Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Stochastic Processes and their Applications 120 (2010) 1535–1562
stochastic processes and their applications
www.elsevier.com/locate/spa

Equilibrium fluctuations for exclusion processes with conductances in random environments[☆]

Jonathan Farfan[a,b], Alexandre B. Simas[a], Fábio J. Valentim[a,c,*]

[a] *IMPA, Estrada Dona Castorina 110, J. Botânico, 22460-320 Rio de Janeiro, Brazil*
[b] *IMCA, Calle los Biólogos 245, Urb. San César Primera Etapa, Lima 12, Peru*
[c] *CEUNES, Universidade Federal do Espirito Santo, Rodovia BR 101 Norte, Km. 60, Bairro Litorâneo, CEP 29932-540, São Mateus, ES, Brazil*

Received 1 October 2009; received in revised form 19 February 2010; accepted 30 March 2010
Available online 8 April 2010

Abstract

Fix a function $W : \mathbb{R}^d \to \mathbb{R}$ such that $W(x_1, \ldots, x_d) = \sum_{k=1}^{d} W_k(x_k)$, where $d \geq 1$, and each function $W_k : \mathbb{R} \to \mathbb{R}$ is strictly increasing, right continuous with left limits. We prove the equilibrium fluctuations for exclusion processes with conductances, induced by W, in random environments, when the system starts from an equilibrium measure. The asymptotic behavior of the empirical distribution is governed by the unique solution of a stochastic differential equation taking values in a certain nuclear Fréchet space.

MSC: 60K35; 35B27; 60F05; 46A11

Keywords: Exclusion processes; Homogenization; Equilibrium fluctuations; Nuclear spaces

1. Introduction

In this article we study the equilibrium fluctuations for exclusion processes with conductances in random environments, which can be viewed as a central limit theorem for the empirical distribution of particles when the system starts from an equilibrium measure.

Let $W : \mathbb{R}^d \to \mathbb{R}$ be a function such that $W(x_1, \ldots, x_d) = \sum_{k=1}^{d} W_k(x_k)$, where $d \geq 1$ and each function $W_k : \mathbb{R} \to \mathbb{R}$ is strictly increasing, right continuous with left limits (càdlàg), and

☆ Research supported by CNPq.
* Corresponding author at: IMPA, Estrada Dona Castorina 110, J. Botânico, 22460-320, Rio de Janeiro, Brazil.
E-mail addresses: jonathan@impa.br (J. Farfan), alesimas@impa.br (A.B. Simas), valentim@impa.br, fjsvalentim@yahoo.com.br (F.J. Valentim).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2010.03.018

periodic in the sense that $W_k(u+1) - W_k(u) = W_k(1) - W_k(0)$, for all $u \in \mathbb{R}$. The inverses of the increments of the function W will play the role of conductances in our system.

The random environment that we considered is governed by the coefficients of the discrete formulation of the model on the lattice. Moreover, we will assume that the underlying random field is ergodic, stationary and satisfies an ellipticity condition.

Informally, the exclusion process with conductances induced by W in random environments is an interacting particle system on the d-dimensional discrete torus $N^{-1}\mathbb{T}_N^d$ in which at most one particle per site is allowed and only nearest neighbor jumps are permitted. Moreover, the rate of jumps in the direction e_j is proportional to the reciprocal of the increments of W with respect to the jth coordinate times a term $a(\omega)$ coming from an elliptic and ergodic random field. Such a system can be understood as a model for diffusion in heterogeneous media. For instance, it may model diffusions of particles in a medium with permeable membranes at the points of discontinuities of W, that tend to reflect particles, creating space discontinuities in the density profiles. A formal description of the stochastic evolution of this process is given in Section 2. Note that these membranes are $(d-1)$-dimensional hyperplanes embedded in a d-dimensional environment. Moreover, if we consider W_j having more than one discontinuity point for more than one j, these membranes will be more sophisticated manifolds, for instance, unions of $(d-1)$-dimensional boxes.

The purpose of this article is to study the density fluctuation field of this system as $N \to \infty$, and also the influence of the randomness in this limit. For any realization of the random environment, the scaling limit depends on the randomness only through some constants which depend on the distribution of the random transition rates, but not on the particular realization of the random environment.

The evolution of one-dimensional exclusion processes with random conductances has attracted some attention recently [2–5,9], with the hydrodynamic limit proved in [9] being also obtained in [2], independently. In all of these papers, a hydrodynamic limit was proved. The hydrodynamic limit may be interpreted as a law of large numbers for the empirical density of the system. Our goal is to go beyond the hydrodynamic limit and provide a new result for such processes, which is the equilibrium fluctuations and can be seen as a central limit theorem for the empirical density of the process.

To prove the equilibrium fluctuations, we would like to call attention to the main tools that we needed: (i) the theory of nuclear spaces and (ii) homogenization of differential operators. The first tool followed the classical approach of Kallianpur and Perez-Abreu [10] and Gel'fand and Vilenkin [6]. Nuclear spaces are very suitable for attaining the existence and uniqueness of solutions for a general class of stochastic differential equations. Furthermore, tightness of processes on such spaces was established by Mitoma [12]. A wide literature on these spaces can be found cited inside the fourth volume of the amazing collection by Gel'fand [6]. The second tool is motivated by several applications in mechanics, physics, chemistry and engineering. We will consider stochastic homogenization. In the stochastic context, several works on homogenization of operators with random coefficients have been published (see, for instance, [13,14] and references therein). In homogenization theory, only the stationarity of such random fields is used. The notion of a stationary random field is formulated in such a manner that it covers many objects of a non-probabilistic nature, e.g., operators with periodic or quasi-periodic coefficients. We follow the approach given in [15], which was introduced by [14].

The focus of our approach is on studying the asymptotic behavior of effective coefficients for a family of random difference schemes whose coefficients can be obtained by the discretization of random high-contrast lattice structures. Furthermore, the introduction of a corrected empirical measure was needed. The corrected empirical measure was used in the literature, for instance, by [9,5,7,16,15]. It can be understood as a version of Tartar's compensated compactness lemma in the context of particle systems. In this situation, the averaging due to the dynamics and the inhomogeneities introduced by the random media factorize after introducing the corrected empirical process, in such a way that we can average them separately. It is noteworthy that we managed to prove an equivalence between the asymptotic behavior with respect to the corrected empirical measure and that with respect to the uncorrected one. This equivalence was helpful in the sense that whenever the calculation with the corrected empirical measure turned cumbersome, we changed to a calculation with respect to the uncorrected one, and vice versa. This whole approach made the proof simpler than the usual one with respect solely to the corrected empirical measure developed in the articles mentioned above.

We now describe the organization of the article. In Section 2 we state the main results of the article; in Section 3 we define the nuclear space needed in our context; in Section 4 we recall some results obtained in [15] concerning homogenization, and then we prove the equilibrium fluctuations by showing that the density fluctuation field converges to a process that solves the martingale problem. We also show that the solution of the martingale problem corresponds to a generalized Ornstein–Uhlenbeck process. In Section 5 we prove tightness of the density fluctuation field, as well as tightness of other related quantities. In Section 6 we prove the Boltzmann–Gibbs principle, which is a key result for proving the equilibrium fluctuations. Finally, the Appendix contains some known results about nuclear spaces and stochastic differential equations evolving on the topological dual of such spaces.

2. Notation and results

Denote by $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d = [0,1)^d$ the d-dimensional torus, and by $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d = \{0,\ldots,N-1\}^d$ the d-dimensional discrete torus with N^d points.

Fix a function $W : \mathbb{R}^d \to \mathbb{R}$ such that

$$W(x_1,\ldots,x_d) = \sum_{k=1}^{d} W_k(x_k), \tag{2.1}$$

where each $W_k : \mathbb{R} \to \mathbb{R}$ is a *strictly increasing* right continuous function with left limits (càdlàg), periodic in the sense that for all $u \in \mathbb{R}$

$$W_k(u+1) - W_k(u) = W_k(1) - W_k(0).$$

Define the generalized derivative ∂_{W_k} of a function $f : \mathbb{T}^d \to \mathbb{R}$ by

$$\partial_{W_k} f(x_1,\ldots,x_k,\ldots,x_d) = \lim_{\epsilon\to 0} \frac{f(x_1,\ldots,x_k+\epsilon,\ldots,x_d) - f(x_1,\ldots,x_k,\ldots,x_d)}{W_k(x_k+\epsilon) - W_k(x_k)}, \tag{2.2}$$

when the above limit exists and is finite. If for a function $f : \mathbb{T}^d \to \mathbb{R}$ the generalized derivatives ∂_{W_k} exist for all $k = 1,\ldots,d$, denote the generalized gradient of f by

$$\nabla_W f = \left(\partial_{W_1} f, \ldots, \partial_{W_d} f\right).$$

Further details on these generalized derivatives can be found in Section 3.1 and in the article [15].

We now introduce the statistically homogeneous rapidly oscillating coefficients that will be used to define the random rates of the exclusion process with conductances for which we want to study the equilibrium fluctuations.

Let $(\Omega, \mathcal{F}, \mu)$ be a standard probability space and $\{T_x : \Omega \to \Omega; x \in \mathbb{Z}^d\}$ be an ergodic group of $\mathcal{F}$-measurable transformations which preserve the measure μ:

- $T_x : \Omega \to \Omega$ is $\mathcal{F}$-measurable for all $x \in \mathbb{Z}^d$,
- $\mu(T_x\mathbf{A}) = \mu(\mathbf{A})$ for any $\mathbf{A} \in \mathcal{F}$ and $x \in \mathbb{Z}^d$,
- $T_0 = I$, $T_x \circ T_y = T_{x+y}$,
- any $f \in L^1(\Omega)$ such that $f(T_x\omega) = f(\omega)$ μ-a.s. for each $x \in \mathbb{Z}^d$ is equal to a constant μ-a.s.

The last condition implies that the group T_x is ergodic.

Let us now introduce the vector-valued $\mathcal{F}$-measurable functions $\{a_j(\omega); j = 1, \dots, d\}$ that satisfy an ellipticity condition: there exists $\theta > 0$ such that

$$\theta^{-1} \le a_j(\omega) \le \theta,$$

for all $\omega \in \Omega$ and $j = 1, \dots, d$. Then, define the diagonal matrices A^N whose elements are given by

$$a_{jj}^N(x) := a_j^N = a_j(T_{Nx}\omega), \quad x \in T_N^d, j = 1, \dots, d. \tag{2.3}$$

Fix a typical realization $\omega \in \Omega$ of the random environment. For each $x \in \mathbb{T}_N^d$ and $j = 1, \dots, d$, define the symmetric rate $\xi_{x,x+e_j} = \xi_{x+e_j,x}$ by

$$\xi_{x,x+e_j} = \frac{a_j^N(x)}{N[W((x+e_j)/N) - W(x/N)]} = \frac{a_j^N(x)}{N[W_j((x_j+1)/N) - W_j(x_j/N)]}, \tag{2.4}$$

where $e_1, \dots, e_d$ is the canonical basis of $\mathbb{R}^d$.

Distribute particles on $\mathbb{T}_N^d$ in such a way that each site of $\mathbb{T}_N^d$ is occupied by at most one particle. Denote by η the configurations of the state space $\{0, 1\}^{\mathbb{T}_N^d}$ so that $\eta(x) = 0$ if site x is vacant, and $\eta(x) = 1$ if site x is occupied.

The exclusion process with conductances in a random environment is the continuous-time Markov process $\{\eta_t : t \ge 0\}$ with state space $\{0, 1\}^{\mathbb{T}_N^d} = \{\eta : \mathbb{T}_N^d \to \{0, 1\}\}$, whose generator L_N acts on functions $f : \{0, 1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ as

$$(L_N f)(\eta) = \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_j} c_{x,x+e_j}(\eta) \{f(\sigma^{x,x+e_j}\eta) - f(\eta)\}, \tag{2.5}$$

where $\sigma^{x,x+e_j}\eta$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+e_j)$:

$$(\sigma^{x,x+e_j}\eta)(y) = \begin{cases} \eta(x+e_j) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+e_j, \\ \eta(y) & \text{otherwise,} \end{cases} \tag{2.6}$$

and

$$c_{x,x+e_j}(\eta) = 1 + b\{\eta(x-e_j) + \eta(x+2\,e_j)\},$$

with $b > -1/2$, and where all sums are modulo N.

