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On a homotopy relation between the 2-local geometry and the Bouc complex for the sporadic group Co_3

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Abstract

We study the homotopy relation between the standard 2-local geometry Δ and the Bouc complex for the sporadic group Co_3. We also give a result concerning the relative projectivity of the reduced Lefschetz module $\tilde{L}(\Delta)$.

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1. Introduction

If G is a Lie group in natural characteristic p, then the Tits building is the simplicial complex on the maximal parabolic subgroups of G. The Bouc complex $|\mathcal{B}_p(G)|$, in this case the complex of unipotent radicals, is the barycentric subdivision of the Tits building.

In the 70s and 80s, Buekenhout [4], Ronan and Smith [11] and Ronan and Stroth [12] constructed various geometries for the sporadic simple groups in an attempt to generalize the Tits buildings for Lie groups. Also, Brown [5], Quillen [10], Bouc [6] and others were considering various collections of p-subgroups related to group cohomology.

In his expository paper, Webb [16] noted the connections between the group geometries and the subgroup complexes, and investigated the associated Lefschetz modules and the correspond-

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ing homology decompositions. Webb's paper contains specific calculations, with details from various authors, on geometries and subgroup complexes.

The relationship between p-local geometries and the Quillen complex of elementary abelian p-subgroups or the Bouc complex of p-radical subgroups for certain sporadic groups was further investigated in papers by Ryba, Smith and Yoshiara [13] and Smith and Yoshiara [14]. In the latter paper, the authors studied the projectivity of the reduced Lefschetz module for some group geometries and observed that the sporadic groups of characteristic p-type behave similarly to the finite Lie groups in defining characteristic p.

There is no uniform approach for p-local geometries. A recent, comprehensive study of 2-local geometries for the 26 sporadic simple groups, with some general results for p-local geometries, can be found in Benson and Smith [3]. In the present paper, it is understood that a p-local geometry for a group G is a simplicial complex whose vertex stabilizers are suitably chosen maximal p-local subgroups of G.

For many of the sporadic simple groups, especially when $p = 2$ and G has characteristic 2-type, there is a 2-local geometry Δ, with vertex stabilizers maximal 2-local subgroups of G, such that the Bouc complex $|\mathcal{B}_2(G)|$ is equal to the barycentric subdivision of the geometry. For the remaining sporadic groups the relationship is not clear anymore; in fact, Δ and $|\mathcal{B}_2(G)|$ are not always homotopy equivalent; see [3] for example.

It is the purpose of this paper to give such a relationship for the Conway's third sporadic group Co_3. Section 2 provides some notation and reviews standard results which will be used in proofs. In Section 3 we describe in detail the 2-local maximal "parabolic" geometry Δ for Co_3. In Section 4 we describe the relevant elements of the Bouc complex $|\mathcal{B}_2|$ of Co_3 in group theoretic and geometric terms. In Section 5 we prove that the 2-local geometry Δ is G-homotopy equivalent with a subcomplex $|\hat{\mathcal{B}}_2|$ of the Bouc complex. In Section 6 we prove that the fixed point set Δ^z of a central involution z is contractible. Then, using a result of Thévenaz [15], we conclude that the reduced Lefschetz module $\tilde{L}(\Delta)$ is projective relative to the collection $\mathfrak{X} = \mathcal{B}_2 \setminus \hat{\mathcal{B}}_2$ of subgroups of Co_3.

2. Notations, terminology and standard results

2.1. Subgroup complexes

Let G be a finite group and p a fixed prime which divides the order of G. A p-subgroup R of G is called p-radical if $R = O_p(N_G(R))$. The p-subgroup R is p-centric if $Z(R)$ is a Sylow p-subgroup of $C_G(R)$.

A collection is a family of subgroups of G which is closed under G-conjugation and it is ordered by inclusion; hence it is a G-poset. The following two collections are standard in the literature:

$$\mathcal{A}_p(G) = \{E \mid E \text{ non-trivial elementary abelian } p\text{-subgroup of } G\},$$

$$\mathcal{B}_p(G) = \{R \mid R \text{ non-trivial } p\text{-radical subgroup of } G\}.$$

In what follows, if $\mathcal{C}$ is a collection of subgroups of G then $|\mathcal{C}|$ will denote the associated simplicial complex, with simplices equal to the flags or chains of subgroups in the poset $\mathcal{C}$. The complex $|\mathcal{A}_p(G)|$ is known as the Quillen complex and $|\mathcal{B}_p(G)|$ is known as the Bouc complex. The two complexes are G-homotopy equivalent.

Define the following subcollection of the Bouc collection:

$$\hat{\mathcal{B}}_p(G) = \{U \in \mathcal{B}_p(G) \mid Z(U) \cap Z(S) \neq 1, \text{ for some } S \in \mathrm{Syl}_p(G)\}.$$

We shall call $|\hat{\mathcal{B}}_p(G)|$ the *distinguished Bouc complex*.

Let $\mathcal{E}_p(G)$ denote the collection of non-trivial elementary abelian p-subgroups of G whose elements lie in the smallest set which contains elements of order p in the center of a Sylow p-subgroup, is closed under conjugation in G and is closed under taking products of commuting elements. This collection was introduced by Benson [2] in order to study the mod-2 cohomology of Co_3.

2.2. Homotopy techniques

Let us assume that Δ is a simplicial complex. We say that G acts admissibly on Δ if whenever $g \in G$ fixes a simplex $\sigma \in \Delta$ then g fixes every face of σ. If v is a vertex of Δ, then $\mathrm{Star}_\Delta(v)$ is the collection of simplices containing it, and the residue of v, denoted $\mathrm{Res}_\Delta(v)$ is the subcomplex of next lower dimension, obtained by deleting v from each simplex of $\mathrm{Star}_\Delta(v)$. Then $\mathrm{Star}_\Delta(v) = v * \mathrm{Res}_\Delta(v)$, the simplicial join.

For $\sigma \in \Delta$, let G_σ be the stabilizer of σ under this action and let U_σ denote the kernel of the action of G_σ on the residue of σ. The following is inspired from a standard property of the finite groups of Lie type acting on their buildings:

The Borel–Tits property (BT). *Let G be a group which acts admissibly on the simplicial complex Δ. For each non-trivial p-subgroup U of G, there exists a simplex σ of Δ such that $N_G(U) \leqslant G_\sigma$.*

Notation 2.1. For $P \leqslant G$, denote by Δ^P the subcomplex of Δ fixed by P.

The following two lemmas are standard results which will be used in the proofs. They are given here in the form used by Ryba, Smith and Yoshiara in [13].

Lemma 2.2. *(See [13, Lemma 2.1].) For a vertex v of a simplicial complex Δ, if the residue $Res_\Delta(v)$ is contractible, then Δ is homotopy equivalent to $\Delta \setminus \mathrm{Star}(v)$.*

Lemma 2.3. *(See [13, Section 2].) Let $\Sigma \in \Delta$ be a simplex of maximal dimension with σ as a face. Assume that Σ is the only simplex of maximal dimension with σ as a face. Then the process of removing Σ from Δ, by collapsing Σ down onto its faces other than σ, is a homotopy equivalence.*

3. The 2-local geometry Δ of Co_3

The sporadic simple group $G = Co_3$ has two classes of involutions [8], which we shall denote $2A$ (the central involutions) whose centralizer is $2{}^{\cdot}S_6(2)$, and $2B$ (the non-central involutions) whose centralizer is $2 \times M_{12}$; using the Atlas [7] notation. Note that central means that the elements are central in some Sylow 2-subgroup of G. Given any two commuting central involutions, their product is a central involution. The product of a central involution and a non-central involution (if an involution) is a non-central involution. The product of two commuting non-central involutions can be either central or non-central.

Let Δ denote the 2-local maximal "parabolic" geometry of G. This geometry was first mentioned in [12]; further details can be found in [3, Section 8.13]. This is a rank 3 geometry, whose objects will be denoted points $\mathcal{P}$, lines $\mathcal{L}$, and $\mathcal{M}$-spaces. The objects of this geometry correspond to pure central elementary abelian subgroups of G; this means that these subgroups contain central involutions only. The points correspond to rank one subgroups 2, lines to 2^2 and $\mathcal{M}$-spaces to 2^4. Incidence is given by containment. Δ can also be regarded as a simplicial complex of dimension two with three types of vertices.

Remark 3.1. There is also a class of pure central elementary abelian subgroups 2^3 in G, which do not correspond to objects in Δ; each subgroup 2^3 is contained in a unique subgroup 2^4; see [3, Section 8.13]. These 2^3 subgroups will be referred to as planes.

The group G acts (faithfully) flag-transitively on the geometry. The stabilizers of the three types of objects are:

$$\begin{aligned} &G_p \simeq 2{\cdot}S_6(2) && \text{for a point } p \in \mathcal{P}; \\ &G_L \simeq 2^{2+6}3(S_3 \times S_3) && \text{for a line } L \in \mathcal{L}; \\ &G_M \simeq 2^{4}{\cdot}L_4(2) && \text{for an } \mathcal{M}\text{-space } M. \end{aligned}$$

The flag stabilizers can be easily determined and they are:

$$\begin{aligned} &G_{pL} \simeq 2^{2+6}(S_3 \times S_3), && G_{pM} \simeq 2_+^{1+6}L_3(2), \\ &G_{LM} \simeq 2^{2+6}(S_3 \times S_3), && G_{pLM} \simeq [2^9].S_3 = [2^{10}3]. \end{aligned}$$

The geometry Δ is pure 2-local, in the sense of [14], that is, all the simplex stabilizers have nontrivial normal 2-subgroups. All vertex stabilizers are maximal 2-local subgroups. Δ has the diagram as shown in Fig. 1.

Notation 3.2. For $X \in \{\mathcal{P}, \mathcal{L}, \mathcal{M}\}$ and F a flag of Δ, we will denote by X_F the collection of all objects in X incident with F.

Notation 3.3. For $p \in \mathcal{P}$ let

$$p^{\perp} = \{q \in \mathcal{P} \mid q \text{ and } p \text{ are incident with some common line}\}.$$

In what follows it will be useful to regard Δ as a point-line geometry, that is, the lines and the $\mathcal{M}$-spaces are identified with the subsets of points they are incident with. We give below some of the properties of Δ:

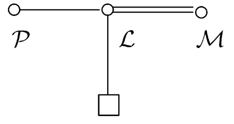

Fig. 1. Diagram for the 2-local geometry of Co_3.

($\Delta 1$) For $p \in \mathcal{P}$ the residue $\operatorname{Res}_\Delta p = (\mathcal{L}_p, \mathcal{M}_p)$ is the collection of all lines and $\mathcal{M}$-spaces containing p. Then $\operatorname{Res}_\Delta p$ is isomorphic with the geometry of isotropic lines and isotropic planes of a polar space of type C_3 over the field $\mathbb{F}_2$. Note there are 315 lines and 135 $\mathcal{M}$-spaces on a given point $p \in \mathcal{P}$.

($\Delta 2$) For $M \in \mathcal{M}$, $\operatorname{Res}_\Delta M = (\mathcal{P}_M, \mathcal{L}_M)$, the collection of points and lines lying in M is the truncation to points and lines of a projective space $PG(3,2)$. It is immediate that there are 15 points and 35 lines incident with M.

($\Delta 3$) For $L \in \mathcal{L}$, $\operatorname{Res}_\Delta L = (\mathcal{P}_L, \mathcal{M}_L)$ is a digon, a complete bipartite graph on the three points and the three $\mathcal{M}$-spaces incident with L.

($\Delta 4$) Given a point p and a line L with $p^\perp \cap L \neq \emptyset$, either $p^\perp \cap L$ is a single point or all of L.

($\Delta 5$) If $(p, M) \in \mathcal{P} \times \mathcal{M}$, either $p \in M$, or $p^\perp \cap M$ is at most a line.

($\Delta 6$) Two distinct $\mathcal{M}$-spaces intersect in at most one line.

The properties ($\Delta 1$)–($\Delta 3$) can be read from the diagram. For completeness we will provide arguments for ($\Delta 4$)–($\Delta 6$).

In order to prove ($\Delta 4$), consider $(p, L) \in \mathcal{P} \times \mathcal{L}$ with $p^\perp \cap L \neq \emptyset$. But $p^\perp \cap L$ cannot consist of two points only, because if the involution corresponding to the point p commutes with two of the involutions in the 2^2 subgroup corresponding to L, then it will commute with their product also. Thus $p^\perp \cap L$ can be a point or the entire line. The latter case occurs if p and L lie in the same $\mathcal{M}$-space.

Next we prove ($\Delta 5$). Let us assume that $p \notin M$. It is easy to see that $p^\perp \cap M$ cannot be the entire M, since there are no projective subspaces of rank 4 or greater in Δ; also, $p^\perp \cap M$ cannot be a plane, by Remark 3.1.

Finally, ($\Delta 6$) is a direct consequence of Remark 3.1.

Remark 3.4. The Benson collection $\mathcal{E}_2 = \mathcal{E}_2(G)$ contains four conjugacy classes of subgroups, of orders $2, 2^2, 2^3, 2^4$; see [2]. Let $\mathcal{E}_2^-$ be the subcollection of $\mathcal{E}_2$ with the subgroups of the form 2^3 removed. It is a direct consequence of Lemma 2.2 and Remark 3.1 that $|\mathcal{E}_2|$ and $|\mathcal{E}_2^-|$ are G-homotopy equivalent. Note that $|\mathcal{E}_2^-| = \Delta$. Simplices in Δ correspond to flags, or chains of subgroups in $\mathcal{E}_2^-$.

Remark 3.5. Let $\tilde{\Delta}$ denote the involution geometry of G, the point-line geometry whose points are the involutions of G. Two involutions are collinear if they commute. Then Δ can be identified with a (geometric) subspace of $\tilde{\Delta}$ whose points correspond to the central involutions, that is every exterior line of $\tilde{\Delta}$ intersects Δ at a point or the empty set. Therefore, we can regard Δ as a "central involution geometry."

4. A geometric description of the distinguished Bouc complex of Co_3

In this section $G = Co_3$ and Δ represents its 2-local geometry; further let $\mathcal{B}_2 = \mathcal{B}_2(Co_3)$ and $\hat{\mathcal{B}}_2 = \hat{\mathcal{B}}_2(Co_3)$. The Bouc complex of Co_3 was determined in [1], from which we reproduce Table 1.

For $v \in \{\mathcal{P}, \mathcal{L}, \mathcal{M}\}$, the notation R_v is suggested by the complex Δ, specifically $R_v = O_2(G_v)$. The subgroup R_L is not equal to a line L in Δ (the pure central subgroups 2^2 are not 2-radical), but contains what we call a "line structure" $\mathbb{L}$; see below for details. Recall that $\operatorname{Res}_\Delta v$, for $v \in \{\mathcal{P}, \mathcal{M}\}$ is a "truncation," obtained from a geometry of type A_3 or C_3 by ignoring one of the usual vertex types. We let $\square$ denote a vertex of that type *in the residue*; but note that

Table 1
Representatives for conjugacy classes of radical 2-subgroups of Co_3

Type	Name	R	$Z(R)$	$N_G(R)$
p	R_p	2	A^1	$2^{\cdot}S_6(2)$
–	R_2	2	B^1	$2 \times M_{12}$
–	R_3	2^2	B^3	$A_4 \times S_5$
–	R_4	2^3	B^7	$(2^3 \times S_3)F_7^3$
M	R_M	2^4	A^{15}	2^4A_8
$p\Box$	$R_{p\Box}$	2^{1+5}	A^1	$2^{1+5}S_6$
pM	R_{pM}	2_+^{1+6}	A^1	$2_+^{1+6}L_3(2)$
$M\Box$	$R_{M\Box}$	2^{3+4}	A^7	$2^{3+4}L_3(2)$
$\mathbb{L}$	R_L	2^{2+6}	A^3	$2^{2+6}3.(S_3 \times S_3)$
pML	R_{pML}	$2^4.2^{2+3}$	A^1	$2^4.2^{2+3}.S_3$
$ML\Box$	$R_{ML\Box}$	$2^4.2^{2+3}$	A^1	$2^4.2^{2+3}.S_3$
$pM\Box$	$R_{pM\Box}$	$2^4.2_+^{1+4}$	A^1	$2^4.2_+^{1+4}.S_3$
$pL\Box$	$R_{pL\Box}$	$[2^9]$	A^1	$[2^9].S_3$
$pML\Box$	$R_{pML\Box}$	$[2^{10}]$	A^1	$[2^{10}]$

$\Box$ does not correspond to a vertex in the *full* geometry Δ. In $\mathrm{Res}_\Delta\, p$, $\Box$ stands for a structure of 15 lines and 15 $\mathcal{M}$-spaces which form a generalized quadrangle $GQ(2,2)$. In $\mathrm{Res}_\Delta\, M$, $\Box$ can be identified with a plane of the projective space $PG(3,2)$. Groups involving $\Box$, such as $R_{p\Box}$ or $R_{M\Box}$ can be constructed as the inverse images under the quotient maps $2^{\cdot}S_6(2) \twoheadrightarrow S_6(2)$ or $2^{4\cdot}L_4(2) \twoheadrightarrow L_4(2)$ of unipotent radicals of corresponding parabolics. The groups R_{pM} and R_{pML} can be constructed in the same manner, or as $O_2(G_{pM})$ and $O_2(G_{pML})$.

