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On the boundary of the numerical range of a matrix

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ABSTRACT

A characterization of real matrices is given for which a diagonal entry of a matrix is a boundary point of its numerical range.

Let $A \in M_n$. The *numerical range* of A is the set of complex numbers

$$W(A) = \left\{x^*Ax : x \in \mathbf{C}^n, |x| = 1\right\}.$$

It is a well-known result due to Toeplitz and Hausdorff that the numerical range $W(A)$ is always a convex set. In particular, for $n = 2$, $W(A)$ is an elliptical disc with foci λ, μ, eigenvalues of A, and semi-major axis $(\|A\|^2 - 2\mathrm{Re}\,\lambda\bar{\mu})^{1/2}/2$. For properties of the numerical range we refer the reader to the books [1,2]. It is clear that every diagonal element of a matrix A lies in $W(A)$. We determine 2×2 real matrices for which an diagonal entry is a boundary point of its numerical range. By using this result, when a diagonal entry or a typical point lies on the boundary of an $n \times n$ real matrix is examined.

Theorem 1. *Let A be a 2×2 real matrix given by*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then diagonal entry a_{11} is a boundary point of the numerical range of A if and only if $a_{12} + a_{21} = 0$, and $a_{12} = a_{21} = 0$ if $a_{11} = a_{22}$.

Proof. Consider the matrix

$$B \equiv A - (a_{11} + a_{22})/2\, I = \begin{pmatrix} (a_{11} - a_{22})/2 & a_{12} \\ a_{21} & (a_{22} - a_{11})/2 \end{pmatrix}.$$

Then $a_{11} \in \partial W(A)$ if and only if $(a_{11} - a_{22})/2 \in \partial W(B)$. Thus we may assume

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}. \tag{1}$$

Suppose $a_{11} \in \partial W(A)$. It is clear that the real number $a_{11} \in \partial W(A)$ if and only if there exists θ such that $\mathrm{Re}\, a_{11}e^{i\theta}$ is the maximal eigenvalue of $H_\theta(A) = (Ae^{i\theta} + A^*e^{-i\theta})/2$. We find that the eigenvalues of $H_\theta(A)$ are

$$\pm\frac{1}{2}\left(4(\mathrm{Re}\, a_{11}e^{i\theta})^2 + |a_{12}e^{i\theta} + \bar{a}_{21}e^{-i\theta}|^2\right)^{1/2}.$$

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Then

$$\frac{1}{2}\left(4(\mathrm{Re}\, a_{11}\mathrm{e}^{\mathrm{i}\theta})^2 + |a_{12}\mathrm{e}^{\mathrm{i}\theta} + \bar{a}_{21}\mathrm{e}^{-\mathrm{i}\theta}|^2\right)^{1/2} = \mathrm{Re}\, a_{11}\mathrm{e}^{\mathrm{i}\theta}. \tag{2}$$

From (2), we obtain

$$\left|a_{12}\mathrm{e}^{\mathrm{i}\theta} + \bar{a}_{21}\mathrm{e}^{-\mathrm{i}\theta}\right|^2 = 0. \tag{3}$$

From (3),

$$a_{21} = -a_{12}\mathrm{e}^{-2\mathrm{i}\theta}. \tag{4}$$

Since a_{21} is real, by (4), it follows that $\mathrm{e}^{-2\mathrm{i}\theta} = \pm 1$. If $\theta = 0$ then, by (4) again, $a_{21} = -a_{12}$, and thus $a_{12} + a_{21} = 0$. If $\theta = \pi/2$ then $a_{21} = a_{12}$; A is Hermitian. Since $a_{11} \in \partial W(A)$, a_{11} is an endpoint of the line segment $W(A)$. Hence a_{11} is an eigenvalue of A. Let μ be another eigenvalue of A. Then $a_{11} + \mu = \mathrm{trace}(A) = 0$; we have $-a_{11} \in \sigma(A)$. This implies that $a_{12} = a_{21} = 0$, $a_{12} + a_{21} = 0$. Suppose $a_{11} = a_{22}$; then $a_{11} = 0$ in (1). In this case, $W(A)$ is the line segment $[-i|a_{12}|,\ i|a_{12}|]$ on y-axis, and thus $a_{12} = 0$.

Conversely, suppose $a_{12} + a_{21} = 0$. We may also assume that A is in the form of (1) and $a_{11} \neq 0$. The eigenvalues of A then become $\pm(a_{11}^2 - a_{12}^2)^{1/2}$, and A is unitarily similar to the upper triangular matrix

$$T = \begin{pmatrix} (a_{11}^2 - a_{12}^2)^{1/2} & \alpha \\ 0 & -(a_{11}^2 - a_{12}^2)^{1/2} \end{pmatrix}.$$

Sine A and T have the same Frobenius norm, we have

$$2a_{11}^2 + 2a_{12}^2 = 2|a_{11}^2 - a_{12}^2| + |\alpha|^2. \tag{5}$$

From (5),

$$|\alpha| = \begin{matrix} 2|a_{12}|, & \text{if } a_{11}^2 \geq a_{12}^2 \\ 2|a_{11}|, & \text{otherwise.} \end{matrix}$$

In either case, $W(A)$ is an elliptical disc centered at the origin with foci $(a_{11}^2 - a_{12}^2)^{1/2}$ and $-(a_{11}^2 - a_{12}^2)^{1/2}$, and a_{11} is a vertex of the ellipse on the real line. □

For general $n \times n$ real matrices, we have the following result.

Theorem 2. *Let $A = (a_{ij}) \in M_n(\mathrm{R})$. If there exists i such that $a_{ii} \in \partial W(A)$ then $a_{ij} + a_{ji} = 0$ for all $1 \leq j \neq i \leq n$.*

Proof. For any $j \neq i$, consider the 2×2 principal submatrix

$$A_{ij} = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}.$$

Suppose $a_{ii} \in \partial W(A)$. Since $W(A_{ij}) \subset W(A)$ and $a_{ii} \in W(A_{ij})$, it follows that $a_{ii} \in \partial W(A_{ij})$. Then, by Theorem 1, $a_{ij} + a_{ji} = 0$. □

It is shown in [3] that

$$W(A) = \cup\, W\left(\begin{pmatrix} u^*Au & u^*Av \\ v^*Au & v^*Av \end{pmatrix}\right),$$

where u and v run over all orthonormal pairs in $\mathbf{C}^n$. We examine some 2×2 compression matrices in the union.

Theorem 3. *Let $A = (a_{ij}) \in M_n(\mathrm{R})$. If x and y are real orthonormal vectors such that $x^*Ax \in \partial W(A)$ then $x^*Ay + y^*Ax = 0$.*

Proof. Suppose $x^*Ax \in \partial W(A)$ and y is orthonormal to x. Consider the 2×2 compression

$$A_{xy} = \begin{pmatrix} x^*Ax & x^*Ay \\ y^*Ax & y^*Ay \end{pmatrix} \in M_2(\mathrm{R}).$$

Then $x^*Ax \in W(A_{xy}) \subset W(A)$, and hence x^*Ax is a boundary point of $W(A_{xy})$. By Theorem 1, $x^*Ay + y^*Ax = 0$. □

Remark. The converse of Theorem 2 is false. For example, consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $W(A)$ is a circular disc centered at the point (1,0) with radius 1. The condition $a_{1j} + a_{j1} = 0$ for $j = 2, 3$ in Theorem 2 is satisfied, but the entry $a_{11} = 1$ does not lie on the boundary of $W(A)$.

This example also provides the invalidity of the converse of Theorem 3 on taking $x = [1, 0, 0]^T$ and $y = [0, 1, 0]^T$.

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