Ghosts in modular representation theory

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Abstract

A *ghost* over a finite p-group G is a map between modular representations of G which is invisible in Tate cohomology. Motivated by the failure of the *generating hypothesis*—the statement that ghosts between finite-dimensional G-representations factor through a projective—we define the *ghost number* of kG to be the smallest integer l such that the composite of any l ghosts between finite-dimensional G-representations factors through a projective. In this paper we study ghosts and the ghost numbers of p-groups. We begin by showing that a weaker version of the generating hypothesis, where the target of the ghost is fixed to be the trivial representation k, holds for all p-groups. We then compute the ghost numbers of all cyclic p-groups and all abelian 2-groups with C_2 as a summand. We obtain bounds on the ghost numbers for abelian p-groups and for all 2-groups which have a cyclic subgroup of index 2. Using these bounds we determine the finite abelian groups which have ghost number at most 2. Our methods involve techniques from group theory, representation theory, triangulated category theory, and constructions motivated from homotopy theory.

1. Introduction

Let G be a finite p-group and let k be a field of characteristic p. Recall that the *stable module category* StMod(kG) is the following tensor triangulated category. The objects are left kG-

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modules and the space of morphisms between kG-modules M and N, denoted $\underline{\mathrm{Hom}}_{kG}(M, N)$, is the k-vector space of kG-module homomorphisms modulo those maps that factor through a projective module. The category stmod(kG) is the full subcategory of finite-dimensional left kG-modules. A *ghost* in the stable module category is a map between kG-modules that is trivial in Tate cohomology. In [3], we formulated the *generating hypothesis* (GH) for kG as the statement that all ghosts between finite-dimensional kG-modules are trivial in the stable module category, i.e., they factor through a projective. (This formulation was motivated by the famous classical generating hypothesis of Peter Freyd [13] in the stable homotopy category which is the conjecture that there are no non-trivial maps between finite spectra that are trivial in stable homotopy groups.) We have shown in [3] that the GH holds for kG, where G is a non-trivial finite p-group and k is a field of characteristic p, if and only if G is either C_2 or C_3.

Motivated by the failure of the GH, we proceed in two natural directions. The first one addresses the GH with the trivial representation k as the target. More precisely, we show that in the stable module category of any p-group, a map $M \to k$ from a finite-dimensional module to the trivial module is stably trivial if it induces the zero map in Tate cohomology. We give two proofs of this result in Section 2. One of them uses Spanier–Whitehead duality and Tate duality to show that a map between finite-dimensional modules over a p-group is a ghost if and only if it induces the zero map on the functor $\underline{\mathrm{Hom}}_{kG}(-, \Omega^* k)$.

The second direction we take measures the degree to which the GH fails in p-groups other than C_2 and C_3. We define the *ghost number* of kG to be the smallest non-negative integer l such that the composite of any l ghosts between finite-dimensional kG-modules is trivial. This is a new invariant for a group algebra. (Note that, in this terminology, C_2 and C_3 are the only p-groups whose group algebras have ghost number 1.) A concept that will be a key to our analysis of ghost numbers and related invariants is that of a *projective class*; see Section 4 for the definition. In Section 3 we prove the existence of universal ghosts which implies that ghosts form part of a projective class. This allows us to derive bounds on the ghost number, such as Theorem 4.7 which states that the ghost number of kG is strictly less than the nilpotency index of the Jacobson radical $J(kG)$ of kG. Using these bounds, in Section 5 we compute the ghost numbers of some p-groups. We show that the ghost number of kC_{p^r} is $\lceil (p^r - 1)/2 \rceil$, where $\lceil x \rceil$ is the smallest integer that is greater than or equal to x. If G is a finite abelian 2-group with C_2 as a summand, then the ghost number of kG is shown to be one less than the nilpotency index of $J(kG)$. Computing the ghost number of an arbitrary group algebra seems to be a hard problem. Experience tells us that finding lower bounds for ghost numbers is much harder than finding upper bounds. We obtain reasonable lower bounds for the ghost numbers of the group algebras of abelian p-groups. We use these bounds to show that the only abelian p-groups with ghost number 2 are C_4, $C_2 \oplus C_2$, and C_5.

The proofs of the aforementioned results involve a pleasant mix of methods from group theory, representation theory and triangulated category theory.

Similar results on *phantom maps* (maps between kG-modules which factor through a projective when restricted to finite-dimensional modules) in the stable module category appear in the work of Benson and Gnacadja; see [2].

We assume throughout that the group G is a finite p-group. The characteristic of the field k is always assumed to divide the order of the group. For example, when we write kC_3, the reader will understand that the characteristic of k is 3. When we speak of suspensions of a kG-module M, we mean $\Omega^i M$ for any integer i, as an object of StMod(kG). We write $\widetilde{\Omega}^i M$ for the projective-free part of $\Omega^i M$, a well-defined kG-module. When we speak of Heller shifts of M, we mean

$\widetilde{\Omega}^i M$ for any integer i. We use standard facts about the stable module category and kG-modules; a good reference is [6].

2. The generating hypothesis

A map $\phi : M \to N$ between kG-modules is said to be a *ghost* if the induced map

$$\underline{\mathrm{Hom}}_{kG}\big(\Omega^i k, M\big) \longrightarrow \underline{\mathrm{Hom}}_{kG}\big(\Omega^i k, N\big)$$

between the Tate cohomology groups is zero for each integer i. (Recall that the Tate group $\widehat{H}^i(G, M)$ of G with coefficients in M is isomorphic to $\underline{\mathrm{Hom}}_{kG}(\Omega^i k, M)$.) If G is a finite p-group and k is a field of characteristic p, then "the generating hypothesis for kG" is the statement that all ghosts between finite-dimensional kG-modules are trivial in the stable module category. In [3] we have shown that the only non-trivial p-groups for which the GH holds are the cyclic groups C_2 and C_3.

