



Equivariant cohomology distinguishes toric manifolds

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Abstract

The equivariant cohomology of a space with a group action is not only a ring but also an algebra over the cohomology ring of the classifying space of the acting group. We prove that toric manifolds (i.e. compact smooth toric varieties) are isomorphic as varieties if and only if their equivariant cohomology algebras are weakly isomorphic. We also prove that quasitoric manifolds, which can be thought of as a topological counterpart to toric manifolds, are equivariantly homeomorphic if and only if their equivariant cohomology algebras are isomorphic.

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1. Introduction

Let T be a \mathbb{C}^* -torus of rank n , i.e., $T = (\mathbb{C}^*)^n$. A toric variety X of complex dimension n is a normal complex algebraic variety with an action of T having an open dense orbit. A fundamental result in the theory of toric varieties says that there is a one-to-one correspondence between toric varieties and fans, and among toric varieties, compact smooth toric varieties, which we call *toric manifolds*, are well studied, see [6,13].

Suppose two toric manifolds X and X' are isomorphic as varieties. Then they are not necessarily equivariantly isomorphic as varieties, but are rather *weakly equivariantly* isomorphic as varieties, i.e. there is a variety isomorphism $\phi: X \rightarrow X'$ together with an automorphism γ of T such that $\phi(tx) = \gamma(t)\phi(x)$ for any $t \in T$ and $x \in X$. This is well known and follows from the fact that the automorphism group of a toric manifold is a linear algebraic group with the acting

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group T as a maximal algebraic torus [13, Section 3.4]. Therefore, classifying toric manifolds up to variety isomorphism is equivalent to classifying them up to weakly equivariant variety isomorphism.

The equivariant cohomology of a toric variety X is by definition

$$H_T^*(X) := H^*(ET \times_T X)$$

where ET is the total space of the universal principal T -bundle and $ET \times_T X$ is the orbit space of $ET \times X$ by the diagonal T -action. $H_T^*(X)$ contains a lot of geometrical information about X , but $H_T^*(X)$ as a ring does not provide a sufficient information to distinguish X from other toric manifolds. In fact, when X is a toric manifold, $H_T^*(X)$ as a ring is the face ring of the underlying simplicial complex Σ of the fan of X and determined by Σ . There are toric manifolds which are not isomorphic as varieties but have the same underlying simplicial complex, so equivariant cohomology as a ring does not distinguish toric manifolds.

However, $H_T^*(X)$ is not only a ring but also an algebra over $H^*(BT)$ through the projection map from $ET \times_T X$ onto $ET/T = BT$. This algebra structure contains more geometrical information about X . If two toric manifolds X and X' are isomorphic as varieties, then they are weakly equivariantly isomorphic as varieties as remarked above, so that $H_T^*(X)$ and $H_T^*(X')$ are *weakly isomorphic* as algebras over $H^*(BT)$, i.e., there is a ring isomorphism $\Phi : H_T^*(X') \rightarrow H_T^*(X)$ together with an automorphism γ of T such that $\Phi(u\omega) = \gamma^*(u)\Phi(\omega)$ for any $u \in H^*(BT)$ and $\omega \in H_T^*(X')$, where γ^* denotes the automorphism of $H^*(BT)$ induced by γ . Our main result asserts that the converse holds.

Theorem 1.1. *Two toric manifolds are weakly equivariantly isomorphic as varieties if and only if their equivariant cohomology algebras are weakly isomorphic.*

As remarked before, toric manifolds are isomorphic as varieties if and only if they are weakly equivariantly isomorphic as varieties. Therefore the theorem above can be restated as follows.

Corollary 1.2. *Two toric manifolds are isomorphic as varieties if and only if their equivariant cohomology algebras are weakly isomorphic.*

The result above leads us to ask how much information ordinary cohomology contains for toric manifolds, in particular we may ask whether two toric manifolds are homeomorphic (or diffeomorphic) if their ordinary cohomology rings are isomorphic. The question is affirmatively solved in some cases [3,11] and the author does not know any counterexample, see [12] for the survey.

This paper is organized as follows. In Section 2 we review how the equivariant cohomology of a toric manifold X is related to the fan of X , and prove Theorem 1.1 in Section 3. In Section 4 we observe that our argument also works with some modification for quasitoric manifolds which are a topological counterpart to toric manifolds.

