ACADEMIC PRESS
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Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
J. Math. Anal. Appl. 276 (2002) 691–708
www.elsevier.com/locate/jmaa

Singular integral operators on function spaces

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Received 19 November 2001

Submitted by L. Chen

Abstract

We study the singular integral operator

$f_{x,t}(y') = f(x - ty'),$

defined on all test functions f, where b is a bounded function, $\alpha \geqslant 0$, Ω is suitable distribution on the unit sphere S^{n-1} satisfying some cancellation conditions. We prove certain boundedness properties of $T_{\Omega,\alpha}$ on the Triebel–Lizorkin spaces and on the Besov spaces. We also use our results to study the Littlewood–Paley functions. These results improve and extend some well-known results.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $n \geqslant 2$, with normalized Lebesgue measure $d\sigma = d\sigma(x')$, and let b be an L^∞ function. In this paper we will study the singular integral operator $T_{\Omega,\alpha}$ defined, on the test function space $\mathcal{S}(\mathbb{R}^n)$, by

$$T_{\Omega,\alpha}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} b\big(|y|\big)\Omega(y')|y|^{-n-\alpha} f(x-y)\,dy, \tag{1.1}$$

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1 Supported by 973 project, Major project of NNSFC, NSFZJ and NECC.

0022-247X/02/$ – see front matter
PII: S0022-247X(02)00419-5

where $\alpha \geqslant 0$, $y' = y/|y|$ for $y \neq 0$, Ω is a distribution in the Hardy space $H^r(S^{n-1})$ with $r = (n-1)/(n-1+\alpha)$ and satisfies

$$\langle \Omega, Y_m \rangle = 0 \tag{1.2}$$

for all spherical harmonic polynomials Y_m with degrees $m \leqslant [\alpha]$. It is easy to check that, for each $x \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$, $|T_{\Omega,\alpha}(f)(x)| < \infty$.

Denote $T_{\Omega,\alpha}$ by T_Ω if $\alpha = 0$. Then the operator T_Ω is the well-known rough singular integral operator initially studied by Calderón and Zygmund in their pioneering papers [5,6]. In [6], using the method of rotation, Calderón and Zygmund proved that if $\Omega \in L\,\mathrm{Log}^+ L(S^{n-1})$ satisfies the mean zero condition (1.2), namely $\alpha = 0$, then the operator T_Ω with kernel $\Omega(x')|x|^{-n}$ is a bounded operator on the Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$. This result was extended and improved by many authors [12,14,15,18]. Particularly, it was discovered by Fefferman [12] that if one adds an additional roughness on the radial direction, namely T_Ω possesses the kernel $b(|x|)\Omega(x')|x|^{-n}$ with $b \in L^\infty$, then the rotation method used by Calderón and Zygmund cannot be adapted. However, by a Fourier transform method, the following result was obtained independently by several authors at an almost same time (see [2,10,17] and also [19] for a survey).

Theorem A. *Suppose $b \in L^\infty$. If $\Omega \in L^r(S^{n-1})$, $r > 1$, and satisfies* (1.2) *for $\alpha = 0$, then T_Ω is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

In a previous paper, we extended Theorem A to the case for all $\alpha \geqslant 0$ and obtained the following result.

Theorem B [1]. *For $1 < p < \infty$, let $\tilde{p} = \max\{p, p/(p-1)\}$. Let $\alpha \geqslant 0$. Suppose that $\Omega \in H^r(S^{n-1})$ with $r = (n-1)/(n-1+\alpha)$ and that Ω satisfies* (1.2) *for all Y_m whose degrees $\leqslant N$ with $2(N+1) > \alpha\tilde{p}$. Then we have*

$$\left\| T_{\Omega,\alpha}(f) \right\|_{L^p(\mathbb{R}^n)} \leqslant C \|f\|_{L^p_\alpha(\mathbb{R}^n)}, \tag{1.3}$$

where L^p_α is the Sobolev space.

The first main purpose of this paper is to establish a more general theorem in the case $\alpha > 0$.

Theorem 1. *For $1 < q, p < \infty$, let $\tilde{p} = \max\{p, p/(p-1)\}$, $\tilde{q} = \max\{q, q/(q-1)\}$. Let $\alpha > 0$. If $\Omega \in H^r(S^{n-1})$ with $r = (n-1)/(n-1+\alpha)$ and Ω satisfies* (1.2) *for all Y_m whose degrees $\leqslant N$ with $4(N+1) > \tilde{p}\alpha\tilde{q}$, then*

$$\left\| T_{\Omega,\alpha}(f) \right\|_{\dot{F}_p^{\beta,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{F}_p^{\beta+\alpha,q}(\mathbb{R}^n)}, \tag{1.4}$$

$$\left\| T_{\Omega,\alpha}(f) \right\|_{\dot{B}_p^{\beta,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{B}_p^{\beta+\alpha,q}(\mathbb{R}^n)}, \tag{1.5}$$

where $\beta \in \mathbb{R}$, $\dot{F}_p^{\beta,q}$ and $\dot{B}_p^{\beta,q}$ are the Triebel–Lizorkin spaces and the Besov spaces, respectively.

The constant C in (1.4) and (1.5) depends on α and $C \cong 1/\alpha$ as $\alpha \to 0^+$. Thus if $\alpha = 0$, we assume on Ω a stronger size condition $\Omega \in L^r(S^{n-1})$ with $r > 1$, for the sake of simplicity. Actually the condition $\Omega \in L \operatorname{Log}^+ L$ might be good enough.

Theorem 2. *Let* $1, p, q < \infty$. *If* $\Omega \in L^r(S^{n-1})$ *with* $r > 1$ *and satisfies* (1.2) *with* $\alpha = 0$, *then*

$$\left\| T_\Omega(f) \right\|_{\dot{F}_p^{\beta,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{F}_p^{\beta,q}(\mathbb{R}^n)}, \tag{1.4$'$}$$

$$\left\| T_\Omega(f) \right\|_{\dot{B}_p^{\beta,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{B}_p^{\beta,q}(\mathbb{R}^n)}. \tag{1.5$'$}$$

Before we recall the definitions of these various function spaces, in order to clarify the relations between Theorems 1, 2 and Theorems A, B, we remark that on the unit sphere S^{n-1}, $L^s \subseteq L \operatorname{Log}^+ L \subseteq H^1 \subseteq L^1 \subseteq H^r$, $0 < r < 1 < s$, and all the inclusions are proper, while $L^q = H^q$ if $1 < q < \infty$. It is known in [13] that on $\mathbb{R}^n$, $\dot{F}_p^{0,2} = L^p$, $\dot{F}_p^{\alpha,2} = L_\alpha^p$, $L^p \subseteq \dot{F}_r^{\alpha,2}$ if $\alpha < 0$ and $1/r = 1/p + \alpha/n$. Letting $X \to Y$ denote that the identity map is a continuous map from X to Y, then $\dot{F}_p^{\beta,q} \to \dot{B}_p^{\beta,q}$. Clearly, our Theorem 1 is an extension of Theorem B and Theorem 2 is an extension of Theorem A.

