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Escape trajectories of solar sails and general relativity

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ABSTRACT

General relativity can have a significant impact on the long-range escape trajectories of solar sails deployed near the sun. For example, spacetime curvature in the vicinity of the sun can cause a solar sail traveling from about 4 solar radii to 2550 AU to be deflected by on the order of a million kilometers, and should therefore be taken into account at the beginning of the mission. There are a number of smaller general relativistic effects, such as frame dragging due to the slow rotation of the sun which can cause a deflection of more than one thousand kilometers.

1. Introduction

The exploration of the solar system's frontiers – the region between 50–2500 astronomical units (AU) from the sun – is a most ambitious and exciting technological challenge. Deep-space missions using chemical propulsion are somewhat limited because they require a long duration, a high launch speed and an enormous amount of fuel. Solar sails are an alternative method of propulsion that could result in a cruise speed that enables the exploration of extra solar space during the span of a human lifetime, and may eventually be applied to interstellar exploration [1–8]. See [8] for additional references on solar sailing. A recent study [9] shows that after sail deployments at parabolic orbit with 0.1 AU perihelion, a 937 m radius beryllium hollow body solar sail with a sail mass of 150 kg and a payload mass of 150 kg reaches 200 AU in 2.5 years, the sun's inner gravitational focus at 550 AU [10,11] in about 6.5 years and the inner Oort Comet Cloud at 2550 AU in 30 years.

A solar sail should be deployed as close to the sun as possible so that the force due to the solar radiation pressure (SRP) is maximized. In order to minimize the perihelion distance, it is necessary to use low density sail materials that are highly reflective and heat tolerant, as well as consider various effects of the environment near the sun [12–14]. In particular, the curvature of spacetime in the region near the sun should be considered. In fact, the perihelion shift of Mercury, located at a mean distance from the sun of about 0.39 AU, was the first experimental verification of general relativity. Perihelion distances as small as 0.02 AU–0.1 AU may be feasible for solar sails in the near future. For example, a trajectory design for a solar probe was presented in [15] that includes repeated pole-to-pole sun flybys at a perihelion of 4 solar radii, which is slightly less than 0.02 AU. The effects of curved spacetime in conjunction with the SRP on solar sails in bound heliocentric and non-Keplerian orbits has recently been considered [16,17], where it was shown to lead to deviations from Kepler's third law. Even though a solar sail in an escape trajectory is close to the sun for only a short time, perturbations to its motion during this period when the outward acceleration due to the SRP is greatest may translate into dramatic effects on long-range trajectories.

Responding to an increasing demand for navigational accuracy, we consider a number of general relativistic effects on the escape trajectories of solar sails. For missions as far as 2550 AU, these effects can deflect a solar sail by as much as a million kilometers. We take the initial conditions to be at closest approach, as depicted in Fig. 1. We will distinguish between the effect of spacetime curvature and special relativistic kinematics. We also find that frame dragging due to the slow rotation of the sun can deflect a solar sail by more than one thousand kilometers.

2. Deflection due to curved spacetime

2.1. Orbital equations

We begin by deriving the general relativistic orbital equations for an object traveling near the sun in the absence of the SRP.

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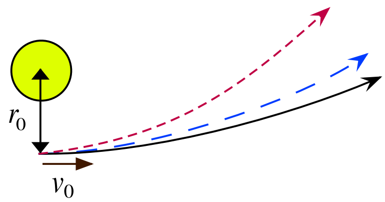

Fig. 1. Escape trajectory in Newtonian theory (solid curve), special relativity with Newtonian gravity (long-dashed curve) and general relativity (short-dashed curve).

We will use the large-distance limit of the Kerr metric [18], which describes the exterior spacetime around a slowly rotating object up to linear order in angular momentum. We do not use the full Kerr metric since it does not seem to describe the external spacetime of a rotating material body, because it does not smoothly fit onto metrics which describe the interior region occupied by physical matter. The large-distance limit of the Kerr metric is given by

$$ds^2 = -f c^2\,dt^2 - \frac{4GJ}{c^2 r}\sin^2\theta\,dt\,d\phi + \frac{dr^2}{f} + r^2\,d\Omega^2,$$
$$f = 1 - \frac{2GM}{c^2 r}, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2, \tag{1}$$

where r, θ, ϕ and t are the heliocentric distance, polar and azimuthal angles, and time as measured by a distant static observer, respectively. Note that an Earth-bound observer at $r = 1$ AU can essentially play the role of a distant observer. In (1), M and J are the mass and angular momentum of the sun. For $J = 0$, this metric reduces to the Schwarzschild metric, which describes the exterior spacetime of a spherical static body. Note that $J > 0$ for a prograde orbit with respect to the sun, while $J < 0$ for a retrograde orbit.

The 4-momentum of the solar sail is $p^\mu = m\,dx^\mu/d\tau$, where $x^\mu = (t, r, \theta, \phi)$ and τ is the proper time measured in the frame of reference of the solar sail. We will restrict ourselves to trajectories that lie within the equatorial plane of the sun, for which the effect of frame dragging is maximized. Then $\theta = \pi/2$ and therefore $p_\theta = 0$. We can define the constants of motion $E \equiv -p_t/m$ and $L \equiv p_\phi/m$, which are the energy and angular momentