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On the deformation quantization of symplectic orbispaces

Markus J. Pflaum

Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, 60054 Frankfurt/Main, Germany

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Abstract

In the first part of this article we provide a geometrically oriented approach to the theory of orbispaces originally introduced by G. Schwarz and W. Chen. We explain the notion of a vector orbibundle and characterize the good sections of a reduced vector orbibundle as the smooth stratified sections. In the second part of the article we elaborate on the quantizability of a symplectic orbispace. By adapting Fedosov's method to the orbispace setting we show that every symplectic orbispace has a deformation quantization. As a byproduct we obtain that every symplectic orbifold possesses a star product.

MSC: 53D55; 57S15; 32S99

Keywords: Orbispaces; Orbifolds; Deformation quantization

Introduction

Deformation quantization has been introduced into mathematical physics by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [1] more than 25 years ago. Since then, various existence and classification results for star products on a symplectic or Poisson manifold have appeared [7,8,11]. A common feature of all these approaches is that the space to be quantized is not allowed to have singularities. But many symplectic or Poisson spaces with strong relevance for mathematical physics are singular. For instance, the phase spaces appearing in gauge theory or obtained by symplectic reduction are in general not smooth and possess singularities. According to the work of Sjamaar and Lerman [22] such singular symplectically reduced spaces are stratified spaces, where each stratum carries a canonical symplectic structure. So the natural question arises, whether an arbitrary symplectic or Poisson stratified spaces has a deformation quantization as well. In this work we consider a particular class of Poisson

E-mail address: pflaum@math.uni-frankfurt.de (M.J. Pflaum).

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spaces with singularities, namely symplectic orbispaces, and show for this class the existence of a star product. We achieve this by generalizing Fedosov's construction to the orbispace setting.

Originally, orbispaces have been introduced by Schwarz [19] in his PhD thesis under the name of "generalized orbit spaces" or "Q-manifolds". The name orbispaces first appeared in the work of Chen [5], where an independent and more topologically oriented approach to a theory of generalized orbit spaces has been set up. By definition, orbispaces are topological spaces which locally look like orbit spaces of compact Lie group actions. Thus, orbispaces comprise a natural generalization of orbifolds, and our results imply in particular that every symplectic orbifold carries a star product.

Our article is set up as follows. In the first section we recall the notion of a stratification and elaborate on the canonical stratification of an orbit space by orbit types. Moreover, we introduce profinite dimensional manifolds and differentiable categories with slices. Both concepts will be needed later in the definition of a (possibly infinite dimensional) orbispace.

In Section 2 we provide an introduction to orbispaces. Since the applications we have in mind are of a differential geometric nature we have adapted the original approaches of Schwarz [19] and Chen [5] to our needs. Moreover, the approach presented here allows infinite dimensional orbispaces. Concerning the subcategory of orbifolds let us mention that we do not make any restrictions on the codimension of the fixed point sets of the local isotropy groups of the orbifold. This entails in particular that manifolds with boundary or with corners can be regarded as orbifolds. In the second part of Section 2 we introduce the notion of a vector orbibundle and of a reduced respectively good orbibundle. The main result is Theorem 2.13, where we show that a continuous section of a reduced vector orbibundle is a good section in the sense of Ruan [16], if and only if it is a stratified section which extends to a (vertical) derivation of the algebra of smooth functions on the orbibundle. Theorem 2.13 is essentially a consequence of the smooth isotopy lifting theorem of Schwarz [21].

In the third section we introduce riemannian and symplectic orbispaces. Moreover, we explain what to understand by a metric respectively symplectic connection and show that for every symplectic orbispace there exist symplectic connections. The explicit definition what to understand by a deformation quantization respectively a star product on a symplectic orbispace is also contained in Section 3. In Section 4 we construct a star-product on a symplectic orbispace by localizing Fedosov's method to the orbispace charts of an appropriate orbispace atlas. It is a consequence of Theorem 2.13 that this idea works, indeed. In some more detail, we introduce the formal Weyl algebra bundle over a symplectic orbispace and, given a symplectic connection, construct a flat connection for this bundle. The fiberwise Weyl–Moyal product on its space of flat sections then gives rise to a star product for the symplectic orbispace.

1. Preliminaries

1.1. Stratifications

In the presentation of the basics of stratification theory we follow Mather [12] (see also Pflaum [15, Chapter 1] for further details).

By a *decomposition* of a paracompact second countable topological Hausdorff space X one understands a locally finite partition $\mathcal{Z}$ of X into locally closed subspaces $S \subset X$ called *pieces* such that the following conditions are satisfied:

(DEC1) Every piece $S \in \mathcal{Z}$ is a smooth manifold in the induced topology.
(DEC2) (*condition of frontier*) If $R \cap \overline{S} \neq \emptyset$ for a pair of pieces $R, S \in \mathcal{Z}$, then $R \subset \overline{S}$. In this case one calls R *incident* to S, or a *boundary stratum* of S.

Obviously, the incidence relation is a partial order on the set of pieces. The set of decompositions of X is partially ordered by the "coarser"-relation. Hereby, a decomposition $\mathcal{Z}_1$ of X is called *coarser* than a decomposition $\mathcal{Z}_2$, if every stratum of $\mathcal{Z}_2$ is contained in a stratum of $\mathcal{Z}_1$.

By a *stratification* of X one now understands a mapping $\mathcal{S}$ which associates to every $x \in X$ the set germ $\mathcal{S}_x$ of a closed subset of X such that the following axiom is satisfied:

(STRA) For every $x \in X$ there is a neighborhood U of x and a decomposition $\mathcal{Z}$ of U such that for all $y \in U$ the germ $\mathcal{S}_y$ coincides with the set germ of the piece of $\mathcal{Z}$ of which y is an element.

The pair $(X, \mathcal{S})$ then is called a *stratified space*. Obviously, a decomposition $\mathcal{Z}$ induces a stratification of X. The following proposition shows that the converse holds true as well; a proof of this result can be found in [15, Proposition 1.2.7].

1.2. Proposition. *Let $\mathcal{S}$ be a stratification on X. Then there exists a coarsest decomposition $\mathcal{Z}_{\mathcal{S}}$ of X inducing $\mathcal{S}$.*

We will denote the decomposition $\mathcal{Z}_{\mathcal{S}}$ by $\mathcal{S}$ as well. Its pieces will be called *strata*.

1.3. Stratification of orbit spaces

Let G be a Lie group acting properly on a smooth manifold M. Denote for every compact subgroup $H \subset G$ by M_H the submanifold of all points of M having isotropy group equal to H and by $M_{(H)}$ the submanifold of all $x \in M$ having isotropy group conjugate to H. If $M_{(H)} \neq \emptyset$, we say that the conjugacy class (H) is an *orbit type* of M. The following propositions hold true.

(1) If M/G is connected, there exists a compact subgroup $G^\circ \subset G$ such that the subsets $M_{(G^\circ)} \subset M$ and $M_{(G^\circ)}/G \subset M/G$ are both open and dense. The set $M_{(G^\circ)}/G$ is connected. Moreover, for every $x \in M$ the group G° is conjugate to a subgroup of the isotropy group G_x.
(2) The mapping $\mathcal{S}$ which associates to every x the set germ of $M_{(G_x)}$ is a stratification of M. Moreover, the mapping which associates to every orbit Gx the set germ of $M_{(G_x)}/G$ is a stratification of the orbit space M/G. The thus defined stratifications are called the stratification of M respectively M/G by *orbit types*. The open stratum $M_{(G^\circ)}$ is called the *principal stratum* of M and will be denoted by M°.
(3) If M/G is connected, then the largest normal subgroup of G contained in G° coincides with the kernel of the canonical homomorphism $G \to \mathrm{Diff}(M)$.
(4) If G is a finite group and M is connected, then G° is a normal subgroup and $G^\circ \subset G_y$ for every $y \in M$. Moreover, G acts effectively on M, if and only if G° is trivial.

Proof. Proposition (1) is the well-known principal orbit type theorem due to Montgomery, Samelson and Zippin [14]; see also Bredon [3] or [15, Section 4.3] for details. A proof of (2) can be found in Bierstone [2, Chapter 2] or [15, Section 4.3].

Let us show (3). To this end consider the canonical homomorphism $G \to \mathrm{Diff}(M)$ of G in the diffeomorphism group of M. Let L be its kernel. By definition of L one has $L \subset gG^\circ g^{-1}$ for all $g \in G$. On the other hand, because $(G^\circ) \subset (G_y)$ for all $y \in M$, the inclusion $\bigcap_{g\in G} gG^\circ g^{-1} \subset L$ holds as well, hence $L = \bigcap_{g\in G} gG^\circ g^{-1}$.

Now we come to (4) and assume that G is finite. We will show that the isotropy groups of all $x \in M^\circ$ coincide. Clearly, this suffices to prove (4). So let M^1 be the stratum of M of codimension 1. Then $M^\circ \cup M^1$ is a connected open subspace of M, as the complement can be decomposed in strata of codimension $\geqslant 2$. According to the slice theorem there exists for every point $x \in M$ an open connected neighborhood U_x which can be mapped by a G_x-equivariant diffeomorphism onto a G_x-invariant open ball around the origin of a G_x-representation space E_x. Now, if $x \in M^\circ$, then every point $z \in U_x$ lies in M° again and has isotropy group equal to G_x. In case $x \in M^1$, we will consider the representation space E_x to prove that the isotropy groups of all elements of $U_x \cap M^\circ$ coincide. By the slice theorem and the assumptions on M^1 the fixed point set $E_x^{G_x}$ is a linear subspace of E_x and of codimension 1. Choose a G_x-invariant metric $\langle\cdot,\cdot\rangle$ on E_x and let v be a unit vector in the orthogonal complement of $E_x^{G_x}$. Then we have $G_x v = \{v, -v\}$. Let $K \subset G_x$ be the kernel of the map $G_x \ni g \mapsto \langle gv, v\rangle$ and h a group element such that $hv = -v$. Then the isotropy group of an element λv with $\lambda > 0$ is identical to K and the isotropy group of $-\lambda v$ is given by hKh^{-1}. But hKh^{-1} is equal to K, as K is normal. Hence the isotropy groups of all elements of $U_x \cap M^\circ$ coincide.

Now, as $M^\circ \cup M^1$ is connected, one can connect any two points $x, x' \in M^\circ$ by a finite chain of U_y with $y \in M^\circ \cup M^1$. In other words this means that there exist $y_0, \ldots, y_n \in M^\circ \cup M^1$ such that $y_0 = x$, $y_n = x'$ and $U_{y_k} \cap U_{y_{k+1}} \neq \emptyset$ for $k \leqslant n$. By the above considerations, the isotropy groups of x and x' then coincide. This proves the claim. □

The proof of (4) entails also the following technical result, which will be needed later.

(5) Let G be finite, x a point of M^1, the stratum of codimension 1, and $U \subset M$ a neighborhood which is G_x-equivariantly diffeomorphic to an open ball around the origin of a G_x-representation space. Then $U \cap M^1$ is connected and $U \cap M^\circ$ has two connected components. Moreover, G_x acts trivially on $U \cap M^1$, and there exists a homomorphism $G_x \to \mathbb{Z}_2$ with kernel G° such that every element of $G_x \setminus G^\circ$ interchanges the connected components of $U \cap M^\circ$.

1.4. Profinite dimensional manifolds

A second countable topological Hausdorff space M is called a *profinite dimensional manifold*, if there exists a projective system $(M_i, \mu_{ij})_{i\leqslant j\in\mathbb{N}}$ of smooth finite dimensional manifolds M_i and surjective submersions $\mu_{ij} : M_j \to M_i$, $i \leqslant j$, such that M coincides with the projective limit, that means

$$M = \varprojlim_{i\in\mathbb{N}} M_i.$$

If M is a profinite dimensional manifold, there exists a unique family of continuous surjections $\mu_i : M \to M_i$ such that $\mu_i = \mu_{ij} \circ \mu_j$ for all $i \leqslant j$ and such that M carries the initial topology with respect to the μ_i.

By a *profinite dimensional vector space* we understand the projective limit

$$V = \varprojlim_{i\in\mathbb{N}} V_i$$

of a projective system $(V_i, \varphi_{ij})_{i \leqslant j \in \mathbb{N}}$ of finite dimensional (real) vector spaces V_i and surjective linear maps $\varphi_{ij} : V_j \to V_i$, $i \leqslant j$. Clearly, every profinite dimensional vector space is a profinite dimensional manifold. Examples of profinite dimensional vector spaces are the projective limit

$$\mathbb{R}^\infty = \lim_{\substack{\longleftarrow \\ n \in \mathbb{N}}} \mathbb{R}^n$$

and the completed symmetric tensor algebra

$$\widehat{\mathrm{Sym}}^\bullet(W) := \lim_{\substack{\longleftarrow \\ n \in \mathbb{N}}} \mathrm{Sym}^\bullet(W)/\mathfrak{m}^n$$

of a finite dimensional real vector space W. Hereby, $\mathrm{Sym}^\bullet(W)$ denotes the (complexified) symmetric tensor algebra of W and $\mathfrak{m}$ the kernel of the canonical homomorphism $\mathrm{Sym}^\bullet(W) \to \mathbb{C}$. Note that $\mathbb{R}^n$ can be naturally embedded as a subspace of $\mathbb{R}^\infty$, since for all $n \leqslant N$, $\mathbb{R}^n$ is canonically embedded in $\mathbb{R}^N$ via the first n coordinates.

The *sheaf of smooth functions* on a profinite dimensional manifold

$$M = \lim_{\substack{\longleftarrow \\ i \in \mathbb{N}}} M_i$$

is defined as the sheaf $\mathcal{C}^\infty_M$ with sectional spaces

$$\mathcal{C}^\infty_M(U) = \left\{ g \in \mathcal{C}(U) \mid \exists\, i \in \mathbb{N} \;\&\; g_i \in \mathcal{C}^\infty\big(\mu_i(U)\big) \text{ s.t. } g_i \circ \pi_{i|U} = g \right\},$$

where U runs through the open subsets of M. Given a second profinite dimensional manifold

$$N = \lim_{\substack{\longleftarrow \\ i \in \mathbb{N}}} N_i,$$

a continuous map $f : M \to N$ is called *smooth*, if $f_* \mathcal{C}^\infty_M \subset \mathcal{C}^\infty_N$. Using Whitney's embedding theorem it is straightforward to check that for every smooth map $f : N \to M$ there exists, possibly only after passing to projective subsystems of (M_i, μ_{ij}) and (N_i, ν_{ij}), a family of smooth maps $f_i : N_i \to M_i$ such $f_i \circ \nu_i = \mu_i \circ f$ for all i. In case the f_i can be chosen to be immersions (respectively submersions, embeddings or diffeomorphisms), one says that f is an immersion (respectively submersion, embedding or diffeomorphism). Using Whitney's embedding theorem again one proves that every profinite dimensional M can be embedded in $\mathbb{R}^\infty$.

