Notes on Jung–Kim–Srivastava integral operator

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Abstract

The object of the present paper is to investigate some properties of certain integral operator $Q^\alpha_{\beta}$ introduced and studied recently by Jung, Kim, and Srivastava [J. Math. Anal. Appl. 176 (1993) 138–147].

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1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

Let $f(z)$ and $g(z)$ be analytic in $U$. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in $U$ such that $|w(z)| < 1$ (for $z \in U$) and $g(z) = f(w(z))$. This relation is denoted $g(z) \prec f(z)$. In case $f(z)$ is univalent in $U$ we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

For each $A$ and $B$ such that $-1 \leq B < A \leq 1$, we define the function

$$h(A, B; z) = \frac{1 + Az}{1 + Bz} \quad (z \in U).$$

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It is well known that $h(A, B; z)$ for $-1 \leq B \leq 1$ is the conformal map of the unit disk onto the disk symmetrical with respect to the real axis having the center $(1 - AB)/(1 - B^2)$ and the radius $(A - B)/(1 - B^2)$. The boundary circle cuts the real axis at the points $(1 - A)/(1 - B)$ and $(1 + A)/(1 + B)$.

Recently, Jung, Kim, and Srivastava [2] introduced the following integral operator:

$$Q^\alpha_\beta f(z) = \left(\frac{\alpha + \beta}{\beta}\right) \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{a-1} t^{\beta-1} f(t) \, dt \quad (\alpha > 0, \beta > -1; \ f \in A(1)).$$

(1.3)

They showed that

$$Q^\alpha_\beta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta + n)\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + \alpha + n)\Gamma(\beta + 1)} a_n z^n.$$  

(1.4)

Some interesting subclasses of analytic function, associated with the operator $Q^\alpha_\beta$, have been investigated recently by Jung, Kim, and Srivastava [2], Aouf et al. [1], Li [3], and Liu [5]. The operator $Q^\alpha_\beta$ is also called Jung–Kim–Srivastava integral operator.

Motivated essentially by Jung, Kim, and Srivastava’s work [2], we now generalize the operator $Q^\alpha_\beta : A(p) \to A(p)$ as follows:

$$Q^\alpha_\beta f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(\beta + n + 1)\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + \alpha + n + 1)\Gamma(\beta + 1)} a_{p+n} z^{p+n}.$$  

(1.5)

It is easily verified from definition (1.5) that

$$z(Q^\alpha_\beta f(z))' = (\alpha + \beta) Q^{\alpha-1}_\beta f(z) - (\alpha + \beta + p) Q^\alpha_\beta f(z).$$

(1.6)

When $p = 1$, the identity (1.6) is given by Jung, Kim, and Srivastava [2]. Also, the identity (1.6) will play an important role in obtaining our results.

In this paper, we shall use the method of differential subordination to derive certain properties of Jung–Kim–Srivastava integral operator $Q^\alpha_\beta$. In [10], Yang and Liu obtained some properties of Ruscheweyh derivatives by using the same method.

2. Main results

In order to derive our main results, we shall need the following lemmas.

**Lemma 1** [6]. Let $h(z)$ be analytic and convex univalent in $U$, $h(0) = 1$, and let $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in $U$. If

$$g(z) + zg'(z)/c < h(z),$$

(2.1)

then for $c \neq 0$ and $\text{Re}\ c \geq 0$,

$$g(z) < cz^{-c} \int_0^z t^{c-1} f(t) \, dt.$$

(2.2)
Let \( P(\gamma) \) denote the class of functions \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \) which are analytic in \( U \) and satisfy the condition \( \text{Re } p(z) > \gamma \quad (z \in U) \). The following lemma is a well-known result.

**Lemma 2** [9]. Let \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P(\gamma) \) \((0 \leq \gamma < 1)\). Then
\[
\text{Re } p(z) > 2\gamma - 1 + \frac{2(1 - \gamma)}{1 + |z|}.
\] (2.3)

**Lemma 3** [8]. The function \((1 - z)^\gamma \equiv e^{\gamma \log(1 - z)}\), \(\gamma \neq 0\), is univalent in \( U \) if and only if \(\gamma\) is either in the closed disk \(|\gamma - 1| \leq 1\) or in the closed disk \(|\gamma + 1| \leq 1\).

**Lemma 4** [7]. Let \( q(z)\) be univalent in \( U \) and let \( \theta(w) \) and \( \phi(w) \) be analytic in a domain \( D \) containing \( q(U) \) with \( \phi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = z q'(z) \phi(q(z)) \), \( h(z) = \theta(q(z)) + Q(z) \) and suppose that
\[
(1) \quad Q(z) \text{ is starlike (univalent) in } U;
\]
\[
(2) \quad \text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re} \left\{ \frac{\theta'(q(z)) + zQ'(z)}{\phi(q(z)) Q(z)} \right\} > 0 \quad (z \in U).
\]

If \( p(z) \) is analytic in \( U \), with \( p(0) = q(0) \), \( p(U) \subset D \), and
\[
\theta(p(z)) + z p'(z)\phi(p(z)) \prec \theta(q(z)) + z q'(z)\phi(q(z)) = h(z),
\]
then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.