2.1. The generating hypothesis with target k

While the generating hypothesis generally fails in the stable module category, we show that a weak version of it holds for all p-groups. We begin with motivation coming from homotopy theory for studying this weak version. Devinatz [11] proved the following partial affirmative result on the generating hypothesis for the p-local stable homotopy category of spectra where p is an odd prime: If $\phi : X \to S^0$ is a map from a finite spectrum to the sphere spectrum such that $\pi_*(\phi) = 0$, then the $K_{(p)}$-localisation of ϕ is trivial, where $K_{(p)}$ is periodic complex K-theory localised at p.

Motivated by this result, we consider "the GH with target k," which is the statement that every map $M \to k$ from a finite-dimensional kG-module M to the trivial representation k that induces the zero map in Tate cohomology is trivial in the stable module category. We show that the GH with target k holds for all p-groups. In fact, we give two proofs of this fact below; see Corollaries 2.2 and 2.8.

Proposition 2.1. *Let G be a p-group and let $f : U \to V$ be a ghost between projective-free kG-modules. Then we have the following*:

(1) U^G *is contained in* $\mathrm{Ker}(f)$. (U^G *is the G-invariant submodule of U.*)
(2) $\mathrm{Im}(f)$ *is contained in* JV. (J *denotes the Jacobson radical of kG.*)

Proof. Since f is a ghost, the induced map in Tate cohomology is zero. In particular, the maps

$$\widehat{H}^0(G, f) : \frac{U^G}{\mathrm{Im}(N)} \longrightarrow \frac{V^G}{\mathrm{Im}(N)}$$

and

$$\widehat{H}^{-1}(G, f) : \frac{\mathrm{Ker}(N)}{JU} \longrightarrow \frac{\mathrm{Ker}(N)}{JV}$$

are zero. (Here $\operatorname{Im}(N)$ and $\operatorname{Ker}(N)$ are respectively the image and kernel of norm maps.) Since the norm map is trivial on any projective-free module, the above maps can be written as

$$\widehat{H}^0(G, f): U^G \longrightarrow V^G \tag{2.1}$$

and

$$\widehat{H}^{-1}(G, f): \frac{U}{JU} \longrightarrow \frac{V}{JV}. \tag{2.2}$$

Both of the above maps are zero. The first part of the proposition follows from (2.1) and the second part from (2.2). □

Corollary 2.2. *The GH with target k holds for p-groups.*

Proof. Let $f : M \to k$ be a ghost in $\operatorname{stmod}(kG)$. M is isomorphic in $\operatorname{stmod}(kG)$ to a projective-free kG-module. Therefore we may assume that M is projective-free. Then, by Proposition 2.1, the image of $f : M \to k$ is contained in $J(k)$, which is zero. So we are done. □

Recall that associated to a kG-module M, one has the socle (ascending) series $\operatorname{Soc}^i M$ and the radical (descending) series $\operatorname{Rad}^i M$. For $i = 1$, $\operatorname{Soc}^1 M := M^G$, the G-invariant submodule of M, and for $i > 1$, $\operatorname{Soc}^i M$ is defined inductively by $\operatorname{Soc}^i M / \operatorname{Soc}^{i-1} M \cong (M / \operatorname{Soc}^{i-1} M)^G$. $\operatorname{Rad}^i M := J^i M$ for all $i \geqslant 0$, where J^i denotes the ith power of the Jacobson radical of kG. See [1] for some properties of these series. Proposition 2.1 can now be generalised as follows.

Corollary 2.3. *Let G be a p-group and let $f : U \to V$ be a composite of l ghosts between projective-free kG-modules. Then we have the following*:

(1) $\operatorname{Soc}^l(U)$ *is contained in* $\operatorname{Ker}(f)$.
(2) $\operatorname{Im}(f)$ *is contained in* $\operatorname{Rad}^l(V)$.

Proof. This follows by a straightforward induction using Proposition 2.1. □

2.2. Ghosts and duality

A map $d : M \to N$ between kG-modules is called a *dual ghost* if the induced map

$$\underline{\operatorname{Hom}}_{kG}\big(M, \Omega^i k\big) \longleftarrow \underline{\operatorname{Hom}}_{kG}\big(N, \Omega^i k\big)$$

is zero for all i. Recall that for every kG-module L, there is a corresponding dual kG-module $L^* := \operatorname{Hom}_k(L, k)$ with the G-action defined as follows: for g in G, ϕ in L^* and x in L, $(g\phi)(x) := \phi(g^{-1}x)$.

Proposition 2.4. *Let G be a finite group. A map $f : M \to N$ between kG-modules is a dual ghost if and only if $f^* : N^* \to M^*$ is a ghost.*

Proof. Consider the natural isomorphism

$$\text{Hom}_{kG}\big(N, \text{Hom}_k(T, k)\big) \cong \text{Hom}_{kG}\big(T, \text{Hom}_k(N, k)\big),$$

where T is a Tate resolution of k and N is any kG-module; see [1, Proposition 3.1.8], for instance. Since $\text{Hom}_k(T, k)$ is a complete injective resolution of k, taking homology of the chain complexes in the last isomorphism gives natural isomorphisms

$$\underline{\text{Hom}}_{kG}\big(N, \Omega^{-n}k\big) \cong \underline{\text{Hom}}_{kG}\big(\Omega^n k, N^*\big).$$

This implies that a map $d : M \to N$ is a dual ghost if and only if $d^* : N^* \to M^*$ is a ghost. $\square$

The second isomorphism used in the proof of the above proposition is *Spanier–Whitehead duality* for the stable module category.

We now use Tate duality to show that one can use group cohomology $(\widehat{H}^i(G, -), i \geqslant 0)$ to detect ghosts.