Obviously, the profinite dimensional manifolds and the smooth maps between them form a category which we will denote by $\mathfrak{Man}_{\mathrm{pf}}$. Similarly, the profinite dimensional vector spaces with smooth linear maps as morphisms form a category.

If a compact Lie group G acts on a profinite dimensional manifold

$$M = \lim_{\substack{\longleftarrow \\ i \in \mathbb{N}}} M_i,$$

one can construct a G-invariant riemannian metric on M. Given a point $x \in M$, such a riemannian metric gives rise to a G-invariant tubular neighborhood of the orbit through x. From this one concludes by a standard argument that the slice theorem holds as well for compact Lie group actions on profinite dimensional manifolds.

1.5. Differentiable categories with slices

Consider a subcategory of the category of profinite dimensional manifolds and smooth maps. In this article we will denote such a subcategory by $\mathfrak{T}$ and always assume that it satisfies the following axioms.

(DCAT1) For every morphism $f : N \to M$ in $\mathfrak{T}$ which is a smooth open embedding of profinite dimensional manifolds, the image $f(N)$ is an open subobject of M. $U \subset M$ being an open subobject hereby means that U is open and that the canonical injection $U \hookrightarrow M$ is a morphism in $\mathfrak{T}$.

(DCAT2) For every object M the set of open subobjects is a topology on M.

The category $\mathfrak{T}^{\mathsf{sym}}$ of *$\mathfrak{T}$-objects with (compact) symmetries* consists of the following object and morphism classes. Objects are given by pairs (M, G), where $M \in \mathrm{Obj}(\mathfrak{T})$ and G is a compact Lie group which acts smoothly on M by elements of the automorphism group $\mathrm{Aut}_{\mathfrak{T}}(M)$. Morphisms are given by equivariant maps $(\varphi, \iota) : (N, H) \to (M, G)$. This means that $\iota : H \to G$ is a continuous group homomorphism and $\varphi : M \to N$ a morphism of $\mathfrak{T}$ such that $\varphi(hy) = \iota(h)\,\varphi(y)$ for all $y \in N$ and $h \in H$. Two equivariant maps $(\varphi, \iota), (\varphi', \iota') : (N, H) \to (M, G)$ are said to be *equivalent*, if there exists an element $g \in G$ such that $(\varphi', \iota') = (g, \mathrm{Ad}_g)(\varphi, \iota)$.

With a view towards symmetries we assume additionally that the category $\mathfrak{T}$ satisfies the axiom (SLC) below; a category for which (DCAT1), (DCAT2) and (SLC) are true will be called a *differentiable category with slices*.

(SLC) Let (M, G) be an object of $\mathfrak{T}^{\mathsf{sym}}$ and $x \in M$ a point. Then there exists a $\mathfrak{T}$-slice for M at x that means an embedding $(\xi, \lambda) : (S, K) \to (M, G)$ with λ injective and a point $s \in S$ such that $\xi(s) = x$ and such that (ξ, λ) is universal in the following sense. Assume to be given an embedding $(\varphi, \iota) : (N, H) \to (M, G)$ and a point $y \in N$ where ι is injective and $x = \varphi(y)$. Then there exists, after passing to appropriate open subobjects, an equivariant automorphism $(\Phi, \mathrm{id}) : (M, G) \to (M, G)$ with $\overline{\Phi} = \mathrm{id}_{M/G}$ and an embedding $(\psi, \kappa) : (S, K) \to (N, H)$ such that $\psi(s) = y$ and such that the following diagram commutes:

$$\begin{array}{ccc} (S, K) & \xrightarrow{(\xi,\lambda)} & (M, G) \\ \big\downarrow{\scriptstyle (\psi,\kappa)} & & \big\downarrow{\scriptstyle (\Phi,\mathrm{id})} \\ (N, H) & \xrightarrow{(\varphi,\iota)} & (M, G). \end{array} \tag{1.1}$$

As typical examples for a differentiable category with slices we have the following in mind; using the slice theorem the reader will easily check that these categories satisfies the above axioms and in particular (SLC):

(1) the category $\mathfrak{Man}$ of finite dimensional smooth manifolds and smooth maps,
(2) the category $\mathfrak{Man}_{\mathsf{pf}}$ of profinite dimensional manifolds and smooth maps,
(3) the category $\mathfrak{VBdl}$ of smooth vector bundles over finite dimensional manifolds; hereby the fiber vector space is allowed to be a profinite dimensional vector space and the morphisms are given by smooth vector bundle maps over smooth maps between the bases.

Given a differentiable category with slices $\mathfrak{T}$, the following properties of a morphism $(\varphi,\iota):(N,H)\to(M,G)$ in $\mathfrak{T}^{\mathrm{sym}}$ are easy to prove:

(SYM0) φ induces a continuous map $\overline{\varphi}:N/H\to M/G$ between orbit spaces.
(SYM1) If φ is surjective and G acts effectively on M, then ι is uniquely determined by φ.
(SYM2) If φ is injective and H acts effectively on N, then ι is a monomorphism.

Let us introduce some useful notation. An object (M,G) of $\mathfrak{T}$ is called *reduced*, if G acts effectively on M. Note that for arbitrary (M,G) there exists a natural equivariant morphism from (M,G) onto the reduced object $(M,G_{M,\mathrm{eff}})$, where $G_{M,\mathrm{eff}}$ is the quotient group of G by the kernel of the homomorphism $G\to\mathrm{Aut}_{\mathfrak{T}}(M)$.

A morphism $(\varphi,\iota):(N,H)\to(M,G)$ between objects of $\mathfrak{T}^{\mathrm{sym}}$ is called an *embedding*, if φ is a smooth embedding and $\overline{\varphi}$ a homeomorphism onto an open subset of the orbit space M/G. If additionally φ is an open map, we say that (φ,ι) is an *open embedding*. Note that for (φ,ι) an embedding, ι need not be a monomorphism. Moreover, (SLC) implies that for every object (M,G) of $\mathfrak{T}^{\mathrm{sym}}$ and every point $x\in M$ there exists an embedding $(\varphi,\mathrm{id}_H):(S,H)\to(M,G)$.

The following further properties hold for finite symmetries in a differentiable slice category $\mathfrak{T}$.

(SYM3) Assume that N is connected and that G,H are finite. Let (φ,ι) and (φ',ι') be two open embeddings from (N,H) to (M,G) with the actions of G and H effective. Then (φ,ι) and (φ',ι') are equivalent, if and only if $\overline{\varphi}=\overline{\varphi'}$.
(SYM4) Assume that N is connected and that G,H are finite. Let $(\varphi,\iota):(N,H)\to(M,G)$ be an open embedding and assume that G acts effectively on M. Then, if $g\varphi(N)\cap\varphi(N)\neq\emptyset$ for $g\in G$, the relation $g\varphi(N)=\varphi(N)$ holds true and g lies in the image of ι.

1.6. Remark. (SYM3) and (SYM4) correspond to Lemma 1 and Lemma 2 in [18], but note that in [18] the additional assumption has been made that (M,G) and (N,H) do not contain strata of codimension 1. In the following we repeat Satake's short proof of (SYM4), which also works in the general case of strata of arbitrary codimension, and provide a new argument showing that (SYM3) is true without any assumptions on the codimension of the strata.

Proof. Let us first prove the claim for the case where $\mathfrak{T}$ is the category of (finite dimensional) smooth manifolds and smooth maps. Denote by M° the open stratum of a G-manifold M and by M^1 the stratum of codimension 1 with respect to the stratification by orbit types. Likewise define N° and N^1 for an H-manifold N. Now, we will show first property (SYM4) and afterwards (SYM3).

So assume that N is connected, (φ,ι) is an open embedding and that $g\varphi(N)\cap\varphi(N)\neq\emptyset$. Then there exist $y,y'\in N^\circ$ such that $\varphi(y)\in M^\circ$ and $\varphi(y')=g\varphi(y)$. As $\overline{\varphi}$ is injective, y and y' have to lie in the same H-orbit, hence $y'=hy$ for some $h\in H$. We then have $\varphi(hz)=g'\varphi(z)$ for all $z\in N$ and $g'=\iota(h)$. As $\varphi(y)\in M^\circ$ and G acts effectively, we have $g=g'=\iota(h)$ and consequently $g\varphi(N)=\varphi(hN)=\varphi(N)$. This shows (SYM4).

Next we consider (SYM3). Assume that $\varphi'(y_\circ)=\varphi(y_\circ)$ for some $y_\circ\in N^\circ$. We will then show that $\varphi'=\varphi$ and $\iota'=\iota$. Clearly, this will prove (SYM3). Using (SYM4) it is straightforward to check that $\varphi(N^\circ)\subset M^\circ$ and $\varphi'(N^\circ)\subset M^\circ$. Let us prove that $\varphi(N^1)\subset M^1$. To this end choose for every point $y\in Y$ an H_y-invariant neighborhood V_y such that $hV_y\cap V_y=\emptyset$ for $h\in H\setminus H_y$ and such that V_y is equivariantly diffeomorphic to an H_y-invariant open ball around the origin in a linear H_y-representation space. In case $y \in N^1$, we know by 1.3 (5) that $V_y \cap N^1$ is connected, that $V_y \cap N^\circ$ has two connected components and that $H_y \cong \mathbb{Z}_2$. Hence, by (SYM2) $\mathbb{Z}_2 \cong \iota(H_y) \subset G_{\varphi(y)}$. The subgroup $\iota(H_y)$ acts trivially on the manifold $U^1_{\varphi(y)} := \varphi(V_y \cap N^1)$, and the non-neutral element interchanges the connected components of $\varphi(V_y \cap N^\circ)$. As a consequence of 1.3 (4), $G_{\varphi(y)}$ acts effectively on a neighborhood of $\varphi(y)$ contained in $U_{\varphi(y)} := \varphi(V_y)$. So, if $\iota(H_y) \neq G_{\varphi(y)}$, one can find by (SYM4) an element $k \in G_{\varphi(y)} \setminus \iota(H_y)$ and a point $x \in U_{\varphi(y)}$ with $kx \in U_{\varphi(y)}$ and $kx \notin \iota(H_y)x$. But this contradicts the fact that $\overline{\varphi}$ is injective. Hence $G_{\varphi(y)} \cong \mathbb{Z}_2$ and consequently $\varphi(N^1) \subset M^1$. The same argument also proves $\psi(N^1) \subset M^1$. We continue with the proof of the equality $\varphi' = \varphi$. Let A be the set $\{y \in N \mid \varphi'(y) = \varphi(y)\}$. Obviously, A is closed in N and nonempty, since $y_\circ \in A$. Let us show that $A \cap N^\circ$ is also open. Let $y \in A \cap N^\circ$ and assume that there exists a sequence $(y_n) \subset N^\circ \setminus A$ converging to y. After transition to an appropriate subsequence there exists $g' \neq e$ such that $\varphi'(y_n) = g'\varphi(y_n)$ for all n. By continuity $\varphi'(y) = g'\varphi(y)$ follows, hence $\varphi(y) = g'\varphi(y)$. But this contradicts $G_{\varphi(y)} = \{e\}$, so $A \cap N^\circ$ must be open indeed. Now let $y \in N^1$ and assume that $A \cap V_y \cap N^\circ \neq \emptyset$. 1.3 (5) entails that V_y can be decomposed in three connected subsets V_y^{N}, V_y^{S} and V_y^1, where the first two are the connected components of $V_y \cap N^\circ$ and V_y^1 is equal to $V_y \cap N^1$. By assumption on y there exists $z_0 \in V_y \cap N^\circ$, let us say $z_0 \in V_y^{\mathrm{N}}$, such that $\varphi'(z_0) = \varphi(z_0)$. By the results proven so far we know that $\varphi'(z) = \varphi(z)$ for all $z \in V_y^{\mathrm{N}} \cup V_y^1$. We now want to show that this holds for $z \in V_y^{\mathrm{S}}$ as well. As it has been shown above, both H_y and $G_{\varphi(y)}$ are isomorphic to $\mathbb{Z}_2$. Let h be the non-neutral element of H_y. Then both $\iota(h)$ and $\iota'(h)$ coincide with the non-neutral element of $G_{\varphi(y)}$; this implies in particular that $\iota'(h) = \iota(h)$. As $hz \in V_y^{\mathrm{N}}$ for $z \in V_y^{\mathrm{S}}$, we obtain

$$\varphi'(z) = \iota'(h)\varphi'\bigl(h^{-1}z\bigr) = \iota(h)\varphi\bigl(h^{-1}z\bigr) = \varphi(z),$$

hence $\varphi'(z) = \varphi(z)$ for all $z \in V_y$. Since every element of $N^\circ \cup N^1$ can be connected with $y_\circ$ by a finite chain of V_y with either $y \in N^\circ$ or $y \in N^1$, this shows that $N^\circ \cup N^1$ is contained in A. As A is closed and N° is dense in N, we thus obtain $A = N$. This proves the relation $\varphi' = \varphi$ under the assumption of finite G and H. To show that $\iota' = \iota$ consider the open set $V = H\,V_{y_\circ} \subset N^\circ$ and the image $U = \varphi'(V) = \varphi(V)$. Obviously, $\operatorname{im}\iota' \subset G_U := \{\tilde g \in G \mid \tilde g U \subset U\}$. Since $\varphi'(hy) = \varphi(hy) = \iota(h)\varphi'(y)$ for $y \in V$ and as G_U acts effectively on U, the relation $\iota' = \iota$ follows. This finishes the proof of axiom (SYM3).