Among the 2-radical subgroups of G all but the first four conjugacy classes of groups are 2-centric. Also it is easy to see that $\hat{\mathcal{B}}_2$ equals the collection obtained by removing from $\mathcal{B}_2$ the subgroups in the conjugacy classes of $\{R_2, R_3, R_4\}$.

As mentioned in the previous section, the standard 2-local geometry Δ can be regarded as a pure central involution geometry. In this section we will describe the subcomplex $|\hat{\mathcal{B}}_2|$ in more geometric terms, using the properties of the central involutions of G.

We start with a weaker version of the Borel–Tits property mentioned in Section 2:

$(\mathrm{BT})^c$ *For each non-trivial 2-subgroup U of G such that $Z(U)$ contains a central involution, there exists a simplex σ of Δ such that $N_G(U) \leqslant G_\sigma$.*

Proposition 4.1. *The pair (G, Δ) satisfies the property* $(\mathrm{BT})^c$.

Proof. Let $U \leqslant G$ be a non-trivial 2-subgroup such that $Z(U)$ contains a central involution. Let H denote the set of central involutions in $Z(U)$. Recall that H is closed under taking products. Thus H is an elementary abelian subgroup of U of rank at most 4. Then $N_G(U) \leqslant N_G(H)$. If $H \simeq 2^3$ then $N_G(H) \leqslant N_G(2^4)$. Otherwise $N_G(H) \in \{G_p, G_L, G_M\}$ and the conclusion follows. $\Box$

Remark 4.2. Note that the pair $(G, \hat{\mathcal{B}}_2)$ satisfies the property $(\mathrm{BT})^c$. To see this let U be a 2-subgroup of G with the property that $Z(U)$ contains a central involution. It is a direct consequence (of Proposition 4.1 and the fact that the stabilizers of vertices in Δ are normalizers of groups R which lie in $\hat{\mathcal{B}}_2$) that there exists $R \in \hat{\mathcal{B}}_2(G)$ with $N_G(U) \leqslant N_G(R)$.

In what follows we shall describe each of the subgroups of $\hat{\mathcal{B}}_2$ in geometric terms; this means that we describe the set of central involutions contained in each distinguished radical 2-subgroup. Recall a central involution is one lying in the center of a Sylow 2-subgroup of G.

Proposition 4.3. *The Sylow* 2*-subgroup* $R_{pML\square}$ *of* G *contains* 55 *central involutions. These points of the geometry* Δ *which are in* $R_{pML\square}$ *lie either in a collection of three* $\mathcal{M}$*-spaces on a common line* (*which we refer to as a "line structure"* $\mathbb{L}$*, consisting of* 39 *points*)*, or in a collection of* 31 *points on* 15 *lines, all containing a common cone point. These* 31 *points also form* 15 *planes, all containing the cone point, and these* 15 *lines and* 15 *planes can be thought of as forming a generalized quadrangle* $GQ(2,2)$*. The overlap* $\mathbb{L} \cap GQ(2,2)$ *consists of* 15 *points, forming* 3 *planes* (*one from each* $\mathcal{M}$*-space, and including their common line*) *and containing* 7 *of the* 15 *lines of* $GQ(2,2)$.

Proof. A computation using GAP [9] verified the existence of the 55 central involutions. The Sylow 2-subgroup $R_{pML\square}$ of G can be described as an extension $2^4.U_4$, where U_4 is the group of upper triangular matrices in $L_4(2)$. Clearly $M = 2^4$ contains 15 central involutions; we are looking for 40 others. Let $z \in R_{pML\square}$ be a central involution such that $z \notin M$, and denote by $\bar{z}$ the image of z in the quotient group U_4. It can be shown that all of the involutions in U_4 lie either in the elementary abelian subgroup

$$\left\{\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

of rank four, or in the extraspecial group

$$2_+^{1+4} = \left\{\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}.$$

Write $\bar{z} = I + N$ where I is the identity matrix and N is nilpotent. In fact, $N^2 = 0$ since $\bar{z}$ has order two.

The geometric condition that $p^\perp \cap M$ is at most a line, see property ($\Delta5$), Section 3, implies that the set of elements of M fixed by the action of the matrix $\bar{z}$ is at most 2^2. Thus $\bar{z}$ cannot be a transvection, which would fix a plane 2^3. Since $\bar{z}v = v$ if and only if $Nv = 0$, we must have $\mathrm{rank}(N) \neq 1$. There are precisely ten upper triangular 4×4 matrices satisfying $\mathrm{rank}(N) = 2$ and $N^2 = 0$, namely:

$$N_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_5 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_7 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_8 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_9 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_{10} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Denote by $\bar{z}_i$, $1 \leqslant i \leqslant 10$, the corresponding unipotent matrix $\bar{z}_i = I + N_i$ (this is an abuse of notation, since we have not yet defined the element z_i in $R_{pML\square}$). Note that $\bar{z}_3 = \bar{z}_1 \cdot \bar{z}_2 = \bar{z}_2 \cdot \bar{z}_1$, and we also have the commuting products $\bar{z}_6 = \bar{z}_4 \cdot \bar{z}_5$, $\bar{z}_9 = \bar{z}_1 \cdot \bar{z}_8 = \bar{z}_5 \cdot \bar{z}_7$ and $\bar{z}_{10} = \bar{z}_1 \cdot \bar{z}_7 = \bar{z}_5 \cdot \bar{z}_8$. Denote $M = 2^4 = \langle a_1, a_2, a_3, a_4 \rangle$ as well as $p = 2 = \langle a_1 \rangle$, $L = 2^2 = \langle a_1, a_2 \rangle$, and $\square = 2^3 = \langle a_1, a_2, a_3 \rangle$. We see that $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_6$ each fix pointwise the line $\langle a_1, a_2 \rangle$, $\bar{z}_7$ and $\bar{z}_8$ fix the line $\langle a_1, a_3 \rangle$, and $\bar{z}_9$ and $\bar{z}_{10}$ fix the line $\langle a_1, a_2 a_3 \rangle$.

If z and z', not in M, are two distinct central involutions in $R_{pML\square}$ with the same image $\bar{z} = \overline{z'}$ in U_4, then $z \cdot z' \in M$, implying $z \cdot z' \in z^\perp \cap M$. Since $z^\perp \cap M$ is at most one line containing 3 points, for each point z there are at most three other points z' with $\bar{z} = \overline{z'}$. We are counting the four elements in a coset $z \cdot 2^2$. Thus we have found at most $40 = 4 \times 10$ central involutions in $z \in R_{pML\square}$ with $z \notin M$. Combining this with the fact that $R_{pML\square}$ contains 55 central involutions (the aforementioned GAP computation) implies that we have described exactly 40 such central involutions not in M.

Let z_i, $1 \leqslant i \leqslant 10$, denote a representative central involution in $R_{pML\square}$ with image the matrix $\bar{z}_i \in U_4$.

Note that $M_1 = M = \langle a_1, a_2, a_3, a_4 \rangle$, $M_2 = \langle a_1, a_2, z_1, z_2 \rangle$, and $M_3 = \langle a_1, a_2, z_4, z_5 \rangle$ are three $\mathcal{M}$-spaces on the common line $L = \langle a_1, a_2 \rangle$. These form the line structure $\mathbb{L}$, containing 39 points.

