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# Vectorial Interpolation Using Radial-Basis-Like Functions

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**Abstract**—This paper deals with vector field interpolation, i.e., the data are $\mathbb{R}^3$ values located in scattered $\mathbb{R}^3$ points, while the interpolating function is a function from $\mathbb{R}^3$ into $\mathbb{R}^3$. In order to take into account possible connections between the components of the interpolant, we derive it by solving a variational spline problem involving the rotational and the divergence of the interpolant, and depending on a parameter $\rho$ significative of the balance of the rotational part and of the divergence part, and on the order $m$ of derivatives of the rotational and divergence involved in the minimized seminorm. We so obtain interpolants whose expression is $\sigma(x) = \sum_{i=1}^{n} \Phi(x - x^i)a^i + p_{m-1}(x)$, where $\Phi$ is some $3 \times 3$ matrix function, $p_{m-1}$ is a degree $m-1$ vectorial polynomial, and where the $a^i$ are $\mathbb{R}^3$-vectors. Besides, the $a^i$ meet a relation generalizing the usual orthogonality to all polynomials of degree at most $m-1$. For $\rho = 1$, we find the usual $m$-harmonic splines in each component of $\sigma$. Numerical examples show the interest of the method, and we compare the so-obtained functions with the ones obtained by Matlab's procedures. © 2002 Elsevier Science Ltd. All rights reserved.

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## 1. INTRODUCTION

Many applications, such as fluid mechanics and meteorology, need interpolate vectorial data located on $\mathbb{R}^3$ points: given $n$ points $(x^i)_{i\in[1:n]}$ in $\mathbb{R}^3$ and $n$ $\mathbb{R}^3$ vectors $(z^i)_{i\in[1:n]}$, we must derive a function $\sigma$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $\forall\, i \in [1 : n]$, $\sigma(x^i) = z^i$.

Of course, this can be done by working independently on each component, i.e., by independently deriving three functions $\sigma_1$, $\sigma_2$, and $\sigma_3$ such that, for all $i \in [1 : n]$, $\sigma_1(x^i) = z_1^i$, $\sigma_2(x^i) = z_2^i$, $\sigma_3(x^i) = z_3^i$. However, the so-obtained results are often of poor quality, since we cannot have in this way any kind of connection between the components of the function $\sigma$, while applications often require such a connection (such as $\operatorname{rot}\sigma = 0$, or $|\operatorname{rot}\sigma|$ "small" in a meaning specified below, or $\operatorname{div}\sigma = 0$, or $|\operatorname{div}\sigma|$ "small"). So we must derive a vectorial function involving a criterion which can take into account such a connection between the components.

Most of the authors who have defined interpolants with a relation between the components did so by using the variational spline theory: Atteia and Benbourhim [1] introduced the "elastic spline manifolds" (in $\mathbb{R}^d$), while Amodei [2,3] determined vectorial interpolants in two variables (deriving a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ interpolating $\mathbb{R}^2$-data located in points in $\mathbb{R}^2$). Handscomb studied, in [4], divergence-free interpolants. Let us also mention the paper [5] where Myers defines

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multivariate interpolants, in any dimension, via the kriging approach. Finally, note that the codes used by Matlab for vectorial interpolants actually provide a scalar interpolant independently for each one of the variables.

In this paper, we choose a seminorm which is based on the Helmholtz decomposition of vector fields into a rotational and a gradient part. Then, using the variational spline technique, we determine the vectorial function which minimizes this seminorm over all the functions in a suitable semi-Hilbert space which interpolates the data. The used seminorm is explicated below in (5). The so-obtained functions *are not*, in general, radial basis functions, not even for each of its components (however, in the particular case when $\rho$ in (5) is equal to 1, we get the $m$-harmonic interpolating spline in each component of $\sigma$, and so have a componentwise radial basis function), but their general form is quite similar to the one of a radial basis function, since there exists a matricial function $\Phi$ (nondependent on the data, and usually nonradial, but which can be expressed in terms of derivatives of a radial function), a degree $m-1$ vectorial polynomial $p_{m-1}$ and $n$ vectorial coefficients $(a^i)_{i\in[1:n]}$ meeting for any $j=1,2,3$, $\forall p \in \mathbb{P}_{m-1}(\mathbb{R}^3)$, $\sum_{i=1}^n a_j^i p(x^i) = 0$, such that the interpolating vectorial function is $\sigma(x) = \sum_{i=1}^n \Phi(x-x^i)a^i + p_{m-1}(x)$.

Now, in order to derive the actual function $\sigma$ interpolating some given data, we must solve a linear system similar to the one used for radial basis function $s$. The dimension of this system is $3n + 3\binom{m+2}{3}$, which may seem a large number, but the number of scalar data is $3n$ since we have $n$ three-dimensional data, and the dimension of the space of three-dimensional polynomials of degree at most $m-1$ is $3\binom{m+2}{3} = m(m+1)(m+2)/2$. This linear system presents the same drawbacks as are usual with radial basis functions (large condition number, zeros on the diagonal, ...). Besides, it is possible to derive B-spline-like matricial functions by discretizing the operator $P_{m,\rho}(D)$ defined below in (10), which allows three-dimensional B-spline like approximation, as presented in [6].

A possible application of this can be to solve partial differential equations (from $\mathbb{R}^3$ to $\mathbb{R}^3$) while controlling the global ratio between the divergence part and the rotational part of the solution, which is specially useful in fluid dynamics problems.

The paper is organized as follows: in Section 2, we introduce our main notations and the minimized seminorm; in Section 3, we present the functional analysis tools necessary to solve the minimization problem; in Section 4, we state the two main theorems of the paper, giving the general form of the interpolant, and the linear system to be solved in order to derive the actual values of the coefficients; in Section 5, we show some numerical results. Finally, in Section 6, we briefly give some possible consequences or extensions of this work. The paper is written in such a way that a reader only interested in the results can drop Section 3 and so avoid the main theoretical development.

## 2. SETTING THE INTERPOLATION PROBLEM

### 2.1. Notations

#### Numbers

$m$ is a fixed integer number such that $m \geq 2$, and $m' = \binom{m+2}{3}$.
$n$ is a fixed integer number such that $n \geq m'$, while $\rho$ is a nonnegative fixed real number.
$p$ and $q$ being two integer numbers, $[p : q]$ is the set of all integers $j$ such that $p \leq j \leq q$.

#### Multivariate tools

$\|\bullet\|$ is the usual Euclidean norm in $\mathbb{R}^3$: $\|u\|^2 = u_1^2 + u_2^2 + u_3^2$.
Multi-integers: $\gamma$ and $\gamma'$ stand for elements in $\mathbb{N}^3 = (\mathbb{Z}_+)^3$. We use standard multi-index notation: $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$; $\gamma! = \gamma_1!\, \gamma_2!\, \gamma_3!$.

If $f$ is a function from $\mathbb{R}^3$ to $\mathbb{R}$, $D^\gamma f$ is the order $\gamma$ derivative of $f$: $D^\gamma f = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} \frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}} f$, and second derivatives of $f$ are also denoted in the following usual way: $\partial^2_{jk} f$ is for $\frac{\partial^2 f}{\partial x_j \partial x_k}$.

If $f$ is a function from $\mathbb{R}^3$ to $\mathbb{R}^3$, $D^\gamma f$ denotes the function from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that, for any $j \in [1 : 3]$, $(D^\gamma f)_j = D^\gamma (f_j)$.

div and rot have their usual meaning,

$$\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}, \qquad \operatorname{rot} f = \begin{pmatrix} \dfrac{\partial f_3}{\partial x_2} - \dfrac{\partial f_2}{\partial x_3} \\ \dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_3}{\partial x_1} \\ \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2} \end{pmatrix}.$$

$\nabla$ is the gradient operator: for any $j$ in $[1 : 3]$, $(\nabla f)_j = \frac{\partial f}{\partial x_j}$.

Applied to a function from $\mathbb{R}^3$ to $\mathbb{R}$, $\Delta = \sum_{|\gamma|=2} D^\gamma$ is the usual Laplacian operator. If $f$ is a function from $\mathbb{R}^3$ to $\mathbb{R}^3$, $\Delta f$ is the "vectorial Laplacian operator", i.e., for any $j$ in $[1 : 3]$, $(\Delta f)_j = \Delta(f_j)$.

$I_3$ is the $3 \times 3$ identity matrix.

### Data and interpolant

$(x^i)_{i \in [1:n]}$ are the locations of the data ($x^i \in \mathbb{R}^3$), while $(z^i)_{i \in [1:n]}$ are the data ($z^i \in \mathbb{R}^3$).

$\sigma_{m,\rho}$ is the wanted interpolant, i.e., $\forall i \in [1 : n]$, $\sigma_{m,\rho}(x^i) = z^i$.

For some functional space $V$, some constraints $C$ and some seminorm $|\bullet|_V$ defined on $V$,

$$g = \operatorname*{Arg\,Min}_{v \in V} \{|v|_V \,;\, C\}$$

means that $g \in V$ and meets the constraints $C$, and that furthermore if $f$ is in $V$ and meets the constraints $C$, we have $|g|_V \le |f|_V$.

### Polynomials

$\mathbb{P}_k(\mathbb{R}^3)$ is the set of scalar polynomials with variable in $\mathbb{R}^3$ and degree at most $k$.

$\mathbb{P}_k(\mathbb{R}^3; \mathbb{R}^3)$ is the set of vectorial polynomials with variable in $\mathbb{R}^3$ and degree at most $k$, i.e., the set of functions $p = (p_1, p_2, p_3)$ where $p_1$, $p_2$, and $p_3$ are in $\mathbb{P}_k(\mathbb{R}^3)$.

$\mathcal{N}$ is short for $\mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$.

For any $k \in \mathbb{N}$, a discrete set $(b^i)_{i \in [1:k']}$ of $k' = \binom{k+3}{3}$ points in $\mathbb{R}^3$ is said to be $\mathbb{P}_k(\mathbb{R}^3)$-unisolvent if and only if for any set of $k'$ real numbers $(y^i)_{i \in [1:k']}$ there exists one and only one $p \in \mathbb{P}_k(\mathbb{R}^3)$ such that $\forall i \in [1 : k']$, $p(b^i) = y^i$.

In the sequel, the set $(x^i)_{i \in [1:n]}$ of the locations of the data is supposed to contain a $\mathbb{P}_{m-1}(\mathbb{R}^3)$-unisolvent set, and without loss of generality, we suppose that the $m'$ first points of the set, i.e., $(x^i)_{i \in [1, m']}$, form a $\mathbb{P}_{m-1}(\mathbb{R}^3)$-unisolvent set.