Theorem 2.5. *Let G be a p-group. A map $f : M \to N$ between finite-dimensional kG-modules is a ghost if and only if the following two conditions hold:*

(1) $\widehat{H}^i(G, f) : \widehat{H}^i(G, M) \to \widehat{H}^i(G, N)$ *is zero for all $i \geqslant 0$.*
(2) $\widehat{H}^i(G, f^*) : \widehat{H}^i(G, N^*) \to \widehat{H}^i(G, M^*)$ *is zero for all $i \geqslant 0$.*

Proof. Clearly it suffices to show that statement (2) is equivalent to the statement:

$$\widehat{H}^{-i-1}(G, f) : \widehat{H}^{-i-1}(G, M) \to \widehat{H}^{-i-1}(G, N)$$

is zero for all $i \geqslant 0$. Recall that Tate duality [5] gives a natural isomorphism

$$\widehat{H}^{-i-1}(G, L) \cong \big(\widehat{H}^i(G, L^*)\big)^*$$

for any finite-dimensional module L. Thus, for each $i \geqslant 0$, we have the following commutative diagram, where the vertical maps are induced by f:

$$
\begin{array}{ccc}
\widehat{H}^{-i-1}(G, M) & \xrightarrow{\;\cong\;} & (\widehat{H}^i(G, M^*))^* \\
\downarrow & & \downarrow \\
\widehat{H}^{-i-1}(G, N) & \xrightarrow{\;\cong\;} & (\widehat{H}^i(G, N^*))^*.
\end{array}
$$

Since the two horizontal maps are isomorphisms, the left vertical map is zero if and only if the right vertical map is zero. Finally, by the faithfulness of the vector space duality functor, the right vertical map is zero if and only if statement (2) holds. So we are done. $\square$

Combining Spanier–Whitehead duality and Tate duality gives us the following interesting result.

Corollary 2.6. *A map* $f : M \to N$ *between finite-dimensional* kG-*modules is a ghost if and only if it is a dual ghost.*

Proof. By Proposition 2.4, we know that $f : M \to N$ is a dual ghost if and only if $f^* : N^* \to M^*$ is a ghost. And by Proposition 2.5, f^* is a ghost if and only if

(1) $\widehat{H}^i(G, f^*) : \widehat{H}^i(G, N^*) \to \widehat{H}^i(G, M^*)$ is zero for all $i \geqslant 0$, and
(2) $\widehat{H}^i(G, f) : \widehat{H}^i(G, M) \to \widehat{H}^i(G, N)$ is zero for all $i \geqslant 0$.

(We get the second statement from the fact that double dual f^{**} is naturally isomorphic to f.) The last two statements are in turn equivalent, again by Proposition 2.5, to the statement that f is a ghost. $\quad\square$

Remark 2.7. The analogue of Corollary 2.6 fails in the derived category of a commutative ring. For example, in $D(\mathbb{Z})$, let X be the cone of the map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$, and let $Y = \Sigma\mathbb{Z}$. Then the map $f : X \to Y$ which projects onto the top class, i.e., the map that is the identity in degree 1 and zero elsewhere, is easily seen to be a non-trivial ghost. However, f is not a dual ghost. In fact, the composite $X \xrightarrow{f} Y \xLongrightarrow{} Y (= \Sigma\mathbb{Z})$ is just f, which is non-trivial.

The first part of the following corollary gives an alternative proof of Corollary 2.2.

Corollary 2.8. *Let* M *be a finite-dimensional* kG-*module. Then we have the following*:

(1) *If* $f : M \to \Omega^i k$ *is a ghost, then* f *is stably trivial. In other words, the GH with target* k *holds.*
(2) *If* $f : \Omega^i k \to M$ *is a dual ghost, then* f *is stably trivial.*

Proof. If $f : M \to \Omega^i k$ is a ghost, then by Corollary 2.6, f is also a dual ghost, so the composite $M \xrightarrow{f} \Omega^i k \xLongrightarrow{} \Omega^i k$, which is just f, is stably trivial.

The proof of the second statement is similar. $\quad\square$

Remark 2.9. Using the results in this section, we can show that "the GH with domain L" (the statement that every ghost in stmod(kG) with domain L is trivial) holds if and only if "the GH with target L^*" holds. This generalises Corollary 2.8. We leave the easy details to the reader.

3. Universal ghosts

A ghost $\Phi : M \to N$ between kG-modules is said to be a *universal ghost* if every ghost out of M factors through Φ. (Such a map should technically be called *weakly* universal, since we do not assume the factorisation is unique.) We will show that there exists a universal ghost out of any given kG-module. Let M be a kG-module. We assemble all the homogeneous elements in $\widehat{H}^*(G, M)$ into a map

$$\bigoplus_{\eta \in \widehat{H}^*(G,M)} \Omega^{|\eta|} k \longrightarrow M,$$

where $|\eta|$ is the degree of η. Completing this map to an exact triangle in StMod(kG), we get

$$\bigoplus_{\eta \in \widehat{H}^*(G,M)} \Omega^{|\eta|}k \longrightarrow M \xrightarrow{\Phi_M} U_M. \tag{3.1}$$

We now recall a couple of easily established facts (see [9] for proofs) which we will need in the sequel.

Proposition 3.1. *(See [9].) The map* $\Phi_M : M \to U_M$ *is a universal ghost out of* M.

Our next proposition characterises modules out of which all ghosts vanish.