2. Equivariant cohomology and fans

Throughout this and the next sections, X will denote a toric manifold of complex dimension n unless otherwise stated. In this section we shall review how the equivariant cohomology of X

is related to the fan of X . The reader will find that most of the arguments in this and the next sections work with a compact torus $(S^1)^n$ instead of $T = (\mathbb{C}^*)^n$.

There are only finitely many T -invariant divisors in X , which we denote by X_1, \dots, X_m . Each X_i is a complex codimension-one invariant closed submanifold of X and fixed pointwise by some \mathbb{C}^* -subgroup of T . Since X and X_i are complex manifolds, they have canonical orientations. Let $\tau_i \in H_T^2(X)$ be the Poincaré dual of X_i viewed as an equivariant cycle in X , in other words, τ_i is the image of the unit $1 \in H_T^0(X_i)$ by the equivariant Gysin homomorphism: $H_T^0(X_i) \rightarrow H_T^2(X)$ induced from the inclusion map: $X_i \rightarrow X$. We call τ_i the *Thom class* of X_i .

We abbreviate a set $\{1, \dots, m\}$ as $[m]$. The invariant divisors X_i intersect transversally, so a cup product $\prod_{i \in I} \tau_i$ for a subset I of $[m]$ is the Poincaré dual of the intersection $\bigcap_{i \in I} X_i$. In particular, $\prod_{i \in I} \tau_i = 0$ if $\bigcap_{i \in I} X_i = \emptyset$. Since $H^*(X)$ is generated by elements in $H^2(X)$ as a ring (see [13, Section 3.3]), we see that $H_T^*(X)$ is generated by τ_i 's as a ring and there are no more relations among τ_i 's than those mentioned above, see [10, Proposition 3.4] for example. Namely we have

Proposition 2.1.

$$H_T^*(X) = \mathbb{Z}[\tau_1, \dots, \tau_m] / \left(\prod_{i \in I} \tau_i \mid \bigcap_{i \in I} X_i = \emptyset \right) \quad \text{as a ring}$$

where I runs all subsets of $[m]$ such that $\bigcap_{i \in I} X_i = \emptyset$.

We set

$$\Sigma := \left\{ I \subset [m] \mid \bigcap_{i \in I} X_i \neq \emptyset \right\}.$$

This is an abstract simplicial complex of dimension $n - 1$ and the proposition above says that $H_T^*(X)$ is the face ring (or Stanley–Reisner ring) of the simplicial complex Σ .

Let $\pi : ET \times_T X \rightarrow ET/T = BT$ be the projection on the first factor. Through $\pi^* : H^*(BT) \rightarrow H_T^*(X)$, one can regard $H_T^*(X)$ as an algebra over $H^*(BT)$. Since T is a torus of rank n , $H^*(BT)$ is a polynomial ring in n variables of degree two, in particular, it is generated by elements of degree two as a ring. Therefore, one can find the algebra structure of $H_T^*(X)$ over $H^*(BT)$ if one knows how elements in $H^2(BT)$ map to $H_T^2(X)$ by π^* .

Proposition 2.2. *To each $i \in [m]$, there is a unique element $v_i \in H_2(BT)$ such that*

$$\pi^*(u) = \sum_{i=1}^m \langle u, v_i \rangle \tau_i \quad \text{for any } u \in H^2(BT) \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between cohomology and homology.

Remark. The identity (2.1) corresponds to the identity in [6, Lemma, p. 61] in algebraic geometry, which describes a principal divisor as a linear combination of the T -invariant divisors X_i .

Proof. The proposition is proved in [7, Lemma 9.3] and [10, Lemma 1.5]. But for the reader’s convenience we shall reproduce the proof given in [7, Lemma 9.3]. By Proposition 2.1 $H_T^2(X)$ is freely generated by τ_1, \dots, τ_m over \mathbb{Z} . Therefore, for each $u \in H^2(BT)$, one can uniquely express $\pi^*(u) \in H_T^2(X)$ as

$$\pi^*(u) = \sum_{i=1}^m v_i(u)\tau_i$$

with integers $v_i(u)$ depending on u . We view $v_i(u)$ as a function of u . Since π^* is a homomorphism, the function $v_i(u)$ is linear; so there is a unique $v_i \in H_2(BT)$ such that $v_i(u) = \langle u, v_i \rangle$. \square