We also will use a transference method to obtain some analogous results on the n-torus $\mathbb{T}^n$. Let $\Omega \in H^r(S^{n-1})$ satisfy (1.2). Define informally

$$\lambda_\Omega(\xi) = \int_{\mathbb{R}^n} b\big(|y|\big)|y|^{-n-\alpha}\Omega(y')e^{-2\pi i\langle y,\xi\rangle}\,dy.$$

We will prove that $\lambda_\Omega(\xi) = O(|\xi|^\alpha)$ in Section 6. Since any $g \in C^\infty(\mathbb{T}^n)$ has the Fourier series $g(x) = \sum_{\ell \in \Lambda} a_\ell e^{2\pi i\langle \ell,x\rangle}$, where $\Lambda = \mathbb{R}^n/\mathbb{T}^n$ is the unit lattice which is an additive group of points in $\mathbb{R}^n$ having integer coordinates, we define $\widetilde{T}_{\Omega,\alpha}$ on all $g \in C^\infty(\mathbb{T}^n)$ by

$$\widetilde{T}_{\Omega,\alpha}(g)(x) = \sum_{\ell \in \Lambda} a_\ell \lambda_\Omega(\ell) e^{2\pi i\langle \ell,x\rangle}.$$

Also denote $\widetilde{T}_{\Omega,0}$ by $\widetilde{T}_\Omega$.

Theorem 3. *Under the conditions of Theorem* 1, *we have that for all* $g \in C^\infty(\mathbb{T}^n)$

$$\left\| \widetilde{T}_{\Omega,\alpha}(g) \right\|_{\dot{F}_p^{\beta,q}(\mathbb{T}^n)} \leqslant C \|g\|_{\dot{F}_p^{\beta+\alpha,q}(\mathbb{T}^n)}, \tag{1.6}$$

$$\left\| \widetilde{T}_{\Omega,\alpha}(g) \right\|_{\dot{B}_p^{\beta,q}(\mathbb{T}^n)} \leqslant C \|g\|_{\dot{B}_p^{\beta+\alpha,q}(\mathbb{T}^n)}, \tag{1.7}$$

where $\beta \in \mathbb{R}$, $\dot{F}_p^{\beta,q}(\mathbb{T}^n)$ *and* $\dot{B}_p^{\beta,q}(\mathbb{T}^n)$ *are the Triebel–Lizorkin spaces and the Besov spaces on the* n*-torus, respectively.*

Theorem 4. *Under the condition of Theorem* 2, *we have that for all* $g \in C^\infty(\mathbb{T}^n)$,

$$\|\widetilde{T}_\Omega(g)\|_{\dot{F}_p^{\beta,q}(\mathbb{T}^n)} \leqslant C\|g\|_{\dot{F}_p^{\beta,q}(\mathbb{T}^n)}, \tag{1.6'}$$

$$\|\widetilde{T}_\Omega(g)\|_{\dot{B}_p^{\beta,q}(\mathbb{T}^n)} \leqslant C\|g\|_{\dot{B}_p^{\beta,q}(\mathbb{T}^n)}. \tag{1.7'}$$

If $\alpha < 0$, then the integral operator defined in (1.1) is the fractional integral operator. This operator was also studied by many authors. The reader can see [7,16] and their references for more information. In Section 7 of this paper, we will obtain a theorem on the fractional integral similar to Theorem 1, in the case $\alpha \in (-1/2, 0)$. We also will obtain some results related to the Littlewood–Paley functions in Section 8.

2. Hardy space $H^r(S^{n-1})$

The Poisson kernel on S^{n-1} is defined by

$$P_{ty'}(x') = (1-t^2)/|ty'-x'|^n,$$

where $0 \leqslant t < 1$ and $x', y' \in S^{n-1}$. For any $\Omega \in \mathcal{S}'(S^{n-1})$, we define the radial maximal function $P^+(\Omega)(x')$ by

$$P^+\Omega(x') = \sup_{0\leqslant t<1} |\Omega, P_{tx'}|,$$

where $\mathcal{S}'(S^{n-1})$ is the space of Schwartz distributions on S^{n-1}.

The Hardy space $H^r(S^{n-1})$, $0 < r < \infty$, is the linear space of distribution $\Omega \in \mathcal{S}'(S^{n-1})$ with the finite norm $\|\Omega\|_{H^r(S^{n-1})} = \|P^+\Omega\|_{L^r(S^{n-1})} < \infty$. It is known in [3] that H^r is the same as the atomic Hardy space $H_a^r(S^{n-1})$. Thus by a standard atomic decomposition method (see [14] or [1]), it is known that to prove Theorems 1 and 3, we can assume that $\Omega(y') = a(y')$ is an (r,∞) atom with support in $B(\mathbf{1},\rho) \cap S^{n-1}$ and prove that the constants C in the theorems are independent of atom $a(y')$, where $\mathbf{1} = (1, 0, \dots, 0)$, and an (r,s) atom is an L^s, $s > 1$, function $a(\cdot)$ that satisfies

$$\operatorname{supp}(a) \subset \left\{x' \in S^{n-1},\ |x'-x_0'| < \rho\right\}$$
$$\text{for some } x_0' \in S^{n-1} \text{ and } \rho > 0, \tag{2.1}$$

$$\int_{S^{n-1}} a(y')Y_m(y')\,\sigma(y') = 0 \tag{2.2}$$

for all spherical harmonic polynomials Y_m with degree $\leqslant N$ with $4(N+1) > \alpha\tilde{p}\tilde{q}$,

$$\|a\|_{L^s(S^{n-1})} \leqslant \rho^{(n-1)(1/s-1/r)}. \tag{2.3}$$

For more information on the Hardy spaces, the reader can see [3,4].

3. The spaces $\dot F_p^{\beta,q}$ and $\dot B_p^{\beta,q}$

Fix a radial function $\Phi \in C^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp}(\Phi) \subseteq \{x,\ 1/2 < |x| \leqslant 2\}$, $0 \leqslant \Phi(x) \leqslant 1$ and $\Phi(x) > c > 0$ if $3/5 \leqslant |x| \leqslant 5/3$. Let $\Phi_j(x) = \Phi(2^j x)$ and require that Φ satisfies

$$\sum_{j=-\infty}^{\infty} \Phi_j(t)^2 = 1 \quad \text{for all } t. \tag{3.1}$$

It is easy to see $\operatorname{supp}(\Phi_j) \subseteq (2^{-j-1}, 2^{-j+1})$. Define the functions Ψ_j by $\widehat{\Psi}_j(\xi) = \Phi_j(\xi)$, so that $(\Psi_j * f)\hat{}(\xi) = \hat f(\xi)\Phi_j(\xi)$. For $1 < p < \infty$, $\beta \in \mathbb{R}$ and $1 < q < \infty$, the Triebel–Lizorkin space $\dot F_p^{\beta,q}(\mathbb{R}^n)$ is the set of all distributions f satisfying

$$\|f\|_{\dot F_p^{\beta,q}(\mathbb{R}^n)} = \left\| \left(\sum_k |2^{-\beta k}\Psi_k * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty; \tag{3.2}$$

the Besov space $\dot B_p^{\beta,q}(\mathbb{R}^n)$ is the set of all distributions f satisfying

$$\|f\|_{\dot B_p^{\beta,q}(\mathbb{R}^n)} = \left\{ \sum_k \left(2^{-\beta k} \|\Psi_k * f\|_{L^p(\mathbb{R}^n)} \right)^q \right\}^{1/q} < \infty. \tag{3.3}$$

