Journal of Combinatorial Theory, Series B **79**, 157–171 (2000)
doi:10.1006/jctb.2000.1953, available online at http://www.idealibrary.com on IDEAL®

# Least-Distortion Euclidean Embeddings of Graphs: Products of Cycles and Expanders

Nathan Linial[1] and Avner Magen

*Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel*
E-mail: {nati,avnerma}@cs.huji.ac.il

Received March 10, 1999

Embeddings of finite metric spaces into Euclidean space have been studied in several contexts: The local theory of Banach spaces, the design of approximation [illegible] constant of the mapping. This question has also been asked for embeddings into other normed spaces. However, when the host space is $l_2$, more can be said: The problem of finding an optimal embedding into $l_2$ can be formulated as a semidefinite program (and can therefore be solved in polynomial time). So far, this elegant statement of the problem has not been applied to any interesting explicit instances. Here we employ this method and examine two families of graphs: (i) products of cycles, and (ii) constant-degree expander graphs. Our results in (i) extend a 30-year-old result of P. Enflo (1969, *Ark. Mat.* **8**, 103–105) on the cube. Our results in (ii) provide an alternative proof to the fact that there are $n$-point metric spaces whose Euclidean distortion is $\Omega(\log n)$. Furthermore, we show that metrics in the class (ii) are $\Omega(\log n)$ far from the class $l_2^2$, namely, the square of the metrics realizable in $l_2$. This is a well studied class which contains all $l_1$ metrics (and therefore also all $l_2$ metrics). Some of our methods may well apply to more general instances where semidefinite programming is used to estimate Euclidean distortions. Specifically, we develop a method for proving the optimality of an embedding. This idea is useful in those cases where it is possible to guess an optimal embedding. © 2000 Academic Press



## 1. INTRODUCTION

Let $(X, d)$ be a finite metric space, and let $n = |X|$ (henceforth, $n$ is always the size of the metric space). Let $f: X \to \mathbb{R}^n$ be an embedding of $X$ into $\mathbb{R}^n$ equipped with the Euclidean norm. Define the following quantities:

[1] Supported in part by grants from the Israeli Academy of Sciences and the Binational Science Foundation Israel–USA.



0095-8956/00 $35.00

- $\text{expansion}(f) = \sup_{x,\, y \in X} \frac{\|f(x) - f(y)\|}{d(x, y)}$,
- $\text{contraction}(f) = \sup_{x,\, y \in X} \frac{d(x, y)}{\|f(x) - f(y)\|}$,
- $\text{distortion}(f) = \text{expansion}(f) \cdot \text{contraction}(f)$.

We denote by $c_2(X, d)$ the least distortion with which $(X, d)$ may be embedded in $l_2$ (note that the dimension plays no role in this discussion, and it is obvious we need no more than $n-1$ dimensions for the best possible embedding). The parameter $c_2(X, d)$ has been studied for the metrics of several classes of graph metrics.

(1) For constant degree expander-graphs it has been shown [2] that $c_2 = \Omega(\log n)$. Thus, by a theorem of Bourgain [4] these are, the metrics with the asymptotically largest $c_2(X, d)$ among all $n$-point metric spaces $X$.

(2) If $X$ is the metric of a tree, $T$, it has been shown [7] that $c_2(X) = O(\sqrt{\log \log n})$.

(3) For the $r$-dimensional cube (i.e., the graph with vertex set $\{0, 1\}^r$ where two vertices are adjacent iff they differ in exactly one coordinate) an exact result is known [1]: $c_2(r\text{-dimensional cube}) = \sqrt{r} = \sqrt{\log n}$. The identity map on $r$-dimensional $l_2$ has this least possible distortion.

In this paper we provide a unified proof for the lower bounds in the cases of expanders and cubes. We also extend the latter result to graphs which are strong products of cycles.

Let $PSD_n$ be the set of positive semidefinite symmetric $n \times n$ matrices. Define $\mathcal{O}_n$ to be the collection of all matrices $Q \in PSD_n$ for which $Q\mathbf{1} = \mathbf{0}$.

*Remark* 1.1. We are mostly concerned with metrics, namely, non-negative symmetric functions $d$ that satisfy the triangle inequality. It is convenient, however, in several instances to consider also functions $d$ which does not necessarily satisfy the triangle inequality.

Define

$$\delta(Q, d) := \begin{cases} \sqrt{\dfrac{\sum_{i,\, j:\, q_{i,j} > 0} d^2(i, j)\, q_{i,j}}{\sum_{i,\, j:\, q_{i,j} < 0} d^2(i, j)\, |q_{i,j}|}} & \text{if the denominator is non-zero} \\ 1 & \text{otherwise.} \end{cases}$$

When it is clear from the context, we use $\delta(Q)$ instead of $\delta(Q, d)$.

Our starting point is the following proposition, which characterizes $c_2(X, d)$ in terms of a semidefinite program. This proposition follows readily from the duality principle in convex programming.