For the case of profinite dimensional manifolds with finite symmetries

$$\Bigl(M = \varprojlim_{i\in\mathbb{N}} M_i, G\Bigr) \quad \text{and} \quad \Bigl(N = \varprojlim_{i\in\mathbb{N}} N_i, H\Bigr)$$

one concludes the claim from the fact that axioms (SYM3) and (SYM4) hold true for the components (M_i, G) and (N_i, H). The details of the corresponding straightforward argument are left to the reader. Finally, an arbitrary differentiable slice category $\mathfrak{T}$ satisfies (SYM3) and (SYM4) since these axioms hold true for $\mathfrak{Man}_{\mathsf{pf}}$. □

2. Orbispaces

2.1. Orbispace charts

Let X be a topological Hausdorff space and $\mathfrak{T}$ a differentiable category. By a *$\mathfrak{T}$-orbispace chart* for X we understand a triple $(\widetilde U, G, \varrho)$ such that $(\widetilde U, G)$ is an object of $\mathfrak{T}^{\mathsf{sym}}$ and $\varrho : \widetilde U \to U \subset X$ a continuous

G-invariant map inducing a homeomorphism $\bar{\varrho}:\widetilde{U}/G \to U$ onto an open subset of X. The set U will be called the *image* of the orbispace chart, $\widetilde{U}$ its *domain*. In case the symmetry group G is finite, $(\widetilde{U}, G, \varrho)$ is called a $\mathfrak{T}$*-orbifold chart* for X. A *morphism* between two $\mathfrak{T}$-orbispace charts $(\widetilde{V}, H, \upsilon)$ and $(\widetilde{U}, G, \varrho)$ is a morphism $(\varphi, \iota):(\widetilde{V}, H) \to (\widetilde{U}, G)$ in $\mathfrak{T}^{\mathsf{sym}}$ such that $\varrho \circ \varphi = \upsilon$. Note that for every $\mathfrak{T}$-orbispace chart $(\widetilde{U}, G, \varrho)$ the triple $(\widetilde{U}, G_{\widetilde{U},\mathrm{eff}}, \varrho)$ is a $\mathfrak{T}$-orbispace chart as well. If $(\widetilde{U}, G, \varrho) = (\widetilde{U}, G_{\widetilde{U},\mathrm{eff}}, \varrho)$, we say that $(\widetilde{U}, G, \varrho)$ is a *reduced* orbispace chart. The category of all $\mathfrak{T}$-orbispace charts for X will be denoted by $\mathfrak{T}_X^{\mathsf{sym}}$.

Two $\mathfrak{T}$-orbispace charts $(\widetilde{U}_1, G_1, \varrho_1)$ and $(\widetilde{U}_2, G_2, \varrho_2)$ are called *germ equivalent* at a point $x \in U_1 \cap U_2$, if there exist two embeddings $(\varphi_i, \iota_i):(\widetilde{V}, H, \upsilon) \to (\widetilde{U}_i, G_i, \varrho_i)$, $i = 1, 2$, and a distinguished point $\tilde{x} \in \widetilde{V}$ such that $\varphi_i(\widetilde{V})$ is a subobject of $\widetilde{U}_i$ and such that $\upsilon(\tilde{x}) = x$. In other words, germ equivalency of orbispace charts means essentially that the slices of $\widetilde{U}_1$ at some point $\tilde{x}_1 \in \varrho_1^{-1}(x)$ and of $\widetilde{U}_2$ at some point $\tilde{x}_2 \in \varrho_2^{-1}(x)$ coincide (up to isomorphy). Using axiom (SLC) it is straightforward to check that the germ equivalence of orbispace charts at a point $x \in X$ is an equivalence relation indeed. By a $\mathfrak{T}$*-orbispace atlas* for X we now understand a covering of X by $\mathfrak{T}$-orbispace charts such that any two of the orbispace charts are germ equivalent at every point of the intersection of their images. If every element of an orbispace atlas is a $\mathfrak{T}$-orbifold chart, we call the atlas a $\mathfrak{T}$-orbifold atlas. Obviously, the set of $\mathfrak{T}$-orbifold atlases for X is partially ordered by inclusion, and for every $\mathfrak{T}$-orbifold atlas $\mathcal{A}$ there exists a unique maximal $\mathfrak{T}$-orbifold atlas $\mathcal{A}_{\max}$ containing $\mathcal{A}$. Clearly, the same holds for orbispace atlases. We arrive at the definition of a $\mathfrak{T}$*-orbifold*; this is just a second countable paracompact topological Hausdorff space X together with a maximal $\mathfrak{T}$-orbifold atlas, usually denoted by $\mathcal{A}_X$. If $\mathfrak{T}$ is the category of finite dimensional manifolds (respectively profinite dimensional manifolds), a $\mathfrak{T}$-orbifold is briefly called an orbifold (respectively profinite dimensional orbifold).

Particularly convenient for the study of orbifolds are the so-called *linear orbifold charts*. These are orbifold charts $(\widetilde{W}, G, \varrho)$, where $\widetilde{W}$ is an open convex neighborhood of the origin of some finite dimensional G-representation space. In this situation we sometimes say that $x = \varrho(0) \in W$ is the *center* of $(\widetilde{W}, G, \varrho)$ or that $(\widetilde{W}, G, \varrho)$ is *centralized* at x. By the slice theorem it is clear that every orbifold germ at x can be represented by a linear orbifold chart centralized at this point.

2.2. Orbispace functors

Let $\mathcal{U}$ be an open covering of X and $\overline{\mathcal{U}}$ the category whose objects are given by connected components of finite intersections $U_1 \cap \cdots \cap U_k$ of elements $U_1, \ldots, U_k \in \mathcal{U}$ and whose morphisms are the canonical inclusions. By a $\mathfrak{T}$*-orbispace functor* we understand a functor X defined on $\overline{\mathcal{U}}$ and with values in the category of orbispace charts of X such that the following conditions hold true:

(OSF1) For every object U of $\overline{\mathcal{U}}$ the orbispace chart $\mathsf{X}(U)$ has image U.
(OSF2) The domain $\widetilde{U}$ of every orbispace chart $\mathsf{X}(U)$, $U \in \overline{\mathcal{U}}$ is connected.
(OSF3) For all objects U, V of $\overline{\mathcal{U}}$ with $V \subset U$ the morphism $\mathsf{X}_{VU} := \mathsf{X}(V \to U)$ is an open embedding.

A $\mathfrak{T}$*-orbispace* now is a second countable and locally connected paracompact topological Hausdorff space X together with a $\mathfrak{T}$-orbispace functor $\mathsf{X}:\overline{\mathcal{U}} \to \mathfrak{T}_X^{\mathsf{sym}}$. Clearly, this functor uniquely determines a maximal atlas $\mathcal{A}_X$ of orbispace charts such that X has image in $\mathcal{A}_X$. From now on only the elements of $\mathcal{A}_X$ will be called orbispace charts for the $\mathfrak{T}$-orbispace X.

If $\mathfrak{T}$ is the category of finite dimensional manifolds (respectively profinite dimensional manifolds), we use the same language like for orbifolds and briefly say orbispace (respectively profinite dimensional orbispace) instead of $\mathfrak{T}$-orbispace.

Using the paracompactness of an orbifold X, the following result can be easily derived from (SYM3) and (SYM4). We leave the details to the reader.

2.3. Proposition. *For every $\mathfrak{T}$-orbifold X there exists a $\mathfrak{T}$-orbispace functor $\mathsf{X}:\overline{\mathcal{U}} \to \mathcal{A}_X \subset \mathfrak{T}_X^{\mathsf{sym}}$.*

2.4. Stratification of orbispaces

Every orbispace X has a canonical stratification. To construct this consider a point x and choose an orbispace chart $(\widetilde{U}, G, \varrho)$ around x. Denote by $\mathcal{S}_x$ the set germ at x of the stratification of $U \cong \widetilde{U}/G$ by orbit types (recall Example 1.3). As a consequence of the slice theorem, $\mathcal{S}_x$ does not depend on the particular choice of $(\widetilde{U}, G, \varrho)$. Since the set germ $\mathcal{S}_x$ is locally induced by a decomposition, we thus obtain a stratification $\mathcal{S}$, called the *canonical stratification* of the orbispace. Proposition 1.2 guarantees the existence of a canonical decomposition of X into smooth manifolds, called the *strata* of the orbispace. Moreover, if X is connected, there exists an open and dense stratum which coincides with the regular part of X and which will be denoted by X°. The *dimension* of X is defined as the dimension of X°.

2.5. Example. Every manifold with boundary M carries in a natural way the structure of a finite dimensional orbifold. To see this choose a smooth collar $c:\partial M \times [0,1) \to M$, denote by $\widetilde{U}_\circ$ the interior $M \setminus \partial M$ and put $\widetilde{U}_1 = \partial M \times (-1,1)$. Then $\mathbb{Z}_2$ acts on $\widetilde{U}_1$ by $(p,t,\pm 1) \mapsto (p, \pm t)$, and the map $\varrho_1:\widetilde{U}_1 \to \operatorname{im} c$, $(p,t) \mapsto c(p,t^2)$ induces a homeomorphism $\widetilde{U}_1/\mathbb{Z}_2 \to \operatorname{im} c$. It is now immediate to check that $(\widetilde{U}_\circ, \{e\}, \mathrm{id})$ and $(\widetilde{U}_1, \mathbb{Z}_2, \varrho_1)$ comprise an orbifold atlas for M. Similarly, though technically somewhat more involving, one proves that every manifold with corners is naturally a finite dimensional orbifold.

Note that in the approach to orbifolds going back to Satake [17], manifolds with boundary or corners are not regarded as orbifolds (or better V-manifolds in the language of [17]), since every orbifold chart around a boundary point possesses a stratum of codimension 1.

2.6. Given an open covering $\mathcal{U}$ of some locally connected topological space Y, any faithful functor $\mathsf{Y}:\overline{\mathcal{U}} \to \mathfrak{T}^{\mathsf{sym}}$ which satisfies axioms (OSF2) and (OSF3) above will be called a $\mathfrak{T}$-*orbispace functor*, too. Hereby, faithful means that the image $\mathsf{Y}_{vu}(\mathsf{Y}(v))$ is properly contained in $\mathsf{Y}(u)$ for all $v, u \in \overline{\mathcal{U}}$ with $v \subsetneq u$. The following proposition shows that this new notation is justified indeed.

2.7. Proposition. *Let $\mathsf{Y}:\overline{\mathcal{U}} \to \mathfrak{T}^{\mathsf{sym}}$ be a faithful functor satisfying axioms* (OSF2) *and* (OSF3)*. Then there exists a $\mathfrak{T}$-orbispace X, an order preserving injective map U from $\overline{\mathcal{U}}$ to the topology of X and a $\mathfrak{T}$-orbispace functor $\mathsf{X}:\mathsf{U}(\overline{\mathcal{U}}) \to \mathfrak{T}_X^{\mathsf{sym}}$, $u \mapsto (\widetilde{U}_u, G_u, \varrho_u)$ such that $\mathsf{Y} = \mathsf{F} \circ \mathsf{X} \circ \mathsf{U}$, where $\mathsf{F}:\mathfrak{T}_X^{\mathsf{sym}} \to \mathfrak{T}^{\mathsf{sym}}$ is the forgetful functor $(\widetilde{U}, G, \varrho) \mapsto (\widetilde{U}, G)$. Moreover, these objects are unique up to isomorphy in the sense that if X', U' and X' also have this property, then there exists a homeomorphism $f:X \to X'$ such that $\mathsf{U}' = f \circ \mathsf{U}$ and $\varrho'_u = f \circ \varrho_u$ for all $u \in \overline{\mathcal{U}}$.*

Proof. To construct X, X and U let us first denote every object $\mathsf{Y}(u)$, $u \in \overline{\mathcal{U}}$ by $(\widetilde{U}_u, G_u)$ and every morhism Y_{vu} for $v \subset u$ by $(\varphi_{vu}, \iota_{uv})$. Then put

$$X := \bigsqcup_{u \in \overline{\mathcal{U}}} \widetilde{U}_u / G_u \Big/ \sim,$$

where two points $x \in \widetilde{U}_u / G_u$ and $x' \in \widetilde{U}_{u'} / G_{u'}$ are in relation $\sim$, if there exists $v \in \overline{\mathcal{U}}$ and a point $y \in \widetilde{U}_v / G_v$ such that $x = \overline{\varphi}_{vu}(y)$ and $x' = \overline{\varphi}_{vu}(y)$. The set X carries a natural topology given by the quotient topology from the (disjoint) topological sum of the orbit spaces $\widetilde{U}_u / G_u$. Now let ϱ_u be the natural map from $\widetilde{U}_u$ to X, denote by U_u the image of ϱ_u, and let X be the functor $u \mapsto (\widetilde{U}_u, G_u, \varrho_u)$, $(v \to u) \mapsto (\varphi_{vu}, \iota_{uv})$. Finally define U by $\mathsf{U}(u) = U_u$. Then the objects X, X and U satisfy the claim of the proposition. The proof of uniqueness up to isomorphy is given by standard arguments, so we will leave it to the reader. □

In the situation of the proposition we say that the $\mathfrak{T}$-orbispace X is *induced* by Y. For convenience we will also notationally identify the functors Y and X.

2.8. Smooth functions on orbispaces

Let $U \subset X$ be an open subset of the $\mathfrak{T}$-orbispace X. A continuous function $g : U \to \mathbb{R}$ is called *smooth*, if for every orbispace chart $(\widetilde{V}, H, \upsilon)$ the composition $\upsilon^*(g) := g \circ \upsilon_{|\upsilon^{-1}(U)}$ is smooth. The algebra of smooth functions $g : U \to \mathbb{R}$ will be denoted by $\mathcal{C}^\infty(U)$. The spaces $\mathcal{C}^\infty(U)$ then form the sectional spaces of a sheaf of algebras on X. We denote this sheaf by $\mathcal{C}^\infty_X$ or briefly $\mathcal{C}^\infty$ and call it the *sheaf of smooth functions* on X. By a *smooth map* between profinite dimensional orbispaces X and Y we understand a continuous map $f : X \to Y$ such that $f^*\mathcal{C}^\infty_Y \subset \mathcal{C}^\infty_X$. It is immediate to check that the $\mathfrak{T}$-orbispaces together with the smooth maps between them form a category. Moreover, it follows by a standard argument that the sheaf of smooth functions on a $\mathfrak{T}$-orbispace is fine.