We consider the Markov process $\{\eta_t : t \ge 0\}$ on the configurations $\{0, 1\}^{\mathbb{T}_N^d}$ associated with the generator L_N on the diffusive scale, i.e., L_N is speeded up by N^2.

We now describe the stochastic evolution of the process. Let $x = (x_1, \dots, x_d) \in \mathbb{T}_N^d$. At rate $\xi_{x,x+e_j} c_{x,x+e_j}(\eta)$ the occupation variables $\eta(x)$, $\eta(x+e_j)$ are exchanged. Note that the random field affects the rate by a multiplicative factor. If W is differentiable at $x/N \in [0,1)^d$, the rate at which particles are exchanged is of order 1 for each direction, but if some W_j is discontinuous at x_j/N, it no longer holds. In fact, assume, to fix ideas, that W_j is discontinuous at x_j/N, and smooth on the segments $(x_j/N, x_j/N + \varepsilon e_j)$ and $(x_j/N - \varepsilon e_j, x_j/N)$. Assume, also, that W_k is differentiable in a neighborhood of x_k/N for $k \neq j$. In this case, the rate at which particles jump over the bonds $\{y - e_j, y\}$, with $y_j = x_j$, is of order $1/N$, whereas in a neighborhood of size N of these bonds, particles jump at rate 1. Thus, note that a particle at site $y - e_j$ jumps to y at rate $1/N$ and jumps at rate 1 to each one of the $2d - 1$ other options. Particles, therefore, tend to avoid the bonds $\{y - e_j, y\}$. However, since time will be scaled diffusively, and since on a time interval of length N^2 a particle spends a time of order N at each site y, particles will be able to cross the slower bond $\{y - e_j, y\}$. Therefore, the conductances are induced by the function W through the inverse of the gradient of W, whereas the random environment is given by the diagonal matrix $A^N := (a_{jj}^N(x))_{d\times d}$.

The effect of the factor $c_{x,x+e_j}(\eta)$ is the following: if the parameter b is positive, the presence of particles in the neighboring sites of the bond $\{x, x+e_j\}$ speeds up the exchange rate by a factor of order 1, and if the parameter b is negative, the presence of particles in the neighboring sites slows down the exchange rate also by a factor of order 1. More details are given in Remark 2.3.

The dynamics informally presented describes a Markov evolution. A computation shows that the Bernoulli product measures $\{\nu_\rho^N : 0 \le \rho \le 1\}$ are invariant, in fact reversible, for the dynamics. The measure ν_ρ^N is obtained by placing a particle at each site, independently from the other sites, with probability ρ. Thus, ν_ρ^N is a product measure over $\{0,1\}^{\mathbb{T}_N^d}$ with marginals given by

$$\nu_\rho^N\{\eta : \eta(x) = 1\} = \rho$$

for x in $\mathbb{T}_N^d$.

Consider the random walk $\{X_t\}_{t\ge 0}$ of a particle in $\mathbb{T}_N^d$ induced by the generator $\mathbb{L}_N$ given as follows. Let $\xi_{x,x+e_j}$ given by (2.4). If the particle is on a site $x \in \mathbb{T}_N^d$, it will jump to $x + e_j$ with rate $N^2 \xi_{x,x+e_j}$. Furthermore, only nearest neighbor jumps are allowed. The generator $\mathbb{L}_N$ of the random walk $\{X_t\}_{t\ge 0}$ acts on functions $f : \mathbb{N}^{-1} T_N^d \to \mathbb{R}$ as

$$\mathbb{L}_N f\left(\frac{x}{N}\right) = \sum_{j=1}^{d} \mathbb{L}_N^j f\left(\frac{x}{N}\right),$$

where

$$\mathbb{L}_N^j f\Big(\frac{x}{N}\Big) = N^2\Big\{\xi_{x,x+e_j}\Big[f\Big(\frac{x+e_j}{N}\Big) - f\Big(\frac{x}{N}\Big)\Big] + \xi_{x-e_j,x}\Big[f\Big(\frac{x-e_j}{N}\Big) - f\Big(\frac{x}{N}\Big)\Big]\Big\}.$$

It is not difficult to see that the following equality holds:

$$\mathbb{L}_N f(x/N) = \sum_{j=1}^{d} \partial_{x_j}^N (a_j^N \partial_{W_j}^N f)(x) := \nabla^N A^N \nabla_W^N f(x), \tag{2.7}$$

where $\partial_{x_j}^N$ is the standard difference operator:

$$\partial_{x_j}^N f\left(\frac{x}{N}\right) = N\left[f\left(\frac{x+e_j}{N}\right) - f\left(\frac{x}{N}\right)\right],$$

and $\partial_{W_j}^N$ is the W_j-difference operator:

$$\partial_{W_j}^N f\left(\frac{x}{N}\right) = \frac{f\left(\frac{x+e_j}{N}\right) - f\left(\frac{x}{N}\right)}{W\left(\frac{x+e_j}{N}\right) - W\left(\frac{x}{N}\right)},$$

for $x \in \mathbb{T}_N^d$. Several properties of the above operator have been obtained in [15].

The counting measure m_N on $N^{-1}\mathbb{T}_N^d$ is reversible for this process. This random walk plays an important role in the proof of the equilibrium fluctuations of the process η_t, as we will see in Section 4.1.

Now we state a central limit theorem for the empirical measure, starting from an equilibrium measure ν_ρ. Fix $\rho > 0$ and denote by $\mathcal{S}_W(\mathbb{T}^d)$ the generalized Schwartz space on $\mathbb{T}^d$, for which the definition and some properties are given in Section 3.

Denote by $Y_\cdot^N$ the *density fluctuation field*, which is the bounded linear functional acting on functions $G \in \mathcal{S}_W(\mathbb{T}^d)$ as

$$Y_t^N(G) = \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} G(x)[\eta_t(x) - \rho]. \tag{2.8}$$

Let $D([0,T],X)$ be the path space of càdlàg trajectories with values in a metric space X. In this way we have defined a process in $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$, where $\mathcal{S}'_W(\mathbb{T}^d)$ is the topological dual of the space $\mathcal{S}_W(\mathbb{T}^d)$.

Theorem 2.1. *Consider the fluctuation field $Y_\cdot^N$ defined above. Then, $Y_\cdot^N$ converges weakly to the unique $\mathcal{S}'_W(\mathbb{T}^d)$-solution, $Y_t \in D([0,T], S'_W(\mathbb{T}^d))$, of the stochastic differential equation*

$$\mathrm{d}Y_t = \phi'(\rho)\nabla A\nabla_W Y_t \mathrm{d}t + \sqrt{2\chi(\rho)\phi'(\rho)A}\,\mathrm{d}N_t, \tag{2.9}$$

where $\chi(\rho) = \rho(1-\rho)$, $\phi(\rho) = \rho + b\rho^2$, and ϕ' is the derivative of ϕ, $\phi'(\rho) = 1 + 2b\rho$; further A is a constant diagonal matrix with jth diagonal element given by $a_j := E(a_j^N)$, for any $N \in \mathbb{N}$; and N_t is an $\mathcal{S}'_W(\mathbb{T}^d)$-valued mean-zero martingale, with quadratic variation

$$\langle N(G)\rangle_t = t\sum_{j=1}^d \int_{\mathbb{T}^d} \left[\partial_{W_j} G(x)\right]^2 d(x^j \otimes W_j),$$

where $d(x^j \otimes W_j)$ is the product measure $\mathrm{d}x_1 \otimes \cdots \otimes \mathrm{d}x_{j-1} \otimes dW_j \otimes \mathrm{d}x_{j+1} \otimes \cdots \otimes \mathrm{d}x_d$. Furthermore, N_t is a Gaussian process with independent increments. More precisely, for each $G \in S_W(\mathbb{T}^d)$, $N_t(G)$ is a time deformation of a standard Brownian motion.

The proof of this theorem is given in Section 4.

Remark 2.2. The process Y_t is known in the literature as the generalized Ornstein–Uhlenbeck process with characteristics $\phi'(\rho)\nabla A\nabla_W$ and $\sqrt{2\chi(\rho)\phi'(\rho)A}\nabla_W$.

Remark 2.3. The specific form of the rates $c_{x,x+e_i}$ is not important, but two conditions must be fulfilled. The rates must be strictly positive, and they may not depend on the occupation variables

$\eta(x)$, $\eta(x+e_i)$, but they have to be chosen in such a way that a symmetric simple exclusion process with this rate is *gradient* (cf. Chapter 7 in [11] for the definition of gradient processes).

We may define rates $c_{x,x+e_i}$ to obtain any polynomial ϕ of the form $\phi(\alpha) = \alpha + \sum_{2\le j\le m} a_j \alpha^j$, $m \ge 1$, with $1 + \sum_{2\le j\le m} j a_j > 0$. Let, for instance, $m = 3$. Then the rates

$$\hat{c}_{x,x+e_i}(\eta) = c_{x,x+e_i}(\eta) + c\,\{\eta(x-2e_i)\eta(x-e_i) + \eta(x-e_i)\eta(x+2e_i) + \eta(x+2e_i)\eta(x+3e_i)\},$$

satisfy the above three conditions, where $c_{x,x+e_i}$ is the rate defined at the beginning of Section 2 and b, c are such that $1 + 2b + 3c > 0$. An elementary computation shows that $\phi(\alpha) = 1 + b\alpha^2 + c\alpha^3$.

3. The space $\mathcal{S}_W(\mathbb{T}^d)$

In this section we build the space $\mathcal{S}_W(\mathbb{T}^d)$, which is associated with the operator $\mathcal{L}_W = \nabla\nabla_W$. This space, as we shall see, is a natural environment in which to attain existence and uniqueness of solutions of the stochastic differential equation (2.9). Several lemmas are obtained to fulfill the conditions to ensure existence and uniqueness of such solutions.

3.1. The operator $\mathcal{L}_W$

Consider the operator $\mathcal{L}_{W_k} : \mathcal{D}_{W_k} \subset L^2(\mathbb{T}) \to \mathbb{R}$ given by

$$\mathcal{L}_{W_k} f = \partial_{x_k} \partial_{W_k} f, \tag{3.1}$$

whose domain $\mathcal{D}_{W_k}$ consists of all functions f in $L^2(\mathbb{T})$ such that

$$f(x) = a + bW_k(x) + \int_{(0,x]} W_k(\mathrm{d}y) \int_0^y \mathfrak{f}(z)\,\mathrm{d}z$$

for some function $\mathfrak{f}$ in $L^2(\mathbb{T})$ that satisfies

$$\int_0^1 \mathfrak{f}(z)\,\mathrm{d}z = 0 \quad \text{and} \quad \int_{(0,1]} W_k(\mathrm{d}y)\left\{b + \int_0^y \mathfrak{f}(z)\,\mathrm{d}z\right\} = 0.$$

In [5] the authors prove that these operators have a countable complete orthonormal system of eigenvectors, which we denote by $\mathcal{A}_{W_k}$. Then, following [16], we define

$$\mathcal{A}_W = \left\{ f : \mathbb{T}^d \to \mathbb{R} : f(x_1,\ldots,x_d) = \prod_{k=1}^d f_k(x_k),\ f_k \in \mathcal{A}_{W_k} \right\},$$

where W is given by (2.1).

We may now build an operator analogous to $\mathcal{L}_{W_k}$, acting on functions on $\mathbb{T}^d$. Let $\mathbb{D}_W$ be the linear space generated by $\mathcal{A}_W$, and define the operator $\mathbb{L}_W : \mathbb{D}_W \to L^2(\mathbb{T}^d)$ as follows: for $f = \prod_{k=1}^d f_k \in \mathcal{A}_W$,

$$\mathbb{L}_W(f)(x_1,\ldots,x_d) = \sum_{k=1}^d \prod_{j=1, j\ne k}^d f_j(x_j)\mathcal{L}_{W_k} f_k(x_k), \tag{3.2}$$

and extend to $\mathbb{D}_W$ by linearity. It is easy to see that if $f \in \mathbb{D}_W$,

$$\mathbb{L}_W f = \sum_{k=1}^{d} \mathcal{L}_{W_k} f, \tag{3.3}$$

where the application of $\mathcal{L}_{W_k}$ to a function $f : \mathbb{T}^d \to \mathbb{R}$ is the natural one, i.e., it considers f only as a function of the kth coordinate, and keeps all the remaining coordinates fixed.