PROPOSITION 1.2 (Linial, London, Rabinovich [2, Corollary 3.5]). *Let* $(X, d)$ *be an n-point metric space. Then* $c_2(X, d) \leqslant C$ *iff for every matrix* $Q \in \mathcal{O}_n$ *the following inequality holds*

$$\sum_{i,\, j:\, q_{i,j} > 0} d^2(i, j)\, q_{i,j} + C^2 \sum_{i,\, j:\, q_{i,j} < 0} d^2(i, j)\, q_{i,j} \leqslant 0.$$

*Remark* 1.3. It will be convenient for us to use Proposition 1.2 in the following equivalent form:

$$c_2(X, d) = \max_{Q \in \mathcal{O}_n} \delta(Q, d).$$

(The expression in Proposition 1.2 cannot be simplified and written in a fraction form only when the metric is Euclidean. This case is taken care of by the second alternative in the formula for $\delta$.)

The task of proving a lower bound $\lambda$ on the Euclidean distortion of metric spaces thus reduces to the problem of finding a *certain* matrix $Q \in \mathcal{O}_n$ with $\delta(Q, d) \geqslant \lambda$. Of course, the greater $\delta(Q, d)$ is, the better the bound is. The next two claims provide some insight on how to search for "good" matrices $Q$. The claim is stated for a situation where a particular embedding is suspected of being optimal, and a matrix $Q \in \mathcal{O}_n$ is sought to prove this.

CLAIM 1.4. *Suppose that* $\phi$ *is an optimal embedding of an n-point metric space* $(X, d)$ *into* $l_2$. *Then a matrix Q that achieves the maximum in Proposition* 1.2 *has the following properties*

- $q_{i,j} \leqslant 0$ *for every pair* $i, j$ *that satisfies* $d(x_i, x_j)/\|\phi(x_i) - \phi(x_j)\| < \text{contraction}(\phi)$. *That is,* $q_{i,j} > 0$ *only for the most contracted pairs* $i, j$.
- $q_{i,j} \geqslant 0$ *for every pair* $i, j$ *that satisfies* $\|\phi(x_i) - \phi(x_j)\|/d(x_i, x_j) < \text{expansion}(\phi)$. *That is,* $q_{i,j} < 0$ *only for the most expanded pairs* $i, j$.

In particular, $q_{i,j} = 0$ for every pair $i, j$ that is neither most contracted nor most expanded by $\phi$.

*Proof.* Let $\phi$ and $Q$ be the embedding and matrix of the claim. We prove the first property, the second follows analogously. Consider a pair of indices $i, j$ with

$$\frac{d(x_i, x_j)}{\|\phi(x_i) - \phi(x_j)\|} < \text{contraction}(\phi).$$

Define $d^*$, which coincides with $d$, apart from the pair $(x_i, x_j)$, where $d^*(x_i, x_j) = \text{contraction}(\phi) \cdot \|\phi(x_i) - \phi(x_j)\|$.

It should be noted that the concept of $c_2$ can be defined for any $d$ that is a symmetric matrices with nonnegative entries, and not necessary a metric. Moreover, Proposition 1.2 applies in this more general context. This implies the validity of the following arguments, regardless of the fact that $d^*$ need not be a metric.

Let distortion$^*(\phi)$ be the distortion of $\phi$ with respect to $d^*$. (Likewise, we define expansion$^*(\phi)$ and contraction$^*(\phi)$).

Clearly, contraction$^*(\phi) = $ contraction$(\phi)$, and expansion$^*(\phi) \leqslant$ expansion$(\phi)$. Therefore, distortion$^*(\phi) \leqslant$ distortion$(\phi)$.

Proposition 1.2, now implies $\delta(Q, d^*) \leqslant$ distortion$^*(\phi)$, and $\delta(Q, d) =$ distortion$(\phi)$, whence also $\delta(Q, d^*) \leqslant \delta(Q, d)$. But $d^*(x_i, x_j) > d(x_i, x_j)$, and otherwise $d$ and $d^*$ are identical. It follows that $q_{ij} \leqslant 0$. ∎

In the embedding problems that we encounter here, it is not difficult to guess an optimal mapping $\phi$. However, even in such circumstances, the previous claim still does not provide us with a complete recipe for constructing the matrix $Q$ to show $\phi$'s optimality, namely, a matrix $Q$ for which $\delta(Q) = \text{distortion}(\phi)$. Note that in order to apply the claim, it is necessary to know which pairs are the most contracted nodes, and which are the most expanded ones (by $\phi$). Even though we may have an optimal $\phi$, this additional information may be hard to derive. We therefore take a somewhat indirect approach, that seems to be of some general interest. The following claim offer a method for proving optimality of an embedding by finding many "good" matrices. Such a matrix is constructed per each pair of points.