**Theorem 1.** Let \( \alpha > 1 \), \( \beta > -1 \), \( \lambda < 1 \), and let \(-1 < B_i < A_i \leq 1\) for \( i = 1, 2 \). If functions \( f_i(z) \in A(p) \) \((i = 1, 2)\) satisfy
\[
(1 - \lambda) \frac{Q^\alpha_{\beta} f_i(z)}{z^p} + \lambda \frac{Q^\alpha_{\beta} f_i(z)}{z^p} < h(A_i, B_i; z),
\] (2.4)
then
\[
(1 - \lambda) \frac{Q^\alpha_{\beta} F(z)}{z^p} + \lambda \frac{Q^\alpha_{\beta} F(z)}{z^p} < h(1 - 2\beta, -1; z),
\] (2.5)
where
\[
F(z) = Q^\alpha_{\beta}(f_1 * f_2)(z)
\] (2.6)

and
\[
\beta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left( 1 - \frac{\alpha + \beta}{1 + \lambda} \int_0^1 \frac{u^{\alpha - 1}}{1 + u} \, du \right).
\] (2.7)

The result is sharp when \( B_1 = B_2 = -1 \).

**Proof.** Suppose that \( f_i(z) \in A(p) \) \((i = 1, 2)\) satisfy the condition (2.4). Set
\[
p_i(z) = (1 - \lambda) \frac{Q^\alpha_{\beta} f_i(z)}{z^p} + \lambda \frac{Q^\alpha_{\beta} f_i(z)}{z^p} \quad (i = 1, 2),
\] (2.8)
then we have \( p_i(z) \in P(\alpha_i) \), where \( \alpha_i = (1 - A_i)/(1 - B_i) \) \((i = 1, 2)\).

By making use of (1.6) and (2.8), we get

\[
Q_\beta^\alpha f_i(z) = \frac{\alpha + \beta}{1 - \lambda} z^{p - \frac{\alpha + \beta}{1 - \lambda}} \int_0^z t^{\frac{\alpha + \beta}{1 - \lambda} - 1} p_i(t) \, dt \quad (i = 1, 2). \tag{2.9}
\]

Now let

\[
F(z) = Q_\beta^\alpha (f_1 \ast f_2)(z). \tag{2.10}
\]

Then, by using (2.9), a simple computation shows that

\[
Q_\beta^\alpha F(z) = \frac{\alpha + \beta}{1 - \lambda} z^{p - \frac{\alpha + \beta}{1 - \lambda}} \int_0^z t^{\frac{\alpha + \beta}{1 - \lambda} - 1} p(t) \, dt, \tag{2.11}
\]

where

\[
p(z) = (1 - \lambda) Q_\beta^{\alpha - 1} F(z) + \frac{\lambda}{z^\beta} Q_\beta^\alpha F(z)
= \frac{\alpha + \beta}{1 - \lambda} z^{p - \frac{\alpha + \beta}{1 - \lambda}} \int_0^z t^{\frac{\alpha + \beta}{1 - \lambda} - 1} (p_1 \ast p_2)(t) \, dt. \tag{2.12}
\]

Note that

\[
p_1(z) \in P(\alpha_1) \quad \text{and} \quad p_3(z) = \frac{p_2(z) - \alpha_2}{2(1 - \alpha_2)} + \frac{1}{2} \in P\left(\frac{1}{2}\right). \tag{2.13}
\]

We obtain \((p_1 \ast p_3)(z) \in P(\alpha_1)\) by using the well-known Herglotz formula. Thus

\[
(p_1 \ast p_2)(z) \in P(\alpha_3), \quad \alpha_3 = 1 - 2(1 - \alpha_1)(1 - \alpha_2). \tag{2.14}
\]

By making use of (2.11), (2.12), (2.14), and Lemma 2, we have

\[
\text{Re } p(z) = \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\alpha + \beta}{1 - \lambda} - 1} \text{Re}((p_1 \ast p_2)(uz)) \, du
\geq \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\alpha + \beta}{1 - \lambda} - 1} \left(2\alpha_3 - 1 + \frac{2(1 - \alpha_3)}{1 + u|z|}\right) \, du
\geq \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\alpha + \beta}{1 - \lambda} - 1} \left(2\alpha_3 - 1 + \frac{2(1 - \alpha_3)}{1 + u}\right) \, du
= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 - \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\alpha + \beta}{1 - \lambda} - 1} \, du\right).\]
When $B_1 = B_2 = -1$, we consider the functions $f_i(z) \in A(p)$ $(i = 1, 2)$ which satisfy the condition (2.4) and is given by

$$Q_\beta^\alpha f_i(z) = \frac{\alpha + \beta}{1 - \lambda} z^{-p} \int_0^z t^{\frac{\alpha + \beta - 1}{1 - \lambda}} \left( \frac{1 + A_i t}{1 - t} \right) dt \quad (i = 1, 2).$$