Let, for each $k = 1, \dots, d$, $f_k \in \mathcal{A}_{W_k}$ be an eigenvector of $\mathcal{L}_{W_k}$ associated with the eigenvalue λ_k. Then $f = \prod_{k=1}^{d} f_k$ belongs to $\mathbb{D}_W$ and is an eigenvector of $\mathbb{L}_W$ with eigenvalue $\sum_{k=1}^{d} \lambda_k$. Moreover, in [16] the following result was proved:

Lemma 3.1. *The following statements hold:*

(a) *The set $\mathbb{D}_W$ is dense in $L^2(\mathbb{T}^d)$.*
(b) *The operator $\mathbb{L}_W : \mathbb{D}_W \to L^2(\mathbb{T}^d)$ is symmetric and non-positive:*

$$\langle -\mathbb{L}_W f, f \rangle \geq 0,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $L^2(\mathbb{T}^d)$.

Also, the set $\mathcal{A}_W$ forms a complete, orthonormal, countable system of eigenvectors for the operator $\mathbb{L}_W$. Let $\mathcal{A}_W = \{\varphi_j\}_{j\geq 1}$, $\{\alpha_j\}_{j\geq 1}$ be the corresponding eigenvalues of $-\mathbb{L}_W$, and consider $\mathcal{D}_W = \{v = \sum_{j=1}^{\infty} v_j \varphi_j \in L^2(\mathbb{T}^d); \sum_{j=1}^{\infty} v_j^2 \alpha_j^2 < +\infty\}$. We define the operator $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ by

$$-\mathcal{L}_W v = \sum_{j=1}^{+\infty} \alpha_j v_j \varphi_j. \tag{3.4}$$

The operator $\mathcal{L}_W$ is clearly an extension of the operator $\mathbb{L}_W$, and we present some properties of this operator in Proposition 3.2, whose proof can be found in [16].

Proposition 3.2. *The operator $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ enjoys the following properties:*

(a) *The domain $\mathcal{D}_W$ is dense in $L^2(\mathbb{T}^d)$. In particular, the set of eigenvectors $\mathcal{A}_W = \{\varphi_j\}_{j\geq 1}$ forms a complete orthonormal system.*
(b) *The eigenvalues of the operator $-\mathcal{L}_W$ form a countable set $\{\alpha_j\}_{j\geq 1}$. All eigenvalues have finite multiplicity, and it is possible to obtain a re-enumeration $\{\alpha_j\}_{j\geq 1}$ such that*

$$0 = \alpha_1 \leq \alpha_2 \leq \cdots \quad \text{and} \quad \lim_{n\to\infty} \alpha_n = \infty.$$

(c) *The operator $\mathbb{I} - \mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ is bijective.*
(d) *$\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ is self-adjoint and non-positive:*

$$\langle -\mathcal{L}_W f, f \rangle \geq 0.$$

(e) *$\mathcal{L}_W$ is dissipative.*

3.2. The nuclear space $\mathcal{S}_W(\mathbb{T}^d)$

Our goal is to build a countably Hilbert nuclear space associated with the self-adjoint operator $\mathcal{L}_W$. The reader is referred to Appendix.

Let $\{\varphi_j\}_{j\geq 1}$ be the complete orthonormal set of the eigenvectors of the operator $\mathcal{L} = \mathbb{I} - \mathcal{L}_W$, and $\{\lambda_j\}_{j\geq 1}$ the associated eigenvalues. Note that $\lambda_j = 1 + \alpha_j$.

Consider the following increasing sequence $\|\cdot\|_n$, $n \in \mathbb{N}$, of Hilbertian norms:

$$\langle f, g\rangle_n = \sum_{k=1}^{\infty} \langle \mathbb{P}_k f, \mathbb{P}_k g\rangle \lambda_k^{2n} k^{2n},$$

where we denote by $\mathbb{P}_k$ the orthogonal projection on the linear space generated by the eigenvector φ_k.

So,

$$\|f\|_n^2 = \sum_{k=1}^{\infty} \|\mathbb{P}_k f\|^2 \lambda_k^{2n} k^{2n},$$

where $\|\cdot\|$ is the $L^2(\mathbb{T}^d)$ norm.

Consider the Hilbert spaces $\mathcal{S}_n$ which are obtained by completing the space $\mathbb{D}_W$ with respect to the inner product $\langle\cdot,\cdot\rangle_n$.

The set

$$\mathcal{S}_W(\mathbb{T}^d) = \bigcap_{n=0}^{\infty} \mathcal{S}_n$$

endowed with the metric (A.2) is our *countably Hilbert space*, and furthermore, it is a countably Hilbert nuclear space; see the Appendix for further details. In fact, for fixed $n \in \mathbb{N}$ and $m > n + 1/2$, we have that $\{\frac{1}{(j\lambda_j)^m}\varphi_j\}_{j\geq 1}$ is a complete orthonormal set in $\mathcal{S}_m$. Therefore,

$$\sum_{j=1}^{\infty} \left\| \frac{1}{(j\lambda_j)^m}\varphi_j \right\|_n^2 \leq \sum_{j=1}^{\infty} \frac{1}{j^{2(m-n)}} < \infty,$$

where the above formula corresponds to formula (A.3) in the Appendix.

Lemma 3.3. *Let $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ be the operator obtained in Proposition 3.2. We have*

(a) *$\mathcal{L}_W$ is the generator of a strongly continuous contraction semigroup $\{P_t : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)\}_{t\geq 0}$;*
(b) *$\mathcal{L}_W$ is a closed operator;*
(c) *for each $f \in L^2(\mathbb{T}^d)$, $t \mapsto P_t f$ is a continuous function from $[0,\infty)$ to $L^2(\mathbb{T}^d)$;*
(d) *$\mathcal{L}_W P_t f = P_t \mathcal{L}_W f$ for each $f \in \mathcal{D}_W$ and $t \geq 0$;*
(e) *$(\mathbb{I} - \mathcal{L}_W)^n P_t f = P_t(\mathbb{I} - \mathcal{L}_W)^n f$ for each $f \in \mathbb{D}_W$, $t \geq 0$ and $n \in \mathbb{N}$.*

Proof. Item (a) follows from Proposition 3.2 and the Hille–Yosida theorem. Items (b), (c) and (d) follow from item (a); see, for instance, [1, chapter 1]. Item (e) follows from item (d) and from the fact that $\mathcal{L}_W f = \mathbb{L}_W f$ if $f \in \mathbb{D}_W$. □

The next lemma permits us to conclude that the semigroup $\{P_t : t \geq 0\}$ acting on the domain $\mathcal{S}_W(\mathbb{T}^d)$ is a $C_{0,1}$-semigroup, whose definition is recalled in Appendix A.2.

Lemma 3.4. *Let $\{P_t : t \geq 0\}$ be the semigroup whose infinitesimal generator is $\mathcal{L}_W$. Then for each $q \in \mathbb{N}$ we have*

$$\|P_t f\|_q \leq \|f\|_q,$$

for all $f \in \mathcal{S}_W(\mathbb{T}^d)$. In particular, $\{P_t : t \geq 0\}$ is a $C_{0,1}$-semigroup.

Proof. Let $f \in \mathbb{D}_W$; then

$$f = \sum_{j=1}^{k} \beta_j \varphi_j,$$

for some $k \in \mathbb{N}$, and some constants $\beta_1, \ldots, \beta_k$. A simple calculation shows that

$$P_t f = \sum_{j=1}^{k} \beta_j \mathrm{e}^{t(1-\lambda_j)} \varphi_j.$$

Therefore, for $f \in \mathbb{D}_W$,

$$\begin{aligned} \|P_t f\|_n^2 &= \|\sum_{j=1}^{k} \beta_j \mathrm{e}^{t(1-\lambda_j)} \varphi_j\|_n \\ &= \sum_{j=1}^{k} \|\beta_j \mathrm{e}^{t(1-\lambda_j)} \varphi_j\|^2 \lambda_j^{2n} j^{2n} \\ &\leq \sum_{j=1}^{k} \|\beta_j \varphi_j\|^2 \lambda_j^{2n} j^{2n} = \|f\|_n^2. \end{aligned}$$

Since $\mathbb{D}_W$ is dense in $\mathcal{S}_W(\mathbb{T}^d)$, we conclude the proof of the lemma. □

Lemma 3.5. *The operator $\mathcal{L}_W$ belongs to $\mathcal{L}(\mathcal{S}_W(\mathbb{T}^d), \mathcal{S}_W(\mathbb{T}^d))$, the space of linear continuous operators from $\mathcal{S}_W(\mathbb{T}^d)$ into $\mathcal{S}_W(\mathbb{T}^d)$.*

Proof. Let $f \in \mathcal{S}_W(\mathbb{T}^d)$, and $\{\varphi_j\}_{j\geq 1}$ be the complete orthonormal set of eigenvectors of $\mathcal{L}_W$, with $\{(1-\lambda_j)\}_{j\geq 1}$ being their respective eigenvalues. We have that

$$f = \sum_{j=1}^{\infty} \beta_j \varphi_j, \quad \text{with } \sum_{j=1}^{\infty} \beta_j^2 < \infty.$$

We also have that

$$\mathcal{L}_W f = \sum_{j=1}^{\infty} (1-\lambda_j) \beta_j \varphi_j.$$

For every $n \in \mathbb{N}$,

$$\begin{aligned} \|\mathcal{L}_W f\|_n^2 &= \sum_{k=1}^{\infty} \|\mathbb{P}_k(\mathcal{L}_W f)\|^2 \lambda_k^{2n} k^{2n} = \sum_{k=1}^{\infty} \|\beta_k (1-\lambda_k) \varphi_k\|^2 \lambda_k^{2n} k^{2n} \\ &= \sum_{k=1}^{\infty} \|\beta_k \varphi_k\|^2 (1-\lambda_k)^2 \lambda_k^{2n} k^{2n} \\ &\leq 2 \sum_{k=1}^{\infty} \|\mathbb{P}_k f\|^2 \lambda_k^{2n} k^{2n} + 2 \sum_{k=1}^{\infty} \|\mathbb{P}_k f\|^2 \lambda_k^{2(n+1)} k^{2(n+1)} \\ &= 2(\|f\|_n + \|f\|_{n+1}). \end{aligned}$$

Therefore, by the definition of $\mathcal{S}_W(\mathbb{T}^d)$, $\mathcal{L}_W f$ belongs to $\mathcal{S}_W(\mathbb{T}^d)$. Furthermore, $\mathcal{L}_W$ is continuous from $\mathcal{S}_W(\mathbb{T}^d)$ to $\mathcal{S}_W(\mathbb{T}^d)$. □

4. Equilibrium fluctuations

We begin by stating some results on the homogenization of differential operators obtained in [15], which will be very useful throughout this section.

Let $L^2_{x^i \otimes W_i}(\mathbb{T}^d)$ be the space of square integrable functions with respect to the product measure $d(x^i \otimes W_i) = dx_1 \otimes \cdots \otimes dx_{i-1} \otimes dW_i \otimes dx_{i+1} \otimes \cdots \otimes dx_d$, and $H_{1,W}(\mathbb{T}^d)$ be the Sobolev space of functions with W-generalized derivatives. More precisely, $H_{1,W}(\mathbb{T}^d)$ is the space of functions $g \in L^2(\mathbb{T}^d)$ such that for each $i = 1, \ldots, d$ there exist functions $G_i \in L^2_{x^i \otimes W_i, 0}(\mathbb{T}^d)$ satisfying the following integration by parts identity.

$$\int_{\mathbb{T}^d} \big(\partial_{x_i} \partial_{W_i} f\big) g \mathrm{d}x = - \int_{\mathbb{T}^d} (\partial_{W_i} f) G_i d(x^i \otimes W_i), \tag{4.1}$$

for every function $f \in \mathcal{S}_W(\mathbb{T}^d)$, where $L^2_{x^j \otimes W_j, 0}(\mathbb{T}^d)$ is the closed subspace of $L^2_{x^j \otimes W_j}(\mathbb{T}^d)$ consisting of the functions that have zero mean with respect to the measure $d(x^j \otimes W_j)$:

$$\int_{\mathbb{T}^d} f d(x^j \otimes W_j) = 0.$$

We write simply G_i for $\partial_{W_i} g$. See [15] for further details and properties of this space.