For $g(x) = \sum a_\ell e^{2\pi i\langle \ell, x\rangle} \in C^\infty(\mathbb{T}^n)$ we define $\widetilde{\Psi}_k * g$ by

$$\widetilde{\Psi}_k * g(x) = \sum_{\ell \in \Lambda} a_\ell \Phi_k(\ell) e^{-2\pi i\langle \ell, x\rangle}. \tag{3.4}$$

In (3.2) and (3.3), replacing $\Psi_k * f$ by $\widetilde{\Psi}_k * g$, and $L^p(\mathbb{R}^n)$ by $L^p(\mathbb{T}^n)$, we similarly define the spaces $\dot F_p^{\beta,q}(\mathbb{T}^n)$ and $\dot B_p^{\beta,q}(\mathbb{T}^n)$. It is well-known that the dual space of $\dot F_p^{\beta,q}$ is $(\dot F_p^{\beta,q})^* = \dot F_{p'}^{-\beta,q'}$, where $1/p + 1/p' = 1/q + 1/q' = 1$. Similarly $(\dot B_p^{\beta,q})^* = \dot B_{p'}^{-\beta,q'}$.

Remark. One also can define the Triebel–Lizorkin spaces and the Besov spaces in a continuous version. Let Ψ and Φ be the same as before and let $\Psi_t(x) = t^{-n}\Psi(x/t)$. Then it is well-known that

$$\|f\|_{\dot F_p^{\beta,q}(\mathbb{R}^n)} \cong \left\| \left\{ \int_0^\infty |t^{-\beta}\Psi_t * f|^q t^{-1}\,dt \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \tag{3.5}$$

$$\|f\|_{\dot B_p^{\beta,q}(\mathbb{R}^n)} \cong \left\{ \int_0^\infty \left(t^{-\beta} \|\Psi_t * f\|_{L^p(\mathbb{R}^n)} \right)^q t^{-1}\,dt \right\}^{1/q}. \tag{3.6}$$

The reader can learn more information on these spaces in [21].

4. Some estimates

Suppose that $a(y')$ is an (r,∞) atom with support in $B(\mathbf{1},\rho)\cap S^{n-1}$ and satisfies the cancellation conditions in Theorem 1. Let I_k be the interval $(2^k,2^{k+1})$, $k=1,2,\ldots,$ and

$$\mathcal{J}_{k,\alpha}f(x)=\int_{\mathbb{R}^n} b\big(|y|\big)|y|^{-n-\alpha}a(y')\chi_{I_k}\big(|y|\big)f(x-y)\,dy.$$

It is easy to see that $\mathcal{J}_{k,\alpha}f=\sigma_{k,\alpha}*f$ so that $(\mathcal{J}_{k,\alpha}f)\hat{}(\xi)=\hat{\sigma}_{k,\alpha}(\xi)\hat{f}(\xi)$, where $\sigma_{k,\alpha}$ is the measure defined by

$$\int_{\mathbb{R}^n} f\,d\sigma_{k,\alpha}=\int_{2^k\leqslant|y|<2^{k+1}} f(y)b\big(|y|\big)|y|^{-n-\alpha}a(y')\,dy.$$

Thus we have

$$\hat{\sigma}_{k,\alpha}(\xi)\cong\int_{2^k}^{2^{k+1}} b\big(|t|\big)|t|^{-1-\alpha}\int_{S^{n-1}} a(y')e^{-2\pi it\langle y',\xi\rangle}\,d\sigma(y')\,dt.$$

By the cancellation condition of $a(y')$, it is easy to see

$$\big|\hat{\sigma}_{k,\alpha}(\xi)\big|\leqslant C\int_{2^k}^{2^{k+1}} t^{-1-\alpha}\left|\int_{S^{n-1}} a(y')\big\{e^{-2\pi it\langle y'-\mathbf{1},\xi\rangle}-1\big\}\,d\sigma(y')\right|dt,$$

because

$$\left|\int_{S^{n-1}} a(y')\big\{e^{-2\pi i\langle y'-\mathbf{1},\xi\rangle}-1\big\}\,d\sigma(y')\right|$$
$$=\left|e^{2\pi i\langle\mathbf{1},\xi\rangle}\int_{S^{n-1}} a(y')\big\{e^{-2\pi i\langle y',\xi\rangle}-e^{-2\pi i\langle\mathbf{1},\xi\rangle}\big\}\,d\sigma(y')\right|.$$

So by the Taylor expansion and the cancellation and support conditions of $a(y')$, we have

$$\big|\hat{\sigma}_{k,\alpha}(\xi)\big|\leqslant C2^{-k\alpha}|2^k\rho\xi|^{N+1}\rho^{(1-1/r)(n-1)}. \tag{4.1}$$

Similarly, by the support and size conditions of $a(y')$ we have

$$\big|\hat{\sigma}_{k,\alpha}(\xi)\big|\leqslant C2^{-k\alpha}\rho^{(1-1/r)(n-1)}. \tag{4.2}$$

The constants C in (4.1) and (4.2) are independent of k, ρ, and ξ.

In the case $\alpha=0$, we need a more precise estimate. If $a(y')=\Omega(y')$ is in $L^r(S^{n-1})$, $r>1$, and satisfies (1.2) with $\alpha=0$, then by [10] we know that there is a $\gamma > 0$ such that

$$\left|\hat{\sigma}_{k,0}(\xi)\right| \leqslant C \min\left\{|2^k\xi|, |2^k\xi|^{-\gamma}\right\}. \tag{4.3}$$

For $\alpha \geqslant 0$, we also have

$$|\sigma_{k,\alpha}| \leqslant C \int\limits_{2^k \leqslant |y| < 2^{k+1}} |y|^{-n-\alpha} \left|\Omega(y')\right| dy. \tag{4.4}$$

It is easy to see that if $\Omega(y') = a(y')$ is an (r, ∞) atom, then for all $1 \leqslant p \leqslant \infty$

$$\|\hat{\sigma}_{k,\alpha}\|_\infty \leqslant C|\sigma_{k,\alpha}| \leqslant C 2^{-k\alpha} \rho^{(1-1/r)(n-1)}, \tag{4.5}$$

$$\left\|\sup_k\left(|\sigma_{k,\alpha}| * f_k\right)\right\|_{L^p(\mathbb{R}^n)} \leqslant C\rho^{(1-1/r)(n-1)} \left\|\sup_k 2^{-k\alpha} f_k\right\|_{L^p(\mathbb{R}^n)}, \tag{4.6}$$

$$\left\||\sigma_{k,\alpha}| * f_k\right\|_{L^p(\mathbb{R}^n)} \leqslant C\rho^{(1-1/r)(n-1)} 2^{-k\alpha} \|f_k\|_{L^p(\mathbb{R}^n)}, \tag{4.6$'$}$$

where C is independent of k and ρ.