Proposition 3.2. *(See [9].) Let M be a kG-module. Then the following are equivalent statements in the stable module category*:

(1) *All ghosts out of M are trivial.*
(2) *The universal ghost $\Phi_M : M \to U_M$ is trivial.*
(3) *M is a retract of a direct sum of suspensions of the trivial representation.*

Moreover, if M is finite-dimensional, (3) *can be replaced with the condition that M is stably isomorphic to a finite direct sum of suspensions of the trivial representation.*

Now suppose that M is a finite-dimensional kG-module such that $\widehat{H}^*(G, M)$ is finitely generated as a graded module over $\widehat{H}^*(G, k)$. (This happens, for example, when k is periodic, that is, $\Omega^i k$ is stably isomorphic to k for some $i \neq 0$. See [7] for interesting and non-trivial examples in the non-periodic case.) We will show that a universal ghost out of M can be constructed in the category stmod(kG); that is, the target module of the universal ghost out of M can be chosen to be finite-dimensional as well. This is done as follows. Let $\{v_j\}$ be a finite set of homogeneous generators for $\widehat{H}^*(G, M)$ as an $\widehat{H}^*(G, k)$-module. These generators can be assembled into a map

$$\bigoplus_j \Omega^{|v_j|}k \longrightarrow M$$

in stmod(kG). This map can then be completed to a triangle

$$\bigoplus_j \Omega^{|v_j|}k \longrightarrow M \xrightarrow{\Psi_M} F_M$$

in stmod(kG). By construction, it is clear that the first map in the above triangle is surjective on the functors $\underline{\mathrm{Hom}}_{kG}(\Omega^l k, -)$ for each l. Therefore, the second map Ψ_M must be a ghost. Universality of Ψ_M is easy to see, so we have proved the following proposition.

Proposition 3.3. *Suppose M is a finite-dimensional kG-module such that $\widehat{H}^*(G, M)$ is finitely generated as a graded module over $\widehat{H}^*(G, k)$. Then a universal ghost out of M can be constructed in* stmod(kG). *In particular, this applies when k is periodic.*

Corollary 3.4. *Let M be a finite-dimensional module such that $\widehat{H}^*(G, M)$ is finitely generated as a graded module over $\widehat{H}^*(G, k)$. Then the following are equivalent statements in the stable module category*:

(1) *All ghosts out of M are trivial.*
(2) *All ghosts out of M into finite-dimensional modules are trivial.*
(3) *The universal ghost $\Psi_M : M \to F_M$ is trivial.*
(4) *M is a finite direct sum of suspensions of the trivial representation.*

We now give some applications of universal ghosts. We begin with a characterisation of finite-dimensional indecomposable projective-free representations that are isomorphic to a Heller shift of the trivial representation.

Corollary 3.5. *Let G be a finite p-group and let M be a finite-dimensional indecomposable projective-free kG-module. Then all ghosts out of M are trivial if and only if $M \cong \widetilde{\Omega}^i k$ for some integer i.*

Proof. This follows from Proposition 3.2 and the Krull–Schmidt theorem. □

Corollary 3.6. *Let G be a finite p-group and let M be a finite-dimensional indecomposable projective-free kG-module. If $\dim(M)$ is not congruent to $+1$ or -1 modulo $|G|$, then there exists a non-trivial ghost out of M.*

Proof. By the previous corollary it suffices to show that $M \not\cong \widetilde{\Omega}^i k$ for any i. This will be shown by proving that the dimensions of the Heller shifts $\widetilde{\Omega}^i k$ are congruent to $+1$ or -1 modulo $|G|$. Recall that $\widetilde{\Omega}^1 k$ is defined to be the kernel of the augmentation map, so we have a short exact sequence

$$0 \longrightarrow \widetilde{\Omega}^1 k \longrightarrow kG \longrightarrow k \longrightarrow 0,$$

which tells us that $\dim(\widetilde{\Omega}^1 k) \equiv -1$ modulo $|G|$. Inductively, it is clear from the short exact sequences

$$0 \longrightarrow \widetilde{\Omega}^{i+1} k \longrightarrow (kG)^t \longrightarrow \widetilde{\Omega}^i k \longrightarrow 0$$

that $\dim(\widetilde{\Omega}^i k) \equiv (-1)^i$ modulo $|G|$ for $i \geqslant 0$. (Here $(kG)^t$, for some t, is a minimal projective cover of $\widetilde{\Omega}^i k$.) Also, since $\widetilde{\Omega}^i k \cong (\widetilde{\Omega}^{-i} k)^*$ in $\mathrm{Mod}(kG)$, it follows that $\dim(\widetilde{\Omega}^i k) \equiv (-1)^i$ modulo $|G|$ for each integer i. In particular, $M \not\cong \widetilde{\Omega}^i k$ for any integer i. □

4. The ghost projective class

In order to measure the degree to which the GH fails, it is natural to consider the smallest integer l such that the composite of any l ghosts between finite-dimensional kG-modules is trivial in the stable module category. We will show (see Theorem 4.7) that such an integer always exists. This integer will be called the *ghost number of* stmod(kG), or, more briefly, the *ghost number of kG* and can be best understood using the concept of a projective class. So we begin with a recollection of the notion of a projective class in a triangulated category. A good reference

for this is [10], where projective classes were studied in the stable homotopy category and the derived category of a ring.

4.1. Projective classes

Let $\mathcal{T}$ denote a triangulated category. A *projective class* in $\mathcal{T}$ is a pair $(\mathcal{P}, \mathcal{G})$ where $\mathcal{P}$ is a class of objects and $\mathcal{G}$ is a class of maps in $\mathcal{T}$ which satisfy the following properties:

(1) The class of all maps $X \to Y$ such that the composite $P \to X \to Y$ is zero for all P in $\mathcal{P}$ and all maps $P \to X$ is precisely $\mathcal{G}$.
(2) The class of all objects P such that the composite $P \to X \to Y$ is zero for all maps $X \to Y$ in $\mathcal{G}$ and all maps $P \to X$ is precisely $\mathcal{P}$.
(3) For each object X there is an exact triangle $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{G}$.

It follows that the maps in $\mathcal{G}$ form an ideal in $\mathcal{T}$. That is, if f and g are parallel maps in $\mathcal{G}$, then $f + g$ is in $\mathcal{G}$, and if f, g, and h are composable with g in $\mathcal{G}$, then both fg and gh are also in $\mathcal{G}$.

