Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 419 (2006) 586–600
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Multiplicative Jordan triple isomorphisms on the self-adjoint elements of von Neumann algebras ☆

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Received 21 April 2006; accepted 1 June 2006
Available online 4 August 2006
Submitted by P. Šemrl

Abstract

In this paper we consider multiplicative Jordan triple isomorphisms between the sets of self-adjoint elements (respectively the sets of positive elements) of von Neumann algebras. These transformations are the bijective maps which satisfy the equality

$$\phi(ABA) = \phi(A)\phi(B)\phi(A)$$

on their domains. We show that all those transformations originate from linear *-algebra isomorphisms and linear *-algebra antiisomorphisms in the case when the underlying von Neumann algebras do not have commutative direct summands. An application of our results concerning non-linear maps which preserve the absolute value of products is also presented.
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AMS classification: Primary: 47B49

Keywords: Multiplicative Jordan triple isomorphism; von Neumann algebra; Self-adjoint operator; Positive operator

1. Introduction and statements of the results

The additivity of multiplicative or multiplicative-like maps on rings and algebras have been studied by a number of mathematicians. The first result in this direction of research is due to

☆ The author was supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant Nos. T043080, T046203 and by a joint Hungarian–Slovene grant (Reg. No. SLO-5/05).

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doi:10.1016/j.laa.2006.06.007

Martindale who presented the surprising result in [20] that every semigroup isomorphism from a prime ring containing a nontrivial idempotent onto an arbitrary ring is automatically additive. Similar problems for operator algebras were treated, for example, in the papers [8,15,31] of Hakeda, Lu and Šemrl. Analogous questions concerning maps on operator algebras which are multiplicative with respect to other products such as the Jordan product $AB + BA$ or the Jordan triple product ABA were investigated in the papers [9,10,13,14,16–18,22,23] by Hakeda and Saitô, Li and Jing, Ling and Lu, and the present author. These products play rather important role in certain parts of ring theory and also in mathematical physics.

The problem that we treat here is closely related to the results presented in the papers [22,25]. To formulate those results we need the following notation. Let $\mathscr{A}$ be a (unital) C^*-algebra. The set of all self-adjoint elements of $\mathscr{A}$ is denoted by $\mathscr{A}_s$. The elements a of $\mathscr{A}_s$ which satisfy $0 \leqslant a \leqslant 1$ are called the effects in $\mathscr{A}$ and they form the set $E(\mathscr{A})$. Both of the structures $\mathscr{A}_s$ and $E(\mathscr{A})$ have serious applications in the mathematical formulations of quantum mechanics. In the case when $\mathscr{A}$ is the algebra $B(H)$ of all bounded linear operators on the Hilbert space H, the elements of $\mathscr{A}_s$ represent observable physical quantities while the effects in $B(H)$ (called Hilbert space effects) represent yes–no measurements which may be unsharp.

The most common multiplicative-like operation that can be defined both on $\mathscr{A}_s$ and on $E(\mathscr{A})$ is the Jordan triple product $(A, B) \mapsto ABA$. We remark that in the case of effects this operation is modified a little to the so-called sequential product $(A, B) \mapsto A^{\frac{1}{2}}BA^{\frac{1}{2}}$ which appears in the quantum theory of measurements [5–7]. In our paper [22] we determined the multiplicative Jordan triple automorphisms of $\mathscr{A}_s$ and $E(\mathscr{A})$ in the case when $\mathscr{A} = B(H)$. It turned out that these transformations originate from the linear *-algebra automorphisms and the linear *-algebra antiautomorphisms of $B(H)$. Using our results, the form of all sequential automorphisms of the set of all Hilbert space effects was obtained in [5]. In [25] we extended this result to the far more general case of von Neumann algebra effects. The main aim of the present paper is to give a similar generalization concerning the multiplicative Jordan triple automorphisms on the self-adjoint elements of von Neumann algebras.

Our key result which follows describes the multiplicative Jordan triple isomorphisms between the positive elements of von Neumann algebras. If $\mathscr{A}$ is a C^*-algebra, then $\mathscr{A}_+$ denotes the set of all positive elements in $\mathscr{A}$.

Theorem 1. *Let $\mathscr{A}$, $\mathscr{B}$ be von Neumann algebras. Suppose that $\mathscr{A}$ does not have a commutative direct summand. Let $\phi : \mathscr{A}_+ \to \mathscr{B}_+$ be a bijective map which satisfies*

$$\phi(ABA) = \phi(A)\phi(B)\phi(A) \quad (A, B \in \mathscr{A}_+).$$

Then we have direct decompositions

$$\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2 \quad \text{and} \quad \mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2$$

in the category of von Neumann algebras and bijective maps

$$\Phi_1 : \mathscr{A}_1 \to \mathscr{B}_1 \quad \text{and} \quad \Phi_2 : \mathscr{A}_2 \to \mathscr{B}_2$$

*such that Φ_1 is a linear *-algebra isomorphism, Φ_2 is a linear *-algebra antiisomorphism and*

$$\phi = \Phi_1 \oplus \Phi_2$$

holds on $\mathscr{A}_+$.

Using this result we can prove the following structural result concerning the multiplicative Jordan triple isomorphisms between the sets of all self-adjoint elements of von Neumann algebras.

Theorem 2. *Let $\mathscr{A}$, $\mathscr{B}$ be von Neumann algebras. Suppose that $\mathscr{A}$ does not have a commutative direct summand. Let $\phi : \mathscr{A}_s \to \mathscr{B}_s$ be a bijective map which satisfies*

$$\phi(ABA) = \phi(A)\phi(B)\phi(A) \quad (A, B \in \mathscr{A}_s).$$

Then we have direct decompositions

$$\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \mathscr{A}_3 \oplus \mathscr{A}_4 \quad \text{and} \quad \mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2 \oplus \mathscr{B}_3 \oplus \mathscr{B}_4$$

and bijective maps

$$\Phi_1 : \mathscr{A}_1 \to \mathscr{B}_1, \quad \Phi_2 : \mathscr{A}_2 \to \mathscr{B}_2, \quad \Phi_3 : \mathscr{A}_3 \to \mathscr{B}_3, \quad \Phi_4 : \mathscr{A}_4 \to \mathscr{B}_4$$

*such that Φ_1, Φ_2 are linear *-algebra isomorphisms, Φ_3, Φ_4 are linear *-algebra antiisomorphisms and*

$$\phi = \Phi_1 \oplus (-\Phi_2) \oplus \Phi_3 \oplus (-\Phi_4)$$

holds on $\mathscr{A}_s$.