If Ω is in $L^r(S^{n-1})$ with $r > 1$, then

$$\left\|\sup_k |\sigma_{k,0}| * f\right\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)} \tag{4.7}$$

for all $1 < p \leqslant \infty$.

5. Proof of Theorems 1 and 2

First, we remark that throughout this section and the next section, the condition $r = (n-1)/(n-1+\alpha)$ in Theorems 1 and 3 is equivalent to $(n-1)(1-1/r) + \alpha = 0$.

To prove Theorem 1, as mentioned in Section 2, we can assume that $\Omega(y') = a(y')$ is an (r, ∞) atom with support in $B(\mathbf{1}, \rho) \cap S^{n-1}$, and prove that the constant C in the theorem is independent of $a(y')$. Let $\{\Phi_j\}$ and $\{\Psi_j\}$ be the same as in Section 3. Following the proof of lemma in [10], we decompose the operator $T_{\Omega,\alpha}(f)$ by

$$T_{\Omega,\alpha}(f) = \sum_j\left(\sum_k S_{j+k}\sigma_{k,\alpha} * S_{j+k} f\right) = \sum_j \Delta_j f, \tag{5.1}$$

where $(S_j f)\hat{}(\xi) = \Phi(2^j\rho\xi)\hat{f}(\xi)$. Let S_j^* be the dual operator of S_j; it is easy to check

$$\|f\|_{\dot{F}_p^{\beta,q}} \cong \left\|\left\{\sum_{j=-\infty}^{\infty} \left|(2^j\rho)^{-\beta} S_j^* f\right|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$
$$\cong \left\|\left\{\sum_{j=-\infty}^{\infty} \left|(2^j\rho)^{-\beta} S_j f\right|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}.$$

For any $g \in \dot{F}_{p'}^{-\beta,q'}$, we have

$$\begin{aligned}\left|\langle \Delta_j f, g\rangle\right| &= \left|\int \sum_k S_{k+j}\sigma_{k,\alpha} * (S_{j+k} f)(x) g(x)\,dx\right| \\ &= \left|\int \sum_k \sigma_{k,\alpha} * (S_{j+k} f)(x) S_{j+k}^* g(x)\,dx\right| \\ &\leqslant \int \left(\sum_k \left|(2^{k+j}\rho)^{-\beta}\sigma_{k,\alpha} * S_{j+k} f(x)\right|^q\right)^{1/q} \\ &\quad \times \left(\sum_k \left|(2^{k+j}\rho)^{\beta} S_{k+j}^* g(x)\right|^{q'}\right)^{1/q'} dx. \qquad (5.2)\end{aligned}$$

Taking supremum over g with $\|g\|_{\dot{F}_{p'}^{-\beta,q'}} \leqslant 1$ and by Hölder's inequality we have

$$\|\Delta_j f\|_{\dot{F}_p^{\beta,q}} \leqslant C \left\| \left(\sum_k \left|(2^{k+j}\rho)^{-\beta}\sigma_{k,\alpha} * S_{k+j} f\right|^q\right)^{1/q} \right\|_{L^p}. \qquad (5.3)$$

Now we use (5.3) to estimate $\|\Delta_j f\|_{\dot{F}_p^{\beta,q}}$ for different pairs (p,q).

For $q = p$, by (5.3) and (4.6′),

$$\begin{aligned}\|\Delta_j f\|_{\dot{F}_q^{\beta,q}} &\leqslant C\left\{\sum_k (2^{k+j}\rho)^{-\beta q} \int_{\mathbb{R}^n} \left|\sigma_{k,\alpha} * S_{k+j} f(x)\right|^q dx\right\}^{1/q} \\ &\leqslant C 2^{j\alpha} \left(\sum_k (2^{k+j}\rho)^{-(\alpha+\beta)q} \int_{\mathbb{R}^n} \left|S_{k+j} f(x)\right|^q dx\right)^{1/q}.\end{aligned}$$

This shows

$$\|\Delta_j f\|_{\dot{F}_q^{\beta,q}} \leqslant C 2^{j\alpha} \|f\|_{\dot{F}_q^{\alpha+\beta,q}}. \qquad (5.4)$$

If $p = q = 2$, we have

$$\begin{aligned}\|\Delta_j f\|^2_{\dot{F}_2^{\beta,2}} \cong \|\Delta_j f\|^2_{L^2_\beta} &\leqslant C \sum_k \int_{\mathbb{R}^n} (2^{k+j}\rho)^{-2\beta} \left|\sigma_{k,\alpha} * (S_{j+k} f)(y)\right|^2 dy \\ &\cong C \sum_k \int_{\mathbb{R}^n} \left|\Phi_{j+k}\big(|\rho\xi|\big)(2^{k+j}\rho)^{-\beta}\hat{\sigma}_{k,\alpha}(\xi)\hat{f}(\xi)\right|^2 d\xi \\ &\leqslant C \sum_k \int_{D_{j+k}} \left|\hat{\sigma}_{k,\alpha}(\xi)\hat{f}(\xi)\right|^2 (2^{k+j}\rho)^{-2\beta}\,d\xi,\end{aligned}$$

where

$$D_j = \left\{\xi\colon 2^{-j-1} \leqslant |\xi\rho| \leqslant 2^{-j+1}\right\}.$$

If $j \geqslant 0$, noting $2^{-k-j} \cong |\xi\rho|$ on D_{j+k}, using (4.1) we have

$$\|\Delta_j f\|^2_{\dot F_2^{\beta,2}} \leqslant C 2^{-2j(N+1)} \rho^{(n-1)(1-1/r)} \sum_k \int_{D_{j+k}} \left|\hat f(\xi)\right|^2 2^{-2k\alpha} |\xi|^{2\beta}\, d\xi$$
$$\leqslant C 2^{-2j(N-\alpha+1)} \sum_k \int_{D_k} \left|\hat f(\xi)\right|^2 |\xi|^{2(\alpha+\beta)}\, d\xi.$$

Therefore, for $j \geqslant 0$, we have

$$\|\Delta_j f\|^2_{\dot F_2^{\beta,2}} \leqslant C 2^{-j(N-\alpha+1)} \|f\|_{\dot F_2^{\alpha+\beta,2}(\mathbb{R}^n)}. \tag{5.5}$$

Similarly, using (4.2), we have for $j < 0$

$$\|\Delta_j f\|_{\dot F_2^{\beta,2}} \leqslant C 2^{j\alpha} \|f\|_{\dot F_2^{\alpha+\beta,2}(\mathbb{R}^n)}. \tag{5.6}$$