Let us now describe the generalized quadrangle $GQ(2,2)$. The fifteen lines, each spanned by $p = \langle a_1 \rangle$ and one element of the following set $\{a_2, a_3, a_2a_3, z_1, a_2z_1, z_5, a_2z_5, z_7, a_3z_7, z_8, a_3z_8, z_9, a_2a_3z_9, z_{10}, a_2a_3z_{10}\}$, are the 15 lines on a common cone point p. The fifteen planes $\square_i$, $1 \leqslant i \leqslant 15$ of the generalized quadrangle $GQ(2,2)$ are spanned by $p = \langle a_1 \rangle$ and one of $\{\langle a_2, a_3 \rangle, \langle a_2, z_1 \rangle, \langle a_2, z_5 \rangle, \langle a_3, z_7 \rangle, \langle a_3, z_8 \rangle, \langle a_2a_3, z_9 \rangle, \langle a_2a_3, z_{10} \rangle, \langle z_1, z_7 \rangle, \langle z_1, z_8 \rangle, \langle z_5, z_7 \rangle, \langle z_5, z_8 \rangle, \langle a_2z_1, a_3z_7 \rangle, \langle a_2z_1, a_3z_8 \rangle, \langle a_2z_5, a_3z_7 \rangle, \langle a_2z_5, a_3z_8 \rangle\}$.

The overlap $\mathbb{L} \cap GQ(2,2)$ equals $\square_1 \cup \square_2 \cup \square_3$ where $\square_1 = \square = \langle a_1, a_2, a_3 \rangle$, $\square_2 = \langle a_1, a_2, z_1 \rangle$, and $\square_3 = \langle a_1, a_2, z_5 \rangle$. Note that $\square_i \subseteq M_i$ for $1 \leqslant i \leqslant 3$. Also $M_1, M_2, M_3, \square_4, \square_5, \ldots, \square_{15}$ are the maximal pure central elementary abelian subgroups of $R_{pML\square}$. □

We now describe the other 2-radical subgroups of G, from both geometric and group theoretic points of view.

(1) $R_p \simeq 2$, generated by a central involution (which the Atlas [7] lists as the conjugacy class $2A$), is a point of the geometry Δ.

(2) $R_M \simeq 2^4$, a pure central elementary abelian 2-group of rank four, is an $\mathcal{M}$-space of the geometry Δ. It contains 15 points, or central involutions, as well as 35 lines and 15 planes.

(3) $R_{p\Box} \simeq 2^{1+5}$ is defined as a subgroup of $G_p = 2^{\cdot}Sp_6(2)$ as the inverse image of the unipotent radical of the parabolic subgroup $2^5.S_6$ in the quotient group $Sp_6(2)$. Under the quotient map, a central involution of type $2A$ lying in $2^{\cdot}Sp_6(2)$ maps either to the identity or to an involution in the conjugacy class $2B$ in $Sp_6(2)$, using the Atlas [7] notation. A computation using GAP [9] shows that $R_{p\Box}$ contains 31 central involutions (there are 15 involutions of type $2B$ in $2^5 \subseteq Sp_6(2)$), and that these 31 points lie on the 15 lines through a cone point, the involution in the center $Z(R_{p\Box})$, and form the 15 planes of the generalized quadrangle $GQ(2,2)$. As a subgroup of a Sylow 2-subgroup $R_{pML\Box} = 2^4.U_4$, $R_{p\Box}$ is generated by $\langle a_1, a_2, a_3, z_1, z_5, z_7\rangle$, with image $\langle \bar{z}_1, \bar{z}_5, \bar{z}_7\rangle$ an elementary abelian subgroup of U_4 of rank three. Thus $R_{p\Box} \simeq 2^3.2^3 \subseteq 2^4.U_4$. $R_{p\Box}$ does not contain any $\mathcal{M}$-spaces. There are a total of 75 lines in $R_{p\Box}$, the 15 lines on the cone point and 60 others, contained in the 15 planes but not through the cone point.

(4) $R_{pM} \simeq 2_+^{1+6}$ can be defined as a subgroup of $G_p = 2^{\cdot}Sp_6(2)$ as the inverse image of the unipotent radical of the parabolic subgroup $2^6.L_3(2)$ in the quotient group $Sp_6(2)$. However, R_{pM} can also be defined as a subgroup of $G_M = 2^4.L_4(2)$ as the inverse image of the unipotent radical of a parabolic subgroup $2^3.L_3(2)$ in the quotient group $L_4(2)$. As a subgroup of a Sylow 2-subgroup $R_{pML\Box}$, $R_{pM} \simeq 2^4.2^3 \subseteq 2^4.U_4$, with quotient group $2^3 \subseteq U_4$ the group of matrices

$$\left\{\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}.$$

Thus R_{pM} contains only the 15 points of $M = \langle a_1, a_2, a_3, a_4\rangle$. Of course, R_M is a subgroup of R_{pM}. Clearly R_{pM} does not contain any subgroup conjugate to $R_{p\Box}$ since the latter contains 31 points.

(5) $R_{M\Box} \simeq 2^{3+4} \simeq 2^4.2^3 \subseteq 2^4.U_4$ can be defined as a subgroup of $G_M = 2^4.L_4(2)$ as the inverse image of the unipotent radical of a parabolic subgroup $2^3.L_3(2)$, stabilizing a plane, in the quotient group $L_4(2)$. The group $2^3 \subseteq U_4$ is the group of matrices

$$\left\{\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\},$$

and thus $R_{M\Box}$ contains only the 15 points (and the $\mathcal{M}$-space) of $M = \langle a_1, a_2, a_3, a_4\rangle$. Note $R_{M\Box}$ is not isomorphic to R_{pM} since their centers have different orders; $Z(R_{M\Box}) = 2^3 = \langle a_1, a_2, a_3\rangle$ and $Z(R_{pM}) = 2 = \langle a_1\rangle$. As above, $R_{M\Box}$ does not contain any conjugate of $R_{p\Box}$.

(6) $R_L \simeq 2^{2+6}$ equals $O_2(G_L) = O_2(N_G(2^2))$, and has center $Z(R_L)$ the line $L = 2^2 = \langle a_1, a_2\rangle$. Note that $G_L = 2^{2+6}3(S_3 \times S_3)$ with the first S_3 permuting the 3 points of L and the second S_3 permuting the three $\mathcal{M}$-spaces on L. Let M be any of the three $\mathcal{M}$-spaces on L; then $G_{LM} = G_L \cap G_M = 2^{2+6}3(S_3 \times 2)$, which is isomorphic to $2^{2+6}(S_3 \times S_3)$. Therefore $R_M \subseteq O_2(G_{LM}) = O_2(G_L) = R_L$. Thus R_L contains all three of the $\mathcal{M}$-spaces on L, which is the line structure $\mathbb{L}$ consisting of 39 points. Also, R_L is the subgroup of $G_M = 2^4.L_4(2)$ which equals the inverse image of the unipotent radical of the parabolic subgroup $2^4(S_3 \times S_3)$ in

$L_4(2)$, stabilizing a line L. As a subgroup of a Sylow 2-subgroup $R_{pML\Box}$, $R_L \simeq 2^4.2^4 \subseteq 2^4.U_4$, with quotient group $2^4 \subseteq U_4$ the group of matrices

$$\left\{\begin{pmatrix}1&0&*&*\\0&1&*&*\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}.$$

It follows that R_L contains precisely 39 central involutions. Also L is generated by its central involutions, $L = \langle a_1, a_2, a_3, a_4, z_1, z_2, z_4, z_5\rangle$. Further R_L does not contain any conjugate of $R_{p\Box}$, since if $R_{p\Box} \subseteq R_L$ then at least 11 of the 31 points of $R_{p\Box}$ would have to lie on one of the $\mathcal{M}$-spaces M of R_L. Since $11 > 7$, these points would span M. But then $M \subseteq R_{p\Box}$, a contradiction, since $R_{p\Box}$ does not contain any $\mathcal{M}$-spaces.