The vectors v_i have a nice geometrical meaning, which we shall explain. The group $\text{Hom}(\mathbb{C}^*, T)$ of homomorphisms from \mathbb{C}^* to T can be identified with $H_2(BT)$ as follows. An element ρ of $\text{Hom}(\mathbb{C}^*, T)$ induces a continuous map $\bar{\rho}: B\mathbb{C}^* \rightarrow BT$ between classifying spaces and $H_2(B\mathbb{C}^*)$ is isomorphic to \mathbb{Z} ; so once we choose and fix a generator, say α , of $H_2(B\mathbb{C}^*)$, we get an element $\bar{\rho}_*(\alpha) \in H_2(BT)$. A correspondence: $\rho \rightarrow \bar{\rho}_*(\alpha)$ gives an isomorphism from $\text{Hom}(\mathbb{C}^*, T)$ to $H_2(BT)$ and we denote by λ_v the element of $\text{Hom}(\mathbb{C}^*, T)$ corresponding to $v \in H_2(BT)$. It turns out that $\lambda_{v_i}(\mathbb{C}^*)$ is the \mathbb{C}^* -subgroup of T fixing X_i pointwise, see [10, Lemma 1.10] for example.

We have obtained two data from X , one is the abstract simplicial complex Σ and the other is the set of vectors v_1, \dots, v_m in $H_2(BT)$. To each $I \in \Sigma$ we form a cone in $H_2(BT) \otimes \mathbb{R} = H_2(BT; \mathbb{R})$ spanned by v_i ’s ($i \in I$). Then the collection of these cones is the fan of X . Precisely speaking, we need to add the 0-dimensional cone consisting of the origin to this collection to satisfy the conditions required in the definition of fan, see [6] or [13]. The 0-dimensional cone corresponds to the empty subset of $[m]$. Although we formed cones using the data Σ and $\{v_i\}$ to define the fan of X , we may think of a pair $(\Sigma, \{v_i\})$ as the fan of X .

As is well known, X can be recovered from the fan of X . There are at least three ways (gluing affine spaces, taking quotient by a \mathbb{C}^* -torus or symplectic reduction) to recover X from the fan of X . We shall recall the quotient construction. For $x = (x_1, \dots, x_m) \in \mathbb{C}^m$ we define $I(x) = \{i \mid x_i = 0\}$. We note that $(\mathbb{C}^*)^m$ acts on \mathbb{C}^m via coordinatewise scalar multiplication.

Proposition 2.3. (See [4].) *Let X be a toric manifold and $(\Sigma, \{v_i\})$ be the fan of X . We consider*

$$Y := \{x \in \mathbb{C}^m \mid I(x) \in \Sigma \cup \{\emptyset\}\}$$

and a homomorphism

$$\mathcal{V}: (\mathbb{C}^*)^m \rightarrow (\mathbb{C}^*)^n = T$$

defined by

$$\mathcal{V}(g_1, \dots, g_m) = \prod_{i=1}^m \lambda_{v_i}(g_i).$$

Then Y is invariant under the $(\mathbb{C}^*)^m$ -action, the kernel $\ker \mathcal{V}$ of \mathcal{V} acts on Y freely and the quotient $Y/\ker \mathcal{V}$ with the induced T -action is a toric manifold equivariantly isomorphic to X .

3. Poof of Theorem 1.1

We continue to use the notation in Section 2. Let X^T denote the set of T -fixed points in X . As is well known, it consists of finitely many points. For $\xi \in H_T^2(X)$, we denote its restriction to $p \in X^T$ by $\xi|_p$ and define

$$Z(\xi) := \{p \in X^T \mid \xi|_p = 0\}.$$

Lemma 3.1. *Express $\xi = \sum_{i=1}^m a_i \tau_i$ with integers a_i . If $a_i \neq 0$ for some i , then $Z(\xi) \subset Z(\tau_i)$. Moreover, if $a_i \neq 0$ and $a_j \neq 0$ for some different i and j , then $Z(\xi) \subsetneq Z(\tau_i)$.*

Proof. Let $p \in X^T$. Since τ_i is the Poincaré dual of X_i viewed as an equivariant cycle in X , $\tau_i|_p = 0$ if $p \notin X_i$. Moreover, if $p \in X_i$, then $\tau_i|_p \in H_T^2(p) = H^2(BT)$ is the equivariant Euler class of the complex one-dimensional normal T -representation at p to X_i . This implies that

$$\tau_i|_p = 0 \quad \text{if and only if } p \notin X_i \tag{3.1}$$

and that there are exactly n number of X_i 's containing p and $\{\tau_i|_p \mid p \in X_i\}$ forms a basis of $H^2(BT)$.