If $p > q$, let $s = (p/q)' = p/(p-q)$. By (5.3), we can take a non-negative $h \in L^s(\mathbb{R}^n)$ with $\|h\|_s = 1$ such that

$$\|\Delta_j f\|^q_{\dot F_p^{\beta,q}} \leqslant C \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \left|(2^{k+j}\rho)^{-\beta} \sigma_{k,\alpha} * (S_{k+j} f)\right|^q h\, dx. \tag{5.7}$$

Since $|\sigma_{k,\alpha} * (S_{k+j} f)|^q$ is bounded by

$$C\left(\rho^{(n-1)(1-1/r)}\right)^{q/q'} 2^{-kq\alpha} \int_{2^k \leqslant |y| < 2^{k+1}} \left|a(y')\right| |y|^{-n} \left|S_{k+j} f(x-y)\right|^q dy$$
$$= C\left(\rho^{(n-1)(1-1/r)}\right)^{q/q'} 2^{-k\alpha q} L_k\left\{|S_{j+k} f|^q\right\}(x),$$

where

$$L_k f(x) = \int_{2^k \leqslant |y| \leqslant 2^{k+1}} \left|a(y')\right| |y|^{-n} f(x-y)\, dy, \tag{5.8}$$

we have that

$$\sum_k \int_{\mathbb{R}^n} \left|(2^{k+j}\rho)^{-\beta} \sigma_{k,\alpha} * (S_{k+j} f)\right|^q h\, dx$$
$$= C\left(\rho^{(n-1)(1-1/r)}\right)^{q/q'}$$
$$\times \int_{\mathbb{R}^n} \left\{\sum_k \left|2^{-k\alpha} (2^{j+k}\rho)^{-\beta} S_{k+j} f(x)\right|^q\right\} N_a h(x)\, dx,$$

where $N_a h(x) = \sup_k (L_k^* h)(x)$, and

$$(L_k^* h)(x) = \int_{2^k \leqslant |y| < 2^{k+1}} |y|^{-n} \left|a(y')\right| h(x+y)\, dy.$$

By the rotation method and the L^p boundedness of the Hardy–Littlewood maximal function, it is easy to see that

$$\|N_a h\|_{L^s} \leqslant C\rho^{(n-1)(1-1/r)}\|h\|_{L^s} \leqslant \rho^{(n-1)(1-1/r)}.$$

Thus by Hölder's inequality and (5.7), we have

$$\|\Delta_j f\|_{\dot{F}_p^{\beta,q}(\mathbb{R}^n)} \leqslant C2^{j\alpha}\left\|\left(\sum_k \left|(2^{k+j}\rho)^{-\alpha-\beta}S_{k+j}f\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)},$$

which, together with (5.4), show that if $p \geqslant q$, then for any $j \in \mathbb{Z}$

$$\|\Delta_j f\|_{\dot{F}_p^{\beta,q}} \leqslant C2^{j\alpha}\|f\|_{\dot{F}_p^{\beta+\alpha,q}}. \tag{5.9}$$

Taking $q = 2$ in (5.9) and by duality, it is easy to check that for all $1 < p < \infty$

$$\|\Delta_j f\|_{\dot{F}_p^{\beta,2}} \leqslant C2^{j\alpha}\|f\|_{\dot{F}_p^{\alpha+\beta,2}}. \tag{5.10}$$

By interpolating (5.5), (5.6) and (5.10) (see [21]), and by the choice of N, we have a positive number θ, which is less than, but arbitrarily close to, $(2(N+1)-\alpha\tilde{p})/\tilde{p}$ such that for $1 < p < \infty$

$$\|\Delta_j f\|_{\dot{F}_p^{\beta,2}} \leqslant 2^{-\theta j}\|f\|_{\dot{F}_p^{\alpha+\beta,2}} \quad \text{if } j \geqslant 0, \tag{5.11}$$

$$\|\Delta_j f\|_{\dot{F}_p^{\beta,2}} \leqslant 2^{\alpha j}\|f\|_{\dot{F}_p^{\alpha+\beta,2}} \quad \text{if } j < 0. \tag{5.11$'$}$$

Interpolating between (5.11), (5.11′) and (5.9), we obtain a positive number $\delta = \min\{\alpha, \gamma\}$, where γ is less than, but arbitrarily close to, $(4(N+1) - \tilde{p}\alpha\tilde{q})/\tilde{p}\tilde{q}$, for $1 < q \leqslant p < \infty$, such that

$$\|\Delta_j f\|_{\dot{F}_p^{\beta,q}} \leqslant C2^{-\delta|j|}\|f\|_{\dot{F}_p^{\alpha+\beta,q}}. \tag{5.12}$$

From (5.1) and (5.12), we have that for $1 < q \leqslant p < \infty$

$$\left\|T_{\Omega,\alpha}(f)\right\|_{\dot{F}_p^{\beta,q}} \leqslant C\|f\|_{\dot{F}_p^{\alpha+\beta,q}}. \tag{5.13}$$

Noting that β is an arbitrary real number, by duality we obtain (5.13) for all $\beta \in \mathbb{R}$, $1 < q, p < \infty$. This proves (1.4) in Theorem 1. Now (1.5) of Theorem 1 follows by an interpolation result $(\dot{F}_r^{\alpha,r}, \dot{F}_s^{\alpha,s})_{\theta,q} \cong \dot{B}_p^{\alpha,q}$ (see [21]).

The proof of Theorem 2 is exactly the same by letting $\alpha = 0$, $\rho = 1$ and using (4.3) instead of (4.1) and (4.2).

6. Proof of Theorems 3 and 4

We prove Theorem 3 only, since the proof of Theorem 4 ($\alpha = 0$) is similar and easier than the case $\alpha > 0$. Similar to the proof of Theorem 1, it suffices to show the boundedness on the Triebel–Lizorkin spaces. Also, we can assume that

$\Omega(y') = a(y')$ is an (r, ∞) atom supported in $B(\mathbf{1}, \rho) \cap S^{n-1}$ and show that the bound is independent of $a(y')$. For $n > \alpha > 0$, let R_α be the Riesz potential on $\mathbb{R}^n$ which is defined by $(R_\alpha f)\hat{}(\xi) = C_\alpha |\xi|^{-\alpha} \hat{f}(\xi)$, and let $\widetilde{R}_\alpha$ be the Riesz potential on $\mathbb{T}^n$ defined by

$$\widetilde{R}_\alpha g(x) = C_\alpha \sum_{\ell \in \Lambda \setminus \{0\}} |\ell|^{-\alpha} a_\ell e^{2\pi i \langle \ell, x \rangle} \quad \text{for } g(x) = \sum_{\ell \in \Lambda} a_\ell e^{2\pi i \langle \ell, x \rangle}, \tag{6.1}$$

where C_α is a constant depending on α. It is known that R_α has the "lift" property, and so does $\widetilde{R}_\alpha$. This means that R_α (also $\widetilde{R}_\alpha$) is an isomorphism between the spaces $\dot{F}_p^{\beta,q}$ and $\dot{F}_p^{\alpha+\beta,q}$ and $\|f\|_{\dot{F}_p^{\beta,q}(\mathbb{R}^n)} \cong \|R_\alpha f\|_{\dot{F}_p^{\alpha+\beta,q}(\mathbb{R}^n)}$ and $\|g\|_{\dot{F}_p^{\beta,q}(\mathbb{T}^n)} \cong \|\widetilde{R}_\alpha g\|_{\dot{F}_p^{\alpha+\beta,q}(\mathbb{T}^n)}$. Thus to prove the theorem, it suffices to show that, for any $\gamma \in R$,

$$\left\| R_\alpha T_{\Omega,\alpha}(f) \right\|_{\dot{F}_p^{\gamma,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{F}_p^{\gamma,q}(\mathbb{R}^n)}$$

implies

$$\left\| \widetilde{R}_\alpha \widetilde{T}_{\Omega,\alpha}(g) \right\|_{\dot{F}_p^{\gamma,q}(\mathbb{T}^n)} \leqslant C \|g\|_{\dot{F}_p^{\gamma,q}(\mathbb{T}^n)}.$$