CLAIM 1.5. *Let $\phi$ be an embedding of a (unweighted) graph metric $X$ into $l_2$. Suppose that $\|\phi(x) - \phi(y)\| = 1$ for every two adjacent vertices $x, y$ (whence expansion$(\phi) = 1$, see Claim 2.2).*

*Further assume that for every two vertices with $d(x, y) > 1$, there exists a matrix $Q^{(x,y)} \in \mathcal{O}_n$ such that*

$$\delta(Q^{(x,y)}) = \left( \frac{\sum_{i,j:\, q_{i,j}^{(x,y)} > 0} d^2(i,j)\, q_{i,j}^{(x,y)}}{\sum_{i,j:\, q_{i,j}^{(x,y)} < 0} d^2(i,j)\, |q_{i,j}^{(x,y)}|} \right)^{\frac{1}{2}} = \frac{d(x,y)}{\|\phi(x) - \phi(y)\|}.$$

*Then $\phi$ is an optimal mapping (i.e., it has the minimal possible distortion).*

*Proof.*

$$c_2(X) \leqslant \text{distortion}(\phi) = \text{contraction}(\phi) = \max_{x,y \in X:\, d(x,y) > 1} \frac{d(x,y)}{\|f(x) - f(y)\|}$$

$$= \max_{x,y \in X:\, d(x,y) > 1} \delta(Q^{(x,y)}) \leqslant \max_{Q \in \mathcal{O}_n} \delta(Q, d) = c_2(X)$$

The last equality comes from Proposition 1.2. We conclude that $c_2(X) = \text{distortion}(\phi)$.

## 2. EMBEDDING PRODUCTS OF CYCLES INTO EUCLIDEAN SPACE

In this section we find optimal Euclidean embeddings for graphs which are products of cycles.

Let $G_1, G_2, ..., G_r$ be graphs. Their *strong product* is the graph $G = \prod_{i=1}^{r} G_i$ whose vertex set is $V(G) = V(G_1) \times V(G_2) \cdots \times V(G_r)$. Two vertices in $G$ are adjacent if they differ in exactly one coordinate, on which their projections are adjacent. Our goal is to find the Euclidean distortion of graphs which are the strong product of cycles. Namely, if $C_n$ denotes the cycle of length $n$, we consider graphs of the form $G = \prod_{i=1}^{r} C_{n_i}$.

Here is an algebraic view of this family of graphs: Let $A$ be a finite Abelian group: $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, (Recall that every finite Abelian group can be thus presented.) Let $g_i$ be a generator of $\mathbb{Z}_{n_i}$, and consider

$$S = \{(g_1, 0, 0, ..., 0), (0, g_2, 0, ..., 0), ..., (0, 0, ..., 0, g_r)\}.$$

the corresponding generating set for $A$. Now let $G = G(A) = G(A, S)$ be the Cayley graph of $A$ with respect to the generating set $S$, and let $X(A)$ be the graph metric associated with $G$. The special case $A = (\mathbb{Z}_2)^m$ is just the cube with its usual graph metric.

### 2.1. *Embedding a Cycle*

We return to the $n$-cycle $C_n$ and assume for convenience that $n \geqslant 4$ and is even (since Claim 2.5 generalizes this section, this loss of generality is justified). Call the embedding of $C_n$ onto the vertices of the regular $n$-gon the *standard embedding*. We claim that the *standard embedding* of $C_n$ into $\mathbb{R}^2$ is optimal. Namely, it has the least possible distortion.

Claim 2.1. *The embedding* $\phi: \{1, 2, ..., n\} \to \mathbb{R}^2$

$$\phi: j \to \frac{1}{2\sin(\pi/n)} \left( \cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n} \right)$$

*is an optimal embedding of* $C_n$ *into* $l_2$.

We start with the following observation.

Claim 2.2. *Let $f$ be an embedding of a graph $G$ into any metric space $M$. Then the expansion of $f$, namely, the maximum of $d_M(f(x), f(y))/d_G(x, y)$ is attained for $x$ and $y$ that are adjacent.*

*Proof.* Let $x = x_0, x_1, ..., x_k = y$ be a shortest path from $x$ to $y$ in $G$. Then

$$\frac{d_M(f(x), f(y))}{d_G(x, y)} \leqslant \frac{\sum_{i<k} d_M(f(x_i), f(x_{i+1}))}{\sum_{i<k} d_G(x_i, x_{i+1})}$$
$$\leqslant \max_{i<k} \frac{d_M(f(x_i), f(x_{i+1}))}{d_G(x_i, x_{i+1})}.$$

A calculation shows that a pair of vertices that are at distance $k$ in $C_n$, is mapped by the standard embedding to points in $\mathbb{R}^2$ that are at distance $\sin(\frac{\pi}{n}k)/\sin\frac{\pi}{n}$. By the above claim expansion$(\phi) = 1$. Since $\sin(\frac{\pi}{n}k)/(k \sin\frac{\pi}{n})$ is minimized for $k = n/2$, we get contraction$(\phi) = \frac{n}{2}\sin\frac{\pi}{n}$.

Therefore distortion$(\phi) =$ expansion$(\phi) \cdot$ contraction$(\phi) = \frac{n}{2}\sin\frac{\pi}{n}$.

We now turn to the lower bound. To prove that the embedding $\phi$ is optimal we seek a matrix $Q \in \mathcal{O}_n$, for which $\delta(Q) =$ distortion$(\phi)$. Claim 1.4 now applies. If indeed $\phi$ is optimal, then the desired matrix must be non positive at entries that correspond to incident vertices, where the maximal expansion occurs. On the other hand, entries corresponding to antipodes (vertices of distance $n/2$) must be non negative, since there, the contraction is maximal. The other off-diagonal entries must vanish. If we further impose symmetry constraints and the equality $\delta(Q) =$ distortion$(\phi)$ the matrix $Q$ is determined up to a multiplicative scalar. To recap: the duality theorem implies that if, as we suspect, $\phi$ is optimal, then the desired matrix exists. The previous discussion yielded sufficiently many constraints to uniquely determine $Q$.