### Distributions

As usual in distribution theory, $\mathcal{D}(\mathbb{R}^3)$ denotes the set of compactly supported, infinitely derivable functions from $\mathbb{R}^3$ to $\mathbb{R}$, and $\mathcal{D}'(\mathbb{R}^3)$ is its topological dual, i.e., the set of distributions on $\mathcal{D}$. In the same way, $\mathcal{D}(\mathbb{R}^3; \mathbb{R}^3) = \mathcal{D}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)$ and $\mathcal{D}'(\mathbb{R}^3; \mathbb{R}^3)$ is its topological dual. Finally, $(\mathcal{C}^0(\mathbb{R}^3; \mathbb{R}^3))'$ is the topological dual of $\mathcal{C}^0(\mathbb{R}^3; \mathbb{R}^3)$, i.e., the set of compactly supported Radon measures.

$\delta$ is the Dirac distribution in $\mathcal{D}'(\mathbb{R}^3)$, $\langle \bullet \mid \bullet \rangle$ denotes the usual duality product, and $\star$ denotes the convolution product of a distribution by a measure.

### Seminorms, norms, and Beppo-Levi spaces

$D^{-m}L^2(\mathbb{R}^3)$ is the scalar Beppo-Levi space of order $m$, i.e., the set of functions (or distributions) from $\mathbb{R}^3$ to $\mathbb{R}$ whose all total order $m$ derivatives (in the sense of distributions) are in $L^2(\mathbb{R}^3)$. Endowed with the semiscalar product $\langle \bullet, \bullet \rangle_m$ defined by

$$\langle f, g \rangle_m = \sum_{|\gamma|=m} \frac{m!}{\gamma!} (D^\gamma f, D^\gamma g)_{L^2(\mathbb{R}^3)}, \tag{1}$$

$D^{-m}L^2(\mathbb{R}^3)$ is a semi-Hilbert space. The associated seminorm is denoted by $|\bullet|_m$.

$D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3) = D^{-m}L^2(\mathbb{R}^3) \times D^{-m}L^2(\mathbb{R}^3) \times D^{-m}L^2(\mathbb{R}^3)$ is the set of functions (or distributions) from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose three components are in $D^{-m}L^2(\mathbb{R}^3)$. Endowed with the semiscalar product $\langle \bullet, \bullet \rangle_m$ defined by

$$\langle f, g \rangle_m = \sum_{j=1}^{3} \langle f_j, g_j \rangle_m = \sum_{|\gamma|=m} \frac{m!}{\gamma!} (D^\gamma f, D^\gamma g)_{L^2(\mathbb{R}^3;\mathbb{R}^3)}, \tag{2}$$

$D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ is a semi-Hilbert space. The associated seminorm is denoted by $|\bullet|_m$. Endowed with the scalar product

$$\langle\langle f, g \rangle\rangle_m = \langle f, g \rangle_m + \sum_{i \in [1\,:\,m']} f\left(x^i\right) \cdot g\left(x^i\right),$$

$D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ is a Hilbert space. The associated norm is denoted by $\| \bullet \|_m$. We will need the following density relation:

$$D^{-m}L^2\left(\mathbb{R}^3;\mathbb{R}^3\right) = \overline{\mathcal{D}\left(\mathbb{R}^3;\mathbb{R}^3\right) + \mathbb{P}_{m-1}\left(\mathbb{R}^3;\mathbb{R}^3\right)}^{\|\bullet\|_m}. \tag{3}$$

## 2.2. The Seminorm to be Minimized

The idea is to be able, via the seminorm to be minimized, to govern in some way the ratio $|\operatorname{div} f|_{\text{div}}/|\operatorname{rot} f|_{\text{rot}}$ of the interpolant, where $|\bullet|_{\text{div}}$ is some scalar (semi-)norm and $|\bullet|_{\text{rot}}$ is some vectorial (semi-)norm. To do so, we need a seminorm which takes into account the rotational and the divergence of the interpolant, giving the user the possibility to stress the one or the other, depending on the type of the desired result. This is why a (semi-)norm involving $\rho|\operatorname{div} f|_{\text{div}}^2 + |\operatorname{rot} f|_{\text{rot}}^2$ seems something reasonable. Obviously, if $\rho$ is large (or tends to infinity), the (seminorm of the) divergence of the so-obtained interpolant will be forced to be small (or tends to zero), while on the opposite if $\rho$ is small (or tends to zero) (the seminorm of) its rotational will be forced to be small (or tends to zero). Note that here $\rho$ is a nondimensional positive real number.

Note also that when $|\bullet|_{\text{div}}$ and $|\bullet|_{\text{rot}}$ are seminorms (and are not norms), a large $\rho$ forces the seminorm of the divergence of the interpolant to be small, which does not imply *a priori* the fact that the divergence of the interpolant is small.

We now have to choose the seminorms $|\bullet|_{\text{div}}$ and $|\bullet|_{\text{rot}}$. Two main motivations will guide us to choose these seminorms: first, we want the so-obtained interpolant to be a continuous function (in order to be uniquely defined on $\mathbb{R}^3$), and so we want the Dirac distribution in $t$ (i.e., $\delta_t$: $f \to f(t)$) to be a continuous one; but we usually want more, since a certain regularity of the so-obtained interpolant is often required from the real problem, and the user often wishes that the oscillations of some quantity linked to the interpolant are as small as possible (both these latter points are usually connected). We propose in this paper to use the two following seminorms, for some arbitrary fixed $m \geq 2$ (we use the seminorm $|\bullet|_{m-1}$ of a scalar function, as defined in (1), and the seminorm $|\bullet|_{m-1}$ of a vectorial function, as defined in (2)):

$$|\bullet|_{\text{div}} = |\bullet|_{m-1} \qquad \text{and} \qquad |\bullet|_{\text{rot}} = |\bullet|_{m-1}. \tag{4}$$

Finally, the seminorm to be minimized is defined by

$$|f|^2_{m,\rho} = \rho |\operatorname{div} f|^2_{m-1} + |\operatorname{rot} f|^2_{m-1}. \tag{5}$$

Of course, the minimization will be done over all functions whose all derivatives of total order $m$ are in $L^2(\mathbb{R}^3;\mathbb{R}^3)$, i.e., over the space $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$.

Now, the interpolant function depends on the two parameters $m$ and $\rho$, fixed by the user. It is denoted by $\sigma_{m,\rho}$ and is defined by

$$\sigma_{m,\rho} = \underset{f \in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)}{\operatorname{Arg\,Min}} \left\{ |f|^2_{m,\rho};\ \forall i \in [1 : n],\ f\left(x^i\right) = z^i \right\}. \tag{6}$$

# 3. FUNCTIONAL ANALYSIS TOOLS

## 3.1. Semi-Hilbertian Subspaces and Associated Semikernels

First, let us recall the definitions of a semi-Hilbertian subspace and of associated semikernels (see [7]).

DEFINITION. *Let $E$ be a vectorial topological space, separated and locally convex; let $E'$ be its topological dual.*

*Let $H$ be a vectorial subspace of $E$, endowed with a semiscalar product $\langle \bullet, \bullet \rangle_H$ (the associated seminorm is $|\bullet|_H$), and $N_H$ the nullspace of the seminorm; $N_H$ is supposed to be closed in $E$. Let $N_H^{\perp}$ be the orthogonal space of $N_H$ in $E'$.*

*$H$ is said to be a semi-Hilbertian subspace of $E$ if the space $H/N_H$, endowed with the quotient norm, is complete, and if the injection from $H/N_H$ into $E/N_H$ is continuous.*

THEOREM 1. *Let $H$ be a semi-Hilbertian subspace of $E$. Then there exists a linear map $\mathtt{h}$ from $E'$ to $E$ meeting*

(i) $\forall f' \in N_H^{\perp},\ \mathtt{h}(f') \in H.$
(ii) $\forall f \in H,\ \langle \mathtt{h}(f'), f \rangle_H = \langle f' \mid f \rangle_{E',E}.$

DEFINITION. $\mathtt{h}$ *is called "semikernel of $H$ in $E$".*

REMARKS.

(a) $\mathtt{h}$ is usually not unique; actually, if $\mathtt{h}$ is a semikernel, then for any linear map $p'$ from $E'$ to $N_H$, $\mathtt{h}_1$ defined by $\mathtt{h}_1 = \mathtt{h} + p'$ is also a semikernel.
(b) The semikernel of a semi-Hilbertian subspace will be of high interest in order to give an expression of the solution $\sigma_{m,\rho}$ of the minimizing problem (6).

## 3.2. A Particular Semi-Hilbertian Subspace

Let us now prove that the minimizing problem (6) can be expressed in the context of semi-Hilbertian spaces. From now on, $\mathcal{N}$ is short for $\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$.

This is done by the following theorem.

THEOREM 2. *Endowed with the seminorm $|\bullet|_{m,\rho}$, $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ is a semi-Hilbertian subspace of $C^0(\mathbb{R}^3;\mathbb{R}^3)$.*

PROOF. The nullspace of the seminorm $|\bullet|_{m,\rho}$ is obviously $\mathcal{N} = \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$. As a finite-dimensional space, $\mathcal{N}$ is closed in $C^0(\mathbb{R}^3;\mathbb{R}^3)$.

Let us prove that $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)/\mathcal{N}$, endowed with the quotient norm, is complete.

Let $(\tilde{f}_n)_{n\in\mathbb{N}}$ be a Cauchy's series of elements of $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)/\mathcal{N}$.

Let $f_n \in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ and $p_n \in \mathcal{N}$ be such that $f_n + p_n \in \tilde{f}_n$.

Using (5), for all $\varepsilon > 0$, there exists some $k \in \mathbb{N}$ such that, for all $p$ and $q$ in $\mathbb{N}$ such that $p > k$ and $q > k$, we have

$$\rho \left|\operatorname{div}(f_p - f_q)\right|_{m-1} < \varepsilon,$$
$$\left|\operatorname{rot}(f_p - f_q)\right|_{m-1} < \varepsilon.$$

So, for any $\gamma$ such that $|\gamma| = m$, $(D^\gamma f_n)_{n\in\mathbb{N}}$ is a Cauchy's series in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ and so converges towards some $f_\gamma \in L^2(\mathbb{R}^3;\mathbb{R}^3)$.