We remark that in [13] Li and Jing obtained a Martindale-type result concerning the automatic additivity of Jordan triple maps in the setting of general rings. Namely, for such maps from a 2-torsion free prime ring containing a nontrivial idempotent onto an arbitrary ring. In fact, they considered more general transformations which are called Jordan elementary maps. To compare their results to ours we note the following. Firstly, von Neumann algebras are generally non-prime (in fact, we have primness exactly for factors). Secondly, our transformation above is defined only on self-adjoint elements. We refer to our recent paper [27] where we have considered Jordan elementary maps on the self-adjoint elements of $B(H)$ to see how much this restriction complicates the solution of the problem.

In the last part of our paper we apply Theorem 1 to obtain a description of bijective maps on von Neumann algebras which preserve the absolute value of products. In [28], Radjabalipour considered additive maps from a von Neumann algebra into $B(H)$ which preserve the absolute value. Below we present a result where we do not assume additivity, instead we suppose a certain mixture of preserving products and absolute values. Preserver problems of similar kinds have been recently investigated in several papers. Here we refer to the articles [2,3,11,24,29,30].

Theorem 3. *Let $\mathscr{A}$, $\mathscr{B}$ be von Neumann algebras. Suppose that $\mathscr{A} \neq \mathbb{C}I$ is a factor. Let $\phi : \mathscr{A} \to \mathscr{B}$ be a bijective map which satisfies*

$$\phi(|AB|) = |\phi(A)\phi(B)| \quad (A, B \in \mathscr{A}). \tag{1}$$

Then ϕ is of the form

$$\phi(A) = \tau(A)\Phi(A) \quad (A \in \mathscr{A}),$$

*where $\Phi : \mathscr{A} \to \mathscr{B}$ is either a linear or a conjugate-linear *-algebra isomorphism and $\tau : \mathscr{A} \to \mathbb{C}$ is a scalar function of modulus 1.*

Observe that in the formulations of our results above we have excluded the case that $\mathscr{A}$ contains a commutative direct summand. It is easy to see that this assumption is really necessary to obtain the conclusions. Indeed, consider a "singular" multiplicative bijection of the set of all positive real numbers. For example, pick a discontinuous bijective additive function $a : \mathbb{R} \to \mathbb{R}$ (consider $\mathbb{R}$ as a linear space over the rationals) and set $m(x) = \exp(a(\ln(x)))$, $x > 0$. Next define $\phi : \mathbb{C} \to \mathbb{C}$ by

$$\phi(z) = \begin{cases} \frac{z}{|z|} m(|z|), & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$$

One can easily verify that this map provides counterexamples concerning all of the above three results.

2. Proofs

The basic idea of the proof of Theorem 1 is quite the same as that of the proof of Theorem 1 in [25] which describes the form of sequential isomorphisms between the sets of von Neumann algebra effects. However, beyond several small deviations there is a point where the difference is really substantial. In order to make the paper easily readable, we have decided to present a rather complete proof, not to repeat such phrases as "the proof goes like in … but with some necessary changes".

So, in what follows let $\mathscr{A}$, $\mathscr{B}$ and ϕ be as in Theorem 1. That is, suppose that $\mathscr{A}$, $\mathscr{B}$ are von Neumann algebras, $\mathscr{A}$ does not have a commutative direct summand and $\phi : \mathscr{A}_+ \to \mathscr{B}_+$ is a bijective map which satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A) \quad (A, B \in \mathscr{A}_+).$$

Like in the case of [25, Theorem 1], the proof of Theorem 1 below is carried out in a series of lemmas and other propositions.

If $\mathscr{C}$ is an arbitrary C^*-algebra, denote $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ $(A, B \in \mathscr{C}_+)$.

Lemma 4. *We have $\phi(I) = I$ and $\phi(0) = 0$. Moreover, $\phi : \mathscr{A}_+ \to \mathscr{B}_+$ is a bijective map which satisfies*

$$\phi(A \circ B) = \phi(A) \circ \phi(B) \quad (A, B \in \mathscr{A}_+).$$

Proof. We first prove that $\phi(I) = I$ and $\phi(0) = 0$. In fact, for any $B \in \mathscr{A}_+$ we have

$$\phi(B) = \phi(IBI) = \phi(I)\phi(B)\phi(I).$$

Choosing $B \in \mathscr{A}_+$ such that $\phi(B) = I$, it follows that $I = \phi(I)^2$. Taking square root, we obtain $\phi(I) = I$. Similarly, for every $B \in \mathscr{A}_+$ we have

$$\phi(0) = \phi(0B0) = \phi(0)\phi(B)\phi(0).$$

Choosing $B \in \mathscr{A}_+$ such that $\phi(B) = 0$, we get $\phi(0) = 0$.

From the equality

$$\phi(A^2) = \phi(AIA) = \phi(A)\phi(I)\phi(A) = \phi(A)^2,$$

we infer that $\phi(A^{\frac{1}{2}}) = \phi(A)^{\frac{1}{2}}$ for every $A \in \mathscr{A}_+$. It follows that

$$\phi(A \circ B) = \phi(A^{\frac{1}{2}} B A^{\frac{1}{2}}) = \phi(A)^{\frac{1}{2}} \phi(B) \phi(A)^{\frac{1}{2}} = \phi(A) \circ \phi(B)$$

holds for all $A, B \in \mathscr{A}_+$. $\square$

If $\mathscr{C}$ is a C^*-algebra, then $P(\mathscr{C})$ denotes the set of all projections in $\mathscr{C}$.

Lemma 5. *The restriction of ϕ onto $P(\mathscr{A})$ is a bijective map from $P(\mathscr{A})$ onto $P(\mathscr{B})$ which preserves the order and the orthogonality in both directions and hence it is completely orthoadditive.*

Proof. It is trivial that for an arbitrary element $A \in \mathscr{A}_+$ we have

$$A \text{ is a projection} \iff A \circ A = A.$$

By Lemma 4 this implies that ϕ preserves the projections in both directions.

Let $P, Q \in P(\mathscr{A})$. Clearly, we have

$$P \leqslant Q \Longleftrightarrow Q \circ P = P.$$

This implies that ϕ preserves the order among projections in both directions. As for orthogonality, we have

$$PQ = 0 \Longleftrightarrow Q \circ P = 0$$

and this clearly yields that ϕ preserves the orthogonality between projections in both directions.