(7) $R_{pML} \simeq 2^4.2^{2+3} \subseteq 2^4.U_4$, with quotient group $2^{2+3} \subseteq U_4$ the group of matrices

$$\left\{\begin{pmatrix}1&*&*&*\\0&1&*&*\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}.$$

R_{pML} has precisely 39 points and three $\mathcal{M}$-spaces of the line structure $\mathbb{L}$ contained in $R_L \subseteq R_{pML}$.

(8) $R_{ML\Box} \simeq 2^4.2^{2+3} \subseteq 2^4.U_4$, with quotient group $2^{2+3} \subseteq U_4$ the group of matrices

$$\left\{\begin{pmatrix}1&0&*&*\\0&1&*&*\\0&0&1&*\\0&0&0&1\end{pmatrix}\right\}.$$

$R_{ML\Box}$ contains precisely the 39 points and the three $\mathcal{M}$-spaces of the line structure $\mathbb{L}$ in $R_L \subseteq R_{pML}$.

(9) $R_{pM\Box} = 2^4.2_+^{1+4} \subseteq 2^4.U_4$ with quotient group $2_+^{1+4} \subseteq U_4$ the group of matrices

$$\left\{\begin{pmatrix}1&*&*&*\\0&1&0&*\\0&0&1&*\\0&0&0&1\end{pmatrix}\right\}.$$

This implies that $R_{pM\Box}$ contains 39 central involutions, with 31 points in $R_{p\Box} \subseteq R_{pM\Box}$ and 15 points in $R_M \subseteq R_{pM}$. These two sets intersect in a set of 7 points, the plane $\Box = \langle a_1, a_2, a_3\rangle \subseteq M$.

(10) $R_{pL\Box} = 2^4.2^{2+3} \subseteq 2^4.U_4$ contains all 55 points of a Sylow 2-subgroup $R_{pML\Box}$ since $R_{p\Box} \subseteq R_{pL\Box}$ and $R_L \subseteq R_{pL\Box}$. Using $z_7 \in R_{pL\Box}$ and $R_L \subseteq R_{pL\Box}$, it follows that the quotient group $2^{2+3} \subseteq U_4$ equals the group of matrices

$$\left\{\begin{pmatrix}1&\epsilon&*&*\\0&1&*&*\\0&0&1&\epsilon\\0&0&0&1\end{pmatrix}\right\},$$

the group of upper triangular 4×4 unipotent matrices (x_{ij}) satisfying $x_{12} = x_{34}$.

(11) $R_{pML\square} = 2^4.U_4$ contains a unique conjugate of each of the groups $R_{pL\square}$, $R_{pM\square}$, R_L and $R_{p\square}$. This can be seen for the groups $R_{pL\square}$, R_L, and $R_{p\square}$ by using their geometric structures, involving the line structure $\mathbb{L}$ and the generalized quadrangle $GQ(2,2)$, and the fact that these three groups are generated by their central involutions. For $R_{pM\square}$, a simple counting argument applies. If $R_{pM\square} \subseteq R_{pML\square}$ then $R_{pML\square} \subseteq N_G(R_{pM\square}) \simeq R_{pM\square}.S_3$, and the latter group contains three Sylow 2-subgroups.

5. The homotopy relation

In this section we prove that the 2-local geometry Δ is G-homotopy equivalent to the distinguished Bouc complex $|\hat{\mathcal{B}}_2|$ of $G = Co_3$. The complex $|\hat{\mathcal{B}}_2|$ can be equivariantly retracted to a subcomplex Δ_1 formed from a subposet of $\hat{\mathcal{B}}_2$. Also Δ_1 can be retracted to a subcomplex Δ_0 which is homeomorphic to Δ.

Let Δ_1 denote the simplicial complex corresponding to the subposet of $\mathcal{B}_2$ with the three types of objects $\{R_p, R_M, R_L\}$. Let Δ_0 denote the subcomplex of Δ_1 obtained by removing those simplices corresponding to the chains $R_p \subseteq R_L$ and $R_p \subseteq R_M \subseteq R_L$ where the subgroup R_p is not normal in the group R_L.

Proposition 5.1.

(a) *Δ_0 is homeomorphic to Δ.*
(b) *Δ_0 is G-homotopy equivalent to Δ_1.*

Proof. (a) The homeomorphism from Δ to Δ_0 is induced by the correspondence $p \to R_p$ (recall $p = R_p = 2$), $L \to R_L$ (recall $R_L = 2^{2+6} = O_2(N_G(L))$ with $L = 2^2 = Z(R_L)$), and $M \to R_M$ (recall $M = R_M = 2^4$). Next, we show that simplices in Δ correspond to simplices in Δ_0. Clearly $p \subseteq M$ corresponds to $R_p \subseteq R_M$. Also, $R_p \trianglelefteq R_L$ if and only if $R_p \subseteq Z(R_L)$ if and only if $p \subseteq L$. Finally, $L \subseteq M$ if and only if $R_M \subseteq R_L$, and $p \subseteq L \subseteq M$ corresponds to $R_p \subseteq R_M \subseteq R_L$ with $R_p \trianglelefteq R_L$.

(b) We use Lemma 2.3 to remove those simplices in Δ_1, not lying in Δ_0, corresponding to chains $R_p \subseteq R_L$ and $R_p \subseteq R_M \subseteq R_L$ where the subgroup R_p is not normal in R_L, so that $R_p \nsubseteq Z(R_L) = L$. Of course R_p does commute with elements of the center $Z(R_L)$, and they generate a group $2^3 = \langle R_p, Z(R_L)\rangle$. This 2^3 lies in a unique $2^4 = M$, an $\mathcal{M}$-space satisfying $L \subseteq M$. Therefore the chain $R_p \subseteq R_L$ is a face of a unique chain $R_p \subseteq R_M \subseteq R_L$, and Lemma 2.3 allows to cancel these two simplices. □

In this section we will prove the main theorem of the paper. The proof of this theorem will be the focus of all of this Section 5.

Theorem 1. *The two complexes Δ and $|\hat{\mathcal{B}}_2|$ are G-homotopy equivalent.*

We will reduce $|\hat{\mathcal{B}}_2|$ down to the subcomplex Δ_1 in a number of steps; at each step we will perform a homotopy retraction, which will use one of the results given in Lemmas 2.2 or 2.3. We first apply Lemma 2.2 to retract $|\hat{\mathcal{B}}_2|$ to a subcomplex $|\hat{\mathcal{B}}_2^I|$ by removing all simplices (flags, or chains of subgroups) which involve groups of type R_{pM} or $R_{M\square}$. Then Lemma 2.2 will allow us to retract $|\hat{\mathcal{B}}_2^I|$ to a subcomplex $|\hat{\mathcal{B}}_2^{II}|$ by removing all simplices which involve groups of type R_{pML} or $R_{ML\square}$.