Suppose $p \in Z(\xi)$. Then $0 = \xi|_p = \sum_{i=1}^m a_i \tau_i|_p$ and it follows from the observation above that $\tau_i|_p = 0$ if $a_i \neq 0$. This proves the former statement in the lemma.

If both a_i and a_j are non-zero, then $Z(\xi) \subset Z(\tau_i) \cap Z(\tau_j)$ by the former statement in the lemma. Therefore, it suffices to prove that $Z(\tau_i) \cap Z(\tau_j)$ is properly contained in $Z(\tau_i)$. Suppose that $Z(\tau_i) \cap Z(\tau_j) = Z(\tau_i)$. Then $Z(\tau_j) \supset Z(\tau_i)$, so $X_j^T \subset X_i^T$ by (3.1). This implies that $X_j = X_i$, a contradiction. \square

Let $S = H^*(BT) \setminus \{0\}$ and let $S^{-1}H_T^*(X)$ denote the localized ring of $H_T^*(X)$ by S . Since $H^{\text{odd}}(X) = 0$, $H_T^*(X)$ is free as a module over $H^*(BT)$. Hence the natural map

$$H_T^*(X) \rightarrow S^{-1}H_T^*(X) \cong S^{-1}H_T^*(X^T) = \bigoplus_{p \in X^T} S^{-1}H_T^*(p)$$

is injective, where the isomorphism above is induced from the inclusion map from X^T to X and is a consequence of the Localization Theorem in equivariant cohomology [8, p. 40]. The annihilator

$$\text{Ann}(\xi) := \{\eta \in S^{-1}H_T^*(X) \mid \eta\xi = 0\}$$

of ξ in $S^{-1}H_T^*(X)$ is nothing but the sum of $S^{-1}H_T^*(p)$ over p with $\xi|_p = 0$. Therefore it is a free $S^{-1}H^*(BT)$ module of rank $|Z(\xi)|$. Since $\text{Ann}(\xi)$ is defined using the algebra structure of $H_T^*(X)$, $|Z(\xi)|$ is an invariant of ξ depending only on the algebra structure of $H_T^*(X)$. We note that $|Z(\xi)|$ is invariant under an algebra isomorphism. We call $|Z(\xi)|$ the *zero-length* of ξ .

Lemma 3.2. *Let X' be another toric manifold (X' might be same as X). If $f : H_T^*(X) \rightarrow H_T^*(X')$ is an algebra isomorphism, then f maps the Thom classes in $H_T^2(X)$ to the Thom classes in $H_T^2(X')$ bijectively up to sign.*

Proof. We classify the Thom classes according to their zero-length. Let \mathcal{T}_1 be the set of Thom classes in $H_T^2(X)$ with largest zero-length, and let \mathcal{T}_2 be the set of Thom classes in $H_T^2(X)$ with second largest zero-length, and so on. Similarly we define $\mathcal{T}'_1, \mathcal{T}'_2$ and so on for the Thom classes in $H_T^2(X')$.

Let m_k (respectively m'_k) be the zero-length of elements in \mathcal{T}_k (respectively \mathcal{T}'_k). Since both f and f^{-1} preserve zero-length and are isomorphisms, $m_1 = m'_1$ and f maps \mathcal{T}_1 to \mathcal{T}'_1 bijectively up to sign by Lemma 3.1. Take an element τ_i from \mathcal{T}_2 . Since \mathcal{T}_1 and \mathcal{T}'_1 are preserved under f and f^{-1} , $f(\tau_i)$ is not a linear combination of elements in \mathcal{T}'_1 . This together with Lemma 3.1 means that $m_2 \leq m'_2$. The same argument for f^{-1} instead of f shows that $m'_2 \leq m_2$, so that $m_2 = m'_2$. Again, this together with Lemma 3.1 implies that f maps \mathcal{T}_2 to \mathcal{T}'_2 bijectively up to sign. The lemma follows by repeating this argument. \square

Now we shall complete the proof of Theorem 1.1. Let X and X' be two toric manifolds whose equivariant cohomology algebras over $H^*(BT)$ are weakly isomorphic. We note that changing the action of T on X through an automorphism of T , we may assume that $H_T^*(X)$ and $H_T^*(X')$ are isomorphic as algebras over $H^*(BT)$.