Now, let (P_α) be an arbitrary collection of mutually orthogonal projections in $\mathscr{A}$ with sum P. Using the above verified properties of ϕ, we deduce that $(\phi(P_\alpha))$ is a collection of mutually orthogonal projections and we have $\phi(P_\alpha) \leqslant \phi(P)$. This implies that

$$\sum_\alpha \phi(P_\alpha) \leqslant \phi(P) = \phi\left(\sum_\alpha P_\alpha\right).$$

Since ϕ^{-1} has similar properties as ϕ, it follows that

$$P = \sum_\alpha \phi^{-1}(\phi(P_\alpha)) \leqslant \phi^{-1}\left(\sum_\alpha \phi(P_\alpha)\right).$$

By the order preserving property of ϕ we obtain that

$$\phi(P) \leqslant \sum_\alpha \phi(P_\alpha).$$

Therefore, we have

$$\sum_\alpha \phi(P_\alpha) = \phi(P),$$

which means that ϕ is completely orthoadditive on $P(\mathscr{A})$. $\square$

Lemma 6. *Let $Z, Z' \in \mathscr{A}_+$ be central elements in $\mathscr{A}$ and let $P, P' \in \mathscr{A}_+$ be mutually orthogonal projections. Then we have*

$$\phi(ZP + Z'P') = \phi(ZP) + \phi(Z'P').$$

Proof. Clearly $ZP + Z'P' = PZP + P'Z'P' \in \mathscr{A}_+$. Next, since $ZP + Z'P'$ commutes with P, P', $P + P'$, the same holds true for its square root. Hence, using Lemma 5 on the orthoadditivity of ϕ on $P(\mathscr{A})$, we compute

$$\begin{aligned}
\phi(ZP + Z'P') &= \phi((ZP + Z'P')^{\frac{1}{2}}(P + P')(ZP + Z'P')^{\frac{1}{2}}) \\
&= \phi(ZP + Z'P')^{\frac{1}{2}}\phi(P + P')\phi(ZP + Z'P')^{\frac{1}{2}} \\
&= \phi(ZP + Z'P')^{\frac{1}{2}}(\phi(P) + \phi(P'))\phi(ZP + Z'P')^{\frac{1}{2}} \\
&= \phi(ZP + Z'P')^{\frac{1}{2}}\phi(P)\phi(ZP + Z'P')^{\frac{1}{2}} \\
&\quad + \phi(ZP + Z'P')^{\frac{1}{2}}\phi(P')\phi(ZP + Z'P')^{\frac{1}{2}} \\
&= \phi((ZP + Z'P')^{\frac{1}{2}}P(ZP + Z'P')^{\frac{1}{2}}) \\
&\quad + \phi((ZP + Z'P')^{\frac{1}{2}}P'(ZP + Z'P')^{\frac{1}{2}}) \\
&= \phi(ZP) + \phi(Z'P').
\end{aligned}$$

The proof is complete. $\square$

If $\mathscr{C}$ is a C^*-algebra, then $\mathscr{Z}(\mathscr{C})$ denotes its center.

Lemma 7. *The map ϕ preserves commutativity in both directions. Therefore, we have $\phi(\mathscr{Z}(\mathscr{A})_+) = \mathscr{Z}(\mathscr{B})_+$.*

Proof. By the result [7, Theorem 2] of Gudder and Nagy, for arbitrary self-adjoint Hilbert space operators A, B we have $AB^2A = BA^2B$ if and only if $BA^2 = A^2B$ and $AB^2 = B^2A$.

This observation has the following immediate corollary. For any $A, B \in \mathscr{A}_+$ we have $A \circ B = B \circ A$ if and only if $AB = BA$. Since, by Lemma 4, ϕ is an isomorphism with respect to the operation $\circ$ of the sequential product, the above characterization of commutativity implies that ϕ preserves commutativity in both directions. $\square$

We say that the collection (E_{ij}) of operators in the von Neumann algebra $\mathscr{A}$ forms a self-adjoint system of $n \times n$ matrix units if n is any cardinal number, the index set in which i, j run has cardinality n, for every i, j, k, l we have

$$E_{ij}E_{kl} = \begin{cases} 0, & j \neq k, \\ E_{il}, & j = k, \end{cases}$$

$\sum_i E_{ii} = I$ in the strong operator topology and $E_{ij}^* = E_{ji}$ holds for all i, j.

Lemma 8. *Suppose that $\mathscr{A}$ has a self-adjoint system of $n \times n$ matrix units for some cardinal number $n \geqslant 2$. Then the restriction of ϕ to $\mathscr{Z}(\mathscr{A})_+$ is a bijective additive map onto $\mathscr{Z}(\mathscr{B})_+$.*

Proof. From Lemma 7 we know that the restriction of ϕ to $\mathscr{Z}(\mathscr{A})_+$ is a bijective map onto $\mathscr{Z}(\mathscr{B})_+$ which preserve the operation $\circ$.

Let (E_{ij}) be a self-adjoint system of $n \times n$ matrix units in $\mathscr{A}$. Let $Z, Z' \in \mathscr{Z}(\mathscr{A})_+$. For temporary use, fix the matrix units $E = E_{ii}$, $V = E_{ij}$, $i \neq j$. Define

$$P = \frac{1}{2}(E+V)^*(E+V), \qquad P' = \frac{1}{2}(E-V)^*(E-V).$$

It is easy to check that P, P' are mutually orthogonal projections and we have

$$E(2ZP)E = Z(E2PE) = ZE, \qquad E(2Z'P')E = Z'(E2P'E) = Z'E. \tag{2}$$

Since ϕ preserves commutativity in both directions, $\phi(Z), \phi(Z'), \phi(Z+Z') \in \mathscr{Z}(\mathscr{B})_+$. Using Lemma 6 and Eq. (2) we can compute

$$\begin{aligned}
\phi(Z+Z')\phi(E) &= \phi(Z+Z')^{\frac{1}{2}}\phi(E)\phi(Z+Z')^{\frac{1}{2}} \\
&= \phi((Z+Z')E) \\
&= \phi(E(2ZP+2Z'P')E) \\
&= \phi(E)\phi(2ZP+2Z'P')\phi(E) \\
&= \phi(E)(\phi(2ZP)+\phi(2Z'P'))\phi(E) \\
&= \phi(E)\phi(2ZP)\phi(E)+\phi(E)\phi(2Z'P')\phi(E) \\
&= \phi(E2ZPE)+\phi(E2Z'P'E) \\
&= \phi(ZE)+\phi(Z'E) \\
&= \phi(Z)\phi(E)+\phi(Z')\phi(E) \\
&= (\phi(Z)+\phi(Z'))\phi(E).
\end{aligned}$$

Therefore, we obtain that

$$\phi(Z + Z')\phi(E_{ii}) = (\phi(Z) + \phi(Z'))\phi(E_{ii}) \tag{3}$$

holds for every i. We know that $\sum_i E_{ii} = I$ and by the complete orthoadditivity of ϕ (see Lemma 5) this yields that $\sum_i \phi(E_{ii}) = I$. Consequently, we deduce from (3) that

$$\phi(Z + Z') = \phi(Z) + \phi(Z')$$

holds for all $Z, Z' \in \mathscr{Z}(\mathscr{A})_+$. $\square$

Lemma 9. *Suppose that $\mathscr{C}$ is a von Neumann algebra, $P \in \mathscr{C}$ is an abelian projection and $A \in \mathscr{C}_+$. Then there is an element $Z \in \mathscr{Z}(\mathscr{C})_+$ such that $PAP = ZP$.*