Note that our definition of smooth maps is in correspondence with the smooth maps between orbit spaces in [2,15,22], but that it is weaker than the notion of smooth maps as defined in [6,16,17] for the case of orbifolds.

A particularly useful characterization of the smooth functions on a finite dimensional orbispace can be given as follows. Let $(\widetilde{U} = G \times_H \widetilde{W}, G, \varrho)$ be a twisted-linear orbispace chart for X that means $H \subset G$ is a closed subgroup and $\widetilde{W}$ an open and convex neighborhood of the origin of some H-representation space $\mathfrak{W}$. Clearly, by the slice theorem there exists an atlas for X consisting of twisted-linear charts. Choose a homogeneous Hilbert basis $p = (p_1, \dots, p_k)$ of the algebra $\mathcal{P}(\mathfrak{W})^H$ of H-invariant polynomials on $\mathfrak{W}$. Since the Hilbert basis p consists of H-invariant functions, the map

$$p_U : U \to \mathbb{R}^k, \ x \mapsto p(v) \quad \text{with } v \in \widetilde{W} \text{ such that } \varrho([e, v]) = x$$

is well-defined and continuous. Moreover, p_U has the following properties:

(1) p_U is a homeomorphism onto its image,
(2) on every stratum of U, p_U restricts to a diffeomorphism onto a smooth submanifold of $\mathbb{R}^k$,
(3) the sheaf $\mathcal{C}^\infty_U$ coincides with the pullback sheaf $p_U^*\mathcal{C}^\infty_{\mathbb{R}^k}$; this is a consequence of the theorem of Schwarz [20].

In other words these properties mean that p_U is a smooth chart for the stratified space X in the sense of [15, Section 1.3]. From that one can derive the following result.

2.9. Proposition. *A continuous map $f : X \to X'$ between orbispaces is smooth, if and only if for all twisted linear charts $(\widetilde{U} = G \times_H \widetilde{W}, G, \varrho)$ of X and $(\widetilde{U}' = G' \times_{H'} \widetilde{W}', G', \varrho')$ of X' such that $f(U) \subset U'$ there exists a smooth map $\hat{f}_{UU'} : O \to \mathbb{R}^{k'}$ defined on an open neighborhood $O \subset \mathbb{R}^k$ of $p_U(U)$ such that*

$$\hat{f}_{UU'} \circ p_U = p'_{U'} \circ f_{|U}.$$

2.10. Vector orbibundles

By a *vector orbibundle* we understand an orbispace E which is induced by an *orbibundle functor* that means by an orbispace functor E having values in the category of vector bundles. We denote an orbibundle functor as follows:

$$\mathsf{E} : \overline{\mathcal{U}} \to \mathfrak{VBdl}^{\mathsf{sym}}, \quad \begin{cases} u \mapsto (\widetilde{E}_u, G_u), \\ (v \to u) \mapsto (\psi_{vu}, \iota_{vu}) : (\widetilde{E}_v, G_v) \to (\widetilde{E}_u, G_u). \end{cases}$$

A $\mathfrak{VBdl}$-orbispace chart for E will be called an *orbibundle chart*. Similarly to the manifold case, a vector orbibundle gives rise to a base orbispace and a canonical projection. Let us show this in more detail. Denote for every $u \in \overline{\mathcal{U}}$ by $\widetilde{U}_u$ the base of the vector bundle $\widetilde{E}_u$ and by $\pi_u : \widetilde{E}_u \to \widetilde{U}_u$ the canonical projection. Moreover, let $\varphi_{vu} : \widetilde{U}_v \to \widetilde{U}_u$ be the embedding on the level of base manifolds induced by the morphism ψ_{vu}. Then

$$\mathsf{X} : \overline{\mathcal{U}} \to \mathfrak{Man}^{\mathsf{sym}}, \quad \begin{cases} u \mapsto (\widetilde{U}_u, G_u), \\ (v \to u) \mapsto (\varphi_{vu}, \iota_{vu}) : (\widetilde{U}_v, G_v) \to (\widetilde{U}_u, G_u) \end{cases}$$

is an orbispace functor. The resulting orbispace X is the *base orbispace* of the vector orbibundle E. Clearly, every orbibundle chart (E, G, η) of E now induces an orbispace chart (X, G, ϱ) on X by the same procedure. Note that even if (E, G, η) is a reduced orbibundle chart, (X, G, ϱ) need not be reduced, in general. Following Chen and Ruan [6] we say that E is a *good* or *reduced vector orbibundle*, if for every reduced orbibundle chart (E, G, η) of E the induced chart (X, G, ϱ) on the base is reduced as well.

Next consider the canonical projections $\pi_u : \widetilde{E}_u \to \widetilde{U}_u$, $u \in \overline{\mathcal{U}}$. Obviously, the π_u induce a unique smooth map $\pi : E \to X$ called *projection* such that

$$\pi \circ \eta_u = \varrho_u \circ \pi_u \quad \text{for all } u \in \overline{\mathcal{U}}. \tag{2.1}$$

Analogously like for vector bundles one defines a *section* of E as a continuous map $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$. We denote the space of continuous (respectively smooth) sections of E by $\Gamma(E)$ (respectively $\Gamma^\infty(E)$). But unlike in the case of vector bundles, an orbibundle $E \to X$ is in general not locally trivial over the base, which implies in particular that the space of continuous respectively smooth sections need not be linear. In the following, we will construct for every vector orbibundle a subspace $\Gamma^\infty_{\mathsf{str}}(E) \subset \Gamma^\infty(E)$ which is a $\mathcal{C}^\infty(X)$-module in a natural way. The elements of $\Gamma^\infty_{\mathsf{str}}(E)$ will be called *smooth stratified sections* of E. To define $\Gamma^\infty_{\mathsf{str}}(E)$ let $(\widetilde{E}, G, \eta)$ be an orbibundle chart for E and $(\widetilde{U}, G, \varrho)$ the induced orbispace chart for the base. For every point $\tilde{x} \in \widetilde{U}$ let $\widetilde{E}^{G_{\tilde{x}}}_{\tilde{x}}$ be the (linear) subspace of $G_{\tilde{x}}$-

invariant elements of the fiber $\widetilde{E}_{\tilde{x}}$. Then for every closed subgroup $H \subset G$

$$\widetilde{E}_{(H)} := \bigcup_{\substack{\tilde{x}\in\widetilde{U}\\ (G_{\tilde{x}})=(H)}} \widetilde{E}_{\tilde{x}}^{G_{\tilde{x}}}$$

is a smooth vector bundle over the stratum $\widetilde{U}_{(H)}$ and $\widetilde{E}_{(H)}/G$ a smooth vector bundle over $\widetilde{U}_{(H)}/G$. Moreover, one concludes easily by the slice theorem that $\widetilde{E}_{(H)}$ can be identified with the pullback bundle of $\widetilde{E}_{(H)}/G \to \widetilde{U}_{(H)}/G$ by the canonical projection $\widetilde{U}_{(H)} \to \widetilde{U}_{(H)}/G$. Now, the union

$$E_u^{\mathsf{str}} := \bigcup_{(H)\subset G} \widetilde{E}_{(H)}/G$$

is a (in general not closed) subspace of $\widetilde{E}/G$ which carries a canonical stratification given by the set germs of the vector bundles $\widetilde{E}_{(H)}/G$. The only nontrivial part in the proof of this is to show that locally, the condition of frontier (DEC2) is satisfied. To this end it suffices to prove that for all orbit types $(K) \subsetneq (H)$ and every point $\tilde{x} \in \widetilde{U}_{(H)} \cap \overline{\widetilde{U}_{(K)}}$ the fiber $\widetilde{E}_{\tilde{x}}^{G_{\tilde{x}}}$ is contained in the closure of $\widetilde{E}_{(K)}$. Let us show this. By the slice theorem we can assume after possibly passing to conjugate subgroups that $G_{\tilde{x}} = H$, $K \subset H$ and that there exists a sequence of points $\tilde{x}_n \in \widetilde{U}_K$ converging to $\tilde{x}$. By passing to an appropriate subsequence of $(\tilde{x}_n)$ we can achieve that the sequence of fibers $\widetilde{E}_{\tilde{x}_n}^K$ converges in the bundle of Grassmannians. By $K \subset H$ one concludes that

$$\widetilde{E}_{\tilde{x}}^H \subset \lim_{n\to\infty} \widetilde{E}_{\tilde{x}_n}^K,$$

whence the condition of frontier holds true.

Next, consider an open embedding $(\psi_{vu}, \iota_{vu}) : (\widetilde{E}_v, G_v, \eta_v) \to (\widetilde{E}_u, G_u, \eta_u)$ between orbibundle charts of E. Then, the induced map between the orbit spaces restricts to a strata preserving open embedding

$$\bar{\psi}_{vu}^{\mathsf{str}} : \widetilde{E}_v^{\mathsf{str}} \to \widetilde{E}_u^{\mathsf{str}}.$$

Restricted to a stratum, $\bar{\psi}_{vu}^{\mathsf{str}}$ is a smooth vector bundle isomorphism onto an open subbundle of the image stratum. Hence, the union

$$E^{\mathsf{str}} = \bigcup_{u\in\overline{\mathcal{U}}} \bar{\eta}_u(E_u^{\mathsf{str}}) \subset E$$

carries a uniquely defined structure of a stratified space such that every one of the topological embeddings $\bar{\eta}_u : \widetilde{E}_u/G \to E$ is an isomorphism of stratified spaces from E_u^{str} onto an open subset of E^{str}. We will say that E^{str} is the *stratified vector bundle associated* to the vector orbibundle E. A smooth section $s : X \to E$ with image in E^{str} now will be called a *smooth stratified section,* if it satisfies the following smooth vertical extension property:

(SVX) For sufficiently small $\varepsilon > 0$ the map

$$E^{\mathsf{str}} \times (-\varepsilon, \varepsilon) \to E, \quad (v, t) \mapsto v + ts\big(\pi(v)\big) \tag{2.2}$$

can be extended to a smooth map $\mathsf{V}_s : E \times (-\varepsilon, \varepsilon) \to E$, which we call a *smooth vertical extension*.

By construction of E^{str}, the map in (2.2) is well-defined and continuous. Clearly, whether it can be extended to a smooth V_s, depends only on the (maximal) orbibundle atlas of E and not on the particular defining orbibundle functor E. The space of smooth stratified sections will be denoted by $\Gamma^\infty_{\mathsf{str}}(E)$ or $\Gamma^\infty(E^{\mathsf{str}})$. The following proposition entails that for a reduced vector orbibundle the vertical extension associated to a smooth stratified section is uniquely defined.

2.11. Proposition. *Let $E \to X$ be a vector orbibundle. Then the following relations are equivalent*:

(1) *E is a reduced vector orbibundle.*
(2) *E^{str} is dense in E.*
(3) *The projection $E^\circ_{|X^\circ} := E^\circ \cap \pi^{-1}(X^\circ) \to X^\circ$ is a smooth vector bundle.*

Proof. Let us first show that (1) implies (3). Let $E \to X$ be reduced and $x \in X^\circ$ be a point. Choose a slice orbibundle chart $(\widetilde{E} \to \widetilde{U}, G, \eta)$ around $0_x \in E$, and $\tilde{x} \in \widetilde{U}$ with $\varrho(\tilde{x}) = x$. By restriction to an appropriate open subbundle of $\widetilde{E}$, we can achieve that $\widetilde{U}/G$ lies in the regular part of X. Moreover, after passing to the reduced orbibundle chart, we can assume that G acts effectively on $\widetilde{E}$. Hence, by assumption, G acts effectively on $\widetilde{U}$. Since $(\widetilde{E}, G, \eta)$ is a slice for the orbibundle germ at $0_{\tilde{x}} \in \widetilde{E}$, the orbichart $(\widetilde{U}, G, \varrho)$ is a slice for the orbispace germ at $\tilde{x}$. Thus $G_{\tilde{y}} = G$ for all $\tilde{y} \in \widetilde{U}$. But G acts effectively on $\widetilde{U}$, so $G = \{e\}$. From this one concludes that $E_{|U} := \pi^{-1}(U) = \widetilde{E}$, hence $E_{|U} \subset E^\circ$. By definition of $E^\circ_{|X^\circ}$, (3) follows.

Clearly, $E^\circ_{|X^\circ} \to X^\circ$ is a vector bundle, if and only if $E^{\mathsf{str}} \cap \pi^{-1}(X^\circ) = E^\circ_{|X^\circ}$. Hence (2) and (3) are equivalent.

For the proof of the implication (3) $\Rightarrow$ (1) let $(\widetilde{E}, G, \varrho)$ be reduced and $v \in E^\circ_{|X^\circ} \cap \eta(\widetilde{E})$. Then $G_{\pi(v)} = G_v$ by definition of E^{str}. Hence $\bigcap_{g\in G} G_{\pi(v)} = \bigcap_{g \in G} G_v = \{e\}$, so 1.3 (3) entails that G acts effectively on $\widetilde{U}$. □

2.12. Example. Let X be an orbispace. Then the *tangent orbibundle functor* $T\mathsf{X}:\overline{\mathcal{U}} \to \mathfrak{VBdl}^{\mathsf{sym}}$ is defined to be the functor which associates to every orbispace chart $(\widetilde{U}, G, \varrho)$ of X the object $(T\widetilde{U}, G)$ and to every morphism $(\varphi, \iota) = \mathsf{X}_{VU} : (\widetilde{V}, H, \upsilon) \to (\widetilde{U}, G, \varrho)$ the morphism $(T\varphi, \iota) : (T\widetilde{V}, H) \to (T\widetilde{U}, G)$. The (finite dimensional) orbibundle defined by $T\mathsf{X}$ will be called the *tangent orbibundle* of X and will be denoted by TX. Similarly, one defines the *cotangent orbibundle* T^*X. Note that both the tangent and cotangent orbibundles are good orbibundles.

More generally, if F is a functor on the category of (finite dimensional) real or complex vector spaces and $\mathsf{E}:\overline{\mathcal{U}} \to \mathfrak{VBdl}^{\mathsf{sym}}$ an orbibundle functor, then the fiberwise application of F to every one of the objects $\mathsf{E}(u)$ leads to a new vector orbibundle functor denoted by FE. Generalizing this even further to covariant and contravariant functors in multiple arguments it is then clear what to understand by the direct sum, the tensor product and so on of vector orbibundles over a common base orbispace X. In the remainder of this work we will use such constructions of vector orbibundles without further explanation.