Let $\lambda > 0$, f be a functional on $H_{1,W}(\mathbb{T}^d)$, u_N be the unique weak solution of

$$\lambda u_N - \nabla^N A^N \nabla^N_W u_N = f,$$

and u_0 be the unique weak solution of

$$\lambda u_0 - \nabla A \nabla_W u_0 = f. \tag{4.2}$$

For more details on existence and uniqueness of such solutions see [15].

In this context, we say that the diagonal matrix $A = \{a_{jj}\} = \{a_j\}$ is a *homogenization* of the sequence of random matrices A^N, denoted by $A^N \xrightarrow{H} A$, if the following conditions hold:

- u_N converges weakly in $H_{1,W}(\mathbb{T}^d)$ to u_0, when $N \to \infty$;
- $a_i^N \partial^N_{W_i} u^N \to a_i \partial_{W_i} u$ weakly in $L^2_{x^i \otimes W_i}(\mathbb{T}^d)$ when $N \to \infty$.

Theorem 4.1. *Let A^N be a sequence of ergodic random matrices, such as the one that defines our random environment. Then, almost surely, $A^N(\omega)$ admits a homogenization where the homogenized matrix A does not depend on the realization ω.*

The following proposition concerns the convergence of energies:

Proposition 4.2. *Let $A^N \xrightarrow{H} A$ as $N \to \infty$, with u_N being the solution of*

$$\lambda u_N - \nabla^N A^N \nabla^N_W u_N = f,$$

where f is a fixed functional on $H_{1,W}(\mathbb{T}^d)$. Then, the following limit relations hold true:

$$\frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} u_N^2(x) \to \int_{\mathbb{T}^d} u_0^2(x) \mathrm{d}x,$$

and

$$\frac{1}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d} a_{jj}^N(x)(\partial_{W_j}^N u_N(x))^2\left[W_j((x_i+1)/N)-W_j(x_i/N)\right]$$
$$\to \sum_{j=1}^{d}\int_{\mathbb{T}^d} a_{jj}(x)(\partial_{W_j}u_0(x))^2 d(x^j\otimes W_j),$$

as $N\to\infty$.

The proofs of Theorem 4.1 and Proposition 4.2 can be found in [15].

4.1. The martingale problem

We say that $Y_t \in \mathcal{S}'_W(\mathbb{T}^d)$ solves the martingale problem with initial condition Y_0 if for any $G\in\mathcal{S}_W(\mathbb{T}^d)$

$$M_t(G)=Y_t(G)-Y_0(G)-\phi'(\rho)\int_0^t Y_s(\nabla A\nabla_W G)\mathrm{d}s \tag{4.3}$$

is a martingale with quadratic variation

$$\langle M_t(G)\rangle = 2t\chi(\rho)\phi'(\rho)\sum_{j=1}^{d}\int_{\mathbb{T}^d} a_{jj}\left(\partial_{W_j}G\right)^2 d(x^j\otimes W_j). \tag{4.4}$$

Observe that if Y_t is the generalized Ornstein–Uhlenbeck process with characteristics $\phi'(\rho)\nabla A\nabla_W$ and $\sqrt{2\chi(\rho)\phi'(\rho)A}\nabla_W$, then Y_t solves the martingale problem above.

Recall the definition of the density fluctuation field $Y_\cdot^N$ given in (2.8), and denote by Q_N the distribution in $D([0,T],\mathcal{S}_W(\mathbb{T}^d))$ induced by $Y_\cdot^N$, with initial distribution ν_ρ. Our goal is to show that any limit point of $Y_\cdot^N$ solves the martingale problem. To this end, let us introduce the *corrected density fluctuation field*:

$$Y_t^{N,\lambda}(G)=\frac{1}{N^{d/2}}\sum_{x\in\mathbb{T}^d} G_N^\lambda(x)\left[\eta_t(x)-\rho\right],$$

where G_N^λ is the weak solution of the equation

$$\lambda G_N^\lambda - L_N G_N^\lambda = \lambda G - \nabla A\nabla_W G, \tag{4.5}$$

that, via homogenization, converges to G which is the trivial solution of the problem

$$\lambda G-\nabla A\nabla_W G=\lambda G-\nabla A\nabla_W G.$$

The processes $Y_\cdot^N$ and $Y_\cdot^{N,\lambda}$ have the same asymptotic behavior, as we will see. But some calculations are simpler with one of them than with the other. In this way, we have defined two processes in $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$.

Given a process $Y_\cdot$ in $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$, and for $t\ge 0$, let $\mathcal{F}_t$ be the σ-algebra generated by $Y_s(H)$ for $s\le t$ and $H\in\mathcal{S}_W(\mathbb{T}^d)$. Furthermore, set $\mathcal{F}_\infty=\sigma\left(\bigcup_{t\ge 0}\mathcal{F}_t\right)$. Denote by Q_N^λ the distribution on $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$ induced by the corrected density fluctuation field $Y_\cdot^{N,\lambda}$ and initial distribution ν_ρ.

Theorem 2.1 is a consequence of the following result concerning the corrected fluctuation field.

Theorem 4.3. *Let Q be the probability measure on $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$ corresponding to the generalized Ornstein–Uhlenbeck process of mean zero and characteristics $\phi'(\rho)\nabla\cdot A\nabla_W$ and $\sqrt{2\chi(\rho)\phi'(\rho)A}\nabla_W$. Then the sequence $\{Q_N^\lambda\}_{N\geq 1}$ converges weakly to the probability measure Q.*

Note also that the above theorem implies that any limit point of $Y_\cdot^N$ solves the martingale problem (4.3)–(4.4).

Before proving Theorem 4.3, we will state and prove a lemma. This lemma shows that tightness of $Y_t^{N,\lambda}$ follows from tightness of Y_t^N, and furthermore, that they have the same limit points. So we can derive our main theorem from Theorem 4.3.

Lemma 4.4. *For all $t\in[0,T]$ and $G\in\mathcal{S}_W(\mathbb{T}^d)$, $\lim_{N\to\infty} E_{\nu_\rho}\big[Y_t^N(G)-Y_t^{N,\lambda}(G)\big]^2=0$.*

Proof. By the convergence of energies, we have that $\lim_{N\to\infty} G_N^\lambda = G$ in $L_N^2(\mathbb{T}^d)$, i.e.

$$\|G_N^\lambda - G\|_N^2 := \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)]^2 \to 0, \quad \text{as } N\to\infty. \tag{4.6}$$

Since ν_ρ is a product measure we obtain

$$\begin{aligned} E_{\nu_\rho}\Big[Y_t^N(G)-Y_t^{N,\lambda}(G)\Big]^2 &= E_{\nu_\rho}\Bigg[\frac{1}{N^d}\sum_{x,y\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)][G_N^\lambda(y/N) \\ &\qquad - G(y/N)](\eta_t(x)-\rho)(\eta_t(y)-\rho)\Bigg] \\ &= E_{\nu_\rho}\Bigg[\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)]^2(\eta_t(x)-\rho)^2\Bigg] \\ &\leq \frac{C(\rho)}{N^d}\sum_{x\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)]^2, \end{aligned}$$

where $C(\rho)$ is a constant that depends on ρ. By (4.6) the last expression vanishes as $N\to\infty$. □

Proof of Theorem 4.3

Consider the martingale

$$M_t^N(G) = Y_t^N(G) - Y_0^N(G) - \int_0^t N^2 L_N Y_s^N(G)\mathrm{d}s \tag{4.7}$$

associated with the original process and the martingale

$$M_t^{N,\lambda}(G) = Y_t^{N,\lambda}(G) - Y_0^{N,\lambda}(G) - \int_0^t N^2 L_N Y_s^{N,\lambda}(G)\mathrm{d}s \tag{4.8}$$

associated with the corrected process.

A long, albeit simple, computation shows that the quadratic variation of the martingale $M_t^{N,\lambda}(G)$, $\langle M^{N,\lambda}(G)\rangle_t$, is given by

$$\frac{1}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d} a_{jj}^N[\partial_{W_j}^N G_N^\lambda(x/N)]^2[W((x+e_j)/N)-W(x/N)]$$
$$\times\int_0^t c_{x,x+e_j}(\eta_s)\,[\eta_s(x+e_j)-\eta_s(x)]^2\,\mathrm{d}s. \tag{4.9}$$

Is not difficult see that the quadratic variation of the martingale $M_t^N(G)$, $\langle M^N(G)\rangle_t$, has the expression (4.9) with G replacing G_N^λ. Further,

$$\begin{aligned}E_{\nu_\rho}\big[c_{x,x+e_j}(\eta)\,[\eta_s(x+e_j)-\eta_s(x)]^2\big] &= E_{\nu_\rho}[1+b(\eta(x-e_j)+\eta(x))]\\ &\quad\times E_{\nu_\rho}[(\eta(x+e_j)-\eta(x))^2]\\ &= 2(1+2b\rho)\rho(1-\rho)\\ &= 2\phi'(\rho)\chi(\rho).\end{aligned}$$

Lemma 4.5. *Fix $G\in\mathcal{S}_W(\mathbb{T}^d)$ and $t>0$, and let $\langle M^{N,\lambda}(G)\rangle_t$ and $\langle M^N(G)\rangle_t$ be the quadratic variations of the martingales $M_t^{N,\lambda}(G)$ and $M_t^N(G)$, respectively. Then,*

$$\lim_{N\to\infty} E_{\nu_\rho}\big[\langle M^{N,\lambda}(G)\rangle_t-\langle M^N(G)\rangle_t\big]^2=0. \tag{4.10}$$

Proof. Fix $G\in\mathcal{S}_W(\mathbb{T}^d)$ and $t>0$. A straightforward calculation shows that

$$E_{\nu_\rho}\big[\langle M^{N,\lambda}(G)\rangle_t-\langle M^N(G)\rangle_t\big]^2 \le \Bigg[k^2t^2\frac{1}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d}a_{jj}^N\Big[\big(\partial_{W_j}^N G_N^\lambda(x/N)\big)^2 - \big(\partial_{W_j}^N G(x/N)\big)^2\Big][W((x+e_j)/N)-W(x/N)]\Bigg]^2,$$

where the constant k comes from the integral term. By the convergence of energies (Proposition 4.2), the last term vanishes as $N\to\infty$. □

Lemma 4.6. *Let $G\in\mathcal{S}_W(\mathbb{T}^d)$ and $d\ge 1$. Then*

$$\lim_{N\to\infty}E_{\nu_\rho}\Bigg[\frac{1}{N^{d-1}}\int_0^t \mathrm{d}s\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d}a_{jj}^N\big(\partial_{W_j}^N G(x/N)\big)^2[W((x+e_j)/N)-W(x/N)]\times\big[c_{x,x+e_j}(\eta_s)\,[\eta_s(x+e_j)-\eta_s(x)]^2-2\chi(\rho)\phi'(\rho)\big]\Bigg]^2=0.$$

Proof. Fix $G\in\mathcal{S}_W(\mathbb{T}^d)$ and $d>1$. The term in the previous expression is less than or equal to

$$\frac{t^2\theta^4C(\rho)}{N^{d-1}}\|\nabla_W^N G\|_{W,N,4}^4, \tag{4.11}$$

where

$$\|\nabla_W^N G\|_{W,N,4}^4 := \frac{1}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d}\big(\partial_{W_j}^N G(x/N)\big)^4[W((x+e_j)/N)-W(x/N)].$$

Thus, since for $G \in \mathcal{S}_W(\mathbb{T}^d)$, $\|\nabla_W^N G\|_{W,N,4}^4$ is bounded, the term in (4.11) converges to zero as $N \to \infty$.

