal schedule. Although OMS is NP-hard in the strong sense problem [46, 63], after having the optimal digraph \( G^s \) constructed we can find the optimal schedule \( s \) in \( O(n^2) \) time [9, 88]. Therefore, the main difficulty of the scheduling problem under consideration is to construct an optimal digraph \( G^s = (C \cup D^s) \), i.e., to define the set \( D^s \) of the chosen arcs. In [87] the set \( D^s \) was called a signature of a schedule \( s \). The combinatorial character of OMS appears through the number, \( \lambda \), of these signatures. Moreover in practice, it is more important to determine, not just a schedule, but the orders, in which operations are to be processed on machines. Let \( \varphi(A) \) denote the set of all indices \( s \) of the optimal digraphs \( G^s \in P(G) \). The question is under what changes in vector \( A \), digraph \( G^s \) will remain optimal. The following definitions have been introduced in [78, 90].

**Definition 6.1.** The closed ball \( \bar{O}_p(A) \) is called a stability ball of \( G^s \), \( s \in \varphi(A) \), if for any vector \( A' \in \bar{O}_p(A) \cap R^+_n \) number \( s \) belongs to \( \varphi(A') \).

**Definition 6.2.** The largest value of the radius \( \rho \) of a stability ball \( \bar{O}_p(A) \) of \( G^s \) is called a stability radius of \( G^s \) and it is denoted by \( \rho_s(A) \).

Let \( H_s \) be the set of all paths in digraph \( G^s \in P(G) \). If \( \mu \in H_s \), then \( \{ \mu \} \) means the set of vertices in the path \( \mu \), and \( l^4(\mu) \) denotes the weight of it:

\[
l^4(\mu) = \sum_{i \in \{ \mu \}} a_i.
\]

The path \( \mu \in H_s \) is called dominant if there is no other path \( v \in H_s \) that \( \{ \mu \} \subseteq \{ v \} \) holds. Let \( H \) and \( H_s \) denote the subsets of all dominant paths in digraphs \((Q, C)\) and \( G^s \in P(G) \), respectively.

If for any real \( \rho > 0 \) the ball \( \bar{O}_p(A) \) is a stability ball of \( G_s \), then we write \( \rho_s(A) = \infty \). The following claims have been proved in [78, 84].

**Theorem 6.1.** We have \( \rho_s(A) = \infty \) iff for any \( \mu \in H_s \setminus H \) and for any digraph \( G^k \in P(G) \) there exists a path \( v \in H_k \) such that inclusion \( \{ \mu \} \subseteq \{ v \} \) holds.

**Theorem 6.2.** The inequality \( \rho_s(A) > 0 \) holds, \( s \in \varphi(A) \), \( A \in R^+_n \), iff for any \( \mu \in H_s \setminus H \) and for any \( k \in \varphi(A) \) there exists a path \( v \in H_k \) such that \( \{ \mu \} \subseteq \{ v \} \).

**Theorem 6.3.** If \( s \in \varphi(A) \), then

\[
\rho_s(A) = \min_{k \neq s} \max_{v \in H_s, \mu \in H \setminus H_s, \beta = 0, 1, \ldots, a_{vr}} \max_{|\{ \mu \} \cup \{ v \}| - |\{ \mu \} \cap \{ v \}| - \beta} l^4(\mu) - l^4(\mu) - \sum_{i=0}^{\beta} a_{\mu v}
\]

where \( a_{0 v}^0 = 0 \) and \((a_{1 v}, a_{2 v}, \ldots, a_{a_{vr}})\) denotes nondecreasing sequence of the operation durations from the set \( \{ v \} \setminus \{ \mu \} \).

**Corollary 6.1.** If \( \rho_s(A) < \infty \), then \( \rho_s(A) < a^* = \max \{ a_i : i \in Q \} \).
Corollary 6.2. If \( \varphi(A) = \{s\} \), then \( \rho_s(A) > 0 \).

Corollary 6.3. If \( s \in \varphi(A) \) and \( H_s \subseteq H \), then \( \rho_s(A) > 0 \), where \( H_s \) denotes a set of all dominant critical paths in \( G^s \in P(G) \).

Theorem 6.1 is valid for any disjunctive graph model, i.e., for any given precedence constraints \( \rightarrow \). Unfortunately, it is difficult to verify the conditions of Theorem 6.1. More simple ones have been obtained in [42, 43] for a so-called job-shop problem, see Theorem 6.4.

Let \( J = \{J_1, J_2, \ldots, J_n\} \) be a set of \( n \) jobs that have to be processed on machine set \( M \). Each job \( J_k \in J \) consists of a sequence of \( n_k \) operations, route (machine order) \( I_k = (l_{k1}, l_{k2}, \ldots, l_{kn_k}) \), \( M_{l_{kj}} \in M \), \( 1 \leq q \leq n_k \), being given. If \( l_k = (1, 2, \ldots, m) \) for all jobs \( J_k \in J \), we have a flow-shop problem indicated by \( n|m|F|C_{\text{max}} \). If \( n_i \) and \( l_k \) may vary per job we have a job-shop problem \( n|m|J|C_{\text{max}} \). If \( l_k \) is not fixed for any job \( J_k \in J \), we have an open-shop problem \( n|m|O|C_{\text{max}} \). Obviously, digraph \( (Q, C) \) for \( n|m|J|C_{\text{max}} \) represents disconnected chains: \( Q = \bigcup_{k=1}^{n} Q^k \), each chain vertices \( Q^k \) being \( n_k \) ordered operations of job \( J_k \in J \). Denote \( L(k) = \{t: i \rightarrow j, i \in Q \setminus Q^k, j \in Q^k\} \), \( R(k) = \{j: i \rightarrow j, j \in Q \setminus Q^k, i \in Q^k\} \) and consider factor sets \( L(k)/J \) and \( R(k)/J \), where two operations \( i \) and \( j \) are equivalent iff \( i \in Q_k \) and \( j \in Q_k \) for some \( k \).