Now, from [8], the equations $(\forall \gamma \in \mathbb{N}^3,\ |\gamma| = m \Rightarrow D^\gamma f = f_\gamma)$ have a solution $f$ in $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ if and only if the usual compatibility conditions on $f_\gamma$ are fulfilled, i.e., if and only if

$$\forall \gamma, \gamma' \in \mathbb{N}^3, \quad |\gamma| = |\gamma'| \quad \Longrightarrow \quad D^{\gamma'} f_\gamma = D^\gamma f_{\gamma'}. \tag{7}$$

Using the continuity of the derivation in $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$, we have, for any $\gamma$ and $\gamma'$ in $\mathbb{N}^3$,

$$\begin{aligned} |\gamma| = |\gamma'| \quad \Rightarrow \quad D^{\gamma'} f_\gamma &= D^{\gamma'} \lim_{n\to\infty} D^\gamma f_n = \lim_{n\to\infty} D^{\gamma'} D^\gamma f_n = \lim_{n\to\infty} D^\gamma D^{\gamma'} f_n \\ &= D^\gamma \lim_{n\to\infty} D^{\gamma'} f_n = D^\gamma f_{\gamma'}. \end{aligned}$$

So there exists a $f \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ such that

$$\forall \gamma \in \mathbb{N}^3, \qquad |\gamma| = m \quad \Longrightarrow \quad D^\gamma f = f_\gamma \in L^2(\mathbb{R}^3;\mathbb{R}^3). \tag{8}$$

So $f \in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$. Using now the unicity of the limit, we get

$$\forall \gamma \in \mathbb{N}^3, \qquad |\gamma| = m-1 \quad \Longrightarrow \quad \begin{cases} \lim\limits_{n\to\infty} D^\gamma \operatorname{div} f_n = D^\gamma \operatorname{div} f, \\ \lim\limits_{n\to\infty} D^\gamma \operatorname{rot} f_n = D^\gamma \operatorname{rot} f, \end{cases}$$

and so $(\tilde{f}_n)_{n\in\mathbb{N}}$ converges in $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)/\mathcal{N}$.

Finally, it is classical (see, for example, [9]) that the injection from $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)/\mathcal{N}$ to $\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3)/\mathcal{N}$ is continuous. ∎

### 3.3. An Associated Semikernel

We will now give the expression of an associated semikernel. We first need the following notation.

### Notation

Let $\langle \bullet, \bullet \rangle_{m,\rho}$ be the semiscalar product associated with the seminorm $|\bullet|_{m,\rho}$, namely, for any $f$ and $g$ in $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$,

$$\langle f, g \rangle_{m,\rho} = \rho \langle \operatorname{div} f, \operatorname{div} g \rangle_{m-1} + \langle \operatorname{rot} f, \operatorname{rot} g \rangle_{m-1}. \tag{9}$$

Let $P_{m,\rho}(D)$ be the homogeneous differential operator of degree $2m$ defined by

$$\begin{aligned} P_{m,\rho}(D) &= (-1)^m \Delta^{m-1} (\rho \nabla \operatorname{div} - \operatorname{rot}\operatorname{rot}) \\ &= (-1)^m \Delta^{m-1} \begin{pmatrix} \rho\partial_{11}^2 + \partial_{22}^2 + \partial_{33}^2 & (\rho - 1)\,\partial_{12}^2 & (\rho - 1)\,\partial_{13}^2 \\ (\rho - 1)\,\partial_{12}^2 & \partial_{11}^2 + \rho\partial_{22}^2 + \partial_{33}^2 & (\rho - 1)\,\partial_{23}^2 \\ (\rho - 1)\,\partial_{13}^2 & (\rho - 1)\,\partial_{23}^2 & \partial_{11}^2 + \partial_{22}^2 + \rho\partial_{33}^2 \end{pmatrix}. \end{aligned} \tag{10}$$

Let $v_{m+1}$ be the function from $\mathbb{R}^3$ to $\mathbb{R}$ defined for any $x$ in $\mathbb{R}^3$ by

$$v_{m+1}(x) = \frac{-1}{4\pi(2m)!} \|x\|^{2m-1}. \tag{11}$$

It is known (see [7,8]) that $v_{m+1}$ meets the relation $\Delta^{m+1} v_{m+1} = \delta$, and so, taking the generalized Fourier transform of both members, $\hat{v}_{m+1}(\zeta) = (-1)^{m+1}/\|2\pi\zeta\|^{2(m+1)}$.

Now, let the tensorial functions $\Phi^{\mathrm{rot}}$, $\Phi^{\mathrm{div}}$, and $\Phi$ be defined by

$$\begin{aligned}
\Phi^{\mathrm{rot}} &= (-1)^m \begin{pmatrix} \partial_{11}^2 v_{m+1} & \partial_{12}^2 v_{m+1} & \partial_{13}^2 v_{m+1} \\ \partial_{12}^2 v_{m+1} & \partial_{22}^2 v_{m+1} & \partial_{23}^2 v_{m+1} \\ \partial_{13}^2 v_{m+1} & \partial_{23}^2 v_{m+1} & \partial_{33}^2 v_{m+1} \end{pmatrix}, \\
\Phi^{\mathrm{div}} &= (-1)^m \left(\Delta v_{m+1} \cdot I_3 - \Phi^{\mathrm{rot}}\right) \\
&= \begin{pmatrix} \left(\partial_{22}^2 + \partial_{33}^2\right) v_{m+1} & -\partial_{12}^2 v_{m+1} & -\partial_{13}^2 v_{m+1} \\ -\partial_{12}^2 v_{m+1} & \left(\partial_{11}^2 + \partial_{33}^2\right) v_{m+1} & -\partial_{23}^2 v_{m+1} \\ -\partial_{13}^2 v_{m+1} & -\partial_{23}^2 v_{m+1} & \left(\partial_{11}^2 + \partial_{22}^2\right) v_{m+1} \end{pmatrix}, \qquad (12) \\
\Phi &= \Phi^{\mathrm{div}} + \frac{1}{\rho}\Phi^{\mathrm{rot}} = \left(\Delta v_{m+1}\right) \cdot I_3 + \frac{1-\rho}{\rho}\Phi^{\mathrm{rot}} \\
&= (-1)^m \begin{pmatrix} \left(\frac{1}{\rho}\partial_{11}^2 + \partial_{22}^2 + \partial_{33}^2\right) v_{m+1} & \frac{1-\rho}{\rho}\partial_{12}^2 v_{m+1} & \frac{1-\rho}{\rho}\partial_{13}^2 v_{m+1} \\ \frac{1-\rho}{\rho}\partial_{12}^2 v_{m+1} & \left(\partial_{11}^2 + \frac{1}{\rho}\partial_{22}^2 + \partial_{33}^2\right) v_{m+1} & \frac{1-\rho}{\rho}\partial_{23}^2 v_{m+1} \\ \frac{1-\rho}{\rho}\partial_{13}^2 v_{m+1} & \frac{1-\rho}{\rho}\partial_{23}^2 v_{m+1} & \left(\partial_{11}^2 + \partial_{22}^2 + \frac{1}{\rho}\partial_{33}^2\right) v_{m+1} \end{pmatrix}.
\end{aligned}$$

The semiscalar product $\langle \bullet, \bullet \rangle_{m,\rho}$, the operator $P_{m,\rho}(D)$ and the tensorial function $\Phi$ are connected by the following relation (13).

THEOREM 3. *$P_{m,\rho}(D)$ and $\Phi$ being defined as above, we have*

$$P_{m,\rho}(D)\Phi = \delta \cdot I_3, \qquad \text{in } \mathcal{D}'\left(\mathbb{R}^3, \mathbb{R}^3\right). \qquad (13)$$

PROOF. Let us take the generalized Fourier transform of (each component of) $\Phi$. We get

$$\hat{\Phi}(\zeta) = \frac{(-1)^m}{(2i\pi)^{2m}\|\zeta\|^{2(m-1)}} \begin{pmatrix} \dfrac{\zeta_1^2 + \rho\left(\zeta_2^2 + \zeta_3^2\right)}{\rho\|\zeta\|^4} & \dfrac{1-\rho}{\rho}\dfrac{\zeta_1\zeta_2}{\|\zeta\|^4} & \dfrac{1-\rho}{\rho}\dfrac{\zeta_1\zeta_3}{\|\zeta\|^4} \\ \dfrac{1-\rho}{\rho}\dfrac{\zeta_1\zeta_2}{\|\zeta\|^4} & \dfrac{\zeta_2^2 + \rho\left(\zeta_1^2 + \zeta_3^2\right)}{\rho\|\zeta\|^4} & \dfrac{1-\rho}{\rho}\dfrac{\zeta_2\zeta_3}{\|\zeta\|^4} \\ \dfrac{1-\rho}{\rho}\dfrac{\zeta_1\zeta_3}{\|\zeta\|^4} & \dfrac{1-\rho}{\rho}\dfrac{\zeta_2\zeta_3}{\|\zeta\|^4} & \dfrac{\zeta_3^2 + \rho\left(\zeta_1^2 + \zeta_2^2\right)}{\rho\|\zeta\|^4} \end{pmatrix}.$$

Inverting this $3 \times 3$ matrix, we get

$$(-1)^m (2i\pi)^{2m}\|\zeta\|^{2(m-1)} \begin{pmatrix} \rho\zeta_1^2 + \zeta_2^2 + \zeta_3^2 & (\rho-1)\zeta_1\zeta_2 & (\rho-1)\zeta_1\zeta_3 \\ (\rho-1)\zeta_1\zeta_2 & \zeta_1^2 + \rho\zeta_2^2 + \zeta_3^2 & (\rho-1)\zeta_2\zeta_3 \\ (\rho-1)\zeta_1\zeta_3 & (\rho-1)\zeta_2\zeta_3 & \zeta_1^2 + \zeta_2^2 + \rho\zeta_3^2 \end{pmatrix} \hat{\Phi}(\zeta) = I_3.$$

Now, taking the inverse Fourier transform of each component, we get

$$(-1)^m \Delta^{m-1} \begin{pmatrix} \rho\partial_{11}^2 + \partial_{22}^2 + \partial_{33}^2 & (\rho-1)\partial_{12}^2 & (\rho-1)\partial_{13}^2 \\ (\rho-1)\partial_{12}^2 & \partial_{11}^2 + \rho\partial_{22}^2 + \partial_{33}^2 & (\rho-1)\partial_{23}^2 \\ (\rho-1)\partial_{13}^2 & (\rho-1)\partial_{23}^2 & \partial_{11}^2 + \partial_{22}^2 + \rho\partial_{33}^2 \end{pmatrix} \Phi = \delta \cdot I_3,$$

which is (13). ∎

REMARK. In the particular case when $\rho = 1$, we have $P_{m,\rho}(D) = (-1)^m \Delta^m \cdot I_3$ and $\Phi = (-1)^m \Delta v_{m+1} \cdot I_3 = (-1)^m v_m \cdot I_3$. So $P_{m,\rho}(D)$ is $(-1)^m$ times the $m^{\mathrm{th}}$ Laplacian operator on each diagonal term, and 0 elsewhere, while $\Phi$ is $(-1)^m$ times $v_m$ on each diagonal term, 0 elsewhere. We so have a polyharmonic spline problem on each component, and no connection between the components.