Table 2
Orbits of flags in $|\hat{\mathcal{B}}_2^{II}|$ which do not lie in Δ_1

Rank 1 flags		
$p\square$	$2^{10}\cdot 3^2\cdot 5$	1
$pM\square$	$2^{10}\cdot 3$	1
$pL\square$	$2^{10}\cdot 3$	1
$pML\square$	2^{10}	1
Rank 2 flags		
$p\subseteq p\square$	$2^{10}\cdot 3^2\cdot 5$	1, 30
$p\subseteq pM\square$	$2^{10}\cdot 3$	1, 6, 8, 24
$p\subseteq pL\square$	$2^{10}\cdot 3$	1, 2, 12, 16, 24
$p\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16, 16
$M\subseteq pM\square$	$2^{10}\cdot 3$	1
$M\subseteq pL\square$	$2^{10}\cdot 3$	3
$M\subseteq pML\square$	2^{10}	1, 2
$\mathbb{L}\subseteq pL\square$	$2^{10}\cdot 3$	1
$\mathbb{L}\subseteq pML\square$	2^{10}	1
$p\square\subseteq pM\square$	$2^{10}\cdot 3$	1
$p\square\subseteq pL\square$	$2^{10}\cdot 3$	1
$p\square\subseteq pML\square$	2^{10}	1
$pM\square\subseteq pML\square$	2^{10}	1
$pL\square\subseteq pML\square$	2^{10}	1
Rank 3 flags		
$p\subseteq M\subseteq pM\square$	$2^{10}\cdot 3$	1, 6, 8
$p\subseteq M\subseteq pL\square$	$2^{10}\cdot 3$	1, 2, 12, 24
$p\subseteq M\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16
$p\subseteq \mathbb{L}\subseteq pL\square$	$2^{10}\cdot 3$	1, 2, 12, 24
$p\subseteq \mathbb{L}\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16
$p\subseteq p\square\subseteq pL\square$	$2^{10}\cdot 3$	1, 2, 12, 16
$p\subseteq p\square\subseteq pM\square$	$2^{10}\cdot 3$	1, 6, 24
$p\subseteq p\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 16
$p\subseteq pM\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16
$p\subseteq pL\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16, 16
$M\subseteq \mathbb{L}\subseteq pL\square$	$2^{10}\cdot 3$	3
$M\subseteq \mathbb{L}\subseteq pML\square$	2^{10}	1, 2
$M\subseteq pM\square\subseteq pML\square$	2^{10}	1
$M\subseteq pL\square\subseteq pML\square$	2^{10}	1, 2
$\mathbb{L}\subseteq pL\square\subseteq pML\square$	2^{10}	1
$p\square\subseteq pL\square\subseteq pML\square$	2^{10}	1
$p\square\subseteq pM\square\subseteq pML\square$	2^{10}	1
Rank 4 flags		
$p\subseteq M\subseteq \mathbb{L}\subseteq pL\square$	$2^{10}\cdot 3$	1, 2, 12, 24
$p\subseteq M\subseteq \mathbb{L}\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16
$p\subseteq M\subseteq pM\square\subseteq pML\square$	2^{10}	1, 2, 4, 8
$p\subseteq M\subseteq pL\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16
$p\subseteq \mathbb{L}\subseteq pL\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16
$p\subseteq p\square\subseteq pM\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 16
$p\subseteq p\square\subseteq pL\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 16
$M\subseteq \mathbb{L}\subseteq pL\square\subseteq pML\square$	2^{10}	1, 2
Rank 5 flags		
$p\subseteq M\subseteq \mathbb{L}\subseteq pL\square\subseteq pML\square$	2^{10}	1, 2, 4, 8, 8, 16

The main part of the proof involves the use of Lemma 2.3 to reduce $|\hat{\mathcal{B}}_2^{II}|$ down to a subcomplex Δ_2 (in a sequence of 35 steps), which in turn can easily be retracted to Δ_1. We will give in Table 2 a list of all orbits of flags that lie in $|\hat{\mathcal{B}}_2^{II}|$ but not in Δ_1, and we proceed to cancel all but 44 of them to arrive at Δ_2. We then apply Lemma 2.3 to remove these 44 flags as well.

Proposition 5.2. *$|\hat{\mathcal{B}}_2|$ is G-homotopy equivalent to the subcomplex $|\hat{\mathcal{B}}_2^I|$, where $\hat{\mathcal{B}}_2^I$ is the subposet of $\hat{\mathcal{B}}_2$ obtained by removing the groups of the form R_{pM} and $R_{M\square}$.*

Proof. There exists a unique R_M in R_{pM}, and there are exactly 15 central involutions in R_{pM}, all contained in R_M. It follows that any group R in $\hat{\mathcal{B}}_2$ which is incident with R_{pM} (either $R\subseteq R_{pM}$ or $R_{pM}\subseteq R$) is also incident with R_M. In the simplicial complex $|\hat{\mathcal{B}}_2|$, the residue of the vertex R_{pM} is a cone on the vertex R_M, so is contractible. According to Lemma 2.2, we can remove the vertices of the form R_{pM} together with their stars.

Similarly, there exist a unique R_M in $R_{M\square}$, containing all 15 central involutions in $R_{M\square}$. Any group R in $\hat{\mathcal{B}}_2$ which is incident with $R_{M\square}$ is also incident with R_M. In $|\hat{\mathcal{B}}_2|$, the residue of the vertex $R_{M\Box}$ is contractible, a cone on the vertex R_M. Another application of Lemma 2.2 gives the result. □

Proposition 5.3. *$|\hat{\mathcal{B}}_2^I|$ is G-homotopy equivalent to a subcomplex $|\hat{\mathcal{B}}_2^{II}|$, where $\hat{\mathcal{B}}_2^{II}$ is the subposet of $\hat{\mathcal{B}}_2^I$ obtained by removing the groups of the form R_{pML} and $R_{ML\Box}$.*

Proof. There exists a unique R_L in R_{pML} which contains all 39 central involutions in R_{pML}. In the poset $\hat{\mathcal{B}}_2^I$, having already removed the groups of type R_{pM}, any group R in $\hat{\mathcal{B}}_2^I$ which is incident with R_{pML} is also incident with R_L. In the simplicial complex $|\hat{\mathcal{B}}_2^I|$, the residue of the vertex R_{pML} is contractible, a cone on R_L. Similarly, there is a unique R_L in $R_{ML\Box}$ which contains the 39 central involutions of R_{pML}. Since we have removed the groups of type $R_{M\Box}$, any group R in $\hat{\mathcal{B}}_2^I$ which is incident with $R_{ML\Box}$ is also incident with R_L. In $|\hat{\mathcal{B}}_2^I|$, the residue of $R_{ML\Box}$ is a cone on R_L. Therefore two applications of Lemma 2.2 prove the proposition. □

Let us remark on the importance of doing these retractions in a particular order. We must remove the groups of type R_{pM} before removing the groups of type R_{pML}, since in the original poset $\hat{\mathcal{B}}_2$, R_{pM} is incident with R_{pML} but not with R_L.

5.1. Flags in $|\hat{\mathcal{B}}_2^{II}|$

Let F: $x_1 \subseteq x_2 \subseteq \cdots \subseteq x_n$ be a flag in $|\hat{\mathcal{B}}_2^{II}|$. Each group x_i will have a type which is a subset of $\{p, M, L, \Box\}$, but we sometimes abbreviate this by saying x_i has "type i." The normalizer of the group x_n acts on the collection of groups of type i contained in x_n; in many cases this action is not transitive. In the first column of Table 2 we give the type of a flag F, for those orbits of flags in $|\hat{\mathcal{B}}_2^{II}|$ but not in Δ_1 (so that each flag contains a group x_i whose type involves □). In the second column of Table 2 we give the order of the normalizer of the group x_n of F, and in the third column we give the number of elements in the orbits of the action of $N_G(x_n)$ on the collection of groups of type 1 contained in x_n.

To prove Theorem 1, we must remove all of these flags from $|\hat{\mathcal{B}}_2^{II}|$ to obtain Δ_1, using Lemma 2.3. In a sequence of 35 steps, we first remove the flags occurring in any orbit after the first orbit listed in the third column, obtaining a subcomplex Δ_2. The remaining 44 orbits of flags in Δ_2 but not in Δ_1 can easily be removed using Lemma 2.3, where each flag Σ with $x_1 = R_p$ is free over a flag σ, the face of Σ with σ not containing the group of type p.

Recall that a maximal simplex Σ is *free* over some maximal face σ, if Σ is the only maximal simplex with σ as a face. In this case we can use Lemma 2.3 to collapse Σ down onto its faces other than σ. This means that we can remove Σ and σ from the complex to obtain a homotopy equivalent subcomplex. Note that Σ is not necessarily a simplex of maximal dimension; it has to be a maximal simplex, in the sense that it is not a face of a larger simplex in the complex.

Since this method depends on the maximality of Σ and since the removal of some simplices might make other simplices maximal, the sequence of homotopy retractions we are performing has to be done in the order indicated here.

The 35 steps mentioned above are given in Tables 3 and 4 below. In the first four steps given in Table 3, we can remove all flags of the stated type. In the next 31 steps given in Table 4, we remove only some of the orbits at each stage.

We begin by homotopically retracting the pairs of simplices listed in Table 3.