Theorem 6.4. For the problem \( n|m|J|C_{\text{max}} \) there exists an optimal digraph \( G^s \in P(G) \) with \( \rho_s(t) = \infty \) iff for any \( M_k \in M \) such that \( |Q_k/J| > 1 \), we have inequalities \( L(k) \leq 1 \) and \( R(k) \leq 1 \) and, moreover, if there exists a job \( J_i \in J \) such that \( L(k) \cap Q^i = g \) and \( R(k) \cap Q^i = f \), then in the digraph \( (Q, C) \) there exists a path from \( f \) to \( g \) or \( g = f \).

In spite of large length of Theorem 6.4, to verify its conditions takes no more than \( O(|Q|^2) \) times [43]. Theorem 6.4 gives the possibility for any given \( n \) and \( m \) to construct a job-shop scheduling problem with an optimal digraph \( G^s \) with \( \rho_s(t) = \infty \). For a flow-shop problem we have the following corollary.

Corollary 6.4. If for the problem \( n|m|F|C_{\text{max}} \) the conditions \( n > 1 \) and \( m > 1 \) hold, then \( \rho_s(a) \leq a^* \) for any optimal schedule.

There exists an open-shop scheduling problem with infinitely large \( \rho_s(t) \) only for smallest \( n \) and \( m \).

Corollary 6.5. If for the problem \( n|m|O|C_{\text{max}} \) either the conditions \( n > 1 \) and \( m > 2 \) or the conditions \( n > 2 \) and \( m > 1 \) hold, then \( \rho_s(a) \leq a^* \) for any optimal digraph \( G^s \).

Real-world examples of schedules \( s \) with \( \rho_s(t) = \infty \) have been presented in [43]. The necessary and sufficient conditions (similar to Theorem 6.1) have been obtained for a job-shop problem with minimization of maximum lateness of the jobs [43]. The obtained conditions can be verified in polynomial time too. It was proved also that
there does not exist an optimal schedule \( s \) with \( \rho_s(t) = \infty \) for a job-shop problem with other nontrivial (traditional) criteria.

In [54] has been studied a solution stability for so-called permutation flow-shop problem. In particular, the necessary and sufficient conditions for stability of optimal permutation have been given. The stability of all optimal schedules (similar to Definitions 3.1 and 3.2) has been considered in [2, 78]. The stability of the solution set for an open-shop problem has been discussed in [91] without large progress.

Some extension of Theorems 6.1–6.3 for scheduling problems with other regular criterion has been presented in [79]. It has been shown that in order to calculate \( \rho_s(A) \) with any given regular criterion it is sufficient to determine an optimal value of the objective function in a nonlinear mathematical programming problem. Some simple properties, similar to (i)–(iv), of a stability region for an optimal digraph have been discussed in [78, 79, 81, 84, 85]. The stability of OMS with both different and identical machines has been investigated in [80]. In [81] it has been shown that a static or dynamic stochastic scheduling system may be represented by means of a mixed graph or mixed multi-graph with abundant vertices, arcs, and edges. Using such models, it is possible to find an optimal schedule under conditions of uncertainty. The paper [1] deals with the applied aspects of solution stability.

7. Conclusion

The problems considered in Sections 3–6 may be formulated in integer programming terms. Within the scope of this review, it is not possible to do full justice to the literature on the stability analysis in mathematical programming. We refer only to some close papers [16, 17, 30, 34, 36, 37, 40, 47, 53, 65, 72, 73, 95], books and surveys [4, 13, 15, 51, 54, 70]. It should be noted again, that stability notions have been used by numerous authors to indicate different things (e.g., the continuity of an objective function in mathematical programming). But this review does not affect results apart from the above-mentioned definitions.

To finish the paper we outline possible meaning of the observed stability results for scheduling theory. In our opinion, stability analysis in sequencing and scheduling may give the means to shorten the gap between scheduling theory and practice. At least, such analysis may be used for transferring some “deterministic” results to scheduling under conditions of uncertainty. Moreover, the stability results may be used as a background to research stochastic scheduling problems [13, 15, 57–59, 61, 62, 93]. The usual assumption that operation durations are known in advance is the most strict one in (deterministic) scheduling theory and essentially restricts its practical aspects (indeed, this assumption is not valid for the most real-world processes). Research on stability of an optimal schedule may help to extend the significance of scheduling theory for some production scheduling problems.

Currently, the work in the field of solution stability is being continued, which includes the study of some practical aspects of our approach. In particular, an
important question is how large \( \rho(A) \) and \( \rho_s(A) \) to be in various practical problems. For different NP-hard problems, the investigation is developed in three directions: To simplify a search in general formulas (5), (6) and (15); to construct heuristic algorithms for calculating \( \rho(A) \), \( \rho_s(x, A) \), and \( \rho_s(A) \); and to consider simple bounds of stability radius.

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References