We can now give the expression of the semikernel of $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ in $\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3)$. To do so, we must use the space $\mathcal{N}^\perp \cap (\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3))'$, which is the space of compactly supported Radon measures orthogonal to $\mathcal{N} = \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$, i.e.,

$$\forall \mu \in \mathcal{N}^\perp \cap \left(\mathcal{C}^0\left(\mathbb{R}^3;\mathbb{R}^3\right)\right)', \quad \forall p \in \mathcal{C}^0\left(\mathbb{R}^3;\mathbb{R}^3\right), \quad \langle \mu \mid p\rangle = \sum_{j=1}^{3} \langle \mu_j \mid p_j\rangle = \sum_{j=1}^{3} \int_{\mathbb{R}^3} p_j(t)\, \mathrm{d}\mu_j(t)$$

and $\langle \mu \mid p\rangle = 0$ whenever $p \in \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$.

THEOREM 4. *Let* h *be the application from* $(\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3))'$ *into* $\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3)$ *defined by*

$$\forall \mu \in \left(\mathcal{C}^0\left(\mathbb{R}^3;\mathbb{R}^3\right)\right)', \qquad \mathtt{h}(\mu) = \Phi \star \mu. \tag{14}$$

*Then we have the following.*

(i) *For any* $\mu$ *in* $\mathcal{N}^\perp \cap (\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3))'$, $\mathtt{h}(\mu)$ *is in* $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$.

(ii) h *is a semikernel of* $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$, *endowed with the seminorm* $|\bullet|_{m,\rho}$, *in* $\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3)$, *and so for all* $\mu \in \mathcal{N}^\perp \cap (\mathcal{C}^0(\mathbb{R}^3;\mathbb{R}^3))'$ *and all* $f \in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$, *we have*

$$\langle \mathtt{h}(\mu), f\rangle_{m,\rho} = \langle \Phi \star \mu, f\rangle_{m,\rho} = \langle \mu \mid f\rangle. \tag{15}$$

PROOF.

(i) Condition (i) is a direct consequence of the following result, due to Amodei [2] and Benbourhim [10]: *for any boundly supported Radon measure* $\nu$, *orthogonal to* $\mathbb{P}_{m-1}(\mathbb{R}^3)$, *and for any* $\gamma \in \mathbb{N}^3$ *such that* $m+1 < |\gamma| < 2m+1$, $D^\gamma v_{m+1} \star \nu \in L^2(\mathbb{R}^3)$.

For completeness we present here the proof of this result. Using the orthogonality of $\nu$ and $\hat{v}_{m+1}(\zeta) = (-1)^{m+1}\|2\pi\zeta\|^{-2(m+1)}$, we have ($C$ and $C_1$ are two real constant numbers)

$$\forall \zeta \in \mathbb{N}^3, \qquad \|\zeta\| < 1 \quad \Longrightarrow \quad \left|(\widehat{D^\gamma v_{m+1} \star \nu})(\zeta)\right| \le C\|\zeta\|^{-m+|\gamma|-2}.$$

Now since $\nu$ is compactly supported, we also have

$$\forall \zeta \in \mathbb{N}^3, \qquad \|\zeta\| \ge 1 \quad \Longrightarrow \quad \left|(\widehat{D^\gamma v_{m+1} \star \nu})(\zeta)\right| \le C_1\|\zeta\|^{-2(m+1)+|\gamma|}.$$

So we have the result by applying the above result to each component of $\nu$.

(ii) Let us first prove (15) when $f \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$

$$\langle \mathtt{h}(\mu) \mid f\rangle_{m,\rho} = \langle \Phi \star \mu, f\rangle_{m,\rho} = \langle \operatorname{div}(\Phi \star \mu), \operatorname{div} f\rangle_{m-1} + \langle \operatorname{rot}(\Phi \star \mu), \operatorname{rot} f\rangle_{m-1}.$$

So we have, by integration by parts (in the sense of distributions),

$$\langle f, g\rangle_{m,\rho} = -\rho\langle \nabla \operatorname{div} f, g\rangle_{m-1} + \langle \operatorname{rot}\operatorname{rot} f, g\rangle_{m-1},$$

and then

$$\begin{aligned}
\langle \mathtt{h}(\mu), f\rangle_{m,\rho} &= -\rho\langle \nabla \operatorname{div}(\Phi \star \mu), f\rangle_{m-1} + \langle \operatorname{rot}\operatorname{rot}(\Phi \star \mu), f\rangle_{m-1} \\
&= -\rho\left\langle (-1)^{m-1}\Delta^{m-1}\nabla \operatorname{div}(\Phi \star \mu) \mid f\right\rangle \\
&\quad + \left\langle (-1)^{m-1}\Delta^{m-1} \operatorname{rot}\operatorname{rot}(\Phi \star \mu) \mid f\right\rangle \\
&= \langle P_{m,\rho}(D)(\Phi \star \mu) \mid f\rangle = \langle (P_{m,\rho}(D)\Phi) \star \mu \mid f\rangle.
\end{aligned}$$

which is, by using (13) $\langle \mathtt{h}(\mu), f\rangle_{m,\rho} = \langle \delta \cdot I_3 \star \mu \mid f\rangle = \langle \mu \mid f\rangle$.

So (15) is true for all $f \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$.

Now, (15) is obviously true for any $f \in \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ since then all members of (15) are trivially zero. So, using the density relation (3), (15) is true for all $f$ in $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$. ∎

### 3.4. Setting a Variational Spline Problem

Let us first recall the general theorem on variational spline (see [9,11] or [12]).

THEOREM 5. *Let $X, Y, Z$ be three Hilbert spaces. Let $u$ (respectively, $v$) be a continuous linear map from $X$ to $Y$ (respectively, from $X$ onto $Z$). Let us suppose $u(X)$ to be closed in $Y$. Let us suppose* $\operatorname{Ker} u + \operatorname{Ker} v$ *is closed in $X$ and* $\operatorname{Ker} u \cap \operatorname{Ker} v = \{0\}$.

(i) *Then, for any $z$ in $Z$, the function $\sigma$ defined by*

$$\sigma = \operatorname*{Arg\,Min}_{f \in X} \left\{ \|u(f)\|_Y^2;\ v(f) = z \right\}$$

*exists and is unique. $\sigma$ is called the interpolating spline relative to $X, Y, Z$, $u, v, z$.*

(ii) *Now, let $\langle \bullet, \bullet \rangle_X$ be the semiscalar product on $X$ defined for any $f$ and $g$ in $X$ by*

$$\langle f, g \rangle_X = \langle u(f), u(g) \rangle_Y .$$

*Let $E$ be some vectorial topological space, separated and locally convex and suppose that $(X, \langle \bullet, \bullet \rangle_X)$ is a semi-Hilbertian subspace of $E$. Let $\mathrm{h}$ be a semikernel of $X$ in $E$. Suppose $Z = \mathbb{R}^k$ for some $k \in \mathbb{N}$ and let $v_i \in E'$ for any $i = 1, \dots, k$, $(v(f))_i = \langle v_i \mid f \rangle_{E', E}$. Then there exists $a = (a_i)_{i \in [1\,:\,k]}$ in $\mathbb{R}^k$ and $p \in$* $\operatorname{Ker} u$ *such that*

$$\begin{aligned} \sigma &= \sum_{i=1}^{k} a_i \mathrm{h}(v_i) + p \\ \forall q \in \operatorname{Ker} u, \quad &\sum_{i=1}^{k} a_i \langle v_i \mid q \rangle_{E', E} = 0. \end{aligned} \tag{16}$$

(iii) *$a$ and $p$ are uniquely defined by the following linear system, where $\ell$ is the dimension of* $\operatorname{Ker} u$ *and the $(q^j)_{j \in [1\,:\,\ell]}$ are a basis of* $\operatorname{Ker} u$*:*

$$\begin{aligned} \forall j \in [1\,:\,k], \qquad & \langle v_j \mid \sigma \rangle_{E', E} = z^j, \\ \forall j \in [1\,:\,\ell], \qquad & \sum_{i=1}^{k} a_i \langle v_i \mid q^j \rangle_{E', E} = 0. \end{aligned} \tag{17}$$

Let us now present the vectorial interpolation problem as a variational spline problem; to do so we need define the space $E$, the spaces $X, Y, Z$ and their norms, the continuous maps $u$ and $v$, and check the two conditions: $\operatorname{Ker} u + \operatorname{Ker} v$ closed in $X$ and $\operatorname{Ker} u \cap \operatorname{Ker} v = \{0\}$.

Let $E = \mathcal{C}^0(\mathbb{R}^3; \mathbb{R}^3)$, and let $X = D^{-m} L^2(\mathbb{R}^3; \mathbb{R}^3)$, endowed with $\| \bullet \|_m$. Let $Y = (L^2(\mathbb{R}^3))^{m'} \times (L^2(\mathbb{R}^3; \mathbb{R}^3))^{m'}$, endowed with the scalar product $\langle \bullet, \bullet \rangle_Y$ defined for any $f, f_1$ in $(L^2(\mathbb{R}^3))^{m'}$ and $g, g_1$ in $(L^2(\mathbb{R}^3; \mathbb{R}^3))^{m'}$ by

$$\langle (f, g), (f_1, g_1) \rangle_Y = \rho \langle f, f_1 \rangle_{(L^2(\mathbb{R}^3))^{m'}} + \langle g, g_1 \rangle_{(L^2(\mathbb{R}^3; \mathbb{R}^3))^{m'}} .$$

Let $Z = (\mathbb{R}^3)^n$, endowed with the Euclidean norm in $\mathbb{R}^{3n}$. Let $u$ (respectively, $v$) be the linear map from $X$ to $Y$ (respectively, from $X$ to $Z$) defined for any $f$ in $X$ by

$$u(f) = \left( (D^\gamma \operatorname{div} f)_{|\gamma| = m-1}, (D^\gamma \operatorname{rot} f)_{|\gamma| = m-1} \right)$$

(respectively, by $\forall i \in [1\,:\,n]$, $v(f)_i = f(x^i)$).

