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Countably P-Concentrative Pairs and the Coincidence Index

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Abstract—In this paper, a new coincidence index is presented for countably P-concentrative pairs. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let X and Y be metric spaces. A continuous single valued map $p : Y \to X$ is called a Vietoris map [1,2] if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic,
(ii) p is a proper map, i.e., for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X, Y)$ be the set of all pairs $X \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). In Section 2, we discuss the coincidence index for pairs $\overline{U} \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} X$ where U is an open subset of a Fréchet space X. In particular, we use the results of [3] together with the notion of r-dominated maps (based on work of Borsuk and Granas) to present a coincidence index for pairs of the above form.

Let Y and Z be topological spaces and V an open subset of Z. Then we say Y is r-dominated by V in Z if there exist a pair of continuous maps $r : V \to Y$, $s : Y \to V$ with $r\,s = 1_Y$. Let ANR, (respectively, AR) denote the class of metrizable absolute neighborhood retracts (respectively, absolute retracts); see [1]. The following result follows immediately from the Arens-Eells theorem (see [4, p. 284]).

THEOREM 1.1. *If $Y \in ANR$, (respectively, AR), then Y is r-dominated by an open subset of a normed space (respectively, by a normed space).*

Let (E, d) be a pseudometric space. For $S \subseteq E$, let $B(S, \epsilon) = \{x \in E : d(x, S) \leq \epsilon\}$, $\epsilon > 0$, where $d(x, S) = \inf_{y \in Y} d(x, y)$. The measure of noncompactness [5] of the set $M \subseteq E$ is defined

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by $\alpha(M) = \inf Q(M)$ where

$$Q(M) = \{\epsilon > 0 : M \subseteq B(A, \epsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Let A and C be two subsets of E. A pair $A \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} C$ is called a countably P-concentrative pair from A to C if for (every) $i \in P$, for each countably bounded non-i-precompact subset Ω of A (i.e., Ω is not precompact in the pseudonormed space (E, i)) we have $\alpha_i(q(p^{-1}(\Omega))) < \alpha_i(\Omega)$; here $\alpha_i(\,.\,)$ denotes the measure of noncompactness in the pseudonormed space (E, i).

In Section 3, a coincidence theory for countably P-concentrative pairs is presented. This section extends and generalizes results presented in [6]. In Section 3, all our pairs are countably P-concentrative. However, it is worth remarking here that in fact all the results in Section 3 hold if our pairs are countably condensing in the sense of [7, pp. 353, 356].

2. COINCIDENCE INDEX FOR COMPACT PAIRS IN FRÉCHET SPACES

A pair (p, q) is called compact if q is compact. Let U be an open subset of a normed space E. By $\mathcal{K}(\overline{U}, E)$ we mean the family of compact pairs (p, q) from $\overline{U}$ to E for which $\text{Fix}\,(p, q) \cap \partial U = \emptyset$ (recall that a pair (p, q) is from $\overline{U}$ to E if there exists a metric space Z for which $\overline{U} \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} E$); here $\text{Fix}\,(p, q) = \{x \in \overline{U} : x \in q\,(p^{-1}(x))\}$. In 1976, Kucharski [3] defined the coincidence index on $\mathcal{K}(\overline{U}, E)$ (the ideas presented in [3] follow closely ideas given by Granas previously [4]).

THEOREM 2.1. *There exists a map $I : \mathcal{K}(\overline{U}, E) \to \mathbf{Q}$ (called the coincidence index) which satisfies the following properties:*

(I) *if $I(p, q) \neq 0$, then $\text{Fix}\,(p, q) \neq \emptyset$;*
(II) *if $q : Z \to E$ given by $q(y) = x_0$ is a constant map, then $I(p, q) \neq 0$ if $x_0 \in U$;*
(III) *if V is an open subset of U, $(p, q) \in \mathcal{K}(\overline{U}, E)$ and $\text{Fix}\,(p, q) \subset V$, then $I(p, q) = I(p_1, q_1)$ where $V \overset{p_1}{\leftarrow} p^{-1}(U) \overset{q_1}{\rightarrow} E$ with $p_1(y) = p(y)$ and $q_1(y) = q(y)$.*

REMARK 2.1. Also the coincidence index has additivity, homotopy, and commutativity properties [4,6].