Then it follows from (2.12) that

$$p(z) = \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\alpha + \beta - 1}{1 - \lambda}} \left( 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - u} \right) du.$$

Therefore,

$$p(z) \to 1 - (1 + A_1)(1 + A_2) \left( 1 - \frac{\alpha + \beta}{1 - \lambda} \int_0^1 \frac{u^{\alpha + \beta - 1}}{1 + u} du \right) \quad \text{as} \quad z \to -1.$$

Now the proof is complete. □

Next we prove our second theorem.

**Theorem 2.** Let $\alpha > 1$, $\beta > -1$, $\lambda < 1$, and $-1 \leq B < A \leq 1$. If $f(z) \in A(p)$ satisfies

$$\left( 1 - \lambda \right) Q_\beta^{\alpha - 1} f(z) + \lambda \frac{Q_\beta^\alpha f(z)}{z^p} < h(A, B; z),$$

then

$$\operatorname{Re} \left( \left( \frac{Q_\beta^\alpha f(z)}{z^p} \right)^{1/m} \right) > \left( \frac{\alpha + \beta}{1 - \lambda} \int_0^1 \frac{u^{\frac{\alpha + \beta - 1}{1 - \lambda}}}{1 - Bu} du \right)^{1/m}. \quad (2.16)$$

The result is sharp.

**Proof.** Let

$$g(z) = \frac{Q_\beta^\alpha f(z)}{z^p} \quad (2.17)$$

for $f(z) \in A(p)$. Then the function $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is analytic in $U$. By making use of (1.6) and (2.17), we obtain

$$Q_\beta^{\alpha - 1} f(z) = g(z) + \frac{z g'(z)}{\alpha + \beta}. \quad (2.18)$$

From (2.15), (2.17), and (2.18) we get

$$g(z) + \frac{1 - \lambda}{\alpha + \beta} zg'(z) < h(A, B; z). \quad (2.19)$$
Now an application of Lemma 1 leads to
\[
g(z) < \frac{\alpha + \beta}{1 - \lambda} z^{-\frac{\mu + \beta}{1 - \lambda}} \int_0^z t^{\frac{\mu + \beta}{1 - \lambda} - 1} \left( \frac{1 + At}{1 + Bt} \right) dt \tag{2.20}
\]
or
\[
\frac{Q^\mu_{\beta} f(z)}{z^p} = \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\mu + \beta}{1 - \lambda} - 1} \left( \frac{1 + Auw(z)}{1 + Buw(z)} \right) du,
\tag{2.21}
\]
where \(w(z)\) is analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1 (z \in U)\).

In view of \(-1 \leq B < A \leq 1\) and \(\alpha + \beta > 0\), it follows from (2.21) that
\[
\text{Re} \frac{Q^\mu_{\beta} f(z)}{z^p} > \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\mu + \beta}{1 - \lambda} - 1} \left( 1 - Au \right) du (z \in U). \tag{2.22}
\]
Therefore, with the aid of the elementary inequality \(\text{Re}(w^{1/m}) \geq (\text{Re} w)^{1/m}\) for \(\text{Re} w > 0\) and \(m \geq 1\), the inequality (2.16) follows directly from (2.22).

To show the sharpness of (2.16), we take \(f(z) \in A(p)\) defined by
\[
\frac{Q^\mu_{\beta} f(z)}{z^p} = \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\mu + \beta}{1 - \lambda} - 1} \left( 1 - Au \right) du (z \in U).
\]
For this function, we find that
\[
(1 - \lambda) \frac{Q^\mu_{\beta}^{-1} f(z)}{z^p} + \lambda Q^\mu_{\beta} f(z) = 1 + Az \quad (1 + Bz)
\]
and
\[
\frac{Q^\mu_{\beta} f(z)}{z^p} \rightarrow \frac{\alpha + \beta}{1 - \lambda} \int_0^1 u^{\frac{\mu + \beta}{1 - \lambda} - 1} \left( 1 - Au \right) du \quad \text{as } z \rightarrow -1.
\]

Hence the proof of the theorem is complete. \(\square\)