If we further use the "lift" property and note that R_α and $\widetilde{R}_\alpha$ satisfy the semigroup property $R_\alpha R_\gamma \cong R_{\alpha+\gamma}$. Then it is easy to see that to prove Theorem 3, we only need to show the following proposition.

Proposition 1. *If* $\|R_\alpha T_{\Omega,\alpha}(f)\|_{\dot{F}_p^{0,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{F}_p^{0,q}(\mathbb{R}^n)}$ *for all* $f \in \mathcal{S}(\mathbb{R}^n)$, *then for all* $g \in C^\infty(\mathbb{T}^n)$

$$\left\| \widetilde{R}_\alpha \widetilde{T}_{\Omega,\alpha}(g) \right\|_{\dot{F}_p^{0,q}(\mathbb{T}^n)} \leqslant C \|g\|_{\dot{F}_p^{0,q}(\mathbb{T}^n)}.$$

Proof. Let $R_\alpha T_{\Omega,\alpha} = T$ and $\widetilde{R}_\alpha \widetilde{T}_{\Omega,\alpha} = \widetilde{T}$. They are convolution operators so that $(Tf)\hat{}(\xi) = \mu(\xi)\hat{f}(\xi)$. By the main theorem in [11], to prove the proposition, we only need to verify $\mu \in L^\infty$, and that $\mu(\xi)$ is continuous at each $\xi \neq 0$. First we show that $\mu(\xi) \in L^\infty(\mathbb{R}^n)$. By the definition and (4.1), (4.2), for any $\xi \neq 0$

$$\begin{aligned}
\left|\mu(\xi)\right| &\leqslant C|\xi|^{-\alpha} \sum_k \left|\hat{\sigma}_{k,\alpha}(\xi)\right| \\
&\leqslant C|\xi|^{-\alpha} \sum_{|2^k \rho \xi| > 1} \left|\hat{\sigma}_{k,\alpha}(\xi)\right| + C|\xi|^{-\alpha} \sum_{|2^k \rho \xi| \leqslant 1} \left|\hat{\sigma}_{k,\alpha}(\xi)\right| \\
&\leqslant C\rho^{(n-1)(1-1/r)} \sum_{|2^k \rho \xi| > 1} |2^k \xi|^{-\alpha} \\
&\quad + C\rho^{(n-1)(1-1/r)+(N+1)} \sum_{2^k \leqslant 1/|\rho \xi|} |2^k \xi|^{(N+1)-\alpha} \leqslant C.
\end{aligned}$$

So $\mu \in L^\infty$.

Next, fix an $\varepsilon > 0$; for any $\xi \neq 0$,

$$\mu(\xi) = C_\alpha |\xi|^{-\alpha} \left\{ \text{p.v.} \int_{|y|<\varepsilon} b\big(|y|\big)|y|^{-n-\alpha} a(y') e^{-2\pi i \langle y,\xi\rangle}\, dy + \int_{|y|\geqslant\varepsilon} |y|^{-n-\alpha} b\big(|y|\big) a(y') e^{-2\pi i\langle y,\xi\rangle}\, dy \right\}.$$

Thus using the cancellation condition on $a(y')$ it is easy to check that $\mu(\xi)$ is continuous at each $\xi \neq 0$. The proposition is proved. □

Remark. In the case $\alpha = 0$ (Theorem 4), by checking the proof of the main theorem in [11], it suffices to prove that the symbol $\mu(\xi)$ of $T_{\Omega,0}$ is bounded and each $\lambda \in \Lambda\backslash\{0\}$ is a Lebesgue point of $m(\xi)$. But this was pointed out on p. 263 in [20].

7. Fractional integral operators

Let $n > \alpha > 0$, $\Omega \in L^1(S^{n-1})$. The fractional integral operator $F_{\Omega,\alpha}$ is defined on all $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$F_{\Omega,\alpha}(f)(x) = \int_{\mathbb{R}^n} |y|^{-n+\alpha} \Omega(y') f(x-y)\, dy.$$

Let $\tau_{\alpha,k}(y) = |y|^{-n+\alpha}\Omega(y')\chi_{I_k}(|y|)$ with $I_k = (2^k, 2^{k+1}]$. Then we have

$$F_{\Omega,\alpha}(f)(x) = \sum_{k=-\infty}^{\infty} \tau_{\alpha,k} * f(x).$$

It is easy to check

$$\big|\hat{\tau}_{\alpha,k}(\xi)\big| \leqslant C2^{k\alpha}. \tag{7.1}$$

By checking p. 551 of [10], we find that if $\Omega \in L^r(S^{n-1})$, $r > 1$, then for any δ less than $1/2r'$

$$\big|\hat{\tau}_{\alpha,k}(\xi)\big| \leqslant C2^{k\alpha}|2^k\xi|^{-\delta}. \tag{7.2}$$

Now replacing (4.1) and (4.2) by (7.1) and (7.2), using the exactly same proof in Theorem 1, we have the following theorem for the fractional integral operator.

Theorem 5. *Let $\Omega \in L^r(S^{n-1})$, $r > 1$. For $1 < q, p < \infty$, let $\tilde{p} = \max\{p, p/(p-1)\}$ and $\tilde{q} = \max\{q, q/(q-1)\}$. If $0 < \alpha < 2/r'\tilde{p}\tilde{q}$ (or $r > 2/(2-\alpha\tilde{p}\tilde{q})$ with $2 - \alpha\tilde{p}\tilde{q} > 0$), then for any real number β*

$$\left\| F_{\Omega,\alpha}(f) \right\|_{\dot{F}_p^{\beta,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{F}_p^{\beta-\alpha,q}(\mathbb{R}^n)},$$
$$\left\| F_{\Omega,\alpha}(f) \right\|_{\dot{B}_p^{\beta,q}(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{B}_p^{\beta-\alpha,q}(\mathbb{R}^n)}.$$

8. Littlewood–Paley functions

For an $L^1(\mathbb{R}^n)$ function ϕ, we define $\phi_t(x) = 2^{-tn}\phi(x/2^t)$, $t \in \mathbb{R}$. Then the Fourier transform of ϕ_t is $\hat{\phi}_t(\xi) = \hat{\phi}(2^t\xi)$. The Littlewood–Paley g-function $g_\phi(f)$ on $\mathbb{R}^n$ is defined on $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$g_\phi f(x) = \left(\int_{\mathbb{R}} \left|\phi_t * f(x)\right|^2 dt \right)^{1/2}. \tag{8.1}$$

The following theorem is the main result in [8].