Note that with these definitions, we have, for any $f$ and $g$ in $X$, $\langle f, g \rangle_{m, \rho} = \langle u(f), u(g) \rangle_Y$. So, we have a variational spline problem, since we have the following theorem.

THEOREM 6. *With the above notation, we have:*

(i) *$u$ is a continuous map;*
(ii) *$v$ is a continuous linear map onto $Z$;*
(iii) *$u(X)$ is closed in $Y$;*
(iv) $\operatorname{Ker} u + \operatorname{Ker} v$ *is closed in $X$;*
(v) $\operatorname{Ker} u \cap \operatorname{Ker} v = \{0\}$.

PROOF.

(i) Condition (i) is a direct consequence of the definitions of $X$, $Y$, and $u$.
(ii) Since $m \geq 2$, the injection from $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ into $C^0(\mathbb{R}^3;\mathbb{R}^3)$ is continuous; so, for all $x$ in $\mathbb{R}^3$, $\delta_x : f \in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3) \to f(x) \in \mathbb{R}^3$ is continuous; as a consequence, $v$ is continuous.

 Let us now prove that $v(X) = Z$. Let $r \leq (1/2)\min_{i\neq j} \|x^i - x^j\|$ and $\psi$ be the map from $\mathbb{R}^3$ to $\mathbb{R}$ such that $\psi(x) = \exp(-\|x\|^2/(r^2 - \|x\|^2))$ if $\|x\| < r$ and $\psi(x) = 0$ if $\|x\| \geq r$. Then for any $z \in Z$, the map $f_z$ defined by $\forall x \in \mathbb{R}^3$, $f_z(x) = \sum_{i=1}^n z^i\psi(x)$ is in $X$ and meets $v(f_z) = z$.
(iii) Let $u(f_n)_{n\in\mathbb{N}}$ be a Cauchy's series of elements of $Y$. By definition of $u$, $(D^\gamma f_n)_{n\in\mathbb{N}}$ is a Cauchy's series of elements of $L^2(\mathbb{R}^3;\mathbb{R}^3)$ for $|\gamma| = m$. So from Schwartz [8], there exists $f \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ such that $D^\gamma f = f_\gamma$ and so $f \in D^{-m}(\mathbb{R}^3;\mathbb{R}^3)$. We obtain

$$\forall \gamma \in \mathbb{N}^3, \qquad |\gamma| = m - 1 \quad \Longrightarrow \quad \begin{cases} \lim\limits_{n\to\infty} D^\gamma \operatorname{div} f_n = D^\gamma \operatorname{div} f, \\ \lim\limits_{n\to\infty} D^\gamma \operatorname{rot} f_n = D^\gamma \operatorname{rot} f, \end{cases}$$

 and so $\lim_{n\to\infty} u(f_n) \in u(X)$.
(iv) We obviously have $\operatorname{Ker} u = \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ whose dimension is finite, so $\operatorname{Ker} u$ is closed in $X$. Besides $\operatorname{Ker} v$ is closed since $v$ is continuous. So $\operatorname{Ker} u + \operatorname{Ker} v$ is closed in $X$.
(v) Besides since the set $(x^i)_{i\in[1:n]}$ is supposed to contain a $\mathbb{P}_{m-1}(\mathbb{R}^3)$-unisolvent set, any $p \in \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ such that $\forall i \in [1 : n]$, $p(x^i) = 0$ is 0. So $\operatorname{Ker} u \cap \operatorname{Ker} v = \{0\}$. ∎

## 4. INTERPOLATING VECTORIAL SPLINE

In this section, we give the general form of the interpolating vectorial spline $\sigma_{m,\rho}$ and the way how to derive its coefficients.

### 4.1. General Form of $\sigma_{m,\rho}$

We can now set the following theorem.

THEOREM 7.

(i) *Over all functions in $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ interpolating the vectorial values $(x^i, z^i)_{i\in[1:n]}$, there is one and only one which minimizes the seminorm $|\bullet|_{m,\rho}$ defined in (5).*

 *Let $\sigma_{m,\rho}$ be this function, i.e.,*

$$\sigma_{m,\rho} = \operatorname*{Arg\,Min}_{f\in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)} \left\{|f|^2_{m,\rho};\ \forall i \in [1 : n],\ f(x^i) = z^i\right\}.$$

(ii) *There exist $n$ $\mathbb{R}^3$-vectors $a^i$ ($i \in [1 : n]$) and a vectorial polynomial $p \in \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ such that ($\Phi$ was defined in (12))*

$$\begin{aligned} \sigma_{m,\rho} &= \sum_{i=1}^n \Phi\left(\bullet - x^i\right) a^i + p, \\ \forall j \in [1 : 3],\ \forall q \in \mathbb{P}_{m-1}(\mathbb{R}^3),\ &\sum_{i=1}^n a^i_j q(x^i) = 0, \end{aligned} \tag{18}$$

*which can be written in the following way, where $\sigma^j_{m,\rho}$ ($j \in [1\,:\,3]$) are the three components of $\sigma_{m,\rho}$ and for all $x$ in $\mathbb{R}^3$, $v_{m+1}(x) = (-1/4\pi(2m)!)\|x\|^{2m-1}$ and where $p_1, p_2, p_3$ are in $\mathbb{P}_{m-1}(\mathbb{R}^3)$*

$$
\begin{aligned}
\sigma^1_{m,\rho} &= (-1)^m \left(\sum_{i=1}^n a^i_1 \left(\frac{1}{\rho}\partial^2_{11} + \partial^2_{22} + \partial^2_{33}\right) v_{m+1}\left(\bullet - x^i\right) + \frac{1-\rho}{\rho}\sum_{i=1}^n a^i_2 \partial^2_{12} v_{m+1}\left(\bullet - x^i\right)\right.\\
&\quad \left. + \frac{1-\rho}{\rho}\sum_{i=1}^n a^i_3 \partial^2_{13} v_{m+1}\left(\bullet - x^i\right)\right) + p_1,\\
\sigma^2_{m,\rho} &= (-1)^m \left(\frac{1-\rho}{\rho}\sum_{i=1}^n a^i_1 \partial^2_{12} v_{m+1}\left(\bullet - x^i\right) + \sum_{i=1}^n a^i_2 \left(\partial^2_{11} + \frac{1}{\rho}\partial^2_{22} + \partial^2_{33}\right) v_{m+1}\left(\bullet - x^i\right)\right.\\
&\quad \left. + \frac{1-\rho}{\rho}\sum_{i=1}^n a^i_3 \partial^2_{23} v_{m+1}\left(\bullet - x^i\right)\right) + p_2,\\
\sigma^3_{m,\rho} &= (-1)^m \left(\frac{1-\rho}{\rho}\sum_{i=1}^n a^i_1 \partial^2_{13} v_{m+1}\left(\bullet - x^i\right) + \frac{1-\rho}{\rho}\sum_{i=1}^n a^i_2 \partial^2_{23} v_{m+1}\left(\bullet - x^i\right)\right.\\
&\quad \left. + \sum_{i=1}^n a^i_3 \left(\partial^2_{11} + \partial^2_{22} + \frac{1}{\rho}\partial^2_{33}\right) v_{m+1}\left(\bullet - x^i\right)\right) + p_3.
\end{aligned}
$$

PROOF. $\sigma_{m,\rho}$ is the interpolating variational spline relative to $X, Y, Z, u, v, z$ as defined in Section 3.4. Equation (18) is then the direct application of (16) by using expression (14) of the semikernel $\mathbf{h}$ of $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$; since $(D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3), |\bullet|_{m,\rho})$ is a semi-Hilbertian subspace of $E = C^0(\mathbb{R}^3;\mathbb{R}^3)$ (Theorem 2). ∎

### 4.2. Deriving $\sigma_{m,\rho}$

The coefficients $a^i$ and the polynomial $p \in \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ used in (18) can be derived via the following theorem.

THEOREM 8. *Let $m' = \binom{m+2}{3}$ be the dimension of $\mathbb{P}_{m-1}(\mathbb{R}^3)$. Let $(q^{i'})_{i'\in[1\,:\,m']}$ be a basis of $\mathbb{P}_{m-1}(\mathbb{R}^3)$, and $b$ be the matrix of the coefficients of $p \in \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ by using this basis for each component of $p$, i.e., $\forall\, j \in [1\,:\,3]$, $p_j = \sum_{i=1}^{m'} b^i_j q^i$. Let $M$ be the $3n \times 3m'$ matrix defined by*

$$
\begin{gathered}
\forall\, i \in [1\,:\,n], \quad \forall\, i' \in [1\,:\,m'], \quad \forall\, j \in [1\,:\,3], \qquad M_{3(i'-1)+j,3(i-1)+j} = q^{i'}\left(x^i\right),\\
M_{k\ell} = 0, \qquad \text{if } l \neq k \pmod 3.
\end{gathered}
$$

*Let $\overline{\Phi^{\mathrm{div}}}$ and $\overline{\Phi^{\mathrm{rot}}}$ be the $3n \times 3n$ matrices defined by*

$$
\begin{aligned}
\forall i, i' \in [1\,:\,n], \quad \forall j, j' \in [1\,:\,3], \qquad \overline{\Phi^{\mathrm{div}}}_{3(i-1)+j,3(i'-1)+j'} &= \left(\Phi^{\mathrm{div}}\left(x^i - x^{i'}\right)\right)_{j,j'},\\
\overline{\Phi^{\mathrm{rot}}}_{3(i-1)+j,3(i'-1)+j'} &= \left(\Phi^{\mathrm{rot}}\left(x^i - x^{i'}\right)\right)_{j,j'}.
\end{aligned}
$$

*Let $\overline{\Phi}$ be the matrix*

$$\overline{\Phi} = \overline{\Phi^{\mathrm{div}}} + \frac{1}{\rho}\overline{\Phi^{\mathrm{rot}}}.$$

*Let $\mathbf{a}$ be the $3n$ vector defined by*

$$\forall\, i \in [1\,:\,n], \quad \forall\, j \in [1\,:\,3], \qquad \mathbf{a}_{3(i-1)+j} = \left(a^i\right)_j,$$

b *be the* $3m'$ *vector defined by*

$$\forall i \in [1 : m'], \quad \forall j \in [1 : 3], \qquad \mathbf{b}_{3(i-1)+j} = \left(b^i\right)_j,$$

z *be the* $3n$ *vector defined by*

$$\forall i \in [1 : n], \quad \forall j \in [1 : 3], \qquad \mathbf{z}_{3(i-1)+j} = \left(z^i\right)_j,$$

*and let* $\mathbf{0}$ *be the* $3m'$ *vector all components of which are 0.*

*Then, the vectors* $a^i$ $(i \in [1 : n])$ *and the coefficients* $b^i_j$ *of the polynomial* $p \in \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ *in equation (18) meet the following linear system, whose size is* $3n + 3m'$:

$$\begin{pmatrix} \overline{\Phi} & M \\ M^t & 0 \end{pmatrix} \begin{pmatrix} \mathrm{a} \\ \mathrm{b} \end{pmatrix} = \begin{pmatrix} \mathbf{z} \\ \mathbf{0} \end{pmatrix}. \tag{19}$$

PROOF. This is the particular case of Theorem 5(iii), applied to the vector $\sigma_{m,\rho}$ defined by Theorem 7. ∎

### 4.3. Remarks

#### Similarity with radial basis functions

It is worthwhile to compare properties (18) of the proposed vectorial interpolant $\sigma_{m,\rho}$ and the radial basis function in the case of scalar interpolation.