Once we have a projective class as defined above, we can form "derived" projective classes in a natural way. The powers $\mathcal{G}^n$ of the ideal $\mathcal{G}$ form a decreasing filtration of the maps in $\mathcal{T}$, and each $\mathcal{G}^n$ is part of a projective class. The corresponding classes of objects are obtained as follows. Let $\mathcal{P}^1 = \mathcal{P}$ and inductively define $\mathcal{P}^n$ to be the collection of retracts of objects M that appear in a triangle

$$A \longrightarrow M \longrightarrow B,$$

where A belongs to $\mathcal{P}^1$ and B belong to $\mathcal{P}^{n-1}$. The classes $\mathcal{P}^n$ form an increasing filtration of the objects in $\mathcal{T}$. It is a fact [10, Theorem 1.1] that $(\mathcal{P}^n, \mathcal{G}^n)$ is a projective class for each n. We set $\mathcal{P}^0$ to be the collection of zero objects in $\mathcal{T}$ and $\mathcal{G}^0$ to be the collection of all maps in $\mathcal{T}$. $(\mathcal{P}^0, \mathcal{G}^0)$ is the *trivial projective class*.

4.2. The ghost projective class

Now we specialise to the stable module category to define the *ghost projective class* in $\mathrm{StMod}(kG)$. The ideal $\mathcal{G}$ consists of the class of ghosts. The associated class $\mathcal{P}$ of objects consists of retracts of direct sums of suspensions of the trivial representation.

Proposition 4.1. *The pair* $(\mathcal{P}, \mathcal{G})$ *is a projective class in* $\mathrm{StMod}(kG)$.

Proof. It is clear that $\mathcal{P}$ and $\mathcal{G}$ are orthogonal, i.e., the composite $P \to M \xrightarrow{h} N$ is zero for all P in $\mathcal{P}$, for all h in $\mathcal{G}$, and all maps $P \to M$. So by [10, Lemma 3.2] it remains to show that for all kG-modules M, there exists a triangle $P \to M \to N$ such that P is in $\mathcal{P}$ and $M \to N$ is in $\mathcal{G}$. The universal ghost (3.1) out of M has this property, so we are done. $\square$

In some special cases one can also build a ghost projective class in $\mathrm{stmod}(kG)$. Let $\mathcal{P}_c$ denote the collection of finite direct sums of suspensions of k and $\mathcal{G}_c$ the class of ghosts in $\mathrm{stmod}(kG)$. (Note that the collection $\mathcal{P}_c$ is already closed under retractions by the Krull–Schmidt theorem.) Then we have the following proposition.

Proposition 4.2. *Let G be a finite p-group such that the trivial module k is periodic. Then $(\mathcal{P}_c, \mathcal{G}_c)$ is a projective class in* stmod(kG).

The proof below only requires that each finite-dimensional module has finitely generated Tate cohomology. It is shown in [7] that this is equivalent to G being cyclic or a generalised quaternion group, and by a result of Artin and Tate [8, p. 262] this is equivalent to k being periodic.

Proof. Orthogonality of $\mathcal{P}_c$ and $\mathcal{G}_c$ is clear. We have already seen in Proposition 3.3 that under the given hypothesis a universal ghost out of a finite-dimensional module can be constructed within stmod(kG). So the proposition follows from [10, Lemma 3.2]. □

Whether or not the pair $(\mathcal{P}_c, \mathcal{G}_c)$ forms a projective class, we can define $\mathcal{P}_c^m$ and $\mathcal{G}_c^m$ as in the previous subsection. A finite-dimensional kG-module is said to have *generating length m* if it belongs to $\mathcal{P}_c^m$ but not to $\mathcal{P}_c^{m-1}$, and *ghost length m* if it is the domain of a non-zero map in $\mathcal{G}_c^{m-1}$ but not in $\mathcal{G}_c^m$.

We now prove a sequence of inequalities.

Proposition 4.3. *Let G be a finite p-group and M a finite-dimensional kG-module. Then*

$$\text{ghost length of } M \leqslant \text{generating length of } M.$$

Moreover, equality holds if $(\mathcal{P}_c, \mathcal{G}_c)$ is a projective class.

Proof. If the generating length of M is one, then M is a finite direct sum of suspensions of k. Then clearly all ghosts out of M vanish, which means the ghost length of M is also one. Suppose that the generating length of M is two. Then M can be chosen to be a direct summand of M' which can be obtained as an extension of finite-dimensional modules

$$0 \longrightarrow \bigoplus_I \Omega^i k \xrightarrow{\alpha} M' \xrightarrow{\beta} \bigoplus_J \Omega^i k \longrightarrow 0.$$

It suffices to show that every two-fold composite of ghosts

$$M' \xrightarrow{f} A \xrightarrow{g} B$$

out of M' is trivial in the stable category, for then the same would be true for M. Consider the following commutative diagram in stmod(kG):

Since f is a ghost, $f\alpha = 0$, therefore the map f factors as $f = \tilde{f}\beta$. But then

$$gf = g(\tilde{f}\beta) = (g\tilde{f})\beta = 0\beta = 0.$$

(The third equality follows because g is a ghost.) Since $gf = 0$, the ghost length of M is at most two. The induction step is similar.

The last statement of the proposition follows directly from [10, Proposition 3.3]. $\quad\square$

Proposition 4.4. *Let G be a finite p-group and M a finite-dimensional kG-module. Then*

$$\text{generating length of } M \leqslant \text{radical length of } M.$$

Proof. The Jacobson radical $J = J(kG)$ of kG is nilpotent. The radical length of M is the smallest integer h such that $J^h M = 0$. This gives the radical or Lowey series for M:

$$M \supsetneq JM \supsetneq \cdots \supsetneq J^{h-1}M \supsetneq J^h M = 0.$$

Note that J annihilates each successive quotient and hence each of them is a direct sum of trivial representations. This shows that the generating length is at most the radical length. $\quad\square$

Remark 4.5. Propositions 4.3 and 4.4 show that if the radical length of M is l, then any map $M \to N$ which is a composite of l ghosts is trivial. In contrast, it follows from Corollary 2.3 that any map $L \to M$ which is a composite of l ghosts is trivial.

Recall that the *nilpotency index* of the Jacobson radical $J(kG)$ of kG is the smallest integer m such that $J(kG)^m = 0$.