Proof. This is Lemma 9 in [25]. $\square$

Lemma 10. *Suppose that $\mathscr{C}$ is a von Neumann algebra of type I_n with $n < \infty$. Let $A \in \mathscr{C}$ be such that $PAP = 0$ holds for every abelian projection $P \in \mathscr{C}$. Then we have $A = 0$.*

Proof. This is Lemma 10 in [25]. $\square$

Lemma 11. *The map ϕ preserves the abelian projections and the equivalence among them in both directions.*

Proof. Clearly, $P \in \mathscr{A}$ is abelian if and only if the set $P\mathscr{A}_+P$ is commutative. As ϕ preserves commutativity in both directions (see Lemma 7), the set $P\mathscr{A}_+P$ is commutative if and only if $\phi(P\mathscr{A}_+P) = \phi(P)\phi(\mathscr{A}_+)\phi(P) = \phi(P)\mathscr{B}_+\phi(P)$ is commutative. This shows that P is abelian if and only if $\phi(P)$ is abelian.

As for the preservation of the equivalence between abelian projections, we recall that two abelian projections are equivalent if and only if their central carriers coincide (see [12, Proposition 6.2.8 and Proposition 6.4.6]). But the notion of the central carrier is expressed by order and commutativity between projections both of them being preserved by ϕ in both directions. This implies the second assertion of the lemma. $\square$

Proposition 12. *Suppose that $\mathscr{A}, \mathscr{B}$ are type I_n algebras with $2 \leqslant n < \infty$. Then ϕ is additive.*

Proof. Let $A, A' \in \mathscr{A}_+$. Pick an arbitrary abelian projection $P \in \mathscr{A}$. By Lemma 9 we have $Z, Z' \in \mathscr{Z}(\mathscr{A})_+$ such that

$$PAP = ZP, \qquad PA'P = Z'P. \tag{4}$$

It is well-known that for $n \geqslant 2$, any type I_n algebra has a self-adjoint system of $n \times n$ matrix units (see [12, Lemma 6.6.3]). Therefore, Lemma 8 can be applied and using (4) we can compute

$$\begin{aligned}
\phi(P)\phi(A + A')\phi(P) &= \phi(P(A + A')P) = \phi(ZP + Z'P)\\
&= \phi(P)\phi(Z + Z')\phi(P)\\
&= \phi(P)(\phi(Z) + \phi(Z'))\phi(P)\\
&= \phi(P)\phi(Z)\phi(P) + \phi(P)\phi(Z')\phi(P)\\
&= \phi(PZP) + \phi(PZ'P) = \phi(PAP) + \phi(PA'P)\\
&= \phi(P)(\phi(A) + \phi(A'))\phi(P).
\end{aligned}$$

Consequently, we obtain that

$$\phi(P)\phi(A+A')\phi(P) = \phi(P)(\phi(A)+\phi(A'))\phi(P)$$

holds for every abelian projection $P \in \mathscr{A}$. By Lemmas 11 and 10 this implies that

$$\phi(A+A') = \phi(A)+\phi(A')$$

proving the additivity of ϕ. □

We have now arrived at the point where the proof of Theorem 1 differs significantly from the proof of [25, Theorem 1]. In fact, to go to the next step we need the following result on the structure of multiplicative bijective maps between the positive parts of commutative C^*-algebras, i.e., between the positive parts of algebras of continuous functions on compact Hausdorff spaces. In the paper [25] we treated maps on effects and in that situation our job was much easier. Namely, we could use a rather simple result describing the form of all bijective multiplicative maps between the spaces of continuous functions with range in the unit interval that preserve a nontrivial constant function (see Proposition 4 in [25]). In the present situation we have to apply a deeper tool.

There is an important result of Milgram [21] describing the structure of all multiplicative bijections between the algebras of all real-valued continuous functions on compact Hausdorff spaces. The following statement is an adaptation of that result for the present situation, i.e. for multiplicative maps on nonnegative functions.

Theorem 13. *Let X, Y be compact Hausdorff spaces. Let $\psi : C(X)_+ \to C(Y)_+$ be a bijective multiplicative map. Then there is a homeomorphism $\varphi : Y \to X$, a continuous strictly positive valued function $p : Y \to \mathbb{R}$, an at most finite set $\{y_1, \ldots, y_n\}$ of isolated points of Y and a corresponding set $\{\tau_1, \ldots, \tau_n\}$ of multiplicative semigroup automorphisms of $\mathbb{R}_+$ such that ψ is of the following form*

$$\psi(f)(y) = \begin{cases} f(\varphi(y))^{p(y)}, & \text{if } y \in Y \setminus \{y_1, \ldots, y_n\}, \\ \tau_i(f(\varphi(y_i))), & \text{if } y = y_i. \end{cases}$$

Proof. The proof of Milgram presented in [21] works without any change. □

For a similar result on the structure of bijective multiplicative maps between the spaces of continuous functions on compact Hausdorff spaces mapping into the unit interval we refer to the recent paper [19] of Marovt.

We use the above statement to verify the following step of the proof of Theorem 1.

Proposition 14. *Suppose that the algebras $\mathscr{A}$, $\mathscr{B}$ are both of type I_n with $2 < n$ or that $\mathscr{A}$, $\mathscr{B}$ have no type I direct summands. Then ϕ is additive.*

Proof. We know from Lemma 5 that ϕ, when restricted onto the set of all projections in $\mathscr{A}$, is orthoadditive. We can apply the deep result of Bunce and Wright [1] stating that every bounded orthoadditive map from the set of all projections of a von Neumann algebra without type I_2 direct summand into a Banach space can be extended to a continuous linear transformation defined on the whole algebra. Therefore, we have a continuous linear transformation $L : \mathscr{A} \to \mathscr{B}$ such that

$$L(P) = \phi(P) \quad (P \in P(\mathscr{A})).$$

We know that $\phi : \mathscr{A}_+ \to \mathscr{B}_+$ is a bijective map which preserves commutativity in both directions. Therefore, we obtain that

$$\phi(\mathscr{A}_+ \cap (\mathscr{A}_+ \cap A')') = \mathscr{B}_+ \cap (\mathscr{B}_+ \cap \phi(A)')',$$

where $'$ stands for the usual commutant.