Form (18) of $\sigma_{m,\rho}$ is very similar to the general form of a radial basis function, say $\sigma = \sum_{i=1}^n \lambda_i \varphi(\bullet - x^i) + q_{m-1}$: instead of the radial function $\varphi$, we now have a $3 \times 3$ tensor function $\Phi$; each component of $\Phi$ is not radial, but is easily derived as a sum of derivatives of the common radial function $v_{m+1}$ (see definitions (12) of $\Phi$ and (11) of $v_{m+1}$). Instead of the scalar coefficients $\lambda_i$, we now have vectorial coefficients $a^i$; for scalar radial basis functions, the vector $\lambda$ meets the orthogonal relation $\forall q \in \mathbb{P}_{m-1}(\mathbb{R}^d)$, $\sum_{i=1}^n \lambda_i q(x^i) = 0$ ($d$ is the dimension of the space), while the second equation in (18) is similar to each of the vectors formed by the first (respectively, the second, the third) component of each vectorial coefficient $a^i$. Lastly, we now have a vectorial polynomial $p$ instead of a scalar polynomial $q_{m-1}$. So we clearly see the difference, but also the similarity between interpolating each component of the $z^i$ data independently from each other (and so find, for example, true radial functions in each component), and interpolating globally all components of the $z^i$ data and taking into account, via the minimized seminorm $|\bullet|_{m-1}$, interactions between them.

As a consequence, the form of the linear system (18) is very similar to the usual one for radial basis functions; it also presents the same drawbacks, such as high condition number when data are numerous, and the way of deriving other linear systems is presently under study (see Section 6.4).

As a consequence of form (18) and as mentioned below (choice of $m$ and $\rho$), $\sigma_{m,\rho}$ is $C^\infty$ everywhere except on the data points, which is an important property of radial basis functions.

As for the differential operator $P_{m,\rho}(D)$, we can compare the relation $P_{m,\rho}(D)\Phi = \delta \cdot I_3$ (see (13)) with the vectorial equivalent of the one obtained for $m$-harmonic splines $\Delta^m \varphi = (-1)^m \delta$ (or its Fourier equivalent $\hat{\varphi} = \| \bullet \|^{2m-d}$), or more generally for a $r$-harmonic spline ($r \in \mathbb{R}_+$, $r > d/2$ where $d$ is the dimension of the space) $\hat{\varphi} = \| \bullet \|^{2r}$.

#### Particular case when $\rho = 1$

When $\rho = 1$, the differential operator $P_{m,\rho}(D)$ reduces to $(-1)^m \Delta^m$ on each diagonal term and to 0 on each nondiagonal term (see (10)). As a consequence, $\Phi(x)$ reduces to $v_m(x) = C\|x\|^{2m-3}$ ($v_m$ is such that $\Delta^m v_m = \delta$, $C$ is a constant) on each diagonal term (see (12)). So $\sigma_{m,\rho}$ reduces to a radial basis function on each component, which is precisely the $m$-harmonic spline interpolating the corresponding component of the data $(z^i)_{i \in [1:n]}$. So when $\rho = 1$, the vectorial interpolation problem reduces to three independent scalar interpolation problems, one for each component of the data, and yields a radial basis function ($m$-harmonic spline) for each one.

### Choice of $m$ and $\rho$

Of course, the choice of $m$ and $\rho$ does influence the resulting $\sigma_{m,\rho}$. The influence of $m$ is quite clear and usual, since $\sigma_{m,\rho}$ is in $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$, which is included in $C^{m-3/2}(\mathbb{R}^3;\mathbb{R}^3)$. So the choice of $m$ mainly depends on the regularity we require for the interpolant. Let us recall that $\sigma_{m,\rho}$ is actually $C^\infty$ everywhere except at the data point where it is actually $C^{m-3/2}$.

The choice of $\rho$ is somewhat less simple. Remember that a small value of $\rho$ implies a small value of $|\operatorname{rot}\sigma_{m,\rho}|_{m-1}$, whereas a big value of $\rho$ implies a small value of $|\operatorname{div}\sigma_{m,\rho}|_{m-1}$. More precisely, it can be proven that $|\operatorname{rot}\sigma_{m,\rho}|_{m-1}$ is a continuous increasing function of $\rho$, while $|\operatorname{div}\sigma_{m,\rho}|_{m-1}$ is a continuous decreasing function of $\rho$, and that we also have $\lim_{\rho\to 0}|\operatorname{rot}\sigma_{m,\rho}|_{m-1} = 0$, $\lim_{\rho\to\infty}|\operatorname{div}\sigma_{m,\rho}|_{m-1} = 0$.

All this can guide the user for the choice of $\rho$. In particular, if the data are supposed to be derived as $z^i = f(x^i)$ where $f$ is some function for which $|\operatorname{div} f|_{m-1}/|\operatorname{rot} f|_{m-1}$ is known, the user can choose $\rho$ iteratively so that the ratio $|\operatorname{div}\sigma_{m,\rho}|_{m-1}/|\operatorname{rot}\sigma_{m,\rho}|_{m-1}$ be sufficiently close to $|\operatorname{div} f|_{m-1}/|\operatorname{rot} f|_{m-1}$. Let us mention that for a given $\sigma_{m,\rho}$, the evaluation of $|\operatorname{div}\sigma_{m,\rho}|_{m-1}$ and $|\operatorname{rot}\sigma_{m,\rho}|_{m-1}$ is quite easy since it can be proved that $|\operatorname{div}\sigma_{m,\rho}|^2_{m-1} = (1/\rho^2)\mathbf{a}^t\overline{\Phi^{\mathrm{rot}}}\mathbf{a}$ and $|\operatorname{rot}\sigma_{m,\rho}|^2_{m-1} = \mathbf{a}^t\overline{\Phi^{\mathrm{div}}}\mathbf{a}$. The monotonicity of the function $\rho \mapsto |\operatorname{div}\sigma_{m,\rho}|^2_{m-1}/|\operatorname{rot}\sigma_{m,\rho}|^2_{m-1}$ makes the iterations easy to be done.

Note, besides, that if the user wants $\operatorname{div} f = 0$ or $\operatorname{rot} f = 0$, he has to use other spaces and other functions, as shown in Section 6.2.

## 5. NUMERICAL RESULTS

We made a great number of tests from which we extract the following two, done by using Matlab. Before giving comments on the graphs, we must specify the problem of the representation of functions from $\mathbb{R}^3$ to $\mathbb{R}^3$.

### 5.1. Graphical Representation of Vectorial $\mathbb{R}^3$-Functions

It is quite a challenge to represent in a significative way a function $f$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ on a sheet. Actually, in order to be meaningful, we must represent only a part of the whole information in $f$. We chose here two different ways.

(a) Represent $\|f(x)\|$ in $\mathbb{R}^3$, or more precisely on some plans of $\mathbb{R}^3$. The representation of $\|f(x)\|$ in each plan is made both by colorscale (or grayscale) and by contour lines. Though the information is not complete (direction of $f(x)$ is not at all represented), it allows rather fair comparisons between close-by functions. This is what we used for the first numerical example, choosing the plans $x = 10$, $y = -3$, $z = 0$, and $z = -3$ since the function is studied in $]0.1, 10] \times [-3, 3]^2$. This uses the Matlab function `slice`.

(b) Represent some planar slices of $f$, each value of $f(x)$ being represented as an arrow proportional to the projection of $f$ on the plan. This is what we used in the second numerical example, since it is the best way to "see" some rotational or divergence in $f$. This uses the Matlab function `quiver3`.

For both representation types, stars represent the interpolation points which are on the shown plans.

### 5.2. The `flow` Function

The `flow` function from Matlab models the speed profile of a submerged jet in an infinite tank. The interpolation points are on a $5 \times 5 \times 5$ grid covering the domain $[0.1, 10] \times [-3, 3]^2$ regularly in each direction. Four graphs are given.

First (Figure 1), the original `flow` as given by Matlab. The second graph (Figure 2) is obtained by the Matlab routine `interp3` with the options "`spline`"; this routine does $\mathbb{R}^3$ scalar interpolation, here by using splines, independent for each component of the function. The third graph

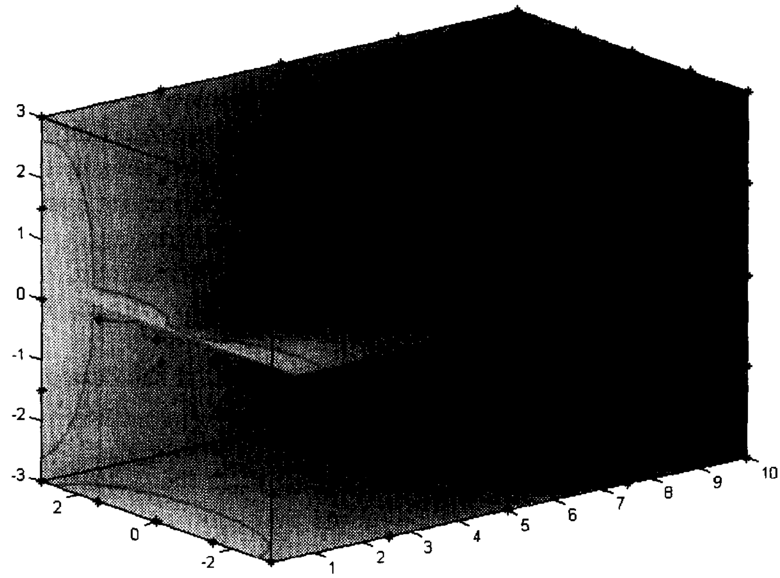


Figure 1. Theoretical **flow**.