2.13. Theorem. *Let E be a reduced orbibundle over an orbispace X. Then the space $\Gamma^\infty_{\mathsf{str}}(E)$ of smooth stratified sections carries a natural structure of a $\mathcal{C}^\infty(X)$-module. Moreover, if $\mathcal{U}$ is an open covering of X and $\mathsf{E}:\overline{\mathcal{U}} \to \mathfrak{VBdl}^{\mathsf{sym}}$ an orbibundle functor of E inducing the orbispace functor X on the base, then a continuous section $s : X \to E$ is a smooth stratified section, if and only if it is a good section. s being a* *good section hereby means that there exists a family* $(s_{\widetilde{U}})_{U\in\overline{\mathcal{U}}}$ *of smooth sections* $s_{\widetilde{U}}:\widetilde{U}\to\widetilde{E}_U$ *such that the following conditions hold true*:

(GSEC1) *For every orbispace chart* $(\widetilde{U},G,\varrho)$ *of* X *the section* $s_{\widetilde{U}}$ *is* G*-equivariant.*
(GSEC2) *If* $(\varphi_{VU},\iota_{VU})=\mathsf{X}_{VU}:(\widetilde{V},H,\upsilon)\to(\widetilde{U},G,\varrho)$ *is a morphism and* $(\psi_{VU},\iota_{VU})=\mathsf{E}_{VU}$ *the corresponding morphism between the vector bundles* $(\widetilde{E}_V,H)$ *and* $(\widetilde{E}_U,G)$*, then*

$$s_{\widetilde{U}}\circ\varphi_{VU}=\psi_{VU}\circ s_{\widetilde{V}}. \tag{2.3}$$

(GSEC3) *For every* $(\widetilde{U},G,\varrho)$ *the following relation holds true*:

$$\eta_U\circ s_{\widetilde{U}}=s\circ\varrho. \tag{2.4}$$

If s *is a smooth stratified section, then the family* $(s_{\widetilde{U}})$ *satisfying* (GSEC1) *to* (GSEC3) *is uniquely determined.*

2.14. Remark. The notion of *good maps* between orbifolds has been introduced by Chen and Ruan [6] in their work on orbifold Gromov–Witten theory. The essential feature hereby is that the pull-back of a vector orbibundle by a good map is a well-defined concept, whereas the pull-back orbibundle of an arbitrary smooth map does in general not exist. Moreover, good maps between orbifolds correspond to the morphisms of orbifolds as defined in the groupoid approach to orbifolds. See Moerdijk [13] for more on this.

Proof. Clearly, the second claim implies the first, so we only show that s is a smooth stratified section if and only if it is a good section. The existence of a family $(s_{\widetilde{U}})$ satisfying (GSEC1) to (GSEC3) is obviously sufficient for s to be a smooth stratified map. Hence it remains to prove that the existence of such a family $(s_{\widetilde{U}})$ is also necessary. For simplicity we assume that $\mathcal{U}$ consists only of one connected open set U or, in other words, that E is the orbit space of the orbibundle chart $(\widetilde{E},G,\eta)=\mathsf{E}(U)$. The general case can easily be deduced from this particular situation. Under the assumption made for E, let s be a smooth stratified section $s:\widetilde{U}/G\to\widetilde{E}/G$. Now, given $f\in\mathcal{C}^\infty(\widetilde{E}/G)$ we define a function $\delta_s f\in\mathcal{C}^\infty(\widetilde{E}/G)$ as follows:

$$\delta_s f(v)=\frac{d}{dt}f\bigl(\mathsf{V}_s(v,t)\bigr)\Big|_{t=0}\quad\text{for all } v\in\widetilde{E}/G, \tag{2.5}$$

where V_s is the uniquely defined smooth vertical extension associated to s. By construction, δ_s is a derivation on $\mathcal{C}^\infty(\widetilde{E}/G)$. Hence, according to the Smooth Lifting Theorem of Schwarz [21, Theorem 0.2], there exists a G-invariant smooth vector field $\xi:\widetilde{E}\to T\widetilde{E}$ such that

$$\xi(f\circ\eta)=\delta_s f\quad\text{for all } f\in\mathcal{C}^\infty(\widetilde{E}/G).$$

Obviously, ξ is a vertical vector field, since the restriction of δ_s to $E^\circ_{|X^\circ}$ is vertical. One concludes $s\circ\varrho=\eta\circ\xi_{|\widetilde{U}}$, where $\widetilde{U}$ has been identified with the zero section of $\widetilde{E}$. Let us put $s_{\widetilde{U}}:=\xi_{|\widetilde{U}}$. Then, $s_{\widetilde{U}}$ is a smooth G-invariant section of $\widetilde{E}$ and satisfies

$$\eta\circ s_{\widetilde{U}}=s\circ\varrho. \tag{2.6}$$

Thus (GSEC1) and (GSEC3) hold true.

Next let us show that the G-invariance and Eq. (2.6) uniquely determine $s_{\widetilde{U}}$. To this end check first that $s_{\widetilde{U}}(\tilde{x}) \in \widetilde{E}_{\tilde{x}}^{G_{\tilde{x}}}$ for all $\tilde{x} \in \widetilde{U}$. Second recall that for every $x \in U$ the fiber E_x^{str} coincides naturally with $\widetilde{E}_{\tilde{x}}^{G_{\tilde{x}}}$, where $\tilde{x} \in \varrho^{-1}(x)$. By Eq. (2.6) this entails that $s_{\widetilde{U}}$ is uniquely determined.

Finally, if $\mathcal{U}$ is an arbitrary open covering of X, axiom (GSEC2) follows immediately from the uniqueness of the sections $s_{\widetilde{U}}$, since for $V, U \in \mathcal{U}$ with $V \subset U$ the composition $\psi_{VU}^{-1} \circ s_{\widetilde{U}} \circ \varphi_{VU}$ is also a G-equivariant section over $\widetilde{V}$ satisfying (GSEC3), hence it must coincide with $s_{\widetilde{V}}$. This proves the claim. □

2.15. Remark. According to the theorem one can identify a smooth stratified section of a reduced vector orbibundle with a family $(s_{\widetilde{U}})_{\widetilde{U} \in \mathcal{U}}$ having properties (GSEC1) to (GSEC3), and every family $(s_{\widetilde{U}})_{\widetilde{U} \in \mathcal{U}}$ which fulfills (GSEC1) and (GSEC2) gives rise to a unique smooth stratified section such that also (GSEC3) holds true. In the rest of this work we will very often make use of these canonical identifications. For example we denote vector fields $\xi : X \to TX$ briefly by $(\xi_{\widetilde{U}})$ and assume from now on that the index $\widetilde{U}$ runs through the domains of the orbispace charts of the defining orbifold functor X. Likewise, we denote differential forms on X, tensor fields and so on.

3. Symplectic orbispaces

3.1. Let X be an orbispace, and $\mathcal{U}, \mathsf{X}$ like before. By a *riemannian metric* (respectively *symplectic form*) on X we understand a family of G-invariant riemannian metrics $g_{\widetilde{U}}$ (respectively symplectic forms $\omega_{\widetilde{U}}$) on $\widetilde{U}$, where $(\widetilde{U}, G, \varrho)$ runs through the charts of X, such that for every morphism $(\varphi, \iota) := \mathsf{X}_{VU} : (\widetilde{V}, H, \upsilon) \to (\widetilde{U}, G, \varrho)$ between two orbispace charts the relation

$$\varphi^* g_{\widetilde{U}} = g_{\widetilde{V}} \quad \text{respectively} \tag{3.1}$$

$$\varphi^* \omega_{\widetilde{U}} = \omega_{\widetilde{V}} \tag{3.2}$$

is satisfied. We will denote such a riemannian metric (respectively symplectic form) by $(g_{\widetilde{U}})$ (respectively $(\omega_{\widetilde{U}})$). An orbispace with a riemannian metric $(g_{\widetilde{U}})$ (respectively symplectic form $(\omega_{\widetilde{U}})$) will be called a *riemannian* (respectively *symplectic*) *orbispace*; likewise one defines *riemannian* and *symplectic orbifolds*. Note that by Theorem 2.13, $(g_{\widetilde{U}})$ (respectively $(\omega_{\widetilde{U}})$) corresponds to a smooth stratified section $g \in \Gamma_{\mathrm{str}}^{\infty}(T^*X \otimes T^*X)$ (respectively $\omega \in \Gamma_{\mathrm{str}}^{\infty}(T^*X \otimes T^*X)$).

Since for every orbispace chart $(\widetilde{U}, G, \varrho)$ there exists a G-invariant riemannian metric on $\widetilde{U}$ and because the sheaf $\mathcal{C}_X^{\infty}$ is fine, it is easy to construct a riemannian metric for X.

Like in the manifold case, natural examples of symplectic orbispaces are given by cotangent bundles. To see this, let T^*X be the cotangent orbibundle of (X, X) and consider the orbispace chart $(T^*\widetilde{U}, G, T^*\varrho)$ induced by $(\widetilde{U}, G, \varrho)$. Then $T^*\widetilde{U}$ carries a canonical symplectic form $\omega_{T^*\widetilde{U}}$ and this symplectic form is invariant with respect to the lifted G-action. Moreover, if $(\varphi, \iota) : (\widetilde{V}, H, \upsilon) \to (\widetilde{U}, G, \varrho)$ is a morphism and $(T^*\varphi, \iota) = (\varphi^{-1^*}, \iota) : (T^*\widetilde{V}, H, T^*\upsilon) \to (T^*\widetilde{U}, G, T^*\varrho)$ the induced morphism of orbispace charts of T^*X, then $(T^*\varphi)^*\omega_{T^*\widetilde{U}} = \omega_{T^*\widetilde{V}}$, hence the $\omega_{T^*\widetilde{U}}$ define a symplectic form on T^*X.

3.2. Example. As a specific example of a symplectic orbifold consider the cotangent orbibundle of the real half line $Y = [0, \infty)$. A global orbifold chart for Y is given by $\mathbb{R}$ with the $\mathbb{Z}_2$-action such that the nonzero element acts by inversion. Therefore, T^*Y is the quotient $\mathbb{R}^2/\mathbb{Z}_2$, where the nonzero element of $\mathbb{Z}_2$ acts again by inversion. A Hilbert basis of the $\mathbb{Z}_2$-invariant polynomials on $\mathbb{R}^2$ is given by the polynomials p^2+q^2, p^2-q^2 and $2pq$, where (p,q) are the coordinates of an element of $\mathbb{R}^2$. Now,

$$\left(p^2+q^2\right)^2 = \left(p^2-q^2\right)^2 + (2pq)^2,$$

hence the orbifold $\mathbb{R}^2/\mathbb{Z}_2$ is diffeomorphic to the standard cone $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1^2+x_2^2=x_3^2\}$. Moreover, the symplectic orbifold $\mathbb{R}^2/\mathbb{Z}_2$ has a natural stratification by two symplectic strata, where the top stratum is given by $\dot{\mathbb{R}}^2/\mathbb{Z}_2$ with $\dot{\mathbb{R}}^2 = \mathbb{R}^2 \setminus \{0\}$ and the second stratum is given by $\{0\}$ or in other words by the cusp of the cone.

3.3. Proposition. *Let X be a symplectic orbispace. Then every stratum of the orbispace stratification carries in a canonical way the structure of a Poisson manifold. Moreover, if X is an orbifold, the strata are symplectic.*

Proof. We show the claim for the case, where the orbispace is given by the orbit space of a symplectic G-action on a symplectic manifold M. Clearly, this suffices to prove the proposition, since the claimed property of X is essentially a local statement. So let us assume that $X = M/G$. Then it is well-known that for every orbit type (H) the manifold M_H of points of M with isotropy group equal to H inherits from M a symplectic structure [10, Proposition 27.5]. Moreover, the canonical projection $\pi_H : M_H \to M_{(H)}/G$ onto the stratum $M_{(H)}/G$ is a principal fiber bundle with typical fiber $N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G. Now, given two functions $f,g\in\mathcal{C}^\infty(M_{(H)}/G)$ the Poisson bracket $\{f\circ\pi_H, g\circ\pi_H\}$ with respect to the canonical symplectic structure on M_H is $N_G(H)$-invariant, hence there exists a unique $\{f,g\}_H \in \mathcal{C}^\infty(M_{(H)}/G)$ such that

$$\{f,g\}_H\circ\pi_H = \{f\circ\pi_H, g\circ\pi_H\}.$$

Clearly, $\{\cdot,\cdot\}_H$ is antisymmetric and satisfies the Jacobi identity, hence is a Poisson bracket on $\mathcal{C}^\infty(M_{(H)}/G)$. Thus, $M_{(H)}/G$ carries the structure of a Poisson manifold and this Poisson structure is natural in the sense that it is invariant under equivariant symplectic diffeomorphisms of M.

Under the assumption that the symmetry group G is finite the zero map $M\to\{0\}$ provides a momentum map for the symplectic G-action, so by Sjamaar and Lerman [22, Theorem 2.1] the strata $M_{(H)}/G$ are symplectic in this case. This proves the proposition. □

3.4. A family $(\nabla_{\widetilde U})$ of G-invariant (affine) connections $\nabla_{\widetilde U}$ defined on $\Gamma^\infty(T\widetilde U)$ is called a *connection* on X, if for every vector field $(\xi_{\widetilde U})$ on X and every morphism $(\varphi,\iota):(\widetilde V, H, \upsilon)\to(\widetilde U, G, \varrho)$ between charts of $\mathcal{U}$ the compatibility relation

$$\varphi^*(\nabla_{\widetilde U}\xi_{\widetilde U}) = \nabla_{\widetilde V}\xi_{\widetilde V} \tag{3.3}$$

holds true. Note that every connection $(\nabla_{\widetilde U})$ on X gives rise to a *covariant derivative*, i.e., a linear map $\nabla:\Gamma^\infty_{\mathrm{str}}(TX)\to\Gamma^\infty_{\mathrm{str}}(T^*X\otimes TX)$ such that

$$\nabla(f\xi) = df\otimes\xi + f\nabla\xi \quad \text{for all } f\in\mathcal{C}^\infty(X) \text{ and } \xi\in\Gamma^\infty_{\mathrm{str}}(TX). \tag{3.4}$$

If $(g_{\widetilde U})$ is a riemannian metric on X, then the family $(\nabla^{\mathrm{LC}}_{\widetilde U})$, which associates to every $\widetilde U$ the Levi-Civita connection with respect to $g_{\widetilde U}$, provides a torsionfree connection on X. Obviously, $(\nabla^{\mathrm{LC}}_{\widetilde U})$ leaves the riemannian metric $(g_{\widetilde U})$ invariant and will be called the *Levi-Civita connection* of $(g_{\widetilde U})$. In case $(\omega_{\widetilde U})$ is a symplectic form on X, a connection $(\nabla_{\widetilde U})$ is called *symplectic*, if $\nabla_{\widetilde U}\omega_{\widetilde U}=0$ holds for all $\widetilde U$.