Table 3
The first four steps of the homotopy retraction from $|\hat{\mathcal{B}}_2^{II}|$ to Δ_2

Step	Σ	σ
1.	$p \subseteq M \subseteq \mathbb{L} \subseteq pL\Box \subseteq pML\Box$	$p \subseteq M \subseteq \mathbb{L} \subseteq pML\Box$
2.	$M \subseteq \mathbb{L} \subseteq pL\Box \subseteq pML\Box$	$M \subseteq \mathbb{L} \subseteq pML\Box$
3.	$p \subseteq \mathbb{L} \subseteq pL\Box \subseteq pML\Box$	$p \subseteq \mathbb{L} \subseteq pML\Box$
4.	$\mathbb{L} \subseteq pL\Box \subseteq pML\Box$	$\mathbb{L} \subseteq pML\Box$

Table 4
The remaining 31 steps of the homotopy retraction from $|\hat{\mathcal{B}}_2^{II}|$ to Δ_2

Step	Σ	σ	Points in σ
5.	(8) $p \subseteq$ (3) $M \subseteq pL\Box \subseteq pML\Box$	(24) $p \subseteq pL\Box \subseteq pML\Box$	$pL\Box \setminus p\Box$
6.	(16) $p \subseteq p\Box \subseteq pL\Box \subseteq pML\Box$	(16) $p \subseteq pL\Box \subseteq pML\Box$	$p\Box \setminus \mathbb{L}$
7.	(24) $p \subseteq p\Box \subseteq pM\Box \subseteq pML\Box$	(24) $p \subseteq pM\Box \subseteq pML\Box$	$p\Box \setminus M_1$
8.	(8) $p \subseteq M \subseteq pM\Box \subseteq pML\Box$	(8) $p \subseteq pM\Box \subseteq pML\Box$	$M_1 \setminus p\Box$
9.	(12) $p \subseteq$ (3) $M \subseteq \mathbb{L} \subseteq pL\Box$	(36) $p \subseteq \mathbb{L} \subseteq pL\Box$	$\mathbb{L} \setminus L$
10.	(8) $p \subseteq$ (3) $M \subseteq pL\Box$	(24) $p \subseteq pL\Box$	$pL\Box \setminus p\Box$
11.	(16) $p \subseteq p\Box \subseteq pL\Box$	(16) $p \subseteq pL\Box$	$p\Box \setminus \mathbb{L}$
12.	(24) $p \subseteq p\Box \subseteq pM\Box$	(24) $p \subseteq pM\Box$	$p\Box \setminus M_1$
13.	(8) $p \subseteq M \subseteq pM\Box$	(8) $p \subseteq pM\Box$	$M_1 \setminus p\Box$
14.	(8) $p \subseteq p\Box \subseteq pL\Box \subseteq pML\Box$	(8) $p \subseteq p\Box \subseteq pML\Box$	$(\Box_2 \cup \Box_3) \setminus L$
15.	(4) $p \subseteq$ (2) $M \subseteq pL\Box \subseteq pML\Box$	(8) $p \subseteq pL\Box \subseteq pML\Box$	$(\Box_2 \cup \Box_3) \setminus L$
16.	(16) $p \subseteq p\Box \subseteq pML\Box$	(16) $p \subseteq pML\Box$	$p\Box \setminus \mathbb{L}$
17.	(8) $p \subseteq$ (3) $M \subseteq pML\Box$	(24) $p \subseteq pML\Box$	$\mathbb{L} \setminus p\Box$
18.	(4) $p \subseteq$ (2) $M \subseteq pML\Box$	(8) $p \subseteq pML\Box$	$(\Box_2 \cup \Box_3) \setminus L$
19.	(3) $p \subseteq$ (2) $M \subseteq pL\Box \subseteq pM_1L\Box$ $M \in \{M_2, M_3\}$	(3) $p \subseteq$ (2) $M \subseteq pML\Box$	L
20.	(4) $p \subseteq p\Box \subseteq pL\Box \subseteq pM_iL\Box$ $i \in \{1, 2, 3\}$	(12) $p \subseteq p\Box \subseteq pL\Box$	$(\Box_1 \cup \Box_2 \cup \Box_3) \setminus L$
21.	(4) $p \subseteq M \subseteq pL\Box \subseteq pML\Box$	(4) $p \subseteq pL\Box \subseteq pML\Box$	$\Box_1 \setminus L$
22.	(4) $p \subseteq p\Box \subseteq pM\Box \subseteq pML\Box$	(4) $p \subseteq p\Box \subseteq pML\Box$	$\Box_1 \setminus L$
23.	(4) $p \subseteq M \subseteq pM\Box \subseteq pML\Box$	(4) $p \subseteq pM\Box \subseteq pML\Box$	$\Box_1 \setminus L$
24.	(2) $p \subseteq M \subseteq pM\Box \subseteq pML_i\Box$ $i \in \{1, 2, 3\}$	(6) $p \subseteq M \subseteq pM\Box$	$\Box_1 \setminus p_1$
25.	(2) $p \subseteq p\Box \subseteq pM\Box \subseteq pML_i\Box$ $i \in \{1, 2, 3\}$	(6) $p \subseteq p\Box \subseteq pM\Box$	$\Box_1 \setminus p_1$
26.	(2) $p \subseteq M_1 \subseteq pL\Box \subseteq pM_1L\Box$	(2) $p \subseteq M_1 \subseteq pM_1L\Box$	$L \setminus p_1$
27.	(2) $p \subseteq p\Box \subseteq pL\Box \subseteq pML\Box$	(2) $p \subseteq p\Box \subseteq pML\Box$	$L \setminus p_1$
28.	(2) $p \subseteq$ (3) $M \subseteq \mathbb{L} \subseteq pL\Box$	(2) $p \subseteq$ (3) $M \subseteq pL\Box$	$L \setminus p_1$
29.	(4) $p \subseteq$ (3) $M \subseteq pL\Box$	(12) $p \subseteq pL\Box$	$(\Box_1 \cup \Box_2 \cup \Box_3) \setminus L$
30.	(2) $p \subseteq pM\Box \subseteq pML_i\Box$ $i \in \{1, 2, 3\}$	(6) $p \subseteq pM\Box$	$\Box_1 \setminus p_1$
31.	(4) $p \subseteq M \subseteq pML\Box$	(4) $p \subseteq pML\Box$	$\Box_1 \setminus L$
32.	(2) $p \subseteq pL\Box \subseteq pML\Box$	(2) $p \subseteq pML\Box$	$L \setminus p_1$
33.	(2)$M \subseteq pL\Box \subseteq pM_1L\Box$ $M \in \{M_2, M_3\}$	(2) $M \subseteq pML\Box$ $M \in \{M_2, M_3\}$	
34.	(2) $p \subseteq p\Box \subseteq pL_i\Box$ $i \in \{1, \dots 15\}$	(30) $p \subseteq p\Box$	$p\Box \setminus p_1$
35.	(2) $p \subseteq \mathbb{L} \subseteq pL\Box$	(2) $p \subseteq pL\Box$	$L \setminus p_1$

We will use the notation: $(k_1)\ x_1 \subseteq \cdots \subseteq (k_{n-1})x_{n-1} \subseteq x_n$ to denote $k_1 \cdot k_2 \cdot \cdots \cdot k_{n-1}$ simplices of the form $x_1 \subseteq \cdots \subseteq x_n$, for which there are k_i choices for the vertices of type $i \in \{1, \ldots, n-1\}$. This notation is needed to record the sizes of the orbits being removed in each step of Table 4.

We will give a detailed description of only one of these steps below. In this description, as well as in the tables, we abbreviate a flag by using the types for its groups. Since $R_L \neq L$, we abbreviate R_L as the line structure $\mathbb{L}$.

Step 5: We will discuss this step in detail since the approach we use here will be repeated. Consider a simplex Σ: $p \subseteq M \subseteq pL\Box \subseteq pML\Box$. We consider here the points $p \in pL\Box \setminus p\Box$. There are 8×3 such points, 8 points in each of the three $\mathcal{M}$-spaces of $pML\Box$. This means that for a fixed $pL\Box \subseteq pML\Box$, there are 24 simplices Σ of the type chosen above. We write this as $(8)p \subseteq (3)M \subseteq pL\Box \subseteq pML\Box$. Fix one such simplex Σ. This is a simplex of maximal dimension and free over the face σ: $p \subseteq pL\Box \subseteq pML\Box$. To see this note that $p \notin L$. Thus $\langle p, L\rangle$ is a plane on L and thus lies in a unique $\mathcal{M}$-space. Note the simplex $p \subseteq \mathbb{L} \subseteq pL\Box \subseteq pML\Box$ was removed in Step 3. Thus Σ is free over σ. Finally we can apply Lemma 2.3 and remove these two simplices.