Theorem C. *For $\phi \in L^1(\mathbb{R}^n)$, if ϕ satisfies*

(i) $\|\sup_{t\in\mathbb{R}} |\phi_t| * f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$ *for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $p \in (1,\infty)$,*
(ii) $|\hat{\phi}(\xi)| \leqslant C \min(|\xi|^\beta, |\xi|^{-\beta})$ *for some $\beta > 0$,*

then we have

$$\left\| g_\phi(f) \right\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n). \tag{8.2}$$

If we define $\phi_t * f(x)$ by $\mathcal{F}(f)(x,t)$, then (8.2) can be written as

$$\left\| \|\mathcal{F}(f)\|_{L^2(\mathbb{R})} \right\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{F}_p^{0,2}(\mathbb{R}^n)}. \tag{8.2$'$}$$

Now we define, for any real number α,

$$\mathcal{F}_\alpha(f)(x,t) = 2^{-t\alpha}\phi_t * f(x) = 2^{-t\alpha}\mathcal{F}(f)(x,t).$$

In this section we extend Theorem C to the following more general theorem.

Theorem 6. *For $1 < p, q < \infty$, let $\tilde{p}$ and $\tilde{q}$ be as in Theorem 5. Suppose that $\phi \in L^1(\mathbb{R}^n)$ satisfies (i), (ii) in Theorem C. If $\alpha \in (-\beta,\beta)$ satisfies $|\alpha| < 4\beta/\tilde{p}\tilde{q}$, then we have*

$$\left\| \|\mathcal{F}_\alpha(f)\|_{L^q(\mathbb{R})} \right\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}. \tag{8.3}$$

Proof. We use the equivalent definition (3.5) to study the Triebel–Lizorkin spaces. Choose a radial function $\Psi \in \mathcal{S}(\mathbb{R}^n)$ as in the definition of the Triebel–Lizorkin spaces, namely Ψ satisfies

$$\int_{\mathbb{R}} \widehat{\Psi}(2^s)\,ds = 1, \qquad \widehat{\Psi}(y) > c > 0 \quad \text{if } 3/5 \leqslant |y| \leqslant 5/3,$$

$$\operatorname{supp}(\widehat{\Psi}) \subseteq \{y \in \mathbb{R}^n\colon\ 2^{-1} < |y| \leqslant 2\}.$$

It is easy to see that for any test function $f \in \mathcal{S}(\mathbb{R}^n)$

$$f \cong \int_{\mathbb{R}} \Psi_s * f\,ds.$$

So by the Minkowski inequality, we have that

$$\begin{aligned}\|\mathcal{F}_\alpha(f)(x,\cdot)\|_{L^q(\mathbb{R})} &= \left(\int_{\mathbb{R}} \left|\int_{\mathbb{R}} (\Psi_{s+t} * 2^{-t\alpha}\phi_t * f)(x)\,ds\right|^q dt\right)^{1/q} \\ &\leqslant \int_{\mathbb{R}} I_{\alpha,s} f(x)\,ds, \end{aligned} \tag{8.4}$$

where

$$I_{\alpha,s} f(x) = \left(\int_{\mathbb{R}} \left|(\Psi_{s+t} * 2^{-t\alpha}\phi_t * f)(x)\right|^q dt\right)^{1/q}.$$

Let

$$L_{\alpha,s}(f)(x,t) = \Psi_{s+t} * 2^{-t\alpha}\phi_t * f(x) = \Psi_{s+t} * \mathcal{F}_\alpha(x,t).$$

Then

$$I_{\alpha,s} f(x) = \|\Psi_{s+t} * \mathcal{F}_\alpha(x,t)\|_{L^q(\mathbb{R},dt)} = \|L_{\alpha,s}(f)(x,\cdot)\|_{L^q(\mathbb{R})}.$$

It is easy to see that

$$\begin{aligned}\big\| \|L_{\alpha,s}(f)\|_{L^q(\mathbb{R})}\big\|_{L^q(\mathbb{R}^n)} &= \big\| \|L_{\alpha,s}(f)\|_{L^q(\mathbb{R}^n)}\big\|_{L^q(\mathbb{R})} \\ &\leqslant C \left\| \left(\int_{\mathbb{R}} |2^{-t\alpha}\Psi_{s+t} * f|^q\,dt\right)^{1/q}\right\|_{L^q(\mathbb{R}^n)}.\end{aligned}$$

This shows

$$\big\| \|L_{\alpha,s}(f)\|_{L^q(\mathbb{R})}\big\|_{L^q(\mathbb{R}^n)} \leqslant C2^{s\alpha}\|f\|_{\dot{F}_q^{\alpha,q}(\mathbb{R}^n)}. \tag{8.5}$$

By the proof of (2.5) in [8] we find that if $s \geqslant 0$,

$$\big\| \|L_{\alpha,s}(f)\|_{L^2(\mathbb{R})}\big\|_{L^2(\mathbb{R}^n)} \leqslant C2^{-s(\beta-\alpha)}\|f\|_{L^2_\alpha} \cong 2^{-s(\beta-\alpha)}\|f\|_{\dot{F}_2^{\alpha,2}(\mathbb{R}^n)}. \tag{8.6}$$

Similarly, by the proof of (2.8) in [8], we find that if $s < 0$, then

$$\big\| \|L_{\alpha,s}(f)\|_{L^2(\mathbb{R})}\big\|_{L^2(\mathbb{R}^n)} \leqslant C2^{s(\alpha+\beta)}\|f\|_{\dot{F}_2^{\alpha,2}(\mathbb{R}^n)}. \tag{8.7}$$

If $p > q$, using the same argument to prove (5.9), we obtain

$$\big\| \|L_{\alpha,s}(f)\|_{L^q(\mathbb{R})} \big\|_{L^p(\mathbb{R}^n)} \leqslant C2^{s\alpha} \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}. \tag{8.8}$$