The case $d = 1$ follows from calculations similar to the ones found in Lemma 12 of [8]. $\square$

So, by Lemmas 4.5 and 4.6, $\langle M^{N,\lambda}(G)\rangle_t$ is given by

$$\frac{2t\chi(\rho)\phi'(\rho)}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d} a_{jj}^N\big(\partial_{W_j}^N G_N^\lambda(x/N)\big)^2[W((x+e_j)/N)-W(x/N)]$$

plus a term that vanishes in $L^2_{\nu_\rho}(\mathbb{T}^d)$ as $N \to \infty$. By the convergence of energies, Proposition 4.2, it converges, as $N \to \infty$, to

$$2t\chi(\rho)\phi'(\rho)\sum_{j=1}^{d}\int_{\mathbb{T}^d} a_{jj}^N\big(\partial_{W_j}G(x)\big)^2 \mathrm{d}x^j \otimes W_j.$$

Our goal now consists in showing that it is possible to write the integral part of the martingale as the integral of a function of the density fluctuation field plus a term that goes to zero in $L^2_{\nu_\rho}(\mathbb{T}^d)$. After some simple computations, we obtain that

$$\begin{aligned} N^2 L_N Y_s^{N,\lambda}(G) = \sum_{j=1}^{d}\Bigg\{ &\frac{1}{N^{d/2}}\sum_{x\in T_N^d}\mathbb{L}_N^j G_N^\lambda(x/N)\,\eta_s(x) \\ &+\frac{b}{N^{d/2}}\sum_{x\in T_N^d}\big[\mathbb{L}_N^j G_N^\lambda((x+e_j)/N)+\mathbb{L}_N^j G_N^\lambda(x/N)\big]\,(\tau_x h_{1,j})(\eta_s) \\ &-\frac{b}{N^{d/2}}\sum_{x\in T_N^d}\mathbb{L}_N^j G_N^\lambda(x/N)(\tau_x h_{2,j})(\eta_s)\Bigg\}, \end{aligned}$$

where $\{\tau_x : x \in \mathbb{Z}^d\}$ is the group of translations such that $(\tau_x\eta)(y) = \eta(x+y)$ for x, y in $\mathbb{Z}^d$, and the sum is understood modulo N. Also, $h_{1,j}, h_{2,j}$ are the cylinder functions

$$h_{1,j}(\eta) = \eta(0)\eta(e_j), \qquad h_{2,j}(\eta) = \eta(-e_j)\eta(e_j).$$

Note that inside the expression $N^2 L_N Y_s^{N,\lambda}$ we may replace $\mathbb{L}_N^j G_N^\lambda$ by $a_j\partial_{x_j}\partial_{W_j}G$. Indeed, the expression

$$\begin{aligned} E_{\nu(\rho)}\Bigg\{\int_0^t \sum_{j=1}^{d}\frac{1}{N^{d/2}}&\sum_{x\in\mathbb{T}_N^d}\Big[\mathbb{L}_N^j G_N^\lambda(x/N)-a_j\partial_{x_j}\partial_{W_j}G(x/N)\Big]\big(\eta_s(x)-\rho\big) \\ &+\frac{b}{N^{d/2}}\sum_{x\in\mathbb{T}_N^d}\Big[\mathbb{L}_N^j G_N^\lambda((x+e_j)/N)-a_j\partial_{x_j}\partial_{W_j}G((x+e_j)/N) \\ &+\mathbb{L}_N^j G_N^\lambda(x/N)-a_j\partial_{x_j}\partial_{W_j}G(x/N)\Big]\big((\tau_x h_{1,j})(\eta_s)-\rho^2\big) \\ &-\frac{b}{N^{d/2}}\sum_{x\in\mathbb{T}_N^d}\Big[\mathbb{L}_N^j G_N^\lambda(x/N)-a_j\partial_{x_j}\partial_{W_j}G(x/N)\Big]\big((\tau_x h_{2,j})(\eta_s)-\rho^2\big)\Bigg\}^2 \end{aligned}$$

is less than or equal to

$$C(\rho, b)\int_0^t \frac{1}{N^d}\sum_{x\in\mathbb{T}^d}\big[L_N G_N^\lambda(x/N) - \nabla A\nabla_W G(x/N)\big]^2.$$

Now, recall that G_N^λ is a solution of Eq. (4.5), and therefore, the previous expression is less than or equal to

$$\frac{tC(\rho, b)}{\lambda^2}\|G_N^\lambda - G\|_N^2,$$

and thus, by homogenization and energy estimates in Theorem 4.1 and Proposition 4.2, respectively, the last expression converges to zero as $N\to\infty$.

By the Boltzmann Gibbs principle, Theorem 6.1 below, we can replace $(\tau_x h_{i,j})(\eta_s) - \rho^2$ by $2\rho[\eta_s(x) - \rho]$ for $i = 1, 2$. Doing this, the martingale (4.8) can be written as

$$M_t^{N,\lambda}(G) = Y_t^{N,\lambda}(G) - Y_0^{N,\lambda}(G) - \int_0^t \frac{1}{N^{d/2}}\sum_{x\in\mathbb{T}^d}\nabla A\nabla_W G(x/N)\phi'(\rho)\big(\eta_s - \rho\big)\mathrm{d}s, \tag{4.12}$$

plus a term that vanishes in $L^2_{\nu_\rho}(\mathbb{T}^d)$ as $N\to\infty$.

Notice that, by (2.8), the integrand in the previous expression is a function of the density fluctuation field Y_t^N. By Lemma 4.4, we can replace the term inside the integral of the above expression by a term which is a function of the corrected density fluctuation field $Y_t^{N,\lambda}$.

From the results of Section 5, the sequence $\{Q_N^\lambda\}_{N\geq 1}$ is tight and we let Q^λ be a limit point of it. Let Y_t be the process in $D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$ induced by the canonical projections under Q^λ. Taking the limit as $N\to\infty$, under an appropriate subsequence, in expression (4.12), we obtain that

$$M_t^\lambda(G) = Y_t(G) - Y_0(G) - \int_0^t Y_s(\phi'(\rho)\nabla\cdot A\nabla_W G)\mathrm{d}s, \tag{4.13}$$

where M_t^λ is some $\mathcal{S}'_W(\mathbb{T}^d)$-valued process, in fact, a martingale. To see this, note that for a measurable set U with respect to the canonical σ-algebra $\mathcal{F}_t$, $E_{Q_N^\lambda}[M_t^{N,\lambda}(G)\mathbf{1}_U]$ converges to $E_{Q^\lambda}[M_t^\lambda(G)\mathbf{1}_U]$. Since $M_\cdot^{N,\lambda}(G)$ is a martingale, $E_{Q_N^\lambda}[M_T^{N,\lambda}(G)\mathbf{1}_U] = E_{Q_N^\lambda}[M_t^{N,\lambda}(G)\mathbf{1}_U]$. Taking a further subsequence if necessary, this last term converges to $E_{Q^\lambda}[M_t^\lambda(G)\mathbf{1}_U]$, which proves that $M_\cdot^\lambda(G)$ is a martingale for any $G\in\mathcal{S}_W(\mathbb{T}^d)$. Since all the projections of M_t^λ are martingales, we conclude that M_t^λ is an $\mathcal{S}'_W(\mathbb{T}^d)$-valued martingale.

Now, we need to obtain the quadratic variation $\langle M^\lambda(G)\rangle_t$ of the martingale $M_t^\lambda(G)$. A simple application of Tchebyshev's inequality shows that $\langle M^{N,\lambda}(G)\rangle_t$ converges in probability to

$$2t\chi(\rho)\phi'(\rho)\sum_{j=1}^d\int_{\mathbb{T}^d} a_j\Big[\partial_{W_j}G\Big]^2 d(x^j\otimes W_j),$$

where $\chi(\rho)$ stands for the static compressibility given by $\chi(\rho) = \rho(1-\rho)$. By the Doob–Meyer decomposition theorem, we need to prove that

$$N_t^\lambda(G) := M_t^\lambda(G)^2 - 2t\chi(\rho)\phi'(\rho)\sum_{j=1}^d\int_{\mathbb{T}^d} a_j\Big[\partial_{W_j}G\Big]^2 d(x^j\otimes W_j)$$

is a martingale. The same argument as we used above applies now if we can show that $\sup_N E_{Q_N^\lambda}[M_T^{N,\lambda}(G)^4] < \infty$ and $\sup_N E_{Q_N^\lambda}[\langle M^{N,\lambda}(G)\rangle_T^2] < \infty$. Both bounds follow easily from the explicit form of $\langle M^{N,\lambda}(G)\rangle_t$ and (4.12).

On the other hand, by a standard central limit theorem, Y_0 is a Gaussian field with covariance

$$E\big[Y_0(G)Y_0(H)\big] = \chi(\rho)\int_{\mathbb{T}^d} G(x)H(x)\mathrm{d}x.$$

Therefore, by Theorem 4.7, Q^λ is equal to the probability distribution Q of a generalized Ornstein–Uhlenbeck process in $D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$ (and it does not depend on λ). By existence and uniqueness of the generalized Ornstein–Uhlenbeck processes (also due to Theorem 4.7), the sequence $\{Q_N^\lambda\}_{N\geq 1}$ has at most one limit point, and from tightness, it does have a unique limit point. This concludes the proof of Theorem 4.3.

4.2. *Generalized Ornstein–Uhlenbeck processes*

In this subsection we show that the generalized Ornstein–Uhlenbeck process obtained as the solution to the martingale problem in which we are interested is also an $\mathcal{S}'_W(\mathbb{T}^d)$-solution of a stochastic differential equation, and then we apply the theory in the Appendix to conclude that there is at most one solution of the martingale problem. Moreover, we also conclude that this process is a Gaussian process.

Theorem 4.7. *Let Y_0 be a Gaussian field on $\mathcal{S}'_W(\mathbb{T}^d)$. Then the unique $\mathcal{S}'_W(\mathbb{T}^d)$-solution, Y_t, of the stochastic differential equation*

$$\mathrm{d}Y_t = \phi'(\rho)\nabla A\nabla_W Y_t \mathrm{d}t + \sqrt{2\chi(\rho)\phi'(\rho)A}\,\mathrm{d}N_t, \tag{4.14}$$

solves the martingale problem (4.3)–(4.4) with initial condition Y_0, where N_t is a mean-zero $\mathcal{S}'_W(\mathbb{T}^d)$-valued martingale with quadratic variation given by

$$\langle N(G)\rangle_t = t\sum_{j=1}^d \int_{\mathbb{T}^d} \big[\partial_{W_j} G\big]^2 d(x^j\otimes W_j).$$

Moreover, Y_t is a Gaussian process.

Proof. In view of the definition of solutions of stochastic differential equations (see Appendix), Y_t is an $\mathcal{S}'_W(\mathbb{T}^d)$-solution of (4.14). In fact, by hypothesis, Y_t satisfies the integral identity (4.3), and is also an additive functional of a Markov process.

We now check the conditions in Proposition A.1 to ensure the uniqueness of $\mathcal{S}'_W(\mathbb{T}^d)$-solutions of (4.14). Since by hypothesis Y_0 is a Gaussian field, condition 1 is satisfied, and since the martingale M_t has quadratic variation given by (4.4), we use Remark A.2 to conclude that condition 2 holds. Condition 3 follows from Lemmas 3.4 and 3.5. Therefore Y_t is unique.