Define

$$q_{i,j} = \begin{cases} 2\cos^2\left(\frac{\pi}{n}\right) & \text{if} \quad i = j \\ -1 & \text{if} \quad d(i,j) = 1 \\ 2\sin^2\left(\frac{\pi}{n}\right) & \text{if} \quad \mathrm{d}(i,j) = n/2 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $Q$ is symmetric, and $Q\mathbf{1} = \mathbf{0}$. To prove that it is positive semi-definite, we exhibit a complete set of eigenvectors for $Q$ all having non-negative eigenvalues. The set of characters of $\mathbb{Z}_n$ provides such a system of eigenvectors. Let $\omega$ be a primitive $n$th root of unity, and consider the orthogonal set of vectors $v_i = (\omega^i, \omega^{2i}, ..., \omega^{ni})$ for $i = 0, ..., n-1$.

CLAIM 2.3. *$v_i$ is an eigenvector of $Q$ with eigenvalue* $\lambda_i = 4\sin^2(\frac{\pi}{n}i) - 2\sin^2(\frac{\pi}{n})(1 - (-1)^i)$.

*Proof.*

$$(Qv_i)_j = \left(2 - 2\sin^2\left(\frac{\pi}{n}\right)\right)\omega^{ij} - (\omega^{i(j+1)} + \omega^{i(j-1)}) + 2\sin^2\left(\frac{\pi}{n}\right)\omega^{i(j+n/2)}$$

$$= \omega^{ij}\left(2 - 2\sin^2\left(\frac{\pi}{n}\right) - 2\cos\left(\frac{2\pi}{n}i\right)2\sin^2\left(\frac{\pi}{n}\right)(-1)^i\right)$$

$$= \omega^{ij}\left(4\sin^2\left(\frac{\pi}{n}i\right) - 2\sin^2\left(\frac{\pi}{n}\right)(1-(-1)^i)\right),$$

and so

$$Qv_i = \left(4\sin^2\left(\frac{\pi}{n}i\right) - 2\sin^2\left(\frac{\pi}{n}\right)(1-(-1)^i)\right)v_i.$$

As $v_i$ are linearly independent, they form a complete set of eigenvectors. Now, for $i$ even, $\lambda_i$ is clearly nonnegative. For $i$ odd, the smallest $\lambda_i$ occurs for $i=1$, and $\lambda_1 = 0$.

Therefore $Q \in \mathcal{O}_n$. Now, $\delta(Q) = \sqrt{n \cdot 2\sin^2(\pi/n)(n/2)^2/2n} = \frac{n}{2}\sin\frac{\pi}{n} = \text{distortion}$ $(\phi)$. By Proposition 1.2 we have shown that $\phi$ is an optimal embedding. ▮

## 2.2. *Embedding a Cube*

The following theorem is due to P. Enflo:

THEOREM 2.4 (Enflo [1]). *Let $X$ be the graph metric of the $r$-dimensional cube. Then $c_2(X) = \sqrt{r}$, the identity map being an optimal embedding.*

We have already noted that the cube is just a simple instance of a graph which is a product of cycles. We use the semidefinite characterization of $c_2$ to reprove the theorem, and as another stage in settling the general case of products of cycles.

*Proof.* The $r$-dimensional cube is the product of $r$ 2-cycles $C_2$. The identity map that sends the vertices of the cube onto $\{0,1\}^r \subseteq \mathbb{R}^r$, is the cartesian product of the one-dimensional standard embeddings.

Clearly, edge lengths are maintained, while the largest contraction, $\sqrt{r}$, occurs at antipodal pairs. More generally, vertices at distance $l$ are mapped to two points at distance $\sqrt{l}$. Now, Claim 1.4 yields the following properties for the matrix $Q$, which, by Proposition 1.2 imply that $\phi$ is optimal: The only nonzero entries reside (i) On the main diagonal (ii) At entries corresponding to neighboring vertices (non positive) and (iii) on antipodal entries (nonnegative). We consider the matrix

$$q_{i,j} = \begin{cases} r-1 & \text{if} \quad i=j \\ -1 & \text{if} \quad d(i,j)=1 \\ 1 & \text{if} \quad d(i,j)=r \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $Q$ is symmetric, and $Q\mathbf{1}=\mathbf{0}$. It is not hard to verify that $Q$ is indeed positive semi-definite. We omit the proof, which is a special case of Claim 2.6.

A simple calculation yields $\delta(Q)=\sqrt{2^r r/2^r}=\sqrt{r}$, which is distortion($\phi$), and we are done.

## 2.3. *The General Case*

Having established the optimality of the standard embedding for $C_n$, we now turn to the general case of $G=\prod C_{n_i}$.