The remaining steps are listed in Table 4. In the first column we number the steps. In the second column we give the maximal simplex Σ and in the third column we give a face σ of Σ such that Σ is free over σ. In the fourth column we specify the collection of points in σ being removed. We use the notation described above.

After these steps we are left with the complex Δ_2. There are still 44 orbits of flags remaining in $\Delta_2 \setminus \Delta_1$, all of these corresponding to the first orbit listed in the third column of Table 2. But these can easily be canceled in pairs, with a flag Σ having $x_1 = R_p$ of type p being free over its face σ which does not involve the group of type p. Recall that by Proposition 5.1, Δ_1 is homotopy equivalent to Δ_0 which, in its turn, is homeomorphic to Δ. This ends the proof of Theorem 1.

6. On the Lefschetz module

Let $G = Co_3$ and let Δ be the 2-local geometry of G described in Section 3.

Proposition 6.1. *Let z be a central involution in G. The set Δ^z is contractible.*

Proof. Recall that $C_G(z) = G_p$ for the point $p = \langle z\rangle$. It follows that $p * \mathrm{Res}(p) = \mathrm{Star}(p) \subseteq \Delta^z$. It is easy to see that a point $q \in \Delta^z$ if and only if $q \in p^\perp$. The main idea of the proof is to use Lemma 2.2 to homotopically remove those lines and $\mathcal{M}$-spaces from Δ^z that do not contain p, as well as those points in $p^\perp$.

It follows from (Δ4), Section 3, that if a line L is an element of Δ^z then p is collinear with one or all the points of L. Thus there are three types of lines we have to consider: lines on p, already included in $\mathrm{Res}(p)$; lines with only one point fixed, the other two points being interchanged; lines with all three points fixed. By (Δ5), given a point, $\mathcal{M}$-space pair (p, M), the set $p^\perp \cap M$ is at most one line or $p \in M$. This implies that an $\mathcal{M}$-space $M \in \Delta^z$ can have one point in Δ^z, three points forming a line in Δ^z or $p \in M$. This is easily seen if we recall that M contains 15 points, thus under the action of z, at least one point is fixed.

Let us consider the case when the line L does not contain p and has only one point q fixed by the action of z. Then the collection $\mathrm{Res}_{\Delta^z}(L) = \{q\} * \{\text{some } \mathcal{M}\text{-spaces on } L\}$ is a cone on q, and thus contractible. It follows that we can remove all such lines from Δ^z, yielding a homotopy equivalent subcomplex $\Delta_1^z \subseteq \Delta^z$.

Let $M \in \Delta_1^z$ be such that $p^\perp \cap M = L$, a line: this means all three points of L are in Δ_1^z. Then $\mathrm{Res}_{\Delta_1^z}(M)$ consists of L and the three points on L, which is a cone on L and thus contractible. The other lines of M are either not stabilized by z or were removed at the previous step, so are not in Δ_1^z. We can remove all such $\mathcal{M}$-spaces from Δ_1^z, yielding a homotopy equivalent complex $\Delta_2^z \subseteq \Delta_1^z \subseteq \Delta^z$.

Let us now consider a line L whose three points are all fixed by the action of z. Then p and L generate a plane $\langle p, L\rangle$ which lies in a unique $\mathcal{M}$-space M. The other $\mathcal{M}$-spaces from Δ^z through L were deleted at the previous step. Thus $\mathrm{Res}_{\Delta_2^z} L = \{M\} * \{3 \text{ points on } L\}$, again a cone on M and thus contractible. Remove these lines, yielding $\Delta_3^z \subseteq \Delta_2^z$. Now all the lines of Δ^z which are not on p have been removed.

Let $M \in \mathcal{M}$ be such that $p^\perp \cap M = \{q\}$, a single point. Then $\mathrm{Res}_{\Delta_3^z} M$ is the point q itself. Remove these $\mathcal{M}$-spaces, yielding $\Delta_4^z \subseteq \Delta_3^z$.

Next, consider the collection of points collinear with p and look at their residues in Δ_4^z. Let $q \in p^\perp$. Since we have already contracted all the lines on q which do not contain p, the residue $\mathrm{Res}_{\Delta_4^z}(q)$ contains a unique line $\langle p, q\rangle$ and the three $\mathcal{M}$-spaces on that line, therefore this residue is contractible. So we can remove these points to retract Δ_4^z to $\mathrm{Star}(p)$. Of course, $\mathrm{Star}(p)$ is contractible, a cone on p. □

In what follows we will use the following results due to Thévenaz; see [15], Theorem 2.1 and Corollary 2.3. A group B is called cyclic mod p if the quotient group $B/O_p(B)$ is cyclic.

Theorem 6.2. *(See [15].)*

(a) *Let $\mathfrak{X}$ be a class of subgroups of A which is closed under subconjugacy. Let Δ be an admissible A-complex such that the reduced Euler characteristic $\tilde{\chi}(\Delta^B) = 0$ for every subgroup B which is cyclic* mod *p and satisfies $O_p(B) \notin \mathfrak{X}$. Then the reduced Lefschetz module $\tilde{L}(\Delta)$ is a $\mathbb{F}_p A$-virtual module projective relative to the collection $\mathfrak{X}$.*

(b) *Let $p^n = |A|_p$ be the p-part of A, and let p^k be the highest power of p dividing the order of some subgroup belonging to $\mathfrak{X}$. Then $\tilde{\chi}(\Delta)$ is a multiple of p^{n-k}.*

We let $\mathfrak{X} = \mathcal{B}_2 \setminus \hat{\mathcal{B}}_2$, the family of the radical 2-subgroups of $G = Co_3$ which are pure non-central; these are subgroups in the conjugacy classes of $\{R_2, R_3, R_4\}$. It is clear that $\mathfrak{X}$ is closed under subconjugacy.

Remark 6.3. The subgroups Q in $G = Co_3$ which are cyclic mod 2 and such that $Z(O_2(Q))$ contains no central involutions have $O_2(Q)$ in $\mathfrak{X}$. To see this, let Q be a cyclic mod 2 subgroup of G; set $R = O_2(Q)$ and $H = \Omega_1 Z(R)$. Thus R is contained in a member of $\mathrm{Syl}_2(N_G(H))$. Let us assume H contains no central involutions; then H is of the form 2, 2^2 or 2^3; see [8, Section 5]. The normalizers of R_2, R_3 and R_4 are given in Table 1, at the beginning of Section 4. It is easy to see from Table 1, that the Sylow 2-subgroup of $N_G(H)$ has the form $H \times K$ with K some 2-subgroup. If $H \leqslant R \leqslant H \times K$ then $R = H \times K_1$ for some subgroup $K_1 \leqslant K$. But $Z(R) = H \times Z(K_1)$, where $Z(K_1)$ is non-trivial if K_1 is non-trivial. Thus $\Omega_1 Z(R) = H \times \Omega_1 Z(K_1) = H$. It follows that $\Omega_1 Z(K_1) = 1$ and also $Z(K_1) = 1$ which, in its turn implies $K_1 = 1$. Finally $R = H$.

Furthermore, G acts admissibly on Δ and $\tilde{\chi}(\Delta^z) = 0$, for every central involution $z \in G$, according to Proposition 6.1. Now, by P.A. Smith theory, it follows that $\tilde{\chi}(\Delta^B) = 0$ also, for every subgroup B which is cyclic mod 2 and such that $O_2(B) \notin \mathfrak{X}$. Thus, all the necessary conditions required by Theorem 6.2 are in place and we can formulate the following:

Theorem 2. *Let $G = Co_3$ and let $\mathfrak{X} = \mathcal{B}_2 \setminus \hat{\mathcal{B}}_2$. Let Δ denote the 2-local geometry of G. The reduced Lefschetz module $\tilde{L}(\Delta)$ is a virtual $\mathbb{F}_2G$-module projective relative to the collection $\mathfrak{X}$. Furthermore $\tilde{\chi}(\Delta)$ is a multiple of 2^7.*

Remark 6.4. Note that $|G|_2 = 2^{10}$ and that the 2-part of $\tilde{\chi}(\Delta)$ is 2^7. Also, the largest subgroup in $\mathfrak{X}$ has order 2^3. Thévenaz' result tells us that the reduced Euler characteristic is a multiple of 2^7, without giving an upper bound.

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