More generally, let us assume now that $E \to X$ is a reduced vector orbibundle, where the typical fiber V is a profinite dimensional vector space. By a *connection* on E we then understand a linear map $D: \Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet X \otimes E) \to \Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet X \otimes E)$ of antisymmetric degree 1 such that

$$D(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge Ds \quad \text{for all } \alpha \in \Gamma^\infty\big(\Lambda^k X\big) \text{ and } s \in \Gamma^\infty_{\mathrm{str}}(E). \tag{3.5}$$

Given a Satake atlas $\mathcal{U}$ for X and a bundle atlas $((E_{\widetilde{U}}, G, \eta_{\widetilde{U}}))_{\widetilde{U}\in\mathcal{U}}$ over $\mathcal{U}$, Theorem 2.13 entails that a connection can be regarded as a family $(D_{\widetilde{U}})$ of connections $D_{\widetilde{U}}: \Gamma^\infty(\Lambda^\bullet \widetilde{U} \otimes E_{\widetilde{U}}) \to \Gamma^\infty(\Lambda^\bullet \widetilde{U} \otimes E_{\widetilde{U}})$ such that for every smooth section $s = (s_{\widetilde{U}})$ one has

$$(Ds)_{\widetilde{U}} = D_{\widetilde{U}} s_{\widetilde{U}} \quad \text{for all } \widetilde{U} \in \mathcal{U}. \tag{3.6}$$

The *curvature* of a connection D is the two-form $R \in \Gamma^\infty_{\mathrm{str}}(\Lambda^2 X \otimes \mathrm{End}(E))$ with

$$R(\xi, \zeta)s = [D_\xi, D_\zeta]s - D_{[\xi,\zeta]}s \quad \text{for all } \xi, \zeta \in \Gamma^\infty_{\mathrm{str}}(TX) \text{ and } s \in \Gamma^\infty_{\mathrm{str}}(E). \tag{3.7}$$

Obviously, $R = (R_{\widetilde{U}})$, where $R_{\widetilde{U}}$ is the curvature of $D_{\widetilde{U}}$.

3.5. Proposition. *For every symplectic orbispace there exists a torsionfree symplectic connection.*

Proof. First fix a riemannian metric $(g_{\widetilde{U}})$ on X and use the corresponding Levi-Civita connection $(\nabla^{\mathrm{LC}}_{\widetilde{U}})$ to define a contravariant 3-tensor field $(\Delta'_{\widetilde{U}})$ on TX:

$$\Delta'_{\widetilde{U}}(\xi_1, \xi_2, \xi_3) := \frac{1}{3}\big(\nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_3, \xi_1, \xi_2) + \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_2, \xi_1, \xi_3)\big), \quad \xi_1, \xi_2, \xi_3 \in T_{\tilde{x}}\widetilde{U},\ \tilde{x} \in \widetilde{U}. \tag{3.8}$$

Note that $(\Delta'_{\widetilde{U}})$ is symmetric in the last two variables. Next lift the first variable of $(\Delta'_{\widetilde{U}})$ with the help of $(\omega_{\widetilde{U}})$ and denote the resulting tensor field by $(\Delta_{\widetilde{U}})$, that means the equality $\omega_{\widetilde{U}}(\,\cdot\,, \Delta_{\widetilde{U}}) = \Delta'_{\widetilde{U}}$ is satisfied over each $\widetilde{U}$. Then by construction, the connection $(\nabla_{\widetilde{U}})$ defined by

$$\nabla_{\widetilde{U}} = \nabla^{\mathrm{LC}}_{\widetilde{U}} + \Delta_{\widetilde{U}} \tag{3.9}$$

consists of G-invariant and torsionfree local connections. Moreover, it is also clear by construction that these connections satisfy the compatibility condition $\varphi^* \nabla_{\widetilde{U}} = \nabla_{\widetilde{V}}$ for every morphism (φ, ι) like above. Finally, $(\nabla_{\widetilde{U}})$ is symplectic by the following computation:

$$\begin{aligned}
\nabla_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_1, \xi_2, \xi_3) &= \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_1, \xi_2, \xi_3) - \omega_{\widetilde{U}}\big(\nabla^{\mathrm{LC}}_{\widetilde{U}}(\xi_1, \xi_2), \xi_3\big) - \omega_{\widetilde{U}}\big(\xi_2, \nabla^{\mathrm{LC}}_{\widetilde{U}}(\xi_1, \xi_3)\big) \\
&= \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_1, \xi_2, \xi_3) - \big(\Delta'_{\widetilde{U}}(\xi_2, \xi_1, \xi_3) - \Delta'_{\widetilde{U}}(\xi_3, \xi_1, \xi_2)\big) \\
&= \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_1, \xi_2, \xi_3) - \frac{1}{3}\big(\nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_3, \xi_2, \xi_1) + \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_1, \xi_2, \xi_3) \\
&\quad - \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_2, \xi_3, \xi_1) - \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_1, \xi_3, \xi_2)\big) \\
&= \frac{1}{3}\big(\nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_1, \xi_2, \xi_3) + \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_2, \xi_3, \xi_1) + \nabla^{\mathrm{LC}}_{\widetilde{U}} \omega_{\widetilde{U}}(\xi_3, \xi_1, \xi_2)\big) \\
&= d\omega_{\widetilde{U}}(\xi_1, \xi_3, \xi_2) = 0. \qquad \square
\end{aligned} \tag{3.10}$$

3.6. Given a symplectic form $(\omega_{\widetilde{U}})$ on X one can define a natural Poisson bracket on the algebra $\mathcal{C}^\infty(X)$ as follows. For every point $x \in X$ choose an orbispace chart $(\widetilde{U}, G, \varrho)$ around x, let $\tilde{x} \in \widetilde{U}$ be a point with $\varrho(\tilde{x}) = x$ and denote by $\{\cdot,\cdot\}_{\widetilde{U}}$ the Poisson bracket on $\mathcal{C}^\infty(\widetilde{U})$. Then define

$$\{f, g\}(x) := \{f \circ \varrho, g \circ \varrho\}_{\widetilde{U}}(\tilde{x}) \quad \text{for } f, g \in \mathcal{C}^\infty(X). \tag{3.11}$$

By the compatibility relation (3.2), the value $\{f, g\}(x)$ is independent of the special choice of the chart $(\widetilde{U}, G, \varrho)$, so $\{f, g\} \in \mathcal{C}^\infty(X)$ is well-defined. Using the corresponding properties of the Poisson brackets $\{\cdot,\cdot\}_{\widetilde{U}}$ one now checks immediately that $\{\cdot,\cdot\}$ is antisymmetric in its arguments and satisfies the Jacobi identity, hence $\{\cdot,\cdot\}$ is a Poisson bracket on $\mathcal{C}^\infty(X)$. Note that the symplectic form $(\omega_{\widetilde{U}})$ also gives rise to the *Poisson bivector field* $\Pi = (\Pi_{\widetilde{U}})$ on X, where $\Pi_{\widetilde{U}}$ is the Poisson bivector field on $\widetilde{U}$ corresponding to $\omega_{\widetilde{U}}$.

The well-known definition of a formal deformation quantization of a symplectic manifold by Bayen, Flato, Lichnerowicz and Sternheimer [1] can be easily extended to the orbispace arena. Let us provide the details. Consider the space $\mathcal{C}^\infty(X)[\![\lambda]\!]$ of formal power series in the variable λ and with coefficients in $\mathcal{C}^\infty(X)$. A $\mathbb{C}[\![\lambda]\!]$-bilinear associative product

$$\star : \mathcal{C}^\infty(X)[\![\lambda]\!] \times \mathcal{C}^\infty(X)[\![\lambda]\!] \to \mathcal{C}^\infty(X)[\![\lambda]\!]$$

is called a *formal deformation quantization* of $\mathcal{C}^\infty(X)$ or a *star product*, if for all $f, g \in \mathcal{C}^\infty(X)$ the following holds:

(DQ1) $f \star g = \sum_{k\in\mathbb{N}} \mu_k(f, g)\lambda^k$, where the $\mu_k : \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ are bilinear maps and $\mu_0 = \mu$ is the pointwise product on $\mathcal{C}^\infty(X)$,
(DQ2) $[f, g]_\star - i\lambda\{f, g\} \in \lambda^2\mathcal{C}^\infty(X)[\![\lambda]\!]$, where $[f, g]_\star$ is the commutator $f \star g - g \star f$,
(DQ3) $f \star 1 = 1 \star f = f$.

The deformation quantization is called *local*, if for all $k \in \mathbb{N}$

$$\operatorname{supp} \mu_k(f, g) \subset \operatorname{supp} f \cap \operatorname{supp} g, \tag{3.12}$$

and *differential*, if all the μ_k are bidifferential operators on X. By a *bidifferential operator* on X we hereby understand an operator $\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ which in every orbispace chart $(\widetilde{U}, G, \varrho)$ is induced by a G-invariant bidifferential on $\widetilde{U}$.

3.7. Example. Consider the symplectic cone $C = \mathbb{R}^2/\mathbb{Z}_2$ of Example 3.2. Let $\star$ be the Moyal–Weyl product on $\mathbb{R}^2$ that means

$$f \star g = \sum_{k\in\mathbb{N}} \left(\frac{-i\lambda}{2}\right)^k \mu\big(\widehat{\Pi}(f \otimes g)\big) \quad \text{for all } f, g \in \mathcal{C}^\infty(\mathbb{R}^2), \tag{3.13}$$

where $\widehat{\Pi}(f \otimes g) = \frac{\partial}{\partial q} f \otimes \frac{\partial}{\partial p} g - \frac{\partial}{\partial p} f \otimes \frac{\partial}{\partial q} g$ and $\mu(f \otimes g) = f\,g$. Since the operator $\widehat{\Pi}$ is $\mathbb{Z}_2$-invariant, $\star$ can be restricted to an associative product on the space $\mathcal{C}^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}[\![\lambda]\!]$, where $\mathcal{C}^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ denotes the algebra of $\mathbb{Z}_2$-invariant smooth functions on $\mathbb{R}^2$. But $\mathcal{C}^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ is canonically isomorphic to $\mathcal{C}^\infty(C)$, hence we obtain a star product for C.

4. Fedosov's quantization for orbispaces

4.1. In this section we will show how Fedosov's method for the construction of a (differentiable) star-product can be transferred to the arena of orbispaces. The essential point hereby is to check that all of Fedosov's constructions can be performed in a manner which is natural with respect to morphisms of orbispace charts and invariant with respect to the involved symmetries. We proceed analogously to Fedosov [9, Chapter 5] (cf. also [4, Section 21]). In particular, we will define the Weyl algebra bundle $\mathbb{W}X$ of a symplectic orbispace X and then construct a flat connection D on the Weyl algebra bundle such that the space of formal power series in $\mathcal{C}^\infty(X)$ can be (linearly) identified with the subalgebra of flat sections of $\mathbb{W}X$. Via this identification, $\mathcal{C}^\infty(X)$ then inherits a star-product from $\mathbb{W}X$.

4.2. Let V be a finite dimensional Poisson vector space and Π its Poisson bivector. One can then associate to V the *formal Weyl algebra* $\mathbb{W}V$ and the *completed formal Weyl algebra* $\widehat{\mathbb{W}}V$ as follows. As a (complex) vector space $\mathbb{W}V$ coincides with $\mathrm{Sym}^\bullet(V^*)[\![\lambda]\!]$, the space of formal power series in λ with coefficients in the algebra of complex valued polynomial functions on V. The completed formal Weyl algebra $\widehat{\mathbb{W}}V$ has $\widehat{\mathrm{Sym}}^\bullet(V^*)[\![\lambda]\!]$ as underlying linear space. Note that $\mathrm{Sym}^\bullet(V^*) = \bigoplus_{s\in\mathbb{N}} \mathrm{Sym}^s(V^*)$ is a graded algebra, where the product is given by μ, the pointwise product of functions, and the homogeneous component $\mathrm{Sym}^s(V^*)$ consists of s-homogeneous polynomials. The profinite dimensional vector space $\widehat{\mathrm{Sym}}^\bullet(V^*)$ coincides with $\prod_{s\in\mathbb{N}} \mathrm{Sym}^s(V^*)$ and carries a natural descending filtration given by the powers $\widehat{\mathfrak{m}}^n$, where $\widehat{\mathfrak{m}}$ is the kernel of the canonical morphism $\widehat{\mathrm{Sym}}^\bullet(V^*) \to \mathbb{C} \cong \mathrm{Sym}^0(V^*)$. Moreover, $\widehat{\mathrm{Sym}}^\bullet(V^*)$ is complete with respect to the topology defined by this filtration.