THEOREM 2.5. *Let $G=\prod C_{n_i}$. The cartesian product of standard embeddings is an optimal embedding of $G$ into $l_2$. That is $\phi : G \to \mathbb{R}^{2n}$ defined by*

$$\phi : (k_1, k_2, ..., k_r) \to$$

$$\left( \frac{\cos \frac{2\pi}{n_1} k_1}{2 \sin \frac{\pi}{n_1}}, \frac{\sin \frac{2\pi}{n_1} k_1}{2 \sin \frac{\pi}{n_1}}, \frac{\cos \frac{2\pi}{n_2} k_2}{2 \sin \frac{\pi}{n_2}}, \frac{\sin \frac{2\pi}{n_2} k_2}{2 \sin \frac{\pi}{n_2}}, ..., \frac{\cos \frac{2\pi}{n_r} k_r}{2 \sin \frac{\pi}{n_r}}, \frac{\sin \frac{2\pi}{n_r} k_r}{2 \sin \frac{\pi}{n_r}} \right)$$

*is an optimal embedding.*

*Proof.* We use the notation $\mathbf{k}=(k_1, k_2, ..., k_r)$ to denote the vertices of $G$, and consider this as a vector in $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$. First note that if $\mathbf{k}$ and $\mathbf{l}$ are neighbors, then $\|\phi(\mathbf{k})-\phi(\mathbf{l})\|=1$ whence expansion($\phi$) $=1$. To determine the contraction of $\phi$, it suffices to consider $\max \frac{d(\mathbf{k},\mathbf{l})}{\|\phi(\mathbf{k})-\phi(\mathbf{l})\|}$ over nonadjacent vertices $\mathbf{k}$ and $\mathbf{l}$.

We now use the symmetry of our construction and show that it suffices to consider the case where $\mathbf{l}=\mathbf{0}$. Indeed $d(\mathbf{k},\mathbf{l})=d(\mathbf{k}-\mathbf{l},\mathbf{0})$ and $\|\phi(\mathbf{k}-\mathbf{l})-\phi(\mathbf{0})\| = \|\phi(\mathbf{k})-\phi(\mathbf{l})\|$. (This latter identity follows from the geometric definition of the embedding, but can also be verified directly.) In other words,

$$\text{distortion}(\phi)=\text{contraction}(\phi)=\max_{\mathbf{h}\,:\, d(\mathbf{h},\mathbf{0})>1} \frac{d(\mathbf{h},\mathbf{0})}{\|\phi(\mathbf{h})-\phi(\mathbf{0})\|}.$$

In view of our discussion in the case of the cube, it seems natural to expect that in general, the largest contraction occurs at antipodal pairs. (We say that $\mathbf{k}$ and $\mathbf{l}$ are antipodal if for every index $i$, the distance between $k_i$ and $l_i$ is as large as possible, namely, $\lfloor n_i/2 \rfloor$.) This is, however, incorrect as we soon explain. There are two sources to the distortion: One that comes from each component in the product $G=\prod C_{n_i}$, while the other arises from the combined effect of many coordinates. In the cube, where each coordinate is mapped isometrically, only the latter effect is observed, and the distortion results only from the disagreement between $l_1$ and $l_2$ metrics. In $d$ dimensions, this yields a distortion of $\sqrt{d}$. What should we expect in the general case? The $l_1$ vs $l_2$ distortion is largest for vectors all of whose coordinates are equal. The standard embedding of each individual cycle $C_k$ with $k \geqslant 4$, has some distortion of its own that is maximized at antipodal vertices. Therefore, when considering products of cycles, two conflicting principles are at play: On one hand, distortion is maximized at individual coordinates for antipodal pairs. On the other hand, to increase the overall distortion, we may want to make individual coordinate differences as equal as possible.

To illustrate the tradeoff between these two principles, let $G$ be the product of a big cycle and a small one, say $G=C_{40} \times C_4$. The antipodal pair $(0, 0)$ and $(20, 2)$ is not as contracted as the pair $(0, 0)$ and $(1, 2)$. In fact the latter is a most contracted pair.

The way to handle this problem is to bypass it using Claim 1.5 as follows: For every $\mathbf{h}$ with $d(\mathbf{h}, \mathbf{0})>1$, define a matrix $Q=Q^{(\mathbf{h})}$.

For $j=1, 2, ..., r$, define

$$\rho_j=\begin{cases} 1 & \text{if} \quad n_j=2 \\ 2 & \text{if} \quad n_j>2. \end{cases}$$

Let $\varepsilon_j=(0, 0, ..., 1, ..., 0)$, the $j$th unit vector, and $\mathscr{H}=\{h' : \forall f, h'_f=\pm h_f\}$.