Proposition 4.6. *Let G be a finite p-group and M a projective-free kG-module. Then*

$$\text{radical length of } M < \text{nilpotency index of } J(kG) \leqslant |G|.$$

Proof. Let m be the nilpotency index of $J(kG)$. We begin by noting that since G is a p-group, the last non-zero power $J(kG)^{m-1}$ of kG is the unique non-zero minimal ideal in kG; see [1, p. 92]. For the first inequality, it is enough to show that $J(kG)^{m-1}M = 0$. Let x be an element of M. Since M is projective-free, $\text{Ann}(x) \neq 0$. Thus $\text{Ann}(x)$ contains the unique non-zero minimal ideal $J(kG)^{m-1}$ of kG. That is, $J(kG)^{m-1}x = 0$.

Since the powers of $J(kG)$ form a strictly decreasing series, the nilpotency index of $J(kG)$ is at most $\dim_k kG = |G|$. $\quad\square$

These inequalities show that for each p-group G, the ghost lengths and generating lengths of finite-dimensional kG-modules are uniformly bounded above. So we define the *generating number* of kG to be the least upper bound of the generating lengths of all finite-dimensional kG-modules, and the *ghost number* of kG to be the least upper bound of the ghost lengths of all finite-dimensional kG-modules. We do not know if the generating number and ghost number of kG depend on the field k.

Combining the above results gives:

Theorem 4.7. *Let G be a finite p-group. Then*

$$\text{ghost number of } kG \leqslant \text{generating number of } kG$$
$$< \text{nilpotency index of } J(kG) \leqslant |G|.$$

In particular, the generating number and ghost number of the group algebra of any finite p-group are finite, and any composite of $|G| - 1$ ghosts in stmod(kG) *is trivial.*

Remark 4.8. A similar argument involving the projective class $(\mathcal{P}, \mathcal{G})$ shows that any composite of $m - 1$ ghosts in StMod(kG) is trivial, where m is the nilpotency index of $J(kG)$.

Proposition 4.9. *Let H be a subgroup of a finite p-group G. Then*

$$\text{ghost number of } kH \leqslant \text{ghost number of } kG.$$

Proof. In [3], we have shown that the induction functor

$$\underline{\text{Ind}} : \text{stmod}(kH) \longrightarrow \text{stmod}(kG),$$

which sends a kH-module M to $M{\uparrow}^G := kG \otimes_{kH} M$, preserves ghosts and non-trivial maps. It follows that the ghost number of kH is no more than that of kG.   □

5. Computing ghost numbers

We now investigate the problem of computing ghost numbers and generating numbers of some specific groups. The following lemma that we learned from Dave Benson [3] will be very helpful in computations.

If an element θ belongs to the centre of kG, there is a natural kG-linear self-map on any kG-module M given by left multiplication by θ. We will denote this map by $\theta : M \to M$.

Proposition 5.1. *Let G be a finite p-group and let M be a kG-module. If an element θ belonging to $J(kG)$ is central in kG, then the map*

$$\theta : M \longrightarrow M$$

is a ghost. In particular, if $g \in G$ is central, then the map

$$g - 1 : M \longrightarrow M$$

is a ghost.

Proof. The proof of [3, Lemma 2.2] applies without change.   □

5.1. Cyclic p-groups

Recall that for cyclic groups, the trivial representation is periodic, and so by Proposition 4.2 $(\mathcal{P}_c, \mathcal{G}_c)$ forms a projective class. So the ghost length and generating length are the same for modules over cyclic p-groups.

Proposition 5.2. *All finite-dimensional kC_{p^r}-modules have ghost length at most $\lceil (p^r - 1)/2 \rceil$.*

Here $\lceil y \rceil$ denotes the smallest integer that is greater than or equal to y.

Proof. Since the characteristic of k is p, we have $kC_{p^r} \cong k[x]/(x^{p^r})$, with x corresponding to $\sigma - 1$, where σ is a generator of C_{p^r}. A finite-dimensional indecomposable projective-free module over $k[x]/(x^{p^r})$ is of the form $k[x]/x^i$ for $1 \leqslant i \leqslant p^r - 1$. It is also clear that $\widetilde{\Omega}(k[x]/(x^i)) \cong k[x]/(x^{p^r - i})$. This tells us that the ghost length of $k[x]/(x^i)$ is the same as that of $k[x]/(x^{p^r - i})$. For $1 \leqslant i \leqslant \lceil (p^r - 1)/2 \rceil$, we show that the ghost length of $k[x]/x^i$ is at most i. For this it is enough to observe that we have short exact sequences

$$0 \longrightarrow k \longrightarrow k[x]/x^i \longrightarrow k[x]/x^{i-1} \longrightarrow 0$$

of modules over $k[x]/x^{p^r}$, for $2 \leqslant i \leqslant \lceil (p^r - 1)/2 \rceil$. □

Proposition 5.3. *There exists a composable sequence of $\lceil (p^r - 1)/2 \rceil - 1$ ghosts in $\mathrm{stmod}(kC_{p^r})$ whose composite is non-trivial.*

Proof. Recall that $kC_{p^r} \cong k[x]/(x^{p^r})$. Let $h : k[x]/x^d \to k[x]/x^d$ be multiplication by $x = \sigma - 1$, where $d = \lceil (p^r - 1)/2 \rceil$. By Proposition 5.1, h is a ghost. To see that h^{d-1} is non-trivial, we have to show that it cannot factor through the projective cover $k[x]/x^{p^r} \twoheadrightarrow k[x]/x^d$, i.e., that we cannot have a commutative diagram

$$
\begin{array}{ccc}
k[x]/x^d & \xrightarrow{\;\;x^{d-1}\;\;} & k[x]/x^d \\
& \searrow \qquad \nearrow & \\
& k[x]/x^{p^r} &
\end{array}
$$
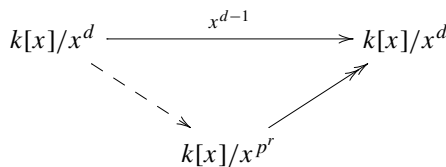
By considering the images of the generator of the left-hand cyclic module in the above diagram, one can easily see that the existence of such a factorisation would mean that

$$(d - 1) + (d - 1) \geqslant p^r - 1,$$

or, equivalently, that

$$\lceil (p^r - 1)/2 \rceil - 1 + \lceil (p^r - 1)/2 \rceil - 1 \geqslant p^r - 1.$$

It is straightforward to verify that this inequality fails for all primes p and all positive integers r. So we are done. □

Combining these two propositions, we get the following theorem.