By construction, $\mathbb{W}V$ is a subspace of $\widehat{\mathbb{W}}V$. Every element $a \in \widehat{\mathbb{W}}V$ now has a unique representation of the form

$$a = \sum_{k\in\mathbb{N},\ s\in\mathbb{N}} a_{sk}\lambda^k, \tag{4.1}$$

where $a_{sk} \in \mathrm{Sym}^s(V^*)$ and where only finitely many a_{sk} do not vanish for fixed k, if $a \in \mathbb{W}V$. Next recall that the Poisson bivector Π can be written as a finite sum $\Pi = \sum_i \Pi_{1i} \otimes \Pi_{2i}$ with $\Pi_{1i}, \Pi_{2i} \in V$ and that the elements of V act by derivations on $\mathrm{Sym}^\bullet(V^*)$. Therefore, the operator

$$\begin{aligned} &\widehat{\Pi} : \mathrm{Sym}^\bullet(V^*) \otimes_{\mathbb{C}} \mathrm{Sym}^\bullet(V^*) \to \mathrm{Sym}^\bullet(V^*) \otimes_{\mathbb{C}} \mathrm{Sym}^\bullet(V^*), \\ &f \otimes g \mapsto \sum_i \Pi_{1i} f \otimes \Pi_{2i} g \end{aligned}$$

is well-defined and continuous with respect to the Krull topology defined by $\mathfrak{m}$. Hence, $\widehat{\Pi}$ can be extended by $\mathbb{C}[\![\lambda]\!]$-linearity and continuity to an operator on $\widehat{\mathbb{W}}V \otimes_{\mathbb{C}} \widehat{\mathbb{W}}V$. The *Moyal–Weyl* product on $\widehat{\mathbb{W}}V$ then is given as follows:

$$a \circ b := \mu\big(\exp(-i\lambda\widehat{\Pi})(a\otimes b)\big) := \sum_{k\in\mathbb{N}} \frac{(-i\lambda)^k}{k!}\mu\big(\widehat{\Pi}^k(a\otimes b)\big) \quad \text{for } a, b \in \widehat{\mathbb{W}}V. \tag{4.2}$$

Thus $(\widehat{\mathbb{W}}V, \circ)$ becomes an associative algebra, and $\mathbb{W}V$ a subalgebra. The (completed) formal Weyl algebra carries a (descending) filtration $(\widehat{\mathbb{W}}_n V)_{n\in\mathbb{N}}$ defined by the *Fedosov-degree*

$$\deg_{\mathrm{F}}(a) = \min\{s + 2k \mid a_{sk} \neq 0\}, \quad a \in \mathbb{W}V. \tag{4.3}$$

This means that $\widehat{\mathbb{W}}_n V$ is the subalgebra $\{a \in \widehat{\mathbb{W}}V \mid \deg_{\mathrm{F}}(a) \geqslant n\}$.

Additionally to $\widehat{\mathbb{W}}V$ we consider the algebra $\Lambda^\bullet\widehat{\mathbb{W}}V := \Lambda^\bullet V \otimes_{\mathbb{R}} \widehat{\mathbb{W}}V$ of alternating forms with values in $\widehat{\mathbb{W}}V$. The product $\circ$ on $\widehat{\mathbb{W}}V$ and the exterior product on $\Lambda^\bullet V$ induce a product on $\Lambda^\bullet\widehat{\mathbb{W}}V$, denoted by $\circ$ as well. Moreover, the filtration of $\widehat{\mathbb{W}}V$ induces a filtration of $\Lambda^\bullet\widehat{\mathbb{W}}V$.

The following result is crucial for all our further considerations. As the proof is obvious, we leave it to the reader.

4.3. Proposition. *Associate to every finite dimensional Poisson vector space V the completed formal Weyl algebra $\widehat{\mathbb{W}}V$ and to every linear Poisson map $f : W \to V$ the linear map*

$$\widehat{\mathbb{W}}f : \widehat{\mathbb{W}}V \to \widehat{\mathbb{W}}W, \quad a = \sum_{k\in\mathbb{N},\ s\in\mathbb{N}} a_{sk}\lambda^k \mapsto \sum_{k\in\mathbb{N},\ s\in\mathbb{N}} f^*(a_{sk})\lambda^k. \tag{4.4}$$

Then, $\widehat{\mathbb{W}}$ is a contravariant functor with values in the category of profinite dimensional vector spaces. Likewise, $\Lambda^\bullet\widehat{\mathbb{W}}$ can be regarded as a functor defined on the category of finite dimensional Poisson vector spaces with values in the category of profinite dimensional vector spaces.

4.4. Next let us consider a symplectic orbispace $(X, (\omega_{\widetilde{U}}))$. Without loss of generality we can assume that every $\widetilde{U}$ appearing as an index of $(\omega_{\widetilde{U}})$ is an orbispace chart of some orbispace functor X such that $(\omega_{\widetilde{U}})$ is an open G-invariant subset of $\mathbb{R}^{2n}$, such that G acts by linear symplectic maps on $\mathbb{R}^{2n}$ and finally such that the symplectic form $\omega_{\widetilde{U}}$ is given by $\sum_{j=1}^n d\tilde{x}_j \wedge d\tilde{x}_{n+j}$, where $(\tilde{x}_1, \dots, \tilde{x}_{2n})$ are the natural coordinate functions over $\widetilde{U} \subset \mathbb{R}^{2n}$. Given an element $(\widetilde{U}, G, \varrho) \in \mathcal{U}$, every fiber of $T\widetilde{U}$ is a Poisson vector space, so we can apply $\widehat{\mathbb{W}}$ fiberwise and thus obtain the Weyl algebra bundle $\widehat{\mathbb{W}}\widetilde{U}$. Likewise, the bundle of forms of the Weyl algebra $\Lambda^\bullet\widehat{\mathbb{W}}\widetilde{U}$ is constructed. Following Fedosov [9, Chapter 5] we will now introduce a convenient representation of the sections of these bundles. Let $(d\tilde{x}_1, \dots, d\tilde{x}_{2n})$ be the local frame of $T^*\widetilde{U}$ corresponding to the coordinates $(\tilde{x}_1, \dots, \tilde{x}_{2n})$ and denote by $\tilde{y}_j$ for $j = 1, \dots, 2n$ the canonical image of $d\tilde{x}_j$ in the sectional space $\Gamma^\infty(\mathrm{Sym}^\bullet(T^*\widetilde{U}))$. Hereby, $\mathrm{Sym}^\bullet$ is regarded as a fiberwise acting functor on the category of finite dimensional vector bundles. As a (topological) $\mathcal{C}^\infty(\widetilde{U})$-module, $\Gamma^\infty(\mathrm{Sym}^\bullet(T^*\widetilde{U}))$ is generated by the sections $\tilde{y}^\alpha = \tilde{y}_1^{\alpha_1} \cdots \tilde{y}_n^{\alpha_n}$, where $\alpha \in \mathbb{N}^n$. With these notational agreements, a section $a_{\widetilde{U}} \in \Gamma^\infty(\Lambda^\bullet\widehat{\mathbb{W}}\widetilde{U})$ respectively an element $a_{\tilde{x}} \in \Gamma^\infty(\widehat{\mathbb{W}}\widetilde{U})$ (with $\tilde{x}$ denoting the footpoint) can be represented in the form

$$a_\diamond = \sum_{k\in\mathbb{N},\ \alpha\in\mathbb{N}^{2n},\ l\in\mathbb{N}} \sum_{1\leqslant j_1<\cdots<j_l\leqslant 2n} a_{\diamond,k\alpha j_1\cdots j_l}\tilde{y}^\alpha\, d\tilde{x}_{j_1} \wedge \cdots \wedge d\tilde{x}_{j_l}\, \lambda^k \tag{4.5}$$

where $\diamond$ is one of the symbols $\widetilde{U}$ or $\tilde{x}$, and the elements $a_{\widetilde{U},k\alpha j_1\cdots j_l} \in \mathcal{C}^\infty(\widetilde{U})$ respectively $a_{\tilde{x},k\alpha j_1\cdots j_l} \in \mathbb{C}$ are uniquely defined. To simplify notation we write $a_{\tilde{x}}$ not only for an element of $\widehat{\mathbb{W}}\widetilde{U}$ with footpoint $\tilde{x}$ but also for the evaluation of a section $a_{\widetilde{U}} \in \Gamma^\infty(\Lambda^\bullet\widehat{\mathbb{W}}\widetilde{U})$ at $\tilde{x}$.

4.5. In the following step we will lift the G-action to $\widehat{\mathbb{W}}\widetilde{U}$. Denote by l_g the action of some group element g on $\widetilde{U}$. Then the derivative $T_{\tilde{x}}l_g$ is a linear Poisson map, so by Proposition 4.3

$$G \times \widehat{\mathbb{W}}\widetilde{U} \to \widehat{\mathbb{W}}\widetilde{U}, \quad (g, a_{\tilde{x}}) \mapsto \widehat{\mathbb{W}}(T_{g\tilde{x}}l_{g^{-1}})(a_{\tilde{x}})$$

is a G-action on $\widehat{\mathbb{W}}\widetilde{U}$. Given a second element $(\widetilde{V}, H, \upsilon) \in \mathcal{U}$ and a morphism of orbispace charts $(\varphi, \iota) : (\widetilde{V}, H, \upsilon) \to (\widetilde{U}, G, \varrho)$ the pair

$$(\widehat{\mathbb{W}}\varphi, \iota) : (\widehat{\mathbb{W}}\widetilde{V}, H) \to (\widehat{\mathbb{W}}\widetilde{U}, G), \quad (a_{\tilde{y}}, h) \mapsto \big(\widehat{\mathbb{W}}(T_{\tilde{y}}\varphi)^{-1}(a_{\tilde{y}}), \iota\big)$$

induces a morphism in the category of profinite dimensional vector spaces with symmetries, since by Proposition 4.3

$$\begin{aligned}\widehat{\mathbb{W}}\varphi(ha_{\tilde y}) &= \widehat{\mathbb{W}}(T_{h\tilde y}\varphi)^{-1}(ha_{\tilde y}) = \widehat{\mathbb{W}}\big(T_{h\tilde y}l_{h^{-1}}\circ(T_{h\tilde y}\varphi)^{-1}\big)(a_{\tilde y})\\ &= \widehat{\mathbb{W}}\big(T_{\tilde y}(\varphi\circ l_h)\big)^{-1}(a_{\tilde y}) = \widehat{\mathbb{W}}\big(T_{\tilde y}(l_{\iota(h)}\circ\varphi)\big)^{-1}(a_{\tilde y})\\ &= \widehat{\mathbb{W}}(T_{\iota(h)\varphi(\tilde y)}l_{\iota(h)^{-1}})\widehat{\mathbb{W}}\varphi(a_{\tilde y}) = \iota(h)\widehat{\mathbb{W}}\varphi(a_{\tilde y}).\end{aligned} \tag{4.6}$$

Thus we obtain an orbibundle functor $\widehat{\mathbb{W}}\mathsf{X}$ which associates to every $\widetilde U$ the pair $(\widehat{\mathbb{W}}\widetilde U, G)$ and to every morphism (φ,ι) between elements of $\mathcal{U}$ the morphism $(\widehat{\mathbb{W}}\varphi,\iota)$. The functor $\widehat{\mathbb{W}}\mathsf{X}$ induces a vector orbibundle $\widehat{\mathbb{W}}X\to X$, called the *Weyl algebra orbibundle* of X, and an orbibundle atlas $(\widehat{\mathbb{W}}\widetilde U, G, \widehat{\mathbb{W}}\varrho)$. Likewise, one constructs the vector orbibundle $\Lambda^\bullet\widehat{\mathbb{W}}X\to X$ of so-called *forms of the Weyl algebra orbibundle*. By construction, the orbibundles $\widehat{\mathbb{W}}X$ and $\Lambda^\bullet\widehat{\mathbb{W}}X$ are reduced, hence Remark 2.15 applies to sections of $\widehat{\mathbb{W}}X$ and $\Lambda^\bullet\widehat{\mathbb{W}}X$.

4.6. Proposition. *The sectional spaces $\Gamma^\infty_{\mathrm{str}}(\widehat{\mathbb{W}}X)$ and $\Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet\widehat{\mathbb{W}}X)$ carry in a natural way a $\mathbb{C}[\![\lambda]\!]$-bilinear associative product $\circ$ such that*

$$(a\circ b)_{\widetilde U} = a_{\widetilde U}\circ b_{\widetilde U}\quad \textit{for all } a,b\in\Gamma^\infty_{\mathrm{str}}(\widehat{\mathbb{W}}X)\ (\textit{respectively } a,b\in\Lambda^\bullet\Gamma^\infty_{\mathrm{str}}(\widehat{\mathbb{W}}X)). \tag{4.7}$$

Moreover, the space $\Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet\widehat{\mathbb{W}}X)$ thus becomes a graded and filtered algebra, where the graduation degree is given by the form degree and the filtration degree by the Fedosov degree. The topology defined by the Fedosov filtration provides $\Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet\widehat{\mathbb{W}}X)$ with the structure of a complete topological vector space.

The graded commutator on $\Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet\widehat{\mathbb{W}}X)$ with respect to the product $\circ$ will be denoted by $[\cdot,\cdot]$.