Let $Q=Q^{(\mathbf{h})}$ be defined by

$$q_{\mathbf{k},\mathbf{l}}=\begin{cases} \sum_j \left(\sin \dfrac{\pi h_j}{n_j} \Big/ \sin \dfrac{\pi}{n_j}\right)^2-1 & \text{if} \quad \mathbf{k}=\mathbf{l} \\ -\dfrac{1}{\rho_j}\left(\sin \dfrac{\pi h_j}{n_j} \Big/ \sin \dfrac{\pi}{n_j}\right)^2 & \text{if} \quad d(\mathbf{k}, \mathbf{l})=1, \quad \text{and they differ on the } j\text{th coordinate} \\ |\mathscr{H}|^{-1} & \text{if} \quad \mathbf{k}-\mathbf{l} \in \mathscr{H} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* As in the special case, it is easily seen that $Q$ is symmetric, and $Q\mathbf{1}=\mathbf{0}$. Also, again we claim that the characters of the underlying group $A=\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\cdots\oplus\mathbb{Z}_{n_r}$ constitute a complete system of eigenvectors for $Q$, all with non negative eigenvalues. Let $v_{\mathbf{a}}$ be the vector corresponding to the character $\chi_{\mathbf{a}}:\mathbf{b}\rightarrow\exp(2\pi i\sum_j a_j b_j/n_j)$. Then

$$
\begin{aligned}
(Qv_{\mathbf{a}})_{\mathbf{b}} = & \left(\sum_j \left(\sin\frac{\pi h_j}{n_j}\Big/\sin\frac{\pi}{n_j}\right)^2 - 1\right) v_{\mathbf{a}}(\mathbf{b}) \\
& -\sum_j \frac{1}{\rho_j}\left(\sin\frac{\pi h_j}{n_j}\Big/\sin\frac{\pi}{n_j}\right)^2 (\rho_j/2)(v_{\mathbf{a}}(\mathbf{b}-\varepsilon_j)+v_{\mathbf{a}}(\mathbf{b}+\varepsilon_j)) \\
& +|\mathscr{H}|^{-1}\sum_{h'\in\mathscr{H}} v_{\mathbf{a}}(\mathbf{b}+h') \\
= & \, v_{\mathbf{a}}(\mathbf{b})(\sum_j \left(\sin\frac{\pi h_j}{n_j}\Big/\sin\frac{\pi}{n_j}\right)^2 - 1 \\
& -\sum_j \left(\sin\frac{\pi h_j}{n_j}\Big/\sin\frac{\pi}{n_j}\right)^2 \cos\frac{2\pi a_j}{n_j} + |\mathscr{H}|^{-1}\sum_{h'\in\mathscr{H}}\exp\left(2\pi i\sum_j\frac{a_j h'_j}{n_j}\right)\Bigg) \\
= & \, v_{\mathbf{a}}(\mathbf{b})(2\sum_j\left(\sin\frac{\pi a_j}{n_j}\right)^2\left(\sin\frac{\pi h_j}{n_j}\Big/\sin\frac{\pi}{n_j}\right)^2 + \prod_j\cos\frac{2\pi a_j h_j}{n_j} - 1\Bigg).
\end{aligned}
$$

In order to show that the eigenvalues are nonnegative we need the following technical statement:

PROPOSITION 2.7.

$$
\left(\sin\frac{\pi a}{n}\right)^2\left(\sin\frac{\pi h}{n}\Big/\sin\frac{\pi}{n}\right)^2 \geqslant \left(\sin\frac{\pi a h}{n}\right)^2
$$

*for* $a, h = 0, 1, ..., n-1$.

*Proof.* We first observe a simple trigonometric fact:

LEMMA 2.8. *The function* $\sin\alpha x/\sin x$ *is decreasing in* $x$ *for* $\alpha>1$, *and* $x$ *in the interval* $(0, \pi/2\alpha)$.

*Proof.* By taking a derivative, the claim reduces to the inequality $\tan\alpha x/\tan x\geqslant\alpha$ with the same ranges for $\alpha$ and $x$. This last inequality follows easily from the convexity of $\tan\theta$ for $0<\theta<\pi/2$.

Note that when $a$ or $h$ equal 0 or 1 the inequality is trivially true. Furthermore, through replacing $a$ by $n-a$ if necessary, and $h$ by $n-h$, it suffices to check only for $1<a,\, h\leqslant\lfloor\frac{n-1}{2}\rfloor$.

Now Lemma 2.8 implies the proposition whenever $ah/n\leqslant\frac{1}{2}$ (taking $\alpha=a$).

The only case not covered is when $a$ and $h$ are as above, whereas $ah/n\geqslant\frac{1}{2}$.

In this range we argue as

$$\left(\sin\frac{\pi a}{n}\right)^2\left(\sin\frac{\pi h}{n}\Big/\sin\frac{\pi}{n}\right)^2\geqslant\left(\sin\frac{\pi}{2h}\right)^2\left(\sin\frac{\pi h}{n}\Big/\sin\frac{\pi}{n}\right)^2$$
$$\geqslant\left(\sin\frac{\pi}{2}\right)^2=1\geqslant\left(\sin\frac{\pi ah}{n}\right)^2.$$

The first inequality expresses the monotonicity of the sin function in $(0,\pi/2)$. The second one follows from Lemma 2.8 (with $\alpha=h$).