We put a prime for notation for X' corresponding to the Thom classes τ_i , the abstract simplicial complex Σ and the vectors v_i , etc., for X . Let $f : H_T^*(X) \rightarrow H_T^*(X')$ be an isomorphism of algebras over $H^*(BT)$. By Lemma 3.2, the number of the Thom classes in $H_T^2(X)$ is same as that in $H_T^2(X')$ and there is a permutation \bar{f} on $[m]$ such that

$$f(\tau_i) = \epsilon_i \tau'_{\bar{f}(i)} \quad \text{with } \epsilon_i = \pm 1. \tag{3.2}$$

If $I \subset [m]$ is an element of Σ , then $\prod_{i \in I} \tau_i$ is non-zero by Proposition 2.1 and hence so is $f(\prod_{i \in I} \tau_i) = \prod_{i \in I} \epsilon_i \tau'_{\bar{f}(i)}$. Therefore a subset $\{\bar{f}(i) \mid i \in I\}$ of $[m]$ is a simplex in Σ' whenever I is a simplex in Σ , which means that \bar{f} induces a simplicial map from Σ to Σ' . Applying the same argument to the inverse of f , we see that the induced simplicial map has an inverse, so that it is an isomorphism.

Since f is an algebra map over $H^*(BT)$, $\pi'^* = f \circ \pi^*$. Therefore, sending the identity (2.1) by f and using (3.2), we have

$$\pi'^*(u) = f(\pi^*(u)) = \sum_{i=1}^m \langle u, v_i \rangle f(\tau_i) = \sum_{i=1}^m \langle u, v_i \rangle \epsilon_i \tau'_{\bar{f}(i)}.$$

Comparing this with the identity (2.1) for X' and noting that \bar{f} is a permutation on $[m]$, we conclude that

$$\epsilon_i v_i = v'_{\bar{f}(i)} \quad \text{for each } i. \tag{3.3}$$

We identify Σ with Σ' through the isomorphism induced by \bar{f} , so that we may think of \bar{f} as the identity map and then the identity (3.3) turns into

$$\epsilon_i v_i = v'_i.$$

By Proposition 2.3 we may assume $X = Y/\ker \mathcal{V}$ and $X' = Y'/\ker \mathcal{V}'$. Since Σ' is identified with Σ , we have $Y = Y'$. Therefore it suffices to check that $\ker \mathcal{V} = \ker \mathcal{V}'$. Since $\lambda_{-v}(g) = \lambda_v(g)^{-1} = \lambda_v(g^{-1})$ for $v \in H_2(BT)$ and $g \in \mathbb{C}^*$, an automorphism ρ of $(\mathbb{C}^*)^m$ defined by

$$\rho(g_1, \dots, g_m) = (g_1^{\epsilon_1}, \dots, g_m^{\epsilon_m})$$

satisfies $\mathcal{V} \circ \rho = \mathcal{V}'$. This implies $\ker \mathcal{V} = \ker \mathcal{V}'$ and completes the proof of Theorem 1.1.

4. Quasitoric manifolds

Davis and Januszkiewicz [5] introduced the notion of what is now called a *quasitoric manifold*, see [1]. A quasitoric manifold is a closed smooth manifold of even dimension, say $2n$, with a smooth action of a compact torus group $(S^1)^n$ of dimension n such that the action is locally isomorphic to a faithful $(S^1)^n$ -representation of real dimension $2n$ and that the orbit space is combinatorially a simple convex polytope. A toric manifold with the action restricted to the maximal compact toral subgroup of T often provides an example of a quasitoric manifold, e.g. this is the case when X is projective. However, there are many quasitoric manifolds which do not arise from a toric manifold. For instance, $\mathbb{C}P^2 \# \mathbb{C}P^2$ with an appropriate action of $(S^1)^2$ is a quasitoric manifold but does not arise from a toric manifold because $\mathbb{C}P^2 \# \mathbb{C}P^2$ does not allow a complex (even almost complex) structure. We note that the equivariant cohomology of a quasitoric manifold of dimension $2n$ is an algebra over $H^*(B(S^1)^n)$ similarly to the toric case. The purpose of this section is to prove the following.