In this section, we extend Theorem 2.1 to the case when our space is a Fréchet space. The strategy presented here is based on an idea of Borsuk and Granas [4]. Let U be an open subset of a Fréchet space X. Let $\mathcal{M}(\overline{U}, X)$ denote the family of all compact pairs (p, q) from $\overline{U}$ to X for which $\text{Fix}\,(p, q) \cap \partial U = \emptyset$. Let $(p, q) \in \mathcal{M}(\overline{U}, X)$. Fix a normed space E which r-dominates X (Theorem 1.1 guarantees the existence of E). Fix $r : X \to E$ and $s : E \to X$ with $r\,s = 1_X$. Notice

$$\overline{r^{-1}(U)} \overset{r^{-1}p}{\longleftarrow} Z \overset{sq}{\rightarrow} E;$$

since $r^{-1}p : Z \to \overline{r^{-1}(U)}$ is a Vietoris map (this follows immediately from the definition, or alternatively use [1, p. 40]). Next, notice

$$\text{Fix}\,\left(r^{-1}p,\, s\,q\right) \neq \emptyset \text{ implies } \text{Fix}\,(p, q) \neq \emptyset. \tag{2.1}$$

To see this, notice if $x \in \text{Fix}\,(r^{-1}p, s\,q)$, then $x \in s\,q\,p^{-1}\,r\,(x)$ with $x \in \overline{r^{-1}(U)}$. Then since $r\,s = 1_X$, we have immediately that $r(x) \in q\,p^{-1}\,r\,(x)$ and $r(x) \in \overline{U}$ (since $r\left(\overline{r^{-1}(U)}\right) \subseteq \overline{r(r^{-1}(U))} \subseteq \overline{U}$), so $r(x) \in \text{Fix}\,(p, q)$.

We now claim that

$$\left(r^{-1}p, s\,q\right) \in \mathcal{K}\left(\overline{r^{-1}(U)}, E\right). \tag{2.2}$$

Certainly $(r^{-1}p,\, s\,q)$ is a compact pair, so it remains to show

$$\text{Fix}\,\left(r^{-1}p,\, s\,q\right) \cap \partial\, r^{-1}(U) = \emptyset.$$

If there exists $x \in \partial r^{-1}(U)$ with $x \in s\,q\,p^{-1}\,r\,(x)$, then since $r\,s = 1_X$, we have

$$r(x) \in q\,p^{-1}\,r\,(x), \qquad \text{with } r(x) \in \partial\,U,$$

a contradiction since $(p,q) \in M(\overline{U},X)$. Thus, (2.2) holds. *Define*

$$I(p,q) = I\left(r^{-1}\,p,\; s\,q\right),$$

where $I(r^{-1}\,p,\; s\,q)$ is as defined in Theorem 2.1.

REMARK 2.2. We note that the definition is independent of the choices involved. Suppose E' is another normed space which r-dominates X, so $s' : X \to E'$ and $r' : E' \to X$ are a pair of continuous maps with $r'\,s' = 1_X$. Then the same ideas as in [1, pp. 241–242] guarantee that

$$I\left((r')^{-1}\;p, s'\,q\right) = I\left(r^{-1}\,p, s\,q\right).$$

The coincidence index also satisfies Properties (I)–(III) (and other well-known properties). For completeness we will establish Property (I). Suppose $(p,q) \in M(\overline{U},X)$ with $I(p,q) \neq 0$. Then there exists E, r, and s as above with

$$I\left(r^{-1}\,p, s\,q\right) = I(p,q) \neq 0.$$

Now Theorem 2.1 guarantees that $\operatorname{Fix}(r^{-1}\,p, s\,q) \neq \emptyset$ so $\operatorname{Fix}(p,q) \neq \emptyset$ from (2.1). Thus, Property (I) holds. The other properties are similar.