We now complete the proof that the eigenvalues are non-negative:

$$\lambda_{\mathbf{a}}=2\sum_j\left(\sin\frac{\pi a_j}{n_j}\right)^2\left(\sin\frac{\pi h_j}{n_j}\Big/\sin\frac{\pi}{n_j}\right)^2+\prod_j\cos\frac{2\pi a_jh_j}{n_j}-1$$
$$\geqslant 2\sum_j\left(\sin\frac{\pi a_jh_j}{n_j}\right)^2+\prod_j\cos\frac{2\pi a_jh_j}{n_j}-1$$

We now define $\beta_j$ to be $2(\sin(\pi a_jh_j/n_j)^2$. It is enough to show that for $\beta_j\in[0,2]$,

$$\prod_j(1-\beta_j)\geqslant 1-\sum_j\beta_j.$$

This is clearly true for $r=1$, and the following inductive step gives the desired result,

$$\sum_j^r\beta_j+\prod_j^r(1-\beta_j)-1=\sum_j^{r-1}\beta_j+\beta_r+(1-\beta_r)\prod_j^{r-1}(1-\beta_j)-1$$
$$\geqslant\beta_r\left(1-\prod_j^{r-1}(1-\beta_j)\right)\geqslant 0.$$

Claim 2.9. *For all* $\mathbf{h}$ *such that* $d(\mathbf{h},\mathbf{0})>1$, $\delta(Q^{(\mathbf{h})})=\frac{d(\mathbf{h},\mathbf{0})}{\|\phi(\mathbf{h})-\phi(0)\|}$

*Proof.*

$$\begin{aligned}\delta(Q^{(\mathbf{h})}) &= \left(\frac{\sum_{\mathbf{k},\mathbf{l}\,:\,q_{\mathbf{k},\mathbf{l}}>0} d^2(\mathbf{k},\mathbf{l})\, q_{\mathbf{k},\mathbf{l}}}{\sum_{\mathbf{k},\mathbf{l}\,:\,q_{\mathbf{k},\mathbf{l}}<0} d^2(\mathbf{k},\mathbf{l})\, |q_{\mathbf{k},\mathbf{l}}|}\right)^{\frac{1}{2}} \\ &= \left(\frac{|G|\,|\mathscr{H}|^{-1}\,|\mathscr{H}|\, d^2(\mathbf{h},\mathbf{0})}{|G| \sum_j (\sin(\pi h_j/n_j)/\sin(\pi/n_j))^2}\right)^{\frac{1}{2}} \\ &= \left(\frac{d^2(\mathbf{h},\mathbf{0})}{\sum_j (\sin(\pi h_j/n_j)/\sin(\pi/n_j))^2}\right)^{\frac{1}{2}} \\ &= \frac{d(\mathbf{h},\mathbf{0})}{\|\phi(\mathbf{h})-\phi(0)\|}.\end{aligned}$$

Claim 1.5 can now be invoked to complete the proof of Theorem 2.5.

## 3. EMBEDDING EXPANDERS IN $L_2$ AND $L_2^2$

Bourgain [4] has proved a major result on the embedding of finite metrics into normed spaces. Namely, that every metric space with $n$ points, can be embedded into Euclidean space with distortion only $O(\log n)$.

It was shown [2] (see also [6]) that this bound is tight and is attained for the graph metric of constant degree expanders. That is, if $(X, d)$ is the metric of a $k$-regular graph of order $n$ whose second largest eigenvalue is $\leqslant k-\varepsilon$ where $\varepsilon$ is a constant not dependent on $n$, then $c_2(X) > c \log n$ where $c$ depends only on $k$ and $\varepsilon$.

Here we provide an alternative proof that is completely different from the previous ones, using Proposition 1.2. It turns out that this technique yields a much stronger result than that. Consider the family of functions $d(\cdot, \cdot)$ that are realizable in $l_2^2$, that is there are $n$ points $x_1, ..., x_n \in \mathbb{R}^n$, and $d(x, y) = \|\phi(x) - \phi(y)\|^2$. The family of such $d$ constitutes a cone that contain the cut-cone, the cone of $l_1$ realizable metrics. It is a standard fact that every $l_2$ metric belongs to the cut-cone. (For this fact as well as a thorough introduction to this area, see [5]). We prove that in fact metrics of expander graphs are $\Omega(\log n)$ far from $l_2^2$. It follows that Bourgain's bound is tight in an even stronger sense. We start by reproving the known result:

THEOREM 3.1. *Let $d$ be the metric of a $k$-regular graph $G$ of order $n$ whose second largest eigenvalue is $\leqslant k-\varepsilon$. Then $c_2(X) > c \log n$ where $c$ depends only on $k$ and $\varepsilon$.*

*Proof.* We start by defining a graph $H$ that has the same vertex set as $G$ and where two vertices are adjacent if their *distance* in $G$ is at least $\log_k n - 2$.

CLAIM 3.2. *All vertices of $H$ have degree $> \frac{1}{2} n$.*

*Proof.* Since $G$ has constant degree $k$, every vertex has at most $k^r(1 + \frac{1}{k-1})$ vertices at distance $\leqslant r$ from it. For $r = \log_k n - 2$, this implies that the $r$-neighborhood of every vertex contains fewer than $\frac{n}{2}$ vertices as claimed.

We now use the following graph theoretic fact: A graph in which all vertices have degree $\geqslant \frac{1}{2} n$, has a matching of $\lfloor \frac{n}{2} \rfloor$ edges. This is a simple consequence of Dirac's sufficient condition for a Hamiltonian circuit (e.g., [3, pp. 106–107]).