Theorem 4.1. *Two quasitoric manifolds are equivariantly homeomorphic if their equivariant cohomology algebras are isomorphic.*

Proof. When X is a quasitoric manifold, we take X_i to be a connected real codimension-two closed submanifold of X fixed pointwise by some circle subgroup of $(S^1)^n$. Then the proof for Theorem 1.1 almost works if we replace \mathbb{C}^* by S^1 (and hence $T = (\mathbb{C}^*)^n$ by $(S^1)^n$). The only problem is that we do not have Proposition 2.3 for quasitoric manifolds, so that the last paragraph in the previous section needs to be modified. In the sequel, it suffices to prove that the existence of an isomorphism $\tilde{f}: \Sigma \rightarrow \Sigma'$ satisfying (3.3) implies that the two quasitoric manifolds X and X' are equivariantly homeomorphic.

Let P be the orbit space of X by the action of $(S^1)^n$ and let $q: X \rightarrow P$ be the quotient map. The orbit space P is a simple convex polytope by the definition of quasitoric manifold. Then $P_i := q(X_i)$ is a facet (i.e., a codimension-one face) of P . The dual polytope P^* of P is a simplicial polytope and its boundary complex agrees with Σ . The vertices of Σ bijectively correspond to the facets of P so that v_i is assigned to P_i . The vectors v_i form a characteristic function on P introduced in [5]. Any (proper) face of P is obtained as an intersection $P_I := \bigcap_{i \in I} P_i$ for some $I \in \Sigma$. We define P_\emptyset to be P itself. For $I \in \Sigma$ we denote by S_I a subgroup of $(S^1)^n$ generated by circle subgroups $\lambda_{v_i}(S^1)$ for $i \in I$. We define S_\emptyset to be the unit group. Associated with a pair $(P, \{v_i\})$ we form a quotient space

$$X(P, \{v_i\}) := P \times (S^1)^n / \sim .$$

Here $(p_1, g_1) \sim (p_2, g_2)$ if and only if $p_1 = p_2$ and $g_1^{-1}g_2 \in S_I$ where $I \in \Sigma \cup \{\emptyset\}$ is determined by the condition that $p_1 = p_2$ is contained in the interior of P_I . The natural action of $(S^1)^n$ on the

product $P \times (S^1)^n$ descends to an action on $X(P, \{v_i\})$ and X is equivariantly homeomorphic to $X(P, \{v_i\})$ (see [5, Proposition 1.8]). As before, we put a prime to denote elements for X' corresponding to P, v_i and Σ . The isomorphism $\tilde{f}: \Sigma \rightarrow \Sigma'$ induces an isomorphism from P^* to P'^* and then a face-preserving homeomorphism from P to P' which we denote by φ . A map $\varphi \times id: P \times (S^1)^n \rightarrow P' \times (S^1)^n$ descends to a map from $X(P, \{v_i\})$ to $X(P', \{v'_i\})$ by virtue of (3.3) and the resulting map is an equivariant homeomorphism, so X is equivariantly homeomorphic to X' . \square

Similarly to the toric case, it would be interesting to ask whether two quasitoric manifolds are homeomorphic (or diffeomorphic) if their ordinary cohomology rings are isomorphic, see [2,11] and [3] for some partial affirmative solutions.

Remark. Davis and Januszkiewicz [5] also introduced the notion of a real version of quasitoric manifold, which they call a *small cover*. A small cover is a closed smooth manifold of dimension, say n , with a smooth action of a rank n mod two torus group $(\mathbb{Z}_2)^n$ such that the action is locally isomorphic to a faithful $(\mathbb{Z}_2)^n$ -representation of real dimension n and that the orbit space is combinatorially a simple convex polytope. Our argument also works for small covers with \mathbb{Z}_2 coefficients, so that small covers are equivariantly homeomorphic if their equivariant cohomology algebras with \mathbb{Z}_2 coefficients are isomorphic. One can also ask whether two small covers are homeomorphic (or diffeomorphic) if their ordinary cohomology rings with \mathbb{Z}_2 coefficients are isomorphic, see [9] for a partial affirmative solution.

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