Finally, by Blumenthal's 0–1 law for Markov processes, M_t and Y_0 are independent, since for measurable sets A and B,

$$\begin{aligned} P(Y_0\in A, M_t\in B) &= E(\mathbf{1}_{Y_0\in A}\mathbf{1}_{M_t\in B}) = E[E(\mathbf{1}_{Y_0\in A}\mathbf{1}_{M_t\in B}|\mathcal{F}_{0+})]\\ &= E[\mathbf{1}_{Y_0\in A}E(\mathbf{1}_{M_t\in B}|\mathcal{F}_{0+})]\\ &= E[\mathbf{1}_{Y_0\in A}P(M_t\in B)] = P(Y_0\in A)P(M_t\in B). \end{aligned}$$

Applying Lévy's martingale characterization of Brownian motions, the quadratic variation of M_t, given by (4.4), yields that M_t is a time deformation of a Brownian motion. Therefore, M_t is a Gaussian process with independent increments. Since Y_0 is a Gaussian field, we apply Proposition A.3 to conclude that Y_t is a Gaussian process in $D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$. □

5. Tightness

In this section we prove tightness of the density fluctuation field $\{Y^N_\cdot\}_N$ introduced in Section 2. We begin by stating Mitoma's criterion [12]:

Proposition 5.1. *Let Φ_∞ be a nuclear Fréchet space and Φ'_∞ its topological dual. Let $\{Q^N\}_N$ be a sequence of distributions in $D([0,T], \Phi'_\infty)$, and for a given function $G \in \Phi_\infty$, let $Q^{N,G}$ be the distribution in $D([0,T],\mathbb{R})$ defined by $Q^{N,G}[y \in D([0,T],\mathbb{R}); y(\cdot) \in A] = Q^N\left[Y \in D([0,T], \Phi'_\infty); Y(\cdot)(G) \in A\right]$. Therefore, the sequence $\{Q^N\}_N$ is tight if and only if $\{Q^{N,G}\}_N$ is tight for any $G \in \Phi_\infty$.*

From Mitoma's criterion, $\{Y^N_\cdot\}_N$ is tight if and only if $\{Y^N_\cdot(G)\}_N$ is tight for any $G \in \mathcal{S}_W(\mathbb{T}^d)$, since $\mathcal{S}_W(\mathbb{T}^d)$ is a nuclear Fréchet space. By Dynkin's formula and after some manipulations, we see that

$$\begin{aligned} Y^N_t(G) = Y^N_0(G) \int_0^t \sum_{j=1}^d \Bigg\{ & \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}^d_N} \mathbb{L}^j_N G_N(x/N)\,\eta_s(x) \\ & + \frac{b}{N^{d/2}} \sum_{x\in\mathbb{T}^d_N} \left[\mathbb{L}^j_N G_N((x+e_j)/N) + \mathbb{L}^j_N G_N(x/N)\right](\tau_x h_{1,j})(\eta_s) \\ & - \frac{b}{N^{d/2}} \sum_{x\in\mathbb{T}^d_N} \mathbb{L}^j_N G_N(x/N)(\tau_x h_{2,j})(\eta_s) \Bigg\} \mathrm{d}s + M^N_t(G), \end{aligned} \tag{5.1}$$

where $M^N_t(G)$ is a martingale of quadratic variation:

$$\begin{aligned} \langle M^N(G)\rangle_t = & \frac{1}{N^{d-1}} \sum_{j=1}^d \sum_{x\in\mathbb{T}^d} a^N_{jj} [\partial^N_{W_j} G_N(x/N)]^2 [W((x+e_j)/N) - W(x/N)] \\ & \times \int_0^t c_{x,x+e_j}(\eta_s)\,[\eta_s(x+e_j) - \eta_s(x)]^2\,\mathrm{d}s. \end{aligned}$$

In order to prove tightness for the sequence $\{Y^N_\cdot(G)\}_N$, it is enough to prove tightness for $\{Y^N_0(G)\}_N$, $\{M^N_\cdot(G)\}_N$ and the integral term in (5.1). The easiest one is the initial condition: from the usual central limit theorem, $Y^N_0(G)$ converges to a normal random variable of mean zero and variance $\chi(\rho)\int G(x)^2 \mathrm{d}x$, where $\chi(\rho) = \rho(1-\rho)$. For the other two terms, we use *Aldous's criterion*:

Proposition 5.2 (*Aldous's Criterion*). *A sequence of distributions $\{P^N\}$ in the path space $D([0,T],\mathbb{R})$ is tight if:*

(i) *For any $t \in [0,T]$ the sequence $\{P^N_t\}$ of distributions in $\mathbb{R}$ defined by $P^N_t(A) = P^N[y \in D([0,T],\mathbb{R}) : y(t) \in A]$ is tight.*

(ii) *For any* $\epsilon > 0$,

$$\lim_{\delta > 0} \limsup_{n \to \infty} \sup_{\substack{\tau \in \Upsilon_T \\ \theta \le \delta}} P^N\big[y \in D([0,T],\mathbb{R}) : |y(\tau+\theta) - y(\tau)| > \epsilon\big] = 0,$$

where Υ_T *is the set of stopping times bounded by* T *and* $y(\tau+\theta) = y(T)$ *if* $\tau + \theta > T$.

Now we prove tightness of the martingale term. By the optional sampling theorem, we have

$$\begin{aligned}
&Q_N\left[\left|M^N_{\tau+\theta}(G) - M^N_\tau(G)\right| > \epsilon\right] \le \frac{1}{\epsilon^2} E_{Q_N}\left[\left\langle M^N_{\tau+\theta}(G)\right\rangle - \left\langle M^N_\tau(G)\right\rangle\right] \\
&= \frac{1}{\epsilon^2}\left[\left\langle M^N_{\tau+\theta}(G)\right\rangle - \left\langle M^N_\tau(G)\right\rangle\right] \\
&= \frac{1}{\epsilon^2 N^{d-1}} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d_N} a_{jj}(x)[\partial^N_{W_j} G(x/N)]^2 [W((x+e_j)/N) - W(x)] \\
&\quad \times \int_t^{t+\delta} c_{x,x+e_j}(\eta_s)[\eta_s(x+e_j) - \eta_s(x)]^2 \mathrm{d}s \\
&\le \frac{\delta}{\epsilon^2}(1+2|b|)\theta \frac{1}{N^{d-1}} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d_N} [\partial^N_{W_j} G(x/N)]^2 [W((x+e_j)/N) - W(x)] \\
&\le \frac{\delta}{\epsilon^2}(1+2|b|)\theta(\|\nabla_W G\|^2_W + \delta),
\end{aligned} \tag{5.2}$$

for N sufficiently large, since the rightmost term on (5.2) converges to $\|\nabla_W G\|^2_W$, as $N \to \infty$, where

$$\|\nabla_W G\|^2_W = \sum_{i=1}^d \int_{\mathbb{T}^d} \left(\partial_{W_i} f\right)^2 d(x^i \otimes W_i).$$

Therefore, the martingale $M^N_t(G)$ satisfies the conditions of Aldous's criterion. The integral term can be handled in a similar way:

$$\begin{aligned}
&E_{Q_N}\Big[\Big(\int_\tau^{\tau+\delta} \frac{1}{N^{d/2}} \sum_{j=1}^d \sum_x \mathbb{L}^j_N G(x/N)(\eta_t - \rho) + b[\mathbb{L}^j_N G((x+e_j)/N) \\
&\quad + \mathbb{L}^j_N G(x/N)](\tau_x h_1 - \rho^2) - b\mathbb{L}^j_N G(x/N)(\tau_x h_2 - \rho^2)\Big)^2 \mathrm{d}t\Big] \\
&\le \delta C(b) \frac{1}{N^d} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d_N} \left(\mathbb{L}^j_N G(x/N)\right)^2 \\
&\le \delta C(G,b),
\end{aligned}$$

where $C(b)$ is a constant that depends on b, and $C(G,b)$ is a constant that depends on $C(b)$ and on the function $G \in \mathcal{S}_W(\mathbb{T}^d)$. Therefore, we conclude, by Mitoma's criterion, that the sequence $\{Y^N_\cdot\}_N$ is tight. Thus, the sequence of $\mathcal{S}'_W(\mathbb{T}^d)$-valued martingales $\{M^N_\cdot\}_N$ is also tight.

6. The Boltzmann–Gibbs principle

We show in this section that the martingales $M^N_t(G)$ introduced in Section 4 can be expressed in terms of the fluctuation field Y^N_t. This replacement of the cylinder function $(\tau_x h_{i,j})(\eta_s) - \rho^2$

by $2\rho[\eta_s(x) - \rho]$ for $i = 1, 2$ constitutes one of the main steps toward the proof of equilibrium fluctuations.

Recall that $(\Omega, \mathcal{F}, \mu)$ is a standard probability space where we consider the vector-valued $\mathcal{F}$-measurable functions $\{a_j(\omega); j = \cdots, d\}$ that form our random environment (see Sections 2 and 4 for more details).

Take a function $f : \Omega \times \{0, 1\}^{\mathbb{T}_N^d} \to \mathbb{R}$. Fix a realization $\omega \in \Omega$, let $x \in \mathbb{T}_N^d$, and define

$$f(x, \eta) = f(x, \eta, \omega) =: f(T_{Nx}\omega, \tau_x \eta),$$

where $\tau_x \eta$ is the shift of η to x: $\tau_x \eta(y) = \eta(x + y)$.

We say that f is local if there exists $R > 0$ such that $f(\omega, \eta)$ depends only on the values of $\eta(y)$ for $|y| \leq R$. For this case, we can consider f as defined in all of the space $\Omega \times \{0, 1\}^{\mathbb{T}_N^d}$ for $N \geq R$.

We say that f is Lipschitz if there exists $c = c(\omega) > 0$ such that for all x, $|f(\omega, \eta) - f(\omega, \eta')| \leq c|\eta(x) - \eta'(x)|$ for any $\eta, \eta' \in \{0, 1\}^{\mathbb{T}_N^d}$ such that $\eta(y) = \eta'(y)$ for any $y \neq x$. If the constant c can be chosen independently of ω, we say that f is uniformly Lipschitz.

Theorem 6.1 (*Boltzmann–Gibbs Principle*)**.** *For every $G \in \mathcal{S}_W(\mathbb{T}^d)$, every $t > 0$ and every local, uniformly Lipschitz function $f : \Omega \times \{0, 1\}^{\mathbb{T}_N^d} \to \mathbb{R}$, it holds that*

$$\lim_{N\to\infty} E_{\nu_\rho}\left[\int_0^t \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} G(x) V_f(x, \eta_s) \mathrm{d}s\right]^2 = 0, \tag{6.1}$$

where

$$V_f(x, \eta) = f(x, \eta) - E_{\nu_\rho}\big[f(x, \eta)\big] - \partial_\rho E\left[\int f(x, \eta) \mathrm{d}\nu_\rho(\eta)\right]\big(\eta(x) - \rho\big).$$

Here, E denotes the expectation with respect to μ, the random environment.

Let $f : \Omega \times \{0, 1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ be a local, uniformly Lipschitz function and take $f(x, \eta) = f(\theta_{Nx}\omega, \tau_x \eta)$. Fix a function $G \in \mathcal{S}_W(\mathbb{T}^d)$ and an integer K that will increase to ∞ after N. For each N, we subdivide $\mathbb{T}_N^d$ into non-overlapping boxes of linear size K. Denote them by $\{B_i, 1 \leq i \leq M^d\}$, where $M = [\frac{N}{K}]$. More precisely,

$$B_i = y_i + \{1, \ldots, K\}^d,$$

where $y_i \in \mathbb{T}_N^d$, and $B_i \cap B_r = \emptyset$ if $i \neq r$. We assume that the points y_i have the same relative position on the boxes.

Let B_0 be the set of points that are not included in any B_i; then $|B_0| \leq dKN^{d-1}$. If we restrict the sum in the expression that appears inside the integral in (6.1) to the set B_0, then its $L^2_{\nu_\rho}(\mathbb{T}^d)$-norm clearly vanishes as $N \to +\infty$, since f is local, ν_ρ is an invariant product measure, and V_f has mean zero with respect to ν_ρ.

Let Λ_{s_f} be the smallest cube centered at the origin that contains the support of f and define s_f as the radius of Λ_{s_f}. Denote by B_i^0 the interior of the box B_i, namely the sites x in B_i that are at a distance at least $s_f + 2$ from the boundary:

$$B_i^0 = \{x \in B_i, d(x, \mathbb{T}_N^d \setminus B_i) > s_f + 2\}.$$

Denote also by B^c the set of points that are not included in any B_i^0. By construction, it is easy to see that $|B^c| \le dN^d(\frac{c(f)}{K} + \frac{K}{N})$, where $c(f)$ is a constant that depends on f.

We have that for continuous $H : \mathbb{T}^d \to \mathbb{R}$,

$$\frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} H(x)V_f(x,\eta_t) = \frac{1}{N^{d/2}} \sum_{x\in B^c} H(x)V_f(x,\eta_t)$$
$$+\frac{1}{N^{d/2}} \sum_{i=1}^{M^d} \sum_{x\in B_i^0} \Big[H(x) - H(y_i)\Big] V_f(x,\eta_t) + \frac{1}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \sum_{x\in B_i^0} V_f(x,\eta_t).$$

Note that we may take H continuous, since the continuous functions are dense in $L^2(\mathbb{T}^d)$. The first step is to prove that

$$\lim_{K\to\infty} \lim_{N\to\infty} E_{\nu_\rho}\Big[\int_o^t \frac{1}{N^{d/2}} \sum_{x\in B^c} H(x)V_f(x,\eta_t)\mathrm{d}s\Big]^2 = 0.$$

As ν_ρ is an invariant product measure and V_f has mean zero with respect to the measure ν_ρ, the last expectation is bounded above by

$$\frac{t^2}{N^d} \sum_{\substack{x,y\in B^c\\|x-y|\le 2s_f}} H(x)H(y)E_{\nu_\rho}\big[V_f(x,\eta)V_f(y,\eta)\big].$$

Since V_f belongs to $L^2_{\nu_\rho}(\mathbb{T}^d)$ and $|B^c| \le dN^d(\frac{c(f)}{K} + \frac{K}{N})$, the last expression vanishes by taking first $N \to +\infty$ and then $K \to +\infty$.