For simplicity we assume that $n$ is even and that $M$ is a perfect matching in $H$.

Let $A$ be the adjacency matrix of $G$ and $B$ the adjacency matrix of the matching $M$, and let $Q = kI - A + \frac{\varepsilon}{2}(B - I)$. We now show that $Q$ is in $\mathcal{O}_n$. Since $Q1 = 0$, it is enough to show for every vector $x$ orthogonal to 1 that $xQx^t \geqslant 0$. Consider a vector $x$ with $x \perp 1$. The assumption on $G$'s eigenvalues now implies $x(kI - A)\, x^t \geqslant \varepsilon \, \|x\|^2$.

Now,

$$x(B - I)\, x^t = \sum_{ij \in M} (2x_i x_j - x_i^2 - x_j^2) \geqslant -2 \sum_{ij \in M} (x_i^2 + x_j^2) = -2\, \|x\|^2,$$

and so

$$xQx^t = x(kI - A)\, x^t + \frac{\varepsilon}{2}\, x(B - I)\, x^t \geqslant \varepsilon\, \|x\|^2 - \varepsilon\, \|x\|^2 = 0.$$

We conclude by evaluating $\delta(Q)$, the lower bound on $c_2(X)$ that $Q$ yields

$$\sum_{i,\, j\, :\, q_{i,j} > 0} d^2(i, j)\, q_{i,j} \geqslant \frac{\varepsilon}{2} \cdot n(\log_k n - 2)^2,$$

since the distances on the entries supported by $B$ are at least $\log_k n - 2$,

$$\sum_{i,\, j:\, q_{i,j} < 0} d^2(i, j)\, |q_{i,j}| = kn,$$

as we have $kn$ ordered pairs of neighbors in the graph. Consequently

$$\delta(Q) \geqslant \sqrt{\frac{\varepsilon}{2k}}\,(\log_k n - 2) \geqslant \Omega(\log n),$$

$$\text{whence } c_2(X) \geqslant \Omega(\log n).$$

DEFINITION 3.3. $l_2^2$ is the class of functions which are the square of Euclidean metrics. (In [5] this class is called $\mathrm{NOR}_2$.) That is, $d \in l_2^2$ iff there exist $n$ points $x_1, ..., x_n$ in $\mathbb{R}^n$ such that $d_{i,j} = \|x_i - x_j\|^2$.

CLAIM 3.4 [5]. *Any metric realizable in $l_1$ is in $l_2^2$.*

We turn to the stronger result and consider the minimal distortion of an embedding of $d$ into $l_2^2$.

THEOREM 3.5. *Let $d$ be the metric of a $k$-regular graph of order $n$ whose second eigenvalue is $\leqslant k - \varepsilon$. Every embedding of $d$ into $l_2^2$ has distortion $\geqslant \Omega(\log n)$. The implicit constants in the $\Omega$ term depend on $k$ and $\varepsilon$ but not on $n$.*

*Proof.* The first thing to observe is that the following two quantities are identical:

- The least distortion in any embedding of $d$ into $l_2^2$.
- $(c_2(\sqrt{d}))^2$, namely, the square of the least Euclidean distortion of the function $\sqrt{d}$.

It follows that we can again use Proposition 1.2 to conclude that the minimal distortion for the class $l_2^2$ is given by

$$\max_{P \in \mathcal{O}_n} \delta^2(P, \sqrt{d}).$$

Again we utilize the matrix $Q \in \mathcal{O}_n$ from the previous proof, and conclude

$$\frac{\sum_{i,j : q_{i,j} > 0} d(i,j)\, q_{i,j}}{\sum_{i,j : q_{i,j} < 0} d(i,j)\, |q_{i,j}|} = \frac{(\varepsilon/2) \cdot n(\log_k n - 2)}{kn} \geqslant \Omega(\log n).$$

*Remark* 3.6. Observe that Claim 3.4 implies that every embedding of $d$ into $l_1$ has distortion $\geqslant \Omega(\log n)$.

# REFERENCES

1. P. Enflo, On the nonexistence of uniform homeomorphisms between $L_p$-spaces, *Ark. Mat.* **8** (1969), 103–105.
2. N. Linial, E. London, and Yu. Rabinovich, The geometry of graphs and some of its algorithmic applications, *Combinatorica* **15** (1995), 215–245.
3. B. Bollobas, "Modern Graph Theory," Springer-Verlag, Berlin/New York, 1998.
4. J. Bourgain, On Lipschitz embedding of finite metric spaces in Hilbert space, *Israel J. Math.* **52** (1985), 46–52.
5. M. Deza and M. Laurent, "Geometry of Cuts and Metrics," Algorithms and Combinatorics, Vol. 15, Springer-Verlag, New York/Berlin, 1997.
6. J. Matousek, On embedding expanders into $l_p$ spaces, *Israel J. Math.* **102** (1997), 189–197.
7. J. Matousek, On embedding trees into uniformly convex Banach spaces, *Israel J. Math.*, in press.
8. L. J. Schulman, Clustering for edge-cost minimization, manuscript.