It is not difficult to verify that

$$\mathscr{A}_+ \cap (\mathscr{A}_+ \cap A')' = (\mathscr{A} \cap (\mathscr{A} \cap A')')_+.$$

Clearly, $\mathscr{A} \cap (\mathscr{A} \cap A')'$ is a von Neumann algebra which contains A. Moreover, it is commutative. In fact, if $X, Y \in \mathscr{A} \cap (\mathscr{A} \cap A')'$, then since $A \in \mathscr{A} \cap A'$, it follows that $YA = AY$ and hence $Y \in \mathscr{A} \cap A'$. This gives us that $XY = YX$. We remark that in general $\mathscr{A} \cap (\mathscr{A} \cap A')'$ is larger than the commutative von Neumann algebra A'' generated by A in the full operator algebra.

From the above argument we obtain that ϕ is a multiplicative bijective map from $(\mathscr{A} \cap (\mathscr{A} \cap A')')_+$ onto $(\mathscr{B} \cap (\mathscr{B} \cap \phi(A)')')_+$. As $(\mathscr{A} \cap (\mathscr{A} \cap A')')_+$ and $(\mathscr{B} \cap (\mathscr{B} \cap \phi(A)')')_+$ are the positive parts of commutative von Neumann algebras, by the Gelfand theory of commutative C^*-algebras we have compact Hausdorff spaces X and Y such that $(\mathscr{A} \cap (\mathscr{A} \cap A')')_+$ and $(\mathscr{B} \cap (\mathscr{B} \cap \phi(A)')')_+$ are isometrically isomorphic to $C(X)_+$ and $C(Y)_+$, respectively. Here an isometric isomorphism means an additive, positive homogeneous, multiplicative and isometric bijective map. Clearly, our map ϕ induces a multiplicative bijective map $\psi : C(X)_+ \to C(Y)_+$. By Lemma 8, ϕ is additive on the central elements of $\mathscr{A}_+$ (notice that by [12, Lemmas 6.5.6 and 6.6.4], $\mathscr{A}$ has a self-adjoint system of $n \times n$ matrix units with $n \geqslant 2$). The positive scalar multiplies of the identity are clearly central in $\mathscr{A}_+$. Therefore, we obtain that the multiplicative bijective map $\psi : C(X)_+ \to C(Y)_+$ has the additional property that it is additive on the constant functions.

Now apply Theorem 13. We see that the semigroups automorphisms τ_i of $\mathbb{R}_+$ are additive. It requires only elementary arguments to verify that we necessarily have that τ_i equals the identity on $\mathbb{R}_+$. At the same time, using the above mentioned restricted additivity property of ψ again, we also get that the positive function p is necessarily identically 1. Therefore, we obtain that ψ is of the form

$$\psi(f) = f \circ \varphi \quad (f \in C(X)_+).$$

Going back to our original transformation, we obtain that ϕ is additive, positive homogeneous and isometric on $(\mathscr{A} \cap (\mathscr{A} \cap A')')_+$.

Pick $A \in \mathscr{A}_+$. The range of the spectral measure of A belongs to A'' and hence also to $(\mathscr{A} \cap (\mathscr{A} \cap A')')_+$. By the above verified properties of ϕ on this set we compute

$$\phi\left(\sum_i \lambda_i P_i\right) = \sum_i \lambda_i \phi(P_i) = \sum_i \lambda_i L(P_i) = L\left(\sum_i \lambda_i P_i\right),$$

where the P_i's are spectral projections of A and the λ_i's are nonnegative real numbers. Using continuity we deduce that $L(A) = \phi(A)$ for all $A \in \mathscr{A}_+$. This verifies the additivity of ϕ on the whole set $\mathscr{A}_+$. $\square$

The next proposition is simply a variant of a well-known result of Kadison (see [12, Exercise 10.5.32]) stating that every unital bijective linear map between C^*-algebras which preserves the order in both directions is a Jordan *-isomorphism (i.e., a bijective linear map which respects the operations of taking adjoints and squares).

Proposition 15. *Let $\psi : \mathscr{A}_+ \to \mathscr{B}_+$ be an additive bijective map which satisfies $\psi(I) = I$. Then ψ can be extended to a Jordan *-isomorphism between the algebras $\mathscr{A}$ and $\mathscr{B}$. Moreover, we have direct decompositions*

$$\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2 \quad \text{and} \quad \mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2$$

and bijective linear maps

$$\Psi_1 : \mathscr{A}_1 \to \mathscr{B}_1 \quad \text{and} \quad \Psi_2 : \mathscr{A}_2 \to \mathscr{B}_2$$

*such that Ψ_1 is a linear *-algebra isomorphism, Ψ_2 is a linear *-algebra antiisomorphism and*

$$\psi = \Psi_1 \oplus \Psi_2$$

holds on $\mathscr{A}_+$.

Proof. The proof that we sketch below goes in a way very similar to the proof of Proposition 3 in [25]. First, one proves that ψ preserves the order in both directions. Using this property and the rational linearity of ψ (a trivial consequence of the additivity of ψ), it follows that ψ is positive homogeneous. Next, one can extend ψ from $\mathscr{A}_+$ onto $\mathscr{A}_s$ to a linear bijection and then onto the whole algebra $\mathscr{A}$ to a bijective linear map Ψ from $\mathscr{A}$ onto $\mathscr{B}$ which preserves the order among self-adjoint elements in both directions. As $\psi(I) = I$, it follows that the extension Ψ is unital. Then one can refer to the above mentioned result of Kadison to show that $\Psi : \mathscr{A} \to \mathscr{B}$ is a Jordan *-isomorphism. It is known that every Jordan *-isomorphism between von Neumann algebras induces a direct sum decomposition of the underlying algebras according to which the Jordan *-isomorphism under consideration splits into the direct sum of a *-algebra isomorphism and a *-algebra antiisomorphism (see [12, Exercise 10.5.26]). This completes the proof. $\square$

Now, we are in a position to prove the first theorem of the present paper.

Proof of Theorem 1. Let $P \in \mathscr{A}$ be a nonzero projection. Then $\phi(P)$ is also a nonzero projection and the restriction of ϕ onto $P\mathscr{A}_+P = (P\mathscr{A}P)_+$ gives rise to a multiplicative Jordan triple isomorphism from $(P\mathscr{A}P)_+$ onto $(\phi(P)\mathscr{B}\phi(P))_+$. By the properties of multiplicative Jordan triple isomorphisms formulated in Lemmas 5, 7, 11, it follows that $P\mathscr{A}P$ is of type I, or of type I_n, or has no direct summand of type I, or has no commutative direct summand if and only if the same holds for $\phi(P)\mathscr{B}\phi(P)$.

Consider the type decomposition of $\mathscr{A}$ (see, for example, [12, Theorem 6.5.2]). For any cardinal number n (not exceeding the dimension of the Hilbert space on which the elements of $\mathscr{A}$ act) let $P_n \in \mathscr{A}$ be a central projection such that the algebra $\mathscr{A}P_n$ is of type I_n (or $P_n = 0$) and in case $Q = I - \sum_n P_n \neq 0$ the algebra $\mathscr{A}Q$ has no type I direct summand. It follows from the first paragraph of the proof that the collection $(\phi(P_n))$ of central projections in $\mathscr{B}$ has the same properties (relating the algebra $\mathscr{B}$ in the place of $\mathcal{A}$, of course).