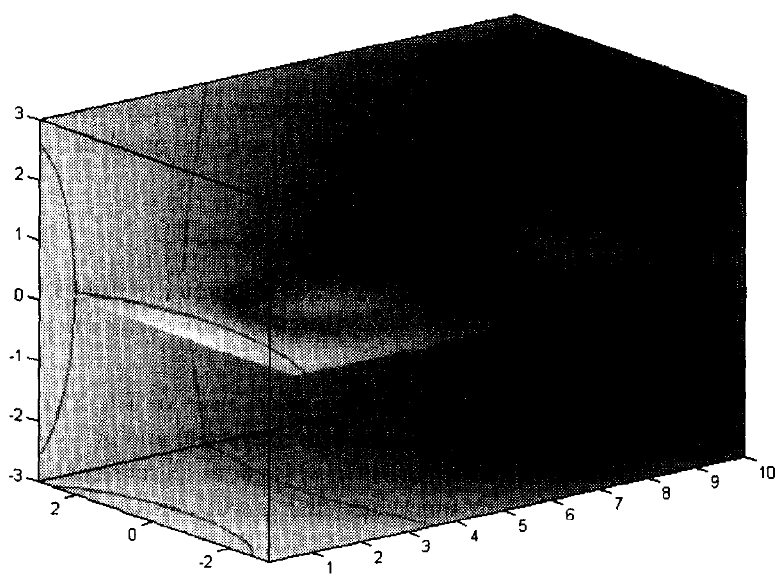


Figure 2. **interp3** with 125 data.

(Figure 3) gives $\sigma_{2,1}$, i.e., does three-dimensional biharmonic spline interpolation independently on each component. Lastly, the fourth graph (Figure 4) gives $\sigma_{2,\overline{\rho}}$, where $\overline{\rho}$ is computed such that $|\operatorname{div}\sigma_{2,\overline{\rho}}|_1/|\operatorname{rot}\sigma_{2,\overline{\rho}}|_1 \simeq |\operatorname{div}\mathtt{flow}|_1/|\operatorname{rot}\mathtt{flow}|_1$ which is $\overline{\rho} \simeq 0.2821$. We clearly see in this result that $\sigma_{2,\overline{\rho}}$ gives a result closer to the original `flow` function than the one given by the method used by **interp3**. $\sigma_{2,1}$ is quite close to $\sigma_{2,\overline{\rho}}$, which is not surprising since $\overline{\rho} \simeq 0.2821$ which can be considered as quite close to one. Results obtained for very small or very large $\rho$ (such as $10^{-8}$ or $10^{8}$) are not shown here, but they were not better than the **interp3** one.

### 5.3. Function $\nabla \sin \| \bullet \|$

In this second example, we wanted to show the interest of using the proposed method when the underlying data is irrotational or divergence free. We generate an irrotational function by $f = \nabla g$ for some function $g$ from $\mathbb{R}^3$ to $\mathbb{R}$. We chose $g = \sin \| \bullet \|$. Interpolation points are $6 \times 6 \times 6$

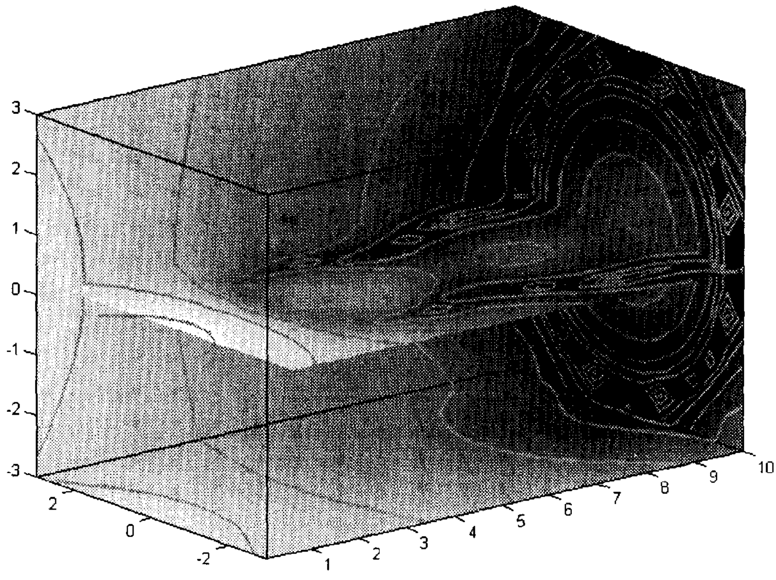


Figure 3. $\sigma_{2,1}$ with 125 data.

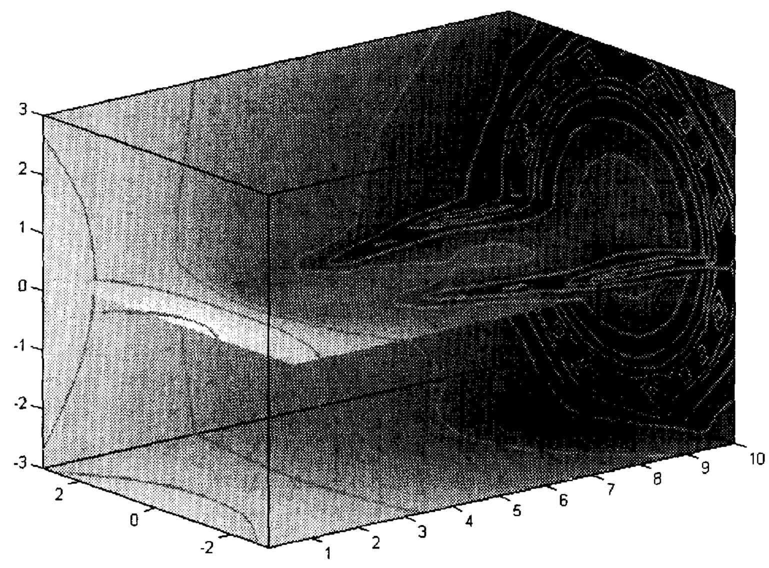


Figure 4. $\sigma_{2,\overline{\rho}}$ with 125 data.

on a regular grid covering $[-\pi,\pi]^3$, and we show the slices of $f$ (Figure 5) and interpolants for $z=\pi/5$ (in order to have interpolation points in the slice).

Choosing $\rho = 10^{-8}$ (Figure 6), we recover quite well the general shape of the function (a source in 0 and a circular sink close to $\|x\| = \pi/2$).

Choosing $\rho = 10^8$, we have Figure 7. No rotational is obvious in this figure, but the divergence seems to be constant (which is correct since $|\operatorname{div}\sigma_{m,10^8}|_1 \simeq 0$), and the circular sink completely disappeared. This shows the influence of $\rho$. Choosing now $\rho = 1$, we have Figure 8. Though we still have the source and the circular sink, they are less important than in the original function, and in $\sigma_{2,10^{-8}}$ and the interpolant recovers the original function in a poorer way.

So we see that, in this example, we get better result when choosing a $\rho$ adapted to the underlying problem and shows the interest of taking into account the interaction between the components of the interpolant to derive a good interpolant.

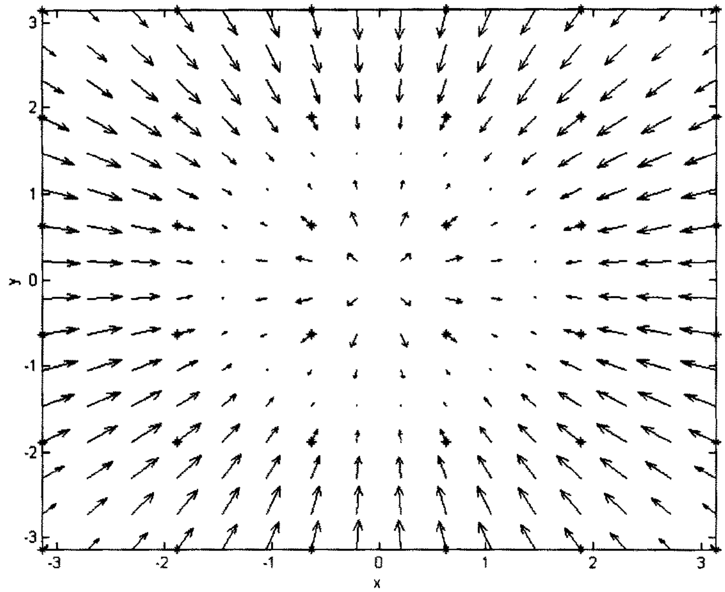


Figure 5. $\nabla \sin(\|x\|)$.

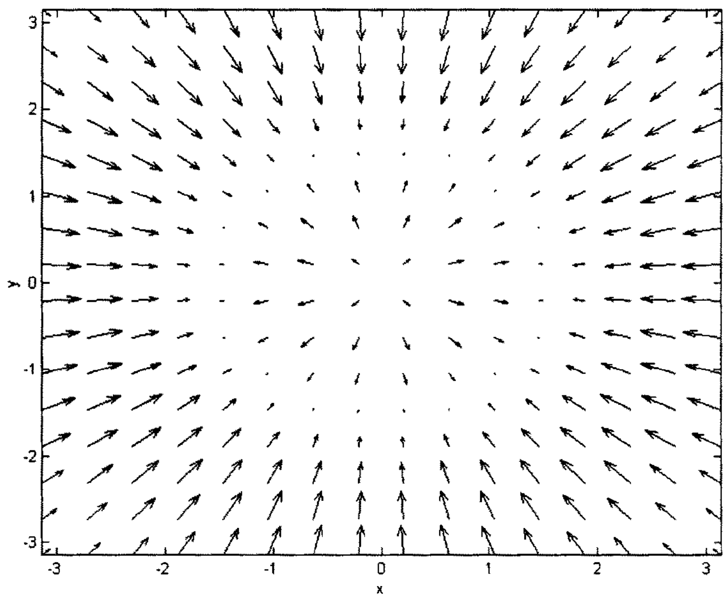


Figure 6. $\sigma_{2,10^{-8}}$ with 216 data.