For our results in Section 3 we will need a slight generalization of the index defined above. Let C be a closed convex subset of a Fréchet space X and U an open subset of C. Let $\mathcal{M}(\overline{U},C)$ denote the family of all compact pairs (p,q) from $\overline{U}$ into C with $\operatorname{Fix}(p,q) \cap \partial\,U = \emptyset$. We now extend the coincidence index from $\mathcal{M}(\overline{U},X)$ to $\mathcal{M}(\overline{U},C)$ using a standard trick [6, p. 5]. For completeness we sketch the extension here (see [6] for complete details). Fix a retraction $r_0 : X \to C$ (notice such a retraction exists by Dugundji's extension theorem). Let $(p,q) \in \mathcal{M}(\overline{U},C)$ so $\overline{U} \overset{p}{\leftarrow} Z \overset{q}{\to} C$. Let

$$\Gamma = \left\{(x,y) \in \overline{(r_0)^{-1}(U)} \times p^{-1}\left(\overline{U}\right) : r_0(x) = p(y)\right\} \qquad \text{and} \qquad g(x,y) = y,$$

with $g : \Gamma \to Z$. Also let

$$\tilde{p}(x,y) = x \qquad \text{and} \qquad \tilde{q} = i \circ q \circ g,$$

where $i : C \to X$ is the inclusion map. It is easy to check that

$$\operatorname{Fix}(p,q) = \operatorname{Fix}\,(\tilde{p},\tilde{q}) \tag{2.3}$$

and

$$\overline{(r_0)^{-1}\,(U)} \overset{\tilde{p}}{\leftarrow} \Gamma \overset{\tilde{q}}{\to} X.$$

Notice $(r_0)^{-1}\,(U)$ is open and from (2.3) we have $\operatorname{Fix}(\tilde{p},\tilde{q}) \cap \partial\,(r_0)^{-1}\,(U) \neq \emptyset$. Also, for fixed $x \in \overline{(r_0)^{-1}(U)}$ notice $(\tilde{p})^{-1}(x)$ is homeomorphic to $p^{-1}(r_0(x))$ so $(\tilde{p})^{-1}(x)$ is acyclic. Define

$$I(p,q) = I\,(\tilde{p},\tilde{q})$$

and again our coincidence index satisfies Properties (I)–(III).

3. COUNTABLY P-CONCENTRATIVE PAIRS

We present a coincidence index for countably P-concentrative pairs (p,q) in this section.

DEFINITION 3.1. *Let A and C be two subsets of a Fréchet space X (P a defining system of seminorms). A pair (see Section 1) $A \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} C$ is called a countably P-concentrative pair from A to C if for (every) $i \in P$, for each countably bounded non-i-precompact subset Ω of A, (i.e., Ω is not precompact in the pseudonormed space (X, i)) we have $\alpha_i(q(p^{-1}(\Omega))) < \alpha_i(\Omega)$; here $\alpha_i(\,.\,)$ denotes the measure of noncompactness in the pseudonormed space (X, i).*

REMARK 3.1. In this section, all our pairs are countably P-concentrative. However, it is worth remarking here that in fact all the results hold if our pairs are countably condensing in the sense of [7, pp. 353, 356].

For the analysis below we assume A is a bounded, closed subset of a Fréchet space X (P is a defining system of seminorms) and C a closed convex subset of X. Consider a countably P-concentrative pair (p, q) from A to C. We *claim* that we can associate with each pair (p, q) a compact pair $(\tilde{p}, \tilde{q})$ such that

$$\operatorname{Fix}(p,q) = \operatorname{Fix}(\tilde{p}, \tilde{q});$$

here $\operatorname{Fix}(p,q) = \{x \in A : x \in q(p^{-1}(x))\}$. To see this let

$$K_1 = \overline{\text{co}}\left(q\left(p^{-1}(A)\right)\right) \quad \text{and} \quad K_n = \overline{\text{co}}\left(q\left(p^{-1}(A \cap K_{n-1})\right)\right), \qquad \text{for } n = 2, 3, \dots.$$