The conditions of the theorem imply that we have $P_1 = 0$. Apply our auxiliary result Proposition 12 for the direct summands $\mathscr{A}P_n$ and $\mathscr{B}\phi(P_n)$ whenever $2 \leqslant n < \infty$. Moreover, apply Proposition 14 for the direct summands $\mathscr{A}P_n$ and $\mathscr{B}\phi(P_n)$ whenever n is an infinite cardinal and do the same for $\mathscr{A}Q$ and $\mathscr{B}\phi(Q)$. Taking direct sums, we obtain that ϕ is additive. Referring to Proposition 15 we complete the proof. $\square$

We now turn to the proof our our second theorem.

Proof of Theorem 2. We first examine the element $\phi(I)$. We know that

$$\phi(I)\phi(B)\phi(I) = \phi(B)$$

holds for every $B \in \mathscr{A}_s$. Choosing $B \in \mathscr{A}_s$ such that $\phi(B) = I$, we obtain $\phi(I)^2 = I$. Next, multiplying the above displayed equation by $\phi(I)$ from the right, we infer that $\phi(I)\phi(B) = \phi(B)\phi(I)$ holds for every $B \in \mathscr{A}_s$. This gives us that $\phi(I)$ is a central element of $\mathscr{B}$ which is self-adjoint and its square is I.

Considering the transformation

$$A \longmapsto \phi(I)\phi(A) \quad (A \in \mathscr{A}_s),$$

we clearly obtain a bijective Jordan triple map from $\mathscr{A}_s$ onto $\mathscr{B}_s$ which maps I to I. So, without serious loss of generality we can assume that this holds already for our orgininal map, i.e., we have $\phi(I) = I$.

Since $\phi(A^2) = \phi(AIA) = \phi(A)\phi(I)\phi(A) = \phi(A)^2$ holds for every $A \in \mathscr{A}_s$, we infer that ϕ maps $\mathscr{A}_+$ onto $\mathscr{B}_+$. Therefore, we can apply Theorem 1 for the restriction of ϕ onto $\mathscr{A}_+$. Hence, we obtain a Jordan *-isomorphism $\Phi : \mathscr{A} \to \mathscr{B}$ such that ϕ and Φ coincide on $\mathscr{A}_+$. In what follows we prove that $\phi = \Phi$ holds also on $\mathscr{A}_s$.

By the already mentioned result [7, Theorem 2] of Gudder and Nagy we know that for arbitrary self-adjoint operators A, B we have $AB^2A = BA^2B$ if and only if $BA^2 = A^2B$ and $AB^2 = B^2A$. In case one of A and B is positive, this is clearly equivalent to $AB = BA$. It follows from what we have already learnt that ϕ preserves the operation $(A, B) \longmapsto AB^2A$ and also preserves the positive elements in both directions. Consequently, for arbitrary $A, B \in \mathscr{A}_s$ one of them being positive we have $AB = BA$ if and only if $\phi(A)\phi(B) = \phi(B)\phi(A)$. Using the preserver properties of ϕ, we thus obtain that

$$\phi(\mathscr{A}_s \cap (\mathscr{A}_+ \cap A')') = \mathscr{B}_s \cap (\mathscr{B}_+ \cap \phi(A)')'.$$

It is not difficult to see that

$$\mathscr{A}_s \cap (\mathscr{A}_+ \cap A')' = \mathscr{A}_s \cap (\mathscr{A} \cap A')' = (\mathscr{A} \cap (\mathscr{A} \cap A')')_s$$

and, of course, similar observation holds in relation with $\mathscr{B}$ as well. In the proof of Proposition 14 we have already seen that $(\mathscr{A} \cap (\mathscr{A} \cap A')')$ is a commutative von Neumann algebra which contains A. Now, we can continue the present proof in a way similar to the argument given there. We have compact Hausdorff spaces X, Y such that the algebras $(\mathscr{A} \cap (\mathscr{A} \cap A')')$ and $(\mathscr{B} \cap (\mathscr{B} \cap \phi(A)')')$ are isometrically *-isomorphic to the algebras $C(X)$ and $C(Y)$ of complex valued continuous functions, respectively. Then our transformation ϕ induces a bijective map $\psi : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ which satisfies

$$\psi(f^2g) = \psi(f)^2\psi(g) = \psi(f^2)\psi(g) \quad (f, g \in C_{\mathbb{R}}(X)) \quad \text{and} \quad \psi(1) = 1. \tag{5}$$

Moreover, as ϕ when considered on $\mathscr{A}_+$ is the restriction of a Jordan *-isomorphism of $\mathscr{A}$ onto $\mathscr{B}$, it follows that ψ on $C(X)_+$ is the restriction of a linear *-algebra isomorphism Ψ of $C(X)$ onto $C(Y)$. We intend to show that $\psi = \Psi$ holds on $C_{\mathbb{R}}(X)$.

The form of the algebra isomorphisms between the algebras of continuous functions on compact Hausdorff spaces is well-known. Namely, any such transformation is a composition operator corresponding to a homeomorphism between the underlying spaces. It is now easy to see that without serious loss of generality we can transform our problem to the following one: Suppose that we have a bijective map $\psi : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(X)$ which satisfies

$$\psi(f^2g) = \psi(f)^2\psi(g) = \psi(f^2)\psi(g) \quad (f, g \in C_{\mathbb{R}}(X)), \qquad \psi(1) = 1 \tag{6}$$

and, in addition, it also satisfies $\psi(f) = f$ for every $f \in C(X)_+$. We claim that $\psi(f) = f$ holds for every $f \in C_{\mathbb{R}}(X)$. To see this, first observe that by (6) we have $\psi(fg) = f\psi(g)$ for every $f \in C(X)_+$ and arbitrary $g \in C_{\mathbb{R}}(X)$. On the other hand, we have $\psi(g)^2 = \psi(g^2) = g^2$ which gives us that $\psi(g)(x) = \pm g(x)$ holds for every $x \in X$. Pick $x_0 \in X$ and suppose that $g(x_0) > 0$. As g is strictly positive in a neighbourhood U of x_0, by Urysohn's lemma we have a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and f vanishes in the complement of U. Therefore, the function fg is nonnegative and $(fg)(x_0) = g(x_0)$. We compute

$$\psi(g)(x_0) = f(x_0)\psi(g)(x_0) = (f\psi(g))(x_0) = \psi(fg)(x_0) = (fg)(x_0) = g(x_0).$$

This means that the positivity of $g(x_0)$ implies the positivity of $\psi(g)(x_0)$. Since ψ^{-1} has similar properties as ψ, it follows that $g(x_0)$ is positive if and only if so is $\psi(g)(x_0)$. These observations result in that $\psi(g)(x) = g(x)$ holds for every $x \in X$ and arbitrary $g \in C_{\mathbb{R}}(X)$. In particular, we obtain that ψ is linear and isometric.