From the continuity of H, and applying similar arguments, one may show that

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\rho}\Big[\int_0^t \frac{1}{N^{d/2}} \sum_{i=1}^{M^d} \sum_{x\in B_i^0} \big[H(x) - H(y_i)\big] V_f(x,\eta_t)\mathrm{d}s\Big]^2 = 0.$$

In order to conclude the proof it remains to be shown that

$$\lim_{K\to\infty} \lim_{N\to\infty} E_{\nu_\rho}\Big[\int_0^t \frac{1}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \sum_{x\in B_i^0} V_f(x,\eta_t)\mathrm{d}s\Big]^2 = 0. \tag{6.2}$$

To this end, recall proposition A 1.6.1 of [11]:

$$E_{\nu_\rho}\left[\int_0^t V(\eta_s)\mathrm{d}s\right] \le 20\theta t \|V\|_{-1}^2, \tag{6.3}$$

where $\|\cdot\|_{-1}$ is given by

$$\|V\|_{-1}^2 = \sup_{F\in L^2(\nu_\rho)} \left\{2\int V(\eta)F(\eta)\mathrm{d}\nu_\rho - \langle F, L_N F\rangle_\rho\right\},$$

and $\langle\cdot,\cdot\rangle_\rho$ denotes the inner product in $L^2(\nu_\rho)$.

Let $\tilde{L}_N$ be the generator of the exclusion process without the random environment, and without the conductances (that is, taking $a(\omega) \equiv 1$, and $W_j(x_j) = x_j$, for $j = 1, \ldots, d$, in (2.5)), and also without the diffusive scaling N^2:

$$\tilde{L}_N g(\eta) = \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} c_{x,x+e_j}(\eta)\big[g(\eta^{x,x+e_j}) - g(\eta)\big],$$

for cylindrical functions g on the configuration space $\{0,1\}^{\mathbb{T}_N^d}$.

For each $i = 1, \ldots, M^d$ denote by ζ_i the configuration $\{\eta(x), x \in B_i\}$ and by $\tilde{L}_{B_i}$ the restriction of the generator $\tilde{L}_N$ to the box B_i, namely,

$$\tilde{L}_{B_i} h(\eta) = \sum_{\substack{x,y \in B_i \\ |x-y|=1/N}} c_{x,y}(\eta)\big[h(\eta^{x,y}) - h(\eta)\big].$$

We would like to emphasize that we introduced the generator $\tilde{L}_N$ because it is translation invariant.

Now we introduce some notation. Let $L^2(P \otimes \nu_\rho)$ be the set of measurable functions g such that $E[\int g(\omega,\eta)^2 \mathrm{d}\nu_\rho] < \infty$. Fix a local function $h : \Omega \times \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ in $L^2(P \otimes \nu_\rho)$, measurable with respect to $\sigma(\eta(x), x \in B_1)$, and let h_i be the translation of h by $y_i - y_1$: $h_i(x,\eta) = h(\theta_{(y_i-y_1)N}\omega, \tau_{y_i-y_1}\eta)$. Consider

$$V_{H,h}^N(\eta) = \frac{1}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \tilde{L}_{B_i} h_i(\zeta_i).$$

The strategy of the proof (6.2) is the following: we show that $V_{H,h}^N$ vanishes in some sense as $N \to \infty$, and then, that the difference between V_f and $V_{H,h}^N$ also vanishes, as $N \to \infty$. The result follows a simple triangle inequality. The first part is done by obtaining estimates on boxes, whereas the second part mainly considers the projections of V_f on some appropriate Hilbert spaces, plus ergodicity of the environment.

Let

$$L_{W,B_i} h(\eta) = \sum_{j=1}^{d} \sum_{x \in B_i} c_{x,x+e_j}(\eta) \frac{N a_j(x)}{W(x+e_j) - W(x)} [h(\eta^{x,x+e_j}) - h(\eta)].$$

Note that the following estimate holds:

$$\sum_{i=1}^{M^d} \langle h, -L_{W,B_i} h \rangle_\rho \leq \langle h, -L_N h \rangle_\rho.$$

Furthermore,

$$\langle f, -\tilde{L}_{B_i} h \rangle \leq \max_{1 \leq k \leq d} \frac{\{W_k(1) - W_k(0)\}}{N} \theta \langle h, -L_{W,B_i} h \rangle_\rho.$$

Using the Cauchy–Schwartz inequality, we have, for each i,

$$\langle \tilde{L}_{B_i} h_i, F \rangle_\rho \leq \frac{1}{2\gamma_i} \langle -\tilde{L}_{B_i} h_i, h_i \rangle_\rho + \frac{\gamma_i}{2} \langle F, -\tilde{L}_{B_i} F \rangle_\rho,$$

where γ_i is a positive constant.

Therefore,

$$2 \int V_{H,h}^N(\eta) F(\eta) \mathrm{d}\nu_\rho \leq \frac{2}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \left[\frac{1}{2\gamma_i} \langle -\tilde{L}_{B_i} h_i, h_i \rangle_\rho + \frac{\gamma_i}{2} \langle F, -\tilde{L}_{B_i} F \rangle_\rho \right]. \quad (6.4)$$

Choose

$$\gamma_i = \frac{N^{1+d/2}}{\theta \max_{1\le k\le d}\{W_k(1)-W_k(0)\}|H(y_i)|},$$

and observe that the generator L_N is already speeded up by the factor N^2. We, thus, obtain

$$\frac{2}{N^{d/2}}\sum_{i=1}^{M^d} H(y_i)\frac{\gamma_i}{2}\langle F, -\tilde{L}_{B_i}F\rangle_\rho \le \langle F, -\tilde{L}_N F\rangle_\rho.$$

The above bound and (6.4) allow us to use inequality (6.2) on $V^N_{H,h}$, with the generator L_{W,B_i}. Therefore, we have that the expectation in (6.3) with $V^N_{H,h}$ is bounded above by

$$\frac{20\theta t}{N^{d/2}}\sum_{i=1}^{M^d}\frac{|H(y_i)|}{\gamma_i}\langle -\tilde{L}_{B_i}h_i, h_i\rangle_\rho,$$

which in turn is less than or equal to

$$\frac{20t\|H\|_\infty M^d\theta^2}{N^{d+1}\max_{1\le k\le d}\{W_k(1)-W_k(0)\}}\sum_{i=1}^{M^d}\frac{1}{M^d}\langle -\tilde{L}_{B_i}h_i, h_i\rangle_\rho.$$

By Birkhoff's ergodic theorem, the sum in the previous expression converges to a finite value as $N\to\infty$. Therefore, this whole expression vanishes as $N\to\infty$. This concludes the first part of the strategy of the proof.

To conclude the proof of the theorem it is enough to show that

$$\lim_{K\to\infty}\inf_{h\in L^2(\nu_\rho\otimes P)}\lim_{N\to\infty} E_{\nu_\rho}\left[\int_0^t \frac{1}{N^{d/2}}\sum_{i=1}^{M^d} H(y_i)\Big\{\sum_{x\in B_i^0} V_f(x,\eta_s) - \tilde{L}_{B_i}h_i(\zeta_i(s))\Big\}\right]^2 = 0.$$

To this end, observe that the expectation in the previous expression is bounded by

$$\frac{t^2}{N^d}\sum_{i=1}^{M^d}\|H\|_\infty^2 E_{\nu_\rho}\Big(\sum_{x\in B_i^0} V_f(x,\eta) - \tilde{L}_{B_i}h_i(\zeta_i)\Big)^2,$$

because the measure ν_ρ is invariant under the dynamics and the supports of $V_f(x,\eta)-\tilde{L}_{B_i}h_i(\zeta_i)$ and $V_f(y,\eta)-\tilde{L}_{B_r}h_r(\zeta_r)$ are disjoint for $x\in B_i^0$ and $y\in B_r^0$, with $i\ne r$.

By the ergodic theorem, as $N\to\infty$, this expression converges to

$$\frac{t^2}{K^d}\|H\|_\infty^2 E\left[\int\Big(\sum_{x\in B_1^0} V_f(x,\eta) - \tilde{L}_{B_1}h(\omega,\eta)\Big)^2 \mathrm{d}\nu_\rho\right]. \tag{6.5}$$

So, it remains to be shown that

$$\lim_{K\to\infty}\frac{t^2}{K^d}\|H\|_\infty^2 \inf_{h\in L^2(\nu_\rho\otimes P)} E\Big[\int\Big(\sum_{x\in B_1^0} V_f(x,\eta) - \tilde{L}_{B_1}h(\omega,\eta)\Big)^2 \mathrm{d}\nu_\rho\Big] = 0.$$

Denote by $R(\tilde{L}_{B_1})$ the range of the generator $\tilde{L}_{B_1}$ in $L^2(\nu_\rho \otimes P)$ and by $R(\tilde{L}_{B_1})^\perp$ the space orthogonal to $R(\tilde{L}_{B_1})$. The infimum of (6.5) over all $h \in L^2(\nu_\rho \otimes P)$ is equal to the projection of $\sum_{x \in B_1^0} V_f(x,\eta)$ into $R(\tilde{L}_{B_1})^\perp$.

The set $R(\tilde{L}_{B_1})^\perp$ is the space of functions that depend on η only through the total number of particles on the box B_1. So, the previous expression is equal to

$$\lim_{K\to\infty} \frac{t^2\|H\|_\infty^2}{K^d} E\Big[\int \Big(E_{\nu_\rho}\Big[\sum_{x\in B_1^0} V_f(x,\eta)|\eta^{B_1}\Big]\Big)^2 \mathrm{d}\nu_\rho\Big], \tag{6.6}$$

where $\eta^{B_1} = K^{-d}\sum_{x\in B_1}\eta(x)$.

Let us call this last expression $\mathcal{I}_0$. Define $\psi(x,\rho) = E_{\nu_\rho}[f(\theta_x\omega)]$. Notice that $V_f(x,\eta) = f(x,\eta) - \psi(x,\rho) - E[\partial_\rho\psi(x,\rho)]\big(\eta(x)-\rho\big)$, since in the last term the partial derivative with respect to ρ commutes with the expectation with respect to the random environment. In order to estimate the expression (6.6), we use the elementary inequality $(x+y)^2 \le 2x^2+2y^2$. Therefore, we obtain $\mathcal{I}_0 \le 4(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3)$, where

$$\mathcal{I}_1 = \frac{1}{K^d} E\Big[\int\Big(\sum_{x\in B_1^0} E_{\nu_\rho}\big[f(x,\eta)|\eta^{B_1}\big] - \psi(x,\eta^{B_1})\Big)^2 \mathrm{d}\nu_\rho\Big],$$

$$\mathcal{I}_2 = \frac{1}{K^d} E\Big[\int\Big(\sum_{x\in B_1^0} \psi(x,\eta^{B_1}) - \psi(x,\rho) - \partial_\rho\psi(x,\rho)[\eta^{B_1}-\rho]\Big)^2 \mathrm{d}\nu_\rho\Big],$$

and

$$\mathcal{I}_3 = \frac{1}{K^d} E\Big[E_{\nu_\rho}\Big[\Big(\sum_{x\in B_1^0}\big(\partial_\rho\psi(x,\rho) - E[\partial_\rho\psi(x,\rho)]\big)\big[\eta^{B_1}-\rho\big]\Big)^2\Big]\Big].$$

Recall the equivalence of ensembles (see Lemma A.2.2.2 in [11]):

Lemma 6.2. *Let $h : \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ be a local uniformly Lipschitz function and $S \in \{1,\dots,N\}$. Then, there exists a constant C that depends on h only through its support and its Lipschitz constant, such that*

$$\Big|E_{\nu_\rho}[h(\eta)|\eta^S] - E_{\nu_{\eta^S}}[h(\eta)]\Big| \le \frac{C}{S^d},$$

and

$$\eta^S(x) = \frac{1}{S^d}\sum_{y\in\Lambda_S}\eta(y),$$

with $\Lambda_S = \{0,\dots,S-1\}^d$.