Proof. Using the G-invariance of the symplectic form $\omega_{\widetilde U}$ it is straightforward to check that

$$ga_{\widetilde U}\circ gb_{\widetilde U} = g(a_{\widetilde U}\circ b_{\widetilde U})\quad \text{for all } a_{\widetilde U}, b_{\widetilde U}\in\Gamma^\infty(\widehat{\mathbb{W}}\widetilde U). \tag{4.8}$$

Moreover, if (φ,ι) is a morphism like above, then

$$\widehat{\mathbb{W}}\varphi(a_{\tilde y}\circ b_{\tilde y}) = \widehat{\mathbb{W}}\varphi(a_{\tilde y})\circ\widehat{\mathbb{W}}\varphi(b_{\tilde y})\quad \text{for all } a_{\tilde y}, b_{\tilde y}\in\widehat{\mathbb{W}}\widetilde V \text{and } \tilde y\in\widetilde V. \tag{4.9}$$

Hence, Eq. (4.7) defines a section $a\circ b\in\Gamma^\infty_{\mathrm{str}}(\widehat{\mathbb{W}}X)$. From the corresponding properties of the product on $\Gamma^\infty(\widehat{\mathbb{W}}\widetilde U)$ one now concludes that $\circ$ is a $\mathbb{C}[\![\lambda]\!]$-bilinear associative product. The same argument proves that $\circ$ is a product on $\Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet\widehat{\mathbb{W}}X)$. The remaining part of the claim is obvious. □

4.7. Let us now choose a symplectic connection $(\nabla_{\widetilde U})$ on X and extend it in a natural way to a connection on $\Lambda^\bullet\widehat{\mathbb{W}}X$ by putting

$$(\nabla b)_{\widetilde U} = \sum_{j=1}^{2n}\sum_{k,\alpha,l}\sum_{1\leqslant j_1<\cdots<j_l\leqslant 2n}\nabla_{\widetilde U,\frac{\partial}{\partial\tilde x_j}}\big(b_{\widetilde U,k\alpha j_1\cdots j_l}\,\tilde y^\alpha\big)\,d\tilde x_j\wedge d\tilde x_{j_1}\wedge\cdots\wedge d\tilde x_{j_l}\lambda^k. \tag{4.10}$$

By construction, $(\nabla b)_{\widetilde U}$ is a G-equivariant section of $\Lambda^\bullet\widehat{\mathbb{W}}\widetilde U$, and $\varphi^*(\nabla b)_{\widetilde U} = (\nabla b)_{\widetilde V}$ holds for every morphism $(\varphi,\iota):(\widetilde V,H,\upsilon)\to(\widetilde U,G,\varrho)$. Hence, the family $((\nabla b)_{\widetilde U})$ gives rise to a section of $\Lambda^\bullet\widehat{\mathbb{W}}X$, and the connection $\nabla:\Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet\widehat{\mathbb{W}}X)\to\Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet\widehat{\mathbb{W}}X)$ is well-defined. Over $\widetilde U$, the components of ∇b are given by

$$(\nabla b)_{\widetilde{U}} = db_{\widetilde{U}} + \frac{i}{\lambda}[\varGamma_{\widetilde{U}}, b_{\widetilde{U}}], \tag{4.11}$$

where $\varGamma_{\widetilde{U}} = \frac{1}{2}\sum_{i,j,k} \varGamma_{\widetilde{U},ijk} \tilde{y}_i \tilde{y}_j \, d\tilde{x}_k$ is a local one-form and the $\varGamma_{\widetilde{U},ijk}$ are the Christoffel symbols of ∇, i.e., $\nabla_{\widetilde{U},\frac{\partial}{\partial \tilde{x}_i}} \frac{\partial}{\partial \tilde{x}_j} = \sum_{k,l} \varGamma_{\widetilde{U},ijk} \omega_{kl} \frac{\partial}{\partial \tilde{x}_l}$. Moreover, the family $R = (R_{\widetilde{U}})$ with $R_{\widetilde{U}} = d\varGamma_{\widetilde{U}} + \frac{1}{2}[\varGamma_{\widetilde{U}}, \varGamma_{\widetilde{U}}]$ defines a smooth section of $\varLambda^2 \widehat{\mathbb{W}} X$. From [9, Lemma 5.1.3] one concludes that

$$\nabla^2 b = \frac{i}{\lambda}[R, b] \quad \text{for all } b \in \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X). \tag{4.12}$$

Hence, R can be interpreted as the *curvature form* of ∇.

We will now employ Fedosov's idea and construct a flat connection D on $\varLambda^\bullet \widehat{\mathbb{W}} X$ of the form

$$Db = \nabla b + \delta b + \frac{i}{\lambda}[r, b] \quad \text{for all } b \in \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X), \tag{4.13}$$

where $r \in \varGamma^\infty_{\mathsf{str}}(\varLambda^1 \widehat{\mathbb{W}} X)$ and $\delta : \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X) \to \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X)$ is a graded derivation which locally is defined by

$$(\delta b)_{\widetilde{U}} = \sum_k d\tilde{x}_k \wedge \frac{\partial b_{\widetilde{U}}}{\partial \tilde{y}_k} = -\frac{i}{\lambda} \sum_{k,l} [\omega_{kl} \, \tilde{y}_k \, d\tilde{x}_l, b_{\widetilde{U}}]. \tag{4.14}$$

Note that Eq. (4.14) gives rise to an operator on the space of smooth stratified sections of $\varLambda^\bullet \widehat{\mathbb{W}} X$ indeed, since the $(\delta b)_{\widetilde{U}}$ are G-equivariant and transform naturally under morphisms of orbispace charts. Similarly one concludes that the operator $\delta^* : \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X) \to \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X)$ is well-defined by putting locally

$$(\delta^* b)_{\widetilde{U}} = \sum_k \tilde{y}_k \cdot \left(\frac{\partial}{\partial \tilde{x}_k} \llcorner b_{\widetilde{U}} \right). \tag{4.15}$$

Finally, δ^* gives rise to a third operator $\delta^- : \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X) \to \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X)$ by the local definition

$$(\delta^- b)_{\widetilde{U}} = \sum_{q+l>0} \frac{1}{q+l} \delta^*(b_{\widetilde{U},ql}), \tag{4.16}$$

where

$$b_{\widetilde{U},ql} = \sum_{k,\,|\alpha|=q} \; \sum_{1 \leqslant j_1 < \cdots < j_l \leqslant 2n} b_{\widetilde{U},k\alpha j_1 \cdots j_l} \tilde{y}^\alpha \, d\tilde{x}_{j_1} \wedge \cdots \wedge d\tilde{x}_{j_l} \lambda^k.$$

The following propositions can now be easily deduced from the corresponding ones in the smooth case.

4.8. Proposition. *For every $b \in \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X)$ one has the so-called Hodge–de Rham decomposition*

$$b = \delta\delta^- b + \delta^- \delta b + \sigma(b), \tag{4.17}$$

where $\sigma : \varGamma^\infty_{\mathsf{str}}(\varLambda^\bullet \widehat{\mathbb{W}} X) \to \mathcal{C}^\infty(X)[\![\lambda]\!]$, $(b_{\widetilde{U}}) \mapsto (b_{\widetilde{U},00})$ is the symbol map.

Proof. This follows immediately from [9, Lemma 5.1.2]. □

4.9. Proposition. *Given $r \in \Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}} X)$ let Ω be the two-form $-\omega + R - \delta r + \nabla r + \frac{i}{\lambda} r^2$ with R the curvature form of ∇. Then Ω is the curvature form of the connection $D = \nabla + \delta b + \frac{i}{\lambda}[r, \cdot]$ that means Ω satisfies*

$$D^2 b = \frac{i}{\lambda}[\Omega, b] \quad \text{for all } b \in \Gamma^\infty_{\mathrm{str}}(\Lambda^\bullet \widehat{\mathbb{W}} X). \tag{4.18}$$

Proof. By [9, Lemma 5.1.5], the equality $D^2_{\widetilde{U}} b_{\widetilde{U}} = \frac{i}{\lambda}[\Omega_{\widetilde{U}}, b_{\widetilde{U}}]$ holds true for all $b_{\widetilde{U}} \in \Gamma^\infty(\Lambda^\bullet \widehat{\mathbb{W}} \widetilde{U})$, hence the claim follows. □

4.10. Proposition. *Given $r_0 \in \Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}} X)$ with $\deg_{\mathrm{F}}(r_0) \geqslant 2$ there exists a unique $r \in \Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}} X)$ with $\deg_{\mathrm{F}}(r) \geqslant \deg_{\mathrm{F}}(r_0)$ such that*

$$r = r_0 + \delta^- \left(\nabla r + \frac{i}{\lambda} r^2 \right). \tag{4.19}$$

Proof. Consider the operator

$$K : \Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}}_2 X) \to \Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}} X), \quad s \mapsto r_0 + \delta^- \left(\nabla s + \frac{i}{\lambda} s^2 \right).$$

It is immediate to check that K has image in $\Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}}_2 X)$ and that K is contractible with respect to the Fedosov filtration in the sense that

$$\deg_{\mathrm{F}}\big(K(s) - K(s')\big) > \deg_{\mathrm{F}}(s - s') \quad \text{for all } s, s' \in \Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}}_2 X).$$

Hence, since $\Gamma^\infty_{\mathrm{str}}(\Lambda^1 \widehat{\mathbb{W}}_2 X)$ is complete with respect to the topology given by the Fedosov filtration, one concludes by a Banach fixed point type argument that there exists a unique r satisfying the claim. □

4.11. Corollary. *Let R be the curvature form of a symplectic connection ∇ on X and $r_0 = \delta^- R$. Then, if r is the solution of* (4.19)*, the curvature Ω of $D = \nabla + \delta b + \frac{i}{\lambda}[r, \cdot]$ is a central element with respect to $\circ$ and satisfies $\Omega = -\omega$. In particular, D then is a flat connection.*

Proof. We follow the argument of [9, Theorem 5.2.2]. First, note that $(\delta^-)^2 = 0$, so one has by the Hodge–de Rham decomposition and Eq. (4.19)

$$\delta^-(\Omega + \omega) = \delta^- \left(R - \delta r + \nabla r + \frac{i}{\lambda} r^2 \right) = r - \delta^- \delta r = \delta (\delta^-)^2 R = 0.$$

Using again the Hodge–de Rham decomposition, the Bianchi identity $D\Omega = 0$ and the equality $D\omega = d\omega = 0$ entail that

$$\Omega + \omega = \delta^-(D + \delta)(\Omega + \omega).$$

Now the operator $\delta^-(D + \delta) = \delta^-(\nabla + \frac{i}{\lambda}[r, \cdot])$ raises the Fedosov degree by 1, hence one concludes that $\Omega + \omega = 0$. But this implies also that Ω is central, so the claim follows. □

For the flat connection D constructed in the corollary let $\widehat{\mathbb{W}}_D X$ be the space of all flat sections, that means the space of all elements $a \in \Gamma^\infty_{\mathrm{str}}(\widehat{\mathbb{W}} X)$ satisfying $Da = 0$. Then $\widehat{\mathbb{W}}_D X$ forms a subalgebra of

$\Gamma^\infty_{\mathrm{str}}(\widehat{\mathbb{W}}X)$, since D is a graded derivation with respect to $\circ$. Using the above results one now proves the following result literally like Theorem 5.2.4 of [9].

4.12. Theorem. *Let X be a symplectic orbispace, ∇ a symplectic connection on X and D the flat connection on $\Lambda^\bullet\widehat{\mathbb{W}}X$ defined above. Then the symbol map induces a linear isomorphism $\sigma:\widehat{\mathbb{W}}_D X \to \mathcal{C}^\infty(X)[\![\lambda]\!]$.*

Proof. Choose $f \in \mathcal{C}^\infty(X)[\![\lambda]\!]$ and consider the equation

$$s = f + \delta^-(D+\delta)s, \quad s \in \Gamma^\infty_{\mathrm{str}}(\widehat{\mathbb{W}}X). \tag{4.20}$$

Since the operator $s \mapsto f + \delta^-(D+\delta)s$ is contractible in the above stated sense, this equation has a unique solution s. Let us show that $s \in \widehat{\mathbb{W}}_D X$ and $\sigma(s) = f$. First check by the Hodge–de Rham decomposition that

$$\delta^- Ds = s - f - \delta^-\delta s = \delta\delta^- s = 0.$$

Using the Hodge–de Rham decomposition again, one gets $\sigma(s) = f$. Applying the Hodge–de Rham decomposition a third time, but now to the argument Ds, one concludes by $D^2 = 0$ and $\delta^- Ds$ that

$$Ds = \delta^-(D+\delta)Ds.$$

But this equation has a unique solution, namely $Ds = 0$, since the operator $\delta^-(D+\delta)$ is contractible. Hence $s \in \widehat{\mathbb{W}}_D X$ and $\sigma(s) = f$. Conversely, every $s \in \widehat{\mathbb{W}}_D X$ with $\sigma(s) = f$ satisfies (Eq. (4.20)) by the Hodge–de Rham decomposition. Thus, the theorem follows. □

Denote by $Q:\mathcal{C}^\infty(X)[\![\lambda]\!] \to \widehat{\mathbb{W}}_D X$ the inverse of the symbol map or in other words the quantization map. The theorem now entails our main result.

4.13. Corollary. *Let $\star:\mathcal{C}^\infty(X)[\![\lambda]\!] \times \mathcal{C}^\infty(X)[\![\lambda]\!] \to \mathcal{C}^\infty(X)[\![\lambda]\!]$ be the uniquely determined $\mathbb{C}[\![\lambda]\!]$-bilinear map such that*

$$f \star g = \sigma\big(Q(f) \circ Q(g)\big) \quad \textit{for all } f, g \in \mathcal{C}^\infty(X).$$

Then $\star$ is a star product for X.

4.14. Corollary. *Every symplectic orbifold possesses a star product.*

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References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantization, Ann. Phys. 111 (1978) 61–151.
[2] E. Bierstone, The Structure of Orbit Spaces and the Singularities of Equivariant Mappings, in: Monografias de Matemática, Vol. 35, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1980.
[3] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
[4] A. Cannas da Silva, A. Weinstein, Geometric Models for Noncommutative Algebras, in: Berkeley Mathematics Lecture Notes, Vol. 10, American Mathematical Society, Providence, RI, 1999.
[5] W. Chen, Homotopy theory of orbispaces, 2001, math.AT/0102020.
[6] W. Chen, Y. Ruan, Orbifold Gromov–Witten theory, 2001, math.AT/0103156.
[7] M. De Wilde, P. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. 7 (1983) 487–496.
[8] B. Fedosov, A simple geometrical construction of deformation quantization, J. Differential Geom. 40 (1994) 213–238.
[9] B. Fedosov, Deformation Quantization and Index Theory, Akademie Verlag, 1996.
[10] V. Guillemin, S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press, Cambridge, 1984.
[11] M. Kontsevich, Deformation quantization of Poisson manifolds, I, 1997, q-alg/9709040.
[12] J.N. Mather, Stratifications and mappings, in: M.M. Peixoto (Ed.), Dynamical Systems, Academic Press, New York, 1973, pp. 195–232.
[13] I. Moerdijk, Orbifolds as groupoids: an introduction, 2002, math.DG/0203100.
[14] D. Montgomery, H. Samelson, L. Zippin, Singular points of a compact transformation group, Ann. of Math. (2) 63 (1956) 1–9.
[15] M.J. Pflaum, Analytic and Geometric Study of Stratified Spaces, in: Lecture Notes in Mathematics, Vol. 1768, Springer, Berlin, 2001.
[16] Y. Ruan, Stringy geometry and the topology of orbifolds, 2000, math.AG/0011149.
[17] I. Satake, On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. USA 42 (1956) 359–363.
[18] I. Satake, The Gauss–Bonnet theorem for V-manifolds, J. Math. Soc. Japan 9 (4) (1957) 464–492.
[19] G.W. Schwarz, Generalized orbit spaces, Ph.D. Thesis, revised version, MIT, 1972, unpublished.
[20] G.W. Schwarz, Smooth functions invariant under the action of a compact Lie group, Topology 14 (1975) 63–68.
[21] G.W. Schwarz, Lifting smooth homotopies of orbit spaces, Inst. Hautes Études Sci. Publ. Math. 51 (1980) 37–135.
[22] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, Ann. of Math. (2) 134 (1991) 375–422.