Going back to our transformation ϕ, we thus deduce that its restriction to $(\mathscr{A} \cap (\mathscr{A} \cap A')')_{\mathrm{s}}$ is linear and isometric.

Let $A \in \mathscr{A}_{\mathrm{s}}$ be arbitrary. The positive and negative parts A_+ and A_- of A belong to A'' and hence also to $(\mathscr{A} \cap (\mathscr{A} \cap A')')_{\mathrm{s}}$. We compute

$$\phi(A) = \phi(A_+ - A_-) = \phi(A_+) - \phi(A_-) = \Phi(A_+) - \Phi(A_-) = \Phi(A),$$

where Φ is the Jordan *-isomorphism which coincides with ϕ on $\mathscr{A}_+$ (see the first part of the proof). Just as in the proof of Theorem 1 we obtain that Φ is the direct sum of a *-algebra isomorphism and a *-algebra antiisomorophism. The proof is complete in the case when $\phi(I) = I$.

In the general case we obtain the above conclusion concerning the structure of the map $A \mapsto \phi(I)\phi(A)$, where $\phi(I)$ is a central element in $\mathscr{B}$, it is self-adjoint and its square is I. It follows from the properties of $\phi(I)$ that $\phi(I) = P - (I - P)$ holds for a central projection P in $\mathscr{B}$. Now, multiplying by $\phi(I)$ from the left, the representation of the map $A \mapsto \phi(I)\phi(A)$ as the direct sum of a *-algebra isomorphism and a *-algebra antiisomorphism splits into a decomposition of ϕ that was stated in the theorem. $\square$

We now turn to the proof of our third theorem in which we shall use the following lemma.

Lemma 16. *Let $\mathscr{A}$ be a von Neumann algebra, $X, Y \in \mathscr{A}$ such that*

$$X^*TX = Y^*TY \quad (T \in \mathscr{A}).$$

Then there exists a central element $Z \in \mathscr{A}$ such that $X = ZY$ and $|Z|$ equals the identity on $\operatorname{rng}|X| = \operatorname{rng}|Y|$. *In particular, if $\mathscr{A}$ is a factor, then $Y = \tau X$ for some complex number τ of modulus* 1.

Proof. First observe that we have $X^*X = Y^*Y$ which implies $|X| = |Y|$. Denote $A = |X| = |Y|$. Consider the polar decompositions of X and Y

$$X = U|X| = UA, \qquad Y = V|Y| = VA,$$

where $U, V \in \mathscr{A}$ are partial isometries and we know that $U^*U = V^*V$ (the ranges of $|X|$ and $|Y|$ coincide). We then have

$$AU^*TUA = AV^*TVA \quad (T \in \mathscr{A}).$$

As the initial spaces of U and V are both equal to the closure of the range of A, we obtain $AU^*TU = AV^*TV$. Taking adjoints, we have $U^*TUA = V^*TVA$ for every $T \in \mathscr{A}$. Just as before, this implies that

$$U^*TU = V^*TV \quad (T \in \mathscr{A}). \tag{7}$$

Let P_X and P_Y be the range projections of X and Y, respectively. We know that $P_X = UU^*$ and $P_Y = VV^*$. It follows from (7) that

$$V^*V = U^*U = U^*P_XU = V^*P_XV.$$

This gives us that

$$\|Vx\|^2 = \|P_X(Vx)\|^2$$

holds for every vector x from the underlying Hilbert space. This easily gives us that Vx belongs to the range of the projection P_X. As the range of V is equal to the range of P_Y, we obtain that $P_Y \leqslant P_X$. In a similar way one can verify the reversed inequality as well, so it follows that $P_X = P_Y$. Denote $P = P_X = P_Y$. Multiplying the equality (7) by U from the left and by V^* from the right, we obtain

$$PTUV^* = UU^*TUV^* = UV^*TVV^* = UV^*TP.$$

Since $PU = U$ and $PV = V$, we get

$$PTP(UV^*) = (UV^*)PTP \quad (T \in \mathscr{A}).$$

This shows that UV^* belongs to the center of the von Neumann algebra $P\mathscr{A}P$. This center is well-known to be equal to $\mathscr{Z}(\mathscr{A})P$. Consequently, there is a central element $Z \in \mathscr{A}$ such that $UV^* = ZP$. We compute

$$U = UU^*U = UV^*V = ZPV = ZV.$$

It then follows that $X = UA = ZVA = ZY$. This yields $|X| = |Z||Y|$ and the proof is complete. $\square$

Proof of Theorem 3. It is easy to see that ϕ maps $\mathscr{A}_+$ onto $\mathscr{B}_+$. Next, if $A \in \mathscr{A}_+$, then we have

$$\phi(A^2) = \phi(|A^2|) = |\phi(A)^2| = \phi(A)^2. \tag{8}$$

We now prove that

$$\phi(ABA) = \phi(A)\phi(B)\phi(A)$$

holds for every $A, B \in \mathscr{A}_+$. In fact, we can compute

$$\phi((AB^2A)^{\frac{1}{2}}) = \phi(|BA|) = |\phi(B)\phi(A)| = (\phi(A)\phi(B)^2\phi(A))^{\frac{1}{2}}.$$

As ϕ respects the square operation (and hence also the square-root operation) on the positive elements (see (8)), it follows that

$$\phi(AB^2A) = \phi(A)\phi(B)^2\phi(A) = \phi(A)\phi(B^2)\phi(A)$$

holds for all $A, B \in \mathscr{A}_+$. This gives us immediately that ϕ is a multiplicative Jordan triple isomorphism from $\mathscr{A}_+$ onto $\mathscr{B}_+$. Applying Theorem 1 we deduce that the restriction of ϕ onto $\mathscr{A}_+$ coincides with the restriction of the direct sum of a linear *-algebra isomorphism and a linear *-algebra antiisomorphism. As $\mathscr{A}$ is a factor, we obtain that one of those components is missing and hence ϕ equals either a linear *-algebra isomorphism or a linear *-algebra antiisomorphism Φ on $\mathscr{A}_+$.