Applying Lemma 6.2, we get

$$\frac{1}{K^d} E\Big[\int\Big(\sum_{x\in B_1^0} E_{\nu_\rho}\big[f(x,\eta)|\eta^{B_1}\big] - \psi(x,\eta^{B_1})\Big)^2 \mathrm{d}\nu_\rho\Big] \le \frac{C}{K^d},$$

which vanishes as $K\to\infty$.

Using a Taylor expansion for $\psi(x,\rho)$, we obtain that

$$\frac{1}{K^d}E\Big[\int\Big(\sum_{x\in B_1^0}\psi(x,\eta^{B_1})-\psi(x,\rho)-\partial_\rho\psi(x,\rho)[\eta^{B_1}-\rho]\Big)^2\mathrm{d}\nu_\rho\Big]\le\frac{C}{K^d},$$

and also goes to 0 as $K\to\infty$.

Finally, we see that

$$\mathcal{I}_3=E_{\nu_\rho}\big[(\eta(0)-\rho)^2\big]\cdot E\Big[\Big(\frac{1}{K^d}\sum_{x\in B_1^0}(\partial_\rho\psi(x,\rho))-E[\partial_\rho\psi(x,\rho)]\Big)^2\Big],$$

and it goes to 0 as $K\to\infty$ by the L^2-ergodic theorem. This concludes the proof of Theorem 6.1.

Acknowledgements

We would like to thank Milton Jara for valuable discussions on nuclear spaces. We also would like to thank an anonymous referee for his/her useful comments and remarks that helped to improve the overall quality of the paper.

Appendix. Stochastic differential equations on nuclear spaces

A.1. Countably Hilbert nuclear spaces

In this subsection we introduce countably Hilbert nuclear spaces which will be the natural environment for the study of the stochastic evolution equations obtained from the martingale problem. We will begin by recalling some basic definitions on these spaces. To this end, we follow the ideas of Kallianpur and Perez-Abreu [10] and Gel'fand and Vilenkin [6].

Let Φ be a (real) linear space, and let $\|\cdot\|_r$, $r\in\mathbb{N}$ be an increasing sequence of Hilbertian norms. Define Φ_r as the completion of Φ with respect to $\|\cdot\|_r$. Since for $n\le m$

$$\|f\|_n\le\|f\|_m,\quad\text{for all } f\in\Phi,\tag{A.1}$$

we have

$$\Phi_m\subset\Phi_n,\quad\text{for all } m\ge n.$$

Let

$$\Phi_\infty=\bigcap_{r=1}^{\infty}\Phi_r.$$

Then Φ_∞ is a Fréchet space with respect to the metric

$$\rho(f,g)=\sum_{r=1}^{\infty}2^{-r}\frac{\|f-g\|_r}{1+\|f-g\|_r},\tag{A.2}$$

and (Φ_∞,ρ) is called a countably Hilbert space.

A countably Hilbert space Φ_∞ is called *nuclear* if for each $n\ge 0$, there exists $m>n$ such that the canonical injection $\pi_{m,n}:\Phi_m\to\Phi_n$ is Hilbert–Schmidt, i.e., if $\{f_j\}_{j\ge1}$ is a complete orthonormal system in Φ_m, we have

$$\sum_{j=1}^{\infty} \|f_j\|_n^2 < \infty. \tag{A.3}$$

We now characterize the topological dual Φ'_∞ of the countably Hilbert nuclear space Φ_∞ in terms of the topological dual of the auxiliary spaces Φ_n.

Let Φ'_n be the dual (Hilbert) space of Φ_n, and for $\phi \in \Phi'_n$ let

$$\|\phi\|_{-n} = \sup_{\|f\|_n \le 1} |\phi[f]|,$$

where $\phi[f]$ means the value of ϕ at f. Equation (A.1) implies that

$$\Phi'_n \subset \Phi'_m \quad \text{for all } m \ge n.$$

Let Φ'_∞ be the topological dual of Φ_∞ with respect to the strong topology, which is given by the complete system of neighborhoods of zero given by sets of the form $\{\phi \in \Phi'_\infty : \|\phi\|_B < \epsilon\}$, where $\|\phi\|_B = \sup\{|\phi[f]| : f \in B\}$ and B is a bounded set in Φ_∞. So,

$$\Phi'_\infty = \bigcup_{r=1}^{\infty} \Phi'_r.$$

A.2. Stochastic differential equations

The aim of this subsection is to recall some results about existence and uniqueness of stochastic evolution equations in nuclear spaces.

We denote by $\mathcal{L}(\Phi_\infty, \Phi_\infty)$ (resp. $\mathcal{L}(\Phi'_\infty, \Phi'_\infty)$) the class of continuous linear operators from Φ_∞ to Φ_∞ (resp. Φ'_∞ to Φ'_∞).

A family $\{S(t) : t \ge 0\}$ of the linear operators on Φ_∞ is said to be a $C_{0,1}$-semigroup if the following three conditions are satisfied:

- $S(t_1)S(t_2) = S(t_1 + t_2)$ for all $t_1, t_2 \ge 0$, $S(0) = I$;
- the map $t \to S(t)f$ is Φ_∞-continuous for each $f \in \Phi_\infty$;
- for each $q \ge 0$ there exist numbers $M_q > 0, \sigma_q > 0$ and $p \ge q$ such that

$$\|S(t)f\|_q \le M_q e^{\sigma_q t} \|f\|_p \quad \text{for all } f \in \Phi_\infty, t > 0.$$

Let A in $\mathcal{L}(\Phi_\infty, \Phi_\infty)$ be infinitesimal generator of the semigroup $\{S(t) : t \ge 0\}$ in $\mathcal{L}(\Phi_\infty, \Phi_\infty)$. The relations

$$\phi[S(t)f] := (S'(t)\phi)[f] \quad \text{for all } t \ge 0,\ f \in \Phi_\infty \text{ and } \phi \in \Phi'_\infty;$$
$$\phi[Af] := (A'\phi)[f] \quad \text{for all } f \in \Phi_\infty \text{ and } \phi \in \Phi'_\infty;$$

define the infinitesimal generator A' in $\mathcal{L}(\Phi'_\infty, \Phi'_\infty)$ of the semigroup $\{S'(t) : t \ge 0\}$ in $\mathcal{L}(\Phi'_\infty, \Phi'_\infty)$.

Let $(\Sigma, \mathcal{U}, P)$ be a complete probability space with a right continuous filtration $(\mathcal{U}_t)_{t\ge 0}$, $\mathcal{U}_0$ containing all the P-null sets of $\mathcal{U}$, and $M = (M_t)_{t\ge 0}$ be a Φ'_∞-valued martingale with respect to $\mathcal{U}_t$, i.e., for each $f \in \Phi_\infty$, $M_t[f]$ is a real-valued martingale with respect to $\mathcal{U}_t$, $t \ge 0$. We are interested in results of existence and uniqueness of the following Φ'_∞-valued stochastic evolution equation:

$$\begin{aligned} &\mathrm{d}\xi_t = A'\xi_t \mathrm{d}t + \mathrm{d}M_t, \quad t > 0, \\ &\xi_0 = \gamma, \end{aligned} \tag{A.4}$$

where γ is a Φ'_∞-valued random variable, and A is the infinitesimal generator of a $C_{0,1}$-semigroup on Φ_∞.

We say that $\xi = (\xi_t)_{t\geq 0}$ is a Φ'_∞-solution of the stochastic evolution equation (A.4) if the following conditions are satisfied:

- ξ_t is Φ'_∞-valued, progressively measurable, and $\mathcal{U}_t$-adapted;
- the following integral identity holds:

$$\xi_t[f] = \gamma[f] + \int_0^t \xi_s[Af]\mathrm{d}s + M_t[f],$$

for all $f \in \Phi_\infty$, $t \geq 0$ a.s.

In [10, Corollary 2.2] the following result on existence and uniqueness of solutions of the stochastic differential equation (A.4) is proved:

Proposition A.1. *Assume the conditions below:*

(1) *γ is a Φ'_∞-valued $\mathcal{U}_0$-measurable random element such that, for some $r_0 > 0$, $E|\gamma|^2_{-r_0} < \infty$;*
(2) *$M = (M_t)_{t\geq 0}$ is a Φ'_∞-valued martingale such that $M_0 = 0$ and, for each $t \geq 0$ and $f \in \Phi$, $E(M_t[f])^2 < \infty$;*
(3) *A is a continuous linear operator on Φ_∞, and is the infinitesimal generator of a $C_{0,1}$-semigroup $\{S(t) : t \geq 0\}$ on Φ_∞.*

Then, the Φ'_∞-valued homogeneous stochastic evolution equation (A.4) has a unique solution $\xi = (\xi_t)_{t\geq 0}$ given explicitly by the "evolution solution":

$$\xi_t = S'(t)\gamma + \int_0^t S'(t-s)\mathrm{d}M_s.$$

Remark A.2. The statement $E(M_t[f])^2 < \infty$ in condition 2 of Proposition A.1 is satisfied if $E(M_t[f])^2 = tQ(f, f)$, where $f \in \Phi_\infty$, and $Q(\cdot,\cdot)$ is a positive definite continuous bilinear form on $\Phi_\infty \times \Phi_\infty$.

We now state a proposition, whose proof can be found in Corollary 2.1 of [10], that gives a sufficient condition for the solution ξ_t of Eq. (A.4) to be a Gaussian process.

Proposition A.3. *Assume γ is a Φ'_∞-valued Gaussian element independent of the Φ'_∞-valued Gaussian martingale with independent increments M_t. Then, the solution $\xi = (\xi_t)$ of (A.4) is a Φ'_∞-valued Gaussian process.*

References

[1] S. Ethier, T. Kurtz, Markov processes: characterization and convergence, in: Wiley Series in Prob. and Math. Stat., 1986.
[2] A. Faggionato, Bulk diffusion of 1D exclusion process with bond disorder, Markov Process. Related Fields 13 (2007) 519–542.
[3] A. Faggionato, Random walks and exclusion processes among random conductances on random infinite clusters: homogenization and hydrodynamic limit, Electron. J. Probab. 13 (2008) 2217–2247.
[4] A. Faggionato, M. Jara, C. Landim, Hydrodynamic behavior of one dimensional subdiffusive exclusion processes with random conductances, Probab. Theory Related Fields 144 (2009) 633–667.

[5] T. Franco, C. Landim, Exclusion processes with conductances—hydrodynamic limit of gradient exclusion processes with conductances, Arch. Ration. Mech. Anal. 195 (2010) 409–439.
[6] I.M. Gel'fand, N.Ya. Vilenkin, Generalized Functions—Vol. 4, Academic Press, New York, 1964.
[7] P. Gonçalves, M. Jara, Scaling limits for gradient systems in random environment, J. Stat. Phys. 131 (2008) 691–716.
[8] M. Jara, Current and density fluctuations for interacting particle systems with anomalous diffusive behavior. arXiv:0901.0229.
[9] M. Jara, C. Landim, Quenched nonequilibrium central limit theorem for a tagged particle in the exclusion process with bond disorder, Ann. Inst. H. Poincaré Probab. Statist. 44 (2008) 341–361.
[10] G. Kallianpur, V. Perez-Abreu, Stochastic evolution equations driven by nuclear-space-valued martingale, Appl. Math. Optim. 17 (1988) 237–272.
[11] C. Kipnis, C. Landim, Scaling limits of interacting particle systems, in: Grundlehren der Mathematischen Wissenschaften, vol. 320, Springer-Verlag, Berlin, 1999.
[12] I. Mitoma, Tightness of probabilities on $C([0,1],\mathcal{S}')$ and $D([0,1],\mathcal{S}')$, Ann. Probab. 11 (1983) 989–999.
[13] G. Papanicolaou, S.R.S. Varadhan, Boundary value problems with rapidly oscillating random coefficients, in: Seria Coll. Math. Soc. Janos Bolyai, vol. 27, North-Holland, 1979.
[14] A. Piatnitski, E. Remy, Homogenization of elliptic difference operators, SIAM J. Math. Anal. 33 (2001) 53–83.
[15] A.B. Simas, F.J. Valentim, W-Sobolev spaces: theory, homogenization and applications, preprint. Available at: arXiv:0911.4177v1.2009.
[16] F.J. Valentim, Hydrodynamic limit of gradient exclusion processes with conductance on $\mathbb{Z}^d$, preprint. Available at: arXiv:0903.4993v1.2009.