For a function \(f(z) \in A(p)\) given by (1.1), we recall here the generalized Bernardi–Libera–Livingston [4] integral operator \(J_c : A(p) \rightarrow A(p)\), defined by
\[
J_c f(z) = \frac{c + p}{z^c} \int_0^z t^{-1} f(t) dt \quad (c > -p). \tag{2.23}
\]

**Theorem 3.** Let \(\alpha > 1, \beta > -1, \lambda < 1, c > -p, \) and \(-1 \leq B < A \leq 1\). Suppose that \(f(z) \in A(p)\) and \(J_c f(z)\) is given by (2.23). If
\[
(1 - \lambda) \frac{Q^\mu_{\beta} f(z)}{z^p} + \lambda \frac{Q^\mu_{\beta} J_c f(z)}{z^p} < h(A, B; z), \tag{2.24}
\]
then...
then
\[
\Re\left(\left(\frac{Q_\beta^a J_c f(z)}{zp^a}\right)^{1/m}\right) > \left(\frac{c + p}{1 - \lambda} \int_0^1 \frac{1 - Au}{1 - Bu} \, du\right)^{1/m} \quad (m \geq 1). \tag{2.25}
\]

The result is sharp.

**Proof.** It follows from (2.23) that

\[
(c + p) Q_\beta^a f(z) = c Q_\beta^a J_c f(z) + z \left( Q_\beta^a J_c f(z) \right)' . \tag{2.26}
\]

Let
\[
g(z) = \frac{Q_\beta^a f(z)}{zp^a} . \tag{2.27}
\]

Then from (2.26), (2.27), and (2.24) we have

\[
(1 - \lambda) \frac{Q_\beta^a f(z)}{zp^a} + \lambda \frac{Q_\beta^a J_c f(z)}{zp^a} = g(z) + \frac{1 - \lambda}{c + p} z g'(z) < h(A, B; z). \quad \square
\]

The remaining part of the proof is similar to that of Theorem 2 and hence we omit it.

**Theorem 4.** Let \( \alpha > 1, \beta > -1, \) and \( 0 \leq \rho < 1. \) Let \( \gamma \) be a complex number with \( \gamma \neq 0 \) and satisfy either \( |2\gamma (1 - \rho)(\alpha + \beta) - 1| \leq 1 \) or \( |2\gamma (1 - \rho)(\alpha + \beta) + 1| \leq 1. \) If \( f(z) \in A(p) \) satisfies the condition

\[
\Re\left\{ \frac{Q_\beta^{-1} f(z)}{Q_\beta^a f(z)} \right\} > \rho \quad (z \in U) , \tag{2.28}
\]

then

\[
\left( \frac{Q_\beta^a f(z)}{zp^a} \right)^\gamma < \frac{1}{(1 - z)^{2\gamma (1 - \rho)(\alpha + \beta)}} = q(z) \quad (z \in U) , \tag{2.29}
\]

where \( q(z) \) is the best dominant.

**Proof.** Let

\[
p(z) = \left( \frac{Q_\beta^a f(z)}{zp^a} \right)^\gamma \quad (z \in U) . \tag{2.30}
\]

Then, by making use of (1.6), (2.28), and (2.30), we have

\[
1 + \frac{zp'(z)}{\gamma(\alpha + \beta)p(z)} \leq \frac{1 + (1 - 2\gamma)z}{1 - z} \quad (z \in U) . \tag{2.31}
\]

If we take

\[
q(z) = \frac{1}{(1 - z)^{2\gamma (1 - \rho)(\alpha + \beta)}}, \quad \theta(w) = 1 \quad \text{and} \quad \phi(w) = \frac{1}{\gamma(\alpha + \beta)w} .
\]
then \( q(z) \) is univalent by the condition of the theorem and Lemma 3. Further, it is easy to show that \( q(z), \theta(w), \) and \( \phi(w) \) satisfy the conditions of Lemma 4. Since

\[
Q(z) = zq'(z)\phi(q(z)) = \frac{2(1 - \rho)z}{1 - z}
\]

is univalent starlike in \( U \) and

\[
h(z) = \theta(q(z)) + Q(z) = \frac{1 + (1 - 2\rho)z}{1 - z},
\]

it may be readily checked that the conditions (1) and (2) of Lemma 4 are satisfied. Thus the result follows from (2.31) immediately. The proof is complete.

**Corollary.** Let \( \alpha > 1, \beta > -1, \) and \( 0 \leq \rho < 1. \) Let \( \gamma \) be a real number and \( \gamma \geq 1. \) If \( f(z) \in A(p) \) satisfies the condition (2.28), then

\[
\text{Re} \left( \frac{Q^\alpha f(z)}{z^\beta} \right)^{1/(2\gamma(1-\rho)(\alpha+\beta)}) > 2^{-1/\gamma} \quad (z \in U).
\]

The bound \( 2^{-1/\gamma} \) is the best possible.

**References**