Theorem 5.4. *The ghost number of kC_{p^r} is $\lceil (p^r - 1)/2 \rceil$.*

Corollary 5.5. *(See [3].) The GH holds for kC_{p^r} if and only if p^r is equal to 2 or 3.*

Proof. Recall that the GH holds for kG precisely when the ghost number of kG is 1. So, by Theorem 5.4, we conclude that the GH holds for kC_{p^r} if and only if $\lceil (p^r - 1)/2 \rceil = 1$. The last equation holds if and only if $p^r = 2$ or 3. $\square$

5.2. The Klein four group V_4

Proposition 5.6. *Let M be a finite-dimensional indecomposable projective-free kV_4-module. Then we have the following*:

(1) *If M is odd-dimensional, then it has generating length one.*
(2) *If M is even-dimensional, then it has generating length two.*

Proof. It is well known that the odd-dimensional indecomposable modules are precisely the Heller shifts of the trivial representation; see [1, Theorem 4.3.2], for instance. So they all have generating length one by definition. Since M is projective-free, one can show using the classification of the indecomposable kV_4-modules (e.g., [1, Theorem 4.3.2]), or directly, that there is a short exact sequence

$$0 \longrightarrow M^{V_4} \longrightarrow M \longrightarrow M_{V_4} \longrightarrow 0,$$

where the invariant submodule M^{V_4} and the coinvariant module M_{V_4} are both direct sums of the trivial representation k. Thus M has generating length at most two. Moreover, if M is even-dimensional and indecomposable, then M is not isomorphic to $\widetilde{\Omega}^i k$ for any i. In particular, M cannot have generating length one. So we are done. $\square$

Theorem 5.7. *The ghost number and the generating number of kV_4 are both two.*

Proof. Since every finite-dimensional module is a sum of indecomposables, the statement about the generating number follows from the above proposition. Since the ghost number is at most the generating number, we only have to show that the ghost number is bigger than one. This follows from [3] because there we showed that the GH fails for stmod(kV_4). $\square$

5.3. The quaternion group

By our main result on the GH in [3], we know that the GH fails for the quaternion group Q_8 of order 8. Now we give bounds on the ghost number. Since the trivial representation of Q_8 is periodic, we know from Proposition 4.2 that the ghost projective class exists in stmod(kQ_8). Therefore the ghost number and the generating number of kQ_8 are the same.

**Proposition 5.8.** *The ghost number of kQ_8 is at least two and at most four.*

Proof. The nilpotency index of $J(kQ_8)$ is 5 (see Section 5.5), so by Theorem 4.7, we know that the ghost number of kQ_8 is at most 4. We have already seen that the GH fails for Q_8, so the ghost number is at least 2. $\square$

In the following example we will give another disproof of the GH for the group Q_8 by exhibiting an explicit finite-dimensional module with ghost length two. This example should also illustrate some of the ideas surrounding projective classes.

Example 5.9. A minimal presentation for Q_8 is given by

$$Q_8 = \langle x, y \mid x^4 = 1, \ x^2 = y^2, \ yxy^{-1} = x^{-1} \rangle.$$

The structure of the left kQ_8-module $J(kQ_8)^3$ can be obtained using Jennings' theorem [14] or otherwise. It is shown in the diagram below:
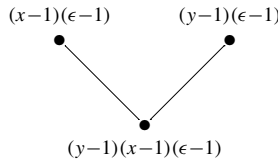
Here a bullet is a one-dimensional k-vector space, the southwest line segment corresponds to the action of $x - 1$, the southeast line segment corresponds to the action of $y - 1$, and if no line segment emanates from a bullet in a given direction, then the corresponding action is trivial. ϵ is the central element $x^2 (= y^2)$.

It is clear from the diagram that the invariant submodule of $J(kQ_8)^3$ is one-dimensional and therefore we conclude that $J(kQ_8)^3$ is indecomposable (see, for example, [9, Lemma 3.2]). Also note that $J(kQ_8)^3$ is projective-free. Moreover, the dimension of $J(kQ_8)^3$ is 3, which is neither $+1$ nor -1 modulo 8 $(= |Q_8|)$. Thus by Corollary 3.6 we know that there exists a non-trivial ghost in StMod(kQ_8) whose domain is $J(kQ_8)^3$. By Corollary 3.4, the target can be chosen to lie in stmod(kQ_8). In particular, the ghost length of $J(kQ_8)^3$ is at least 2. On the other hand the generating length of $J(kQ_8)^3$ is at most 2 because we have a short exact sequence of kQ_8-modules

$$0 \longrightarrow k \longrightarrow J(kQ_8)^3 \longrightarrow k \oplus k \longrightarrow 0.$$

Since the ghost length is always less than or equal to the generating length, we conclude that the ghost length and the generating length of $J(kQ_8)^3$ are both 2.

5.4. Ghost numbers of abelian groups

We begin with an extremely useful proposition. This is a slight generalisation of a result we learned from Dave Benson that appeared in [3].