Notice since $q(p^{-1}(A \cap K_n)) \subseteq \overline{\text{co}}(q(p^{-1}(A \cap K_n)))$ that we have

$$q\left(p^{-1}(A \cap K_n)\right) \subseteq K_{n+1}, \qquad \text{for each } n. \tag{3.1}$$

In addition, we note

$$\operatorname{Fix}(p,q) \subseteq K_n, \qquad \text{for each } n. \tag{3.2}$$

First, we show $\operatorname{Fix}(p,q) \subseteq K_1$. If $x \in \operatorname{Fix}(p,q)$, then $x \in A$ and $x \in q(p^{-1}(A))$, so $\operatorname{Fix}(p,q) \subseteq K_1$. Suppose $\operatorname{Fix}(p,q) \subseteq K_k$ for some k. Let $x \in \operatorname{Fix}(p,q)$ so $x \in A \cap K_k$ and as a result

$$x \in q\left(p^{-1}(A \cap K_k)\right) \subseteq \overline{\text{co}}\left(q\left(p^{-1}(A \cap K_k)\right)\right) = K_{k+1}.$$

Now there are two possibilities that can occur, namely

$$K_n \neq \emptyset, \qquad \text{for each } n \tag{3.3}$$

or

$$K_i \neq \emptyset, \qquad \text{for } i = 1, \dots, m \quad \text{and} \quad K_{m+j} = \emptyset, \qquad \text{for each } j \in \{1, 2, \dots\}. \tag{3.4}$$

If (3.4) holds, then we can choose $x_0 \in K_m$ and let

$$\tilde{q} : Z \to C \text{ by } q(z) = x_0 \qquad \text{and} \qquad \tilde{p} = p. \tag{3.5}$$

Clearly $(\tilde{p}, \tilde{q})$ is a compact pair. Now (3.2) guarantees that $\operatorname{Fix}(p,q) = \emptyset$. Also, if $x \in A$ with $x \in \tilde{q}(\tilde{p}^{-1}(x))$, then it is immediate that $x = x_0$ so $K_{m+1} \neq \emptyset$, a contradiction. Thus, $\operatorname{Fix}(\tilde{p}, \tilde{q}) = \emptyset$ and so

$$\operatorname{Fix}(p,q) = \operatorname{Fix}(\tilde{p}, \tilde{q}) = \emptyset.$$

Next suppose (3.3) holds. Then [7, Theorem 2.2] (note A is bounded and $q(p^{-1}(\,.\,))$ takes relatively compact sets into relatively compact sets) guarantees that

$$K_\infty = \bigcap_{n=1}^{\infty} K_n$$

is compact. Also, from (3.2) we have that $\operatorname{Fix}(p,q) \subseteq K_\infty$. Now define a compact (note p is a proper map) pair $(\tilde{p}, \tilde{q})$ as follows:

$$A \cap K_\infty \overset{\tilde{p}}{\leftarrow} p^{-1}(A \cap K_\infty) \overset{\tilde{q}}{\rightarrow} K_\infty,$$

with

$$\tilde{p}(u) = p(u) \quad \text{and} \quad \tilde{q} = q(u), \qquad \text{for all } u. \tag{3.6}$$

Also note

$$\text{Fix}\,(\tilde{p},\tilde{q}) = \left\{x \in A \cap K_\infty : x \in \tilde{q}\,\left(\tilde{p}^{-1}(x)\right)\right\} \subseteq \text{Fix}\,(p,q).$$

In addition, since $q\left(p^{-1}\left(A \cap K_\infty\right)\right) \subseteq K_\infty$ and $\text{Fix}\,(p,q) \subseteq K_\infty$ (from (3.2)) we have that

$$\text{Fix}\,(p,q) \subseteq \text{Fix}\,(\tilde{p},\tilde{q}).$$

Thus, $\text{Fix}\,(p,q) = \text{Fix}\,(\tilde{p},\tilde{q})$.