Suppose first that Φ is a *-isomorphism. Pick $A \in \mathscr{A}$ and $B \in \mathscr{A}_+$. We compute

$$\begin{aligned}\phi(A^*BA) &= \phi(|B^{\frac{1}{2}}A|^2) = \phi(|B^{\frac{1}{2}}A|)^2 = |\phi(B^{\frac{1}{2}})\phi(A)|^2 \\ &= |\phi(B)^{\frac{1}{2}}\phi(A)|^2 = \phi(A)^*\phi(B)\phi(A).\end{aligned}$$

Denote $\Psi = \Phi^{-1} \circ \phi$. Clearly, Ψ is the identity on $\mathscr{A}_+$ and it satisfies

$$\Psi(T^*ST) = \Psi(T)^*\Psi(S)\Psi(T) \quad (T \in \mathscr{A}, S \in \mathscr{A}_+).$$

If A, B are as before, then A^*BA and B are positive, so we obtain that

$$A^*BA = \Psi(A^*BA) = \Psi(A)^*\Psi(B)\Psi(A) = \Psi(A)^*B\Psi(A).$$

Let now B run through $\mathscr{A}_+$. We obtain that $A^*TA = \Psi(A)^*T\Psi(A)$ $(T \in \mathscr{A})$. Applying Lemma 16 we infer that there is a scalar $\tau(A) \in \mathbb{C}$ of modulus 1 such that $\Psi(A) = \tau(A)A$. By the linearity of Φ this gives us that

$$\phi(A) = \tau(A)\Phi(A) \quad (A \in \mathscr{A}).$$

In the case when Φ is a *-antiisomorphism, a similar argument leads to the existence of a scalar function $\tau : \mathscr{A} \to \mathbb{C}$ of modulus 1 for which

$$\phi(A) = \tau(A)\Phi(A^*) \quad (A \in \mathscr{A}).$$

Since the map $A \mapsto \Phi(A^*)$ is a conjugate-linear *-algebra isomorphism, the proof is complete. □

We remark that it can be seen from the argument above that one could get a structural result for the maps satisfying (1) also in the non-factor case. However, the formulation of the statement would not be as simple as in the present situation and this was the reason why we have set that assumption.

To conclude the paper we note that in our recent paper [26] we have investigated multiplicative Jordan triple isomorphisms on the set of all invertible positive operators and also on the set of all invertible self-adjoint operators in $B(H)$. Under the assumption of continuity we could present the complete description of those maps. It has turned out that in that case the inverse operation also shows up in the form of our transformations and, in addition, if the underlying Hilbert space is finite dimensional, a certain power of the determinant appears as well. So, in that case our transformations can be highly non-additive. Since there is a theory of determinants in von Neumann algebras (more precisely, in finite factors; see [4]), regarding our present results we believe that it would be interesting to know how the theorems in [26] can survive in the general context of von Neumann algebras. We pose this as an open problem.

References

[1] L.J. Bunce, J.D.M. Wright, The Mackey–Gleason problem, Bull. Amer. Math. Soc. 26 (1992) 288–293.
[2] J.T. Chan, C.K. Li, N.S. Sze, Mappings on matrices: invariance of functional values of matrix products, J. Austral. Math. Soc., in press.
[3] J.T. Chan, C.K. Li, N.S. Sze, Mappings preserving spectra of product of matrices, Proc. Amer. Math. Soc., in press.
[4] B. Fuglede, R.V. Kadison, Determinant theory in finite factors, Ann. Math. 55 (1952) 520–530.
[5] S. Gudder, R. Greechie, Sequential products on effect algebras, Rep. Math. Phys. 49 (2002) 87–111.
[6] S. Gudder, G. Nagy, Sequential quantum measurements, J. Math. Phys. 42 (2001) 5212–5222.

[7] S. Gudder, G. Nagy, Sequentially independent effects, Proc. Amer. Math. Soc. 130 (2002) 1125–1130.
[8] J. Hakeda, Additivity of *-semigroup isomorphisms among *-algebras, Bull. London Math. Soc. 18 (1986) 51–56.
[9] J. Hakeda, Additivity of Jordan *-maps on AW^*-algebras, Proc. Amer. Math. Soc. 96 (1986) 413–420.
[10] J. Hakeda, K. Saitô, Additivity of Jordan *-maps between operator algebras, J. Math. Soc. Japan 38 (1986) 403–408.
[11] J. Hou, Q. Di, Maps preserving numerical ranges of operator products, Proc. Amer. Math. Soc. 134 (2006) 1435–1446.
[12] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, Academic Press, 1986.
[13] P. Li, W. Jing, Jordan elementary maps on rings, Linear Algebra Appl. 382 (2004) 237–245.
[14] Z. Ling, F. Lu, Jordan maps of nest algebras, Linear Algebra Appl. 387 (2004) 361–368.
[15] F. Lu, Multiplicative mappings of operator algebras, Linear Algebra Appl. 347 (2002) 283–291.
[16] F. Lu, Additivity of Jordan maps on standard operator algebras, Linear Algebra Appl. 357 (2002) 121–131.
[17] F. Lu, Jordan maps on associative algebras, Commun. Algebra 31 (2003) 2273–2286.
[18] F. Lu, Jordan triple maps, Linear Algebra Appl. 375 (2003) 311–317.
[19] J. Marovt, Multiplicative bijections of $C(\mathscr{X},\mathscr{I})$, Proc. Amer. Math. Soc. 134 (2006) 1065–1075.
[20] W.S. Martindale III, When are multiplicative mappings additive? Proc. Amer. Math. Soc. 21 (1969) 695–698.
[21] A.N. Milgram, Multiplicative semi-groups of continuous functions, Duke Math. J. 16 (1949) 377–383.
[22] L. Molnár, On some automorphisms of the set of effects on Hilbert space, Lett. Math. Phys. 51 (2000) 37–45.
[23] L. Molnár, On isomorphisms of standard operator algebras, Studia Math. 142 (2000) 295–302.
[24] L. Molnár, Some characterizations of the automorphisms of $B(H)$ and $C(X)$, Proc. Amer. Math. Soc. 130 (2002) 111–120.
[25] L. Molnár, Sequential isomorphisms between the sets of von Neumann algebra effects, Acta Sci. Math. (Szeged) 69 (2003) 755–772.
[26] L. Molnár, Non-linear Jordan triple automorphisms of sets of self-adjoint matrices and operators, Studia Math. 173 (2006) 39–48.
[27] L. Molnár, P. Šemrl, Elementary operators on self-adjoint operators, J. Math. Anal. Appl., in press.
[28] M. Radjabalipour, Additive mappings on von Neumann algebras preserving absolute values, Linear Algebra Appl. 368 (2003) 229–241.
[29] N.V. Rao, A.K. Roy, Multiplicatively spectrum-preserving maps of function algebras, Proc. Amer. Math. Soc. 133 (2005) 1135–1142.
[30] N.V. Rao, A.K. Roy, Multiplicatively spectrum-preserving maps of function algebras II, Proc. Edinb. Math. Soc. 48 (2005) 219–229.
[31] P. Šemrl, Isomorphisms of standard operator algebras, Proc. Amer. Math. Soc. 123 (1995) 1851–1855.