Proposition 5.10. *Let G be a finite p-group and let H be a non-trivial subgroup of G. Let $\theta = \sum_g \alpha_g g$ be a central element of kG such that $\sum_{h \in H} \alpha_h \neq 0$. Then multiplication by θ on $k_H {\uparrow}^G$ is stably non-trivial, where k_H is the trivial kH-module. In particular, if g is a central element in $G - H$, then multiplication by $g - 1$ on $k_H {\uparrow}^G$ is a non-trivial ghost.*

We include a proof, since this is slightly more general than [3, Lemma 2.3]: we do not require H to be normal and we include more general θ.

Proof. Recall that $k_H{\uparrow}^G$ denotes the induced module $kG \otimes_{kH} k_H$, and that induction is both left and right adjoint to restriction. These adjunctions give rise to natural kH-linear maps $k_H \to k_H{\uparrow}^G{\downarrow}_H$, sending x to $1 \otimes x$, and $k_H{\uparrow}^G{\downarrow}_H \to k_H$, sending $g \otimes x$ to x if $g \in H$ and to 0 otherwise.

To show that $\theta : k_H{\uparrow}^G \to k_H{\uparrow}^G$ is stably non-trivial, it is enough to show that

$$\theta{\downarrow}_H : k_H{\uparrow}^G{\downarrow}_H \to k_H{\uparrow}^G{\downarrow}_H$$

is stably non-trivial. For this, it is enough to show that the composite

$$k_H \to k_H{\uparrow}^G{\downarrow}_H \xrightarrow{\theta{\downarrow}_H} k_H{\uparrow}^G{\downarrow}_H \to k_H$$

is stably non-trivial. But this composite is multiplication by $\sum_{h \in H} \alpha_h$, which is non-zero by assumption. And since H is non-trivial, all non-zero maps $k_H \to k_H$ are stably non-trivial.

The last statement follows from the first part of this proposition, combined with Proposition 5.1. $\quad\square$

Theorem 5.11. *Let G be an abelian p-group and let m denote the nilpotency index of $J(kG)$. Then we have*

$$m - p^{r-1}(p-1) \leqslant \text{ghost number of } kG \leqslant m - 1,$$

where p^r is the order of the smallest cyclic summand of G.

Proof. We have already seen the upper bound, so we only have to establish the lower bound. Let C_{p^r} be the smallest cyclic summand of G, so that for some integers $r_i \geqslant r$,

$$G = C_{p^r} \oplus C_{p^{r_1}} \oplus \cdots \oplus C_{p^{r_t}}.$$

Let H be the subgroup of order p in C_{p^r} and let k_H be the trivial kH-module. Set $M = k_H{\uparrow}^G$. We will produce $m - p^{r-1}(p-1) - 1$ ghosts $M \to M$ whose composite is stably non-trivial. This will give the desired lower bound for the ghost number. Let g be a generator for C_{p^r} and let g_i be a generator for $C_{p^{r_i}}$ for each i. By Proposition 5.1, the map

$$\theta = (g-1)^{p^{r-1}-1}(g_1-1)^{p^{r_1}-1} \cdots (g_t-1)^{p^{r_t}-1} : M \to M$$

is a composite of

$$\left(p^{r-1}-1\right) + \left(p^{r_1}-1\right) + \cdots + \left(p^{r_t}-1\right)$$

ghosts. One can see easily that the nilpotency index of $J(kG)$ is

$$m = 1 + \left(p^r - 1\right) + \left(p^{r_1} - 1\right) + \cdots + \left(p^{r_t} - 1\right).$$

Thus θ is a composite of $m - p^{r-1}(p-1) - 1$ ghosts. Now to see that θ is stably non-trivial on M, it is enough to note that if $\theta \in kG$ is written $\sum_{g \in G} \alpha_g g$, then $\alpha_h = 0$ for $h \in H$ unless $h = e$. So Proposition 5.10 applies. $\quad\square$

We derive some easy corollaries.

Corollary 5.12. *Let G be an abelian 2-group which has C_2 as a summand. Then the ghost number of kG is one less than the nilpotency index of $J(kG)$.*

Proof. In this case, both the lower bound and the upper bound for the ghost number of kG are one less than the nilpotency index of $J(kG)$. □

Corollary 5.13. *Let G be an elementary abelian 2-group of rank l, i.e., $G \cong (C_2)^l$. Then the ghost number of kG is l.*

Proof. The nilpotency index of $J(k(C_2)^l)$ is easily shown to be $l + 1$. □

Theorem 5.14. *Let G be an abelian p-group. The ghost number of kG is 2 if and only if G is C_4, $C_2 \oplus C_2$, or C_5.*

Proof. By Theorem 5.4 and Corollary 5.13 we know that the three given groups have ghost number 2. Now we prove the converse. An easy exercise using the structure theorem tells us that if $|G| > 5$, then G contains one of the following groups as a subgroup: C_{p^l} $(p^l > 5)$, $C_2 \oplus C_2 \oplus C_2$, $C_p \oplus C_p$ $(p > 2)$, or $C_2 \oplus C_4$. It is easily seen using the lower bound in Theorem 5.11 that the ghost number of each of the above groups is at least 3. Therefore, by Proposition 4.9, the ghost number of G is also at least 3. So if the ghost number of G is at most 2, then $|G|$ should be at most 5. We know that C_2 and C_3 have ghost number 1, and the only remaining groups of order at most 5 are C_4, C_5 and $C_2 \oplus C_2$. □

5.5. Ghost numbers for non-abelian groups

Computing ghost numbers for non-abelian groups is much harder. However, the nilpotency indices of $J(kG)$ can be computed using a formula of Jennings [14]. For example, if G is a non-cyclic group of order 2^n which has a cyclic subgroup of index 2, then the nilpotency index of $J(kG)$ is $2^{n-1} + 1$; see [12] for instance. Therefore the composite of any 2^{n-1} ghosts in stmod(kG) is trivial. By a well-known theorem [4, Chapter 8, pp. 134–135] every non-abelian 2-group that has a cyclic subgroup of index 2 is a dihedral, semidihedral, modular or quaternion group. Thus we have upper bounds for the ghost numbers of their group algebras.

Acknowledgments

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