If $q > p$, then $p' > q'$. Now for all $g(x,t)$ satisfying $\|\|g\|_{L^{q'}(\mathbb{R})}\|_{L^{p'}(\mathbb{R}^n)} = 1$, we have

$$\begin{aligned}\big|\langle L_{\alpha,s}(f), g\rangle\big| &\leqslant \big\| \|\mathcal{F}^*(g)\|_{L^{q'}(\mathbb{R})} \big\|_{L^{p'}(\mathbb{R}^n)} \\ &\quad\times \left\| \left(\int_{\mathbb{R}} |2^{-t\alpha}\Psi_{s+t} * f|^q \, dt \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\cong \big\| \|\mathcal{F}^*(g)\|_{L^{q'}(\mathbb{R})} \big\|_{L^{p'}(\mathbb{R}^n)} 2^{s\alpha} \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)},\end{aligned}$$

where

$$\mathcal{F}^*(g)(x,t) = \int_{\mathbb{R}^n} \phi_t(y) g(x+y,t)\, dy.$$

Let $s = p'/q' > 1$ and let s' be the dual exponent of s. There is a positive function $h \in L^{s'}(\mathbb{R}^n)$, $\|h\|_{L^{s'}(\mathbb{R}^n)} = 1$, such that

$$\begin{aligned}&\big\| \|\mathcal{F}^*(g)\|_{L^{q'}(\mathbb{R})} \big\|_{L^{p'}(\mathbb{R}^n)}^{q'} \\ &\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \left| \int_{\mathbb{R}^n} \phi_t(y) g(x+y,t)\, dy \right|^{q'} dt\, h(x)\, dx \\ &\quad \leqslant C \int_{\mathbb{R}^n} \left(\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^n} \big|\phi_t(x-y)\big| h(x)\, dx \right) \int_{\mathbb{R}^n} \big|g(y,t)\big|^{q'} dt\, dy \\ &\quad \leqslant C \left\| \sup_{t\in\mathbb{R}} |\phi_t| * h \right\|_{L^{s'}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} \big|g(y,t)\big|^{q'} dt \right)^s dy \right)^{1/s} \\ &\quad \leqslant C\|h\|_{L^{s'}(\mathbb{R}^n)} \big\| \|g\|_{L^{q'}(\mathbb{R})} \big\|_{L^{p'}(\mathbb{R}^n)}^{q'} = C.\end{aligned}$$

This shows, for all $q > p$,

$$\big\| \|L_{\alpha,s}(f)\|_{L^q(\mathbb{R})} \big\|_{L^p(\mathbb{R}^n)} \leqslant C2^{s\alpha} \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}. \tag{8.8$'$}$$

By interpolating among (8.5)–(8.8′) and the condition $|\alpha| < 4\beta/\tilde{p}\tilde{q}$, we obtain a $\delta > 0$:

$$\big\| \|L_{\alpha,s}(f)\|_{L^q(\mathbb{R})} \big\|_{L^p(\mathbb{R}^n)} \leqslant C2^{-|s|\delta} \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}. \tag{8.9}$$

Thus by (8.4) and (8.9),

$$\big\| \|\mathcal{F}_\alpha(f)\|_{L^q(\mathbb{R})} \big\|_{L^p(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}} \|I_{\alpha,s}\|_{L^p(\mathbb{R}^n)}\,ds$$
$$\leqslant \int_{\mathbb{R}} \big\| \|L_{\alpha,s}(f)\|_{L^q(\mathbb{R})} \big\|_{L^p(\mathbb{R}^n)}\,ds \leqslant C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}.$$

The theorem is proved. □

Next we give two applications of Theorem 6.

Let $\Omega \in L^1(S^{n-1})$ satisfy $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$ and let

$$\phi(x) = \chi_B(x)|x|^{-n+1}\Omega(x'), \tag{8.10}$$

where χ_B is the characteristic function on the unit ball $B = \{x\colon\ |x| < 1\}$. Let $M(f)(x,t) = \phi_t * f(x)$; then

$$\mu_\Omega(f) = \left\{ \int_{\mathbb{R}} \big|M(f)(x,t)\big|^2\,dt \right\}^{1/2}$$

is the well-known n-dimensional Marcinkiewicz integral defined by Stein. It is known in [8] that if $\Omega \in L^r$, $r > 1$, then for all $1 < p < \infty$

$$\|\mu_\Omega\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}.$$

But by using the same argument on p. 551 of [10], we find that $|\hat{\phi}(\xi)| \leqslant C\min\{|\xi|, |\xi|^{-\gamma}\}$, where γ is any positive number less than $1/2r'$. Thus we have the following corollary of Theorem 6.

Corollary 1. *Let $\tilde{p}, \tilde{q}$ be the same as in Theorem* 6 *and $\Omega \in L^r(S^{n-1})$. If $|\alpha| < 2/(r'\tilde{p}\tilde{q})$, then*

$$\left\| \left\{ \int_{\mathbb{R}} \big|2^{-t\alpha} M(f)(\cdot,t)\big|^q\,dt \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}.$$

We can also define a g_λ^*-function $G_{\phi,\lambda,\alpha,q}(f)$ by

$$G_{\phi,\lambda,\alpha,q}(f)(x) = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}} 2^{-tn}\big\{2^t/\big(2^t + |x-y|\big)\big\}^{n\lambda} \times \big|\mathcal{F}_\alpha(f)(y,t)\big|^q\,dt\,dy \right)^{1/q}.$$

By the same proof as Theorem 2 in [9] and the above Theorem 6, we have

Corollary 2. *Let $1 < q \leqslant p < \infty$, $\lambda > 1$ and α, ϕ be the same as in Theorem* 6. *Then we have*

$$\big\|G_{\phi,\lambda,\alpha,q}(f)\big\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}.$$

This corollary recovers Theorem 2 in [9] if $\alpha = 0$, $q = 2$ and ϕ is defined as in (8.10).

9. A final remark

Following the definition of (3.5) we can define the Triebel–Lizorkin spaces on the product space $\mathbb{R}^n \times \mathbb{R}^m$. Let $U \in C^\infty(\mathbb{R}^n)$ and $V \in C^\infty(\mathbb{R}^m)$ satisfy support$(U) \subseteq \{x, 1/2 < |x| \leqslant 2\}$, support$(V) \subseteq \{y,\ 1/2 < |y| \leqslant 2\}$ and $U(x) > c > 0$, $V(y) > c > 0$ if $3/5 \leqslant |x| \leqslant 5/3$, $3/5 \leqslant |y| \leqslant 5/3$. Let Φ and Ψ be the Fourier inverse of U and V, respectively. For $\alpha, \beta \in \mathbb{R}$, $1 < p_1, p_2, q < \infty$, let $s = (\alpha, \beta)$ and $\mathbf{p} = (p_1, p_2)$. The Triebel–Lizorkin spaces $\dot{F}_{\mathbf{p}}^{s,q}(\mathbb{R}^n \times \mathbb{R}^m)$ is the set of all distributions f on $\mathbb{R}^n \times \mathbb{R}^m$ such that

$$\|f\|_{\dot{F}_{\mathbf{p}}^{s,q}(\mathbb{R}^n \times \mathbb{R}^m)} = \left\| \left\{ \int_0^\infty \int_0^\infty \left| (\Phi_t \otimes \Psi_s) * f \right|^q t^{-\alpha q} s^{-\beta q} \, ds \, dt \right\}^{1/q} \right\|_{L^{\mathbf{p}}(\mathbb{R}^n \times \mathbb{R}^m)} < \infty,$$

where $\|\cdot\|_{L^{\mathbf{p}}(\mathbb{R}^n \times \mathbb{R}^m)}$ is the mixed norm.

It is possible to extend the results in this paper to the product spaces.

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