## 6. CONCLUSION

The family of functions $\sigma_{m,\rho}$, for various $m$ and $\rho$, are an extension of splines for vectorial interpolation. Their form is rather close to the form of radial basis functions. The parameter $m$ governs the regularity of $\sigma_{m,\rho}$, while $\rho$ governs the quantity of the divergence part or of the rotational part in it (more precisely the ratio $|\mathrm{div}\, \sigma_{m,\rho}|_{m-1}/|\mathrm{rot}\, \sigma_{m,\rho}|_{m-1}$). For any fixed integer $m \geq 2$ and nonnegative real number $\rho$, and for any data points $(x^i, z^i)_{i\in[1:n]}$ whose locations $(x^i)_{i\in[1:n]}$ form a $\mathbb{P}_{m-1}(\mathbb{R}^3)$-unisolvent set, there exists one and only one $\sigma_{m,\rho}$ interpolating the data.

For $m$ and $\rho$ fixed (and eventually $(x^i)_{i\in[1:n]}$ fixed), the set of $\sigma_{m,\rho}$ form a vectorial space which can be used for most vector approximation problems, as are the spline spaces and spaces of the radial basis functions.

We now present some extensions of this work.

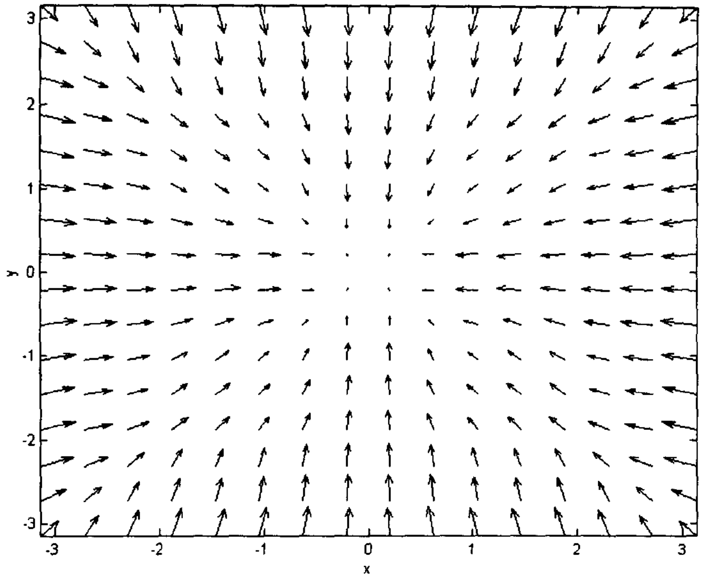


Figure 7. $\sigma_{2,10^8}$ with 216 data.

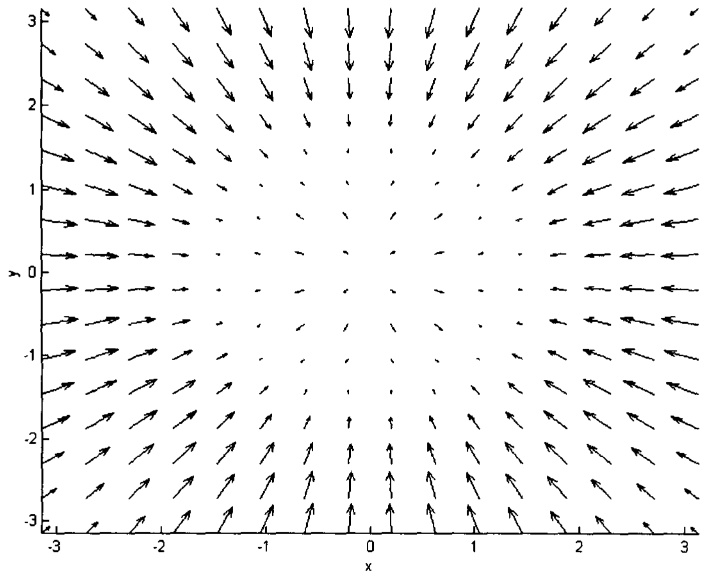


Figure 8. $\sigma_{2,1}$ with 216 data.

## 6.1. Extension Similar to the Radial Basis Functions: "Shifted $\sigma_{m,\rho}$"

As it has been done at the beginning of radial basis functions theory for scalar functions, the solution shown in this paper can be regularized by using the infinitely derivable function $v_{m+1,c} = (\|x\|^2 + c^2)^{m-1/2}$ instead of the function $v_{m+1}$ in the corresponding tensor functions $\Phi^{\text{div}}$, $\Phi^{\text{rot}}$, and $\Phi$, deriving so a $\mathcal{C}^\infty$ function $\sigma_{m,\rho,c}$ in the form (18).

The vectorial coefficients $(a^i)_{i\in[1:n]}$ and the coefficients $(b^i)_{[1:m']}$ of the vectorial polynomial $p_{m-1}$ are then derived from the interpolating conditions $\sigma_{m,\rho}(x^i) = z^i$ and the orthogonality conditions in (18), i.e., by the linear system (19) (where $v_{m+1}$ is replaced by $v_{m+1,c}$). The same reflections as usual can be made about the choice of $c$ and its influence on the regularity and the form of the so-obtained $\sigma_{m,\rho,c}$, on the condition number of the linear system (19).

Most of the interpolating properties of the vectorial spline presented in this paper should still be valid, but of course, by such a modification, we lose two important properties: the minimization property and the influence of $\rho$ on $\operatorname{div}\sigma_{m,\rho}$ and $\operatorname{rot}\sigma_{m,\rho}$.

### 6.2. Interpolant with the Condition $\operatorname{div}\sigma = 0$ or $\operatorname{rot}\sigma = 0$

The method and the interpolant presented in this paper uses $|\operatorname{div} f|_{m-1}$ and $|\operatorname{rot} f|_{m-1}$, which allows the user to govern in some way these quantities, and, in particular, for extreme values of $\rho$ (0 or $\infty$), to ensure that $\operatorname{rot}\sigma_{m,\rho}$ or $\operatorname{div}\sigma_{m,\rho}$ is polynomial. This method cannot derive $\sigma_{m,\rho}$ such that $\operatorname{div}\sigma_{m,\rho} \equiv 0$ or $\operatorname{rot}\sigma_{m,\rho} \equiv 0$ (independently of the regularity we may require for the interpolating function; besides $m = 1$ is not allowed since the elements of $D^{-1}L^2(\mathbb{R}^3;\mathbb{R}^3)$ are not all continuous, and so we have no more Hilbertian subspaces, variational splines, :, and everything collapses).

However, by considering for $m \geq 2$ the spaces $D^{-m}L^2_{\operatorname{div}=0} = \{f \in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3);\ \operatorname{div} f = 0\}$ and $D^{-m}L^2_{\operatorname{rot}=0} = \{f \in D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3);\ \operatorname{rot} f = 0\}$, we can prove they are Hilbertian subspaces of $D^{-m}L^2(\mathbb{R}^3;\mathbb{R}^3)$ and we can derive their semikernels, and so obtain divergence-free or rotational-free interpolating functions. The method to solve this problem is different from the one presented in this paper since the equivalent of the differential operator $P_{m,\rho}(D)$ is not invertible and so we cannot derive a fundamental solution of it (the equivalent of $\Phi$). This will be presented in a forthcoming paper.

### 6.3. Towards PDEs

It is expected that the vectorial spline $\sigma_{m,\rho}$ (and the function $\sigma_{m,\rho,c}$ as defined in Section 5.1) can be used in order to solve vectorial PDEs, in a similar way as radial basis functions are used to solve scalar PDEs. The coefficient $\rho$ can be used in order to govern the ratio $|\operatorname{div} f|_{m-1}/|\operatorname{rot} f|_{m-1}$ independently of the PDE. No experimentation has yet been done in that direction.

### 6.4. Quasi-Interpolant and Improvement of the Condition Number

Looking at the general form of $\sigma_{m,\rho}$ (18) and the relation $P_{m,\rho}(D)\Phi = \delta \cdot I_3$ (see (13)), we may think that we can derive a quasi-interpolant by applying a discretization of $P_{m,\rho}(D)$, say $P_{m,\rho}(D)_h$, to $\Phi$.

This is what actually happens on equidistant grids: as shown in [6], let for $k < m$ the tensor function $B^{m\rho}_{h,k} = P_{m,\rho}(D)_h\Phi$, where $P_{m,\rho}(D)_h$ is a $\mathbb{P}_{2k+1}$-exact discretization of $P_{m,\rho}(D)$, i.e., $P_{m,\rho}(D)_h$ is in the form $P_{m,\rho}(D)_h f = \sum_{\ell\in\mathbb{Z}^3, |\ell|\leq k} \lambda_\ell f(\bullet - \ell h)$ where the $\lambda_\ell$ are real coefficients, and for any $p \in \mathbb{P}_{2k+1}(\mathbb{R}^3;\mathbb{R}^3)$, $P_{m,\rho}(D)_h p = P_{m,\rho}(D)p$. Then $B^{m,\rho}_{h,k}$ has many properties of cardinal B-spline functions (or cardinal quasi-interpolants), as for example if $\sigma^{m,\rho}_{h,k;f} = \sum_{i\in\mathbb{Z}^3} B^{m,\rho}_{h,k}(\bullet - ih)f(ih)$, then for any $f \in \mathcal{C}^{k'}(\mathbb{R}^3;\mathbb{R}^3)$, with $k' \leq 2k+1$, $\|\sigma^{m,\rho}_{h,k;f} - f\|_\infty \underset{h\to 0}{=} \mathcal{O}(h^{k'})$, and $\forall p \in \mathbb{P}_{2k+1}$, $\sigma^{m,\rho}_{h,k;p} \equiv p$.

As for scattered data, the results given in [13] show that in one dimension, a discretization based on the $(x^i)_{i\in[1:n]}$, of the equivalent of $P_{m,\rho}(D)$ applied to the equivalent of $v_{m,\rho}$ or $v_{m,\rho,c}$ (see Section 6.1) gives bell-shaped functions which can be used to improve the form of the linear system to be used for deriving a $\sigma_{m,\rho}$ or $\sigma_{m,\rho,c}$ interpolating the data, in a similar way as proposed by David *et al.* in [14]. We expect to obtain similar results by discretizing $P_{m,\rho}(D)$ on $(x^i)_{i\in[1:n]}$.

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