REMARK 3.2. If $A = C$, then it is easy to see that (3.4) cannot occur.

THEOREM 3.1. *Let A be a bounded, closed, convex subset of a Fréchet space X (P is a defining system of seminorms) and (p,q) a countably P-concentrative pair from A to A. Then $\text{Fix}\,(p,q) \neq \emptyset$.*

PROOF. From the above there exists a compact pair $(\tilde{p},\tilde{q})$ with $\text{Fix}\,(p,q) = \text{Fix}\,(\tilde{p},\tilde{q})$. Now $\text{Fix}\,(\tilde{p},\tilde{q}) \neq \emptyset$ from [1, p. 59]. ∎

Next, we show that it is easy using the results in Section 2 to define the coincidence index for countably P-concentrative pairs. Let X be a Fréchet space (P is a defining system of seminorms), C a closed convex subset of X and U an open subset of C with $\overline{U}$ bounded. By $M(\overline{U},C)$ we denote the family of all countably P-concentrative pairs from $\overline{U}$ to C with no fixed points on ∂U, (i.e., $\text{Fix}\,(p,q) \cap \partial U = \emptyset$ for all $(p,q) \in M(\overline{U},C)$).

DEFINITION 3.2. *Let $(p,q) \in M(\overline{U},C)$. We define the coincidence index $I(p,q)$ of (p,q) by putting*

$$I(p,q) = I\left(\tilde{p},\tilde{q}\right),$$

where $(\tilde{p},\tilde{q})$ is the compact pair associated with (p,q) by (3.5) or (3.6).

REMARK 3.3. Recall the coincidence index $I(\tilde{p},\tilde{q})$ is well defined (see Section 2).

THEOREM 3.2. *Let $(p,q) \in M(\overline{U},C)$ and $I(p,q) \neq 0$. Then $\text{Fix}\,(p,q) \neq \emptyset$.*

PROOF. Now $I(\tilde{p},\tilde{q}) \neq 0$, where $(\tilde{p},\tilde{q})$ is the compact pair associated with (p,q). Now Section 2 (Property (I)) guarantees that $\text{Fix}\,(\tilde{p},\tilde{q}) \neq \emptyset$, so $\text{Fix}\,(p,q) \neq \emptyset$. ∎

Essentially the same argument as in [6, p. 9] guarantees the following homotopy result.

THEOREM 3.3. *Let X, C, and U be as above and $p : Z \to \overline{U}$ a Vietoris map with $h : Z \times [0,1] \to C$ a continuous map. Suppose the following two conditions are satisfied:*

$$\text{Fix}\,(p,h) \cap \partial U = \emptyset \tag{3.7}$$

and

$$\begin{array}{l}\text{for (every) } i \in P, \text{ for each countably non-}i\text{-precompact subset } \Omega \text{ of } \overline{U} \\ \text{we have } \alpha_i\left(h\,\left(p^{-1}(\Omega) \times [0,1]\right)\right) < \alpha_i(\Omega).\end{array} \tag{3.8}$$

Then $I(p,h_0) = I(p,h_1)$, where $h_i(x) = h(x,i)$, $i = 0,1$.

Let X be a Fréchet space (P is a defining system of seminorms), C a closed convex subset of X, U an open subset of C with $\overline{U}$ bounded, and let $\phi \in Ad(\overline{U},C)$ be countably P-concentrative with $\text{Fix}\,\phi \cap \partial U = \emptyset$; here $\text{Fix}\,\phi = \{x \in U : x \in \phi(x)\}$ and ϕ is called countably P-concentrative if there exists a selected pair (p,q) of ϕ which is countably P-concentrative. Of course we can now define the coincidence index $I(\phi)$ by putting

$$I(\phi) = \{I(p,q) : (p,q) \subset \phi \text{ such that } (p,q) \text{ is countably } P\text{-concentrative}\};$$

note $\text{Fix}\,\phi = \text{Fix}\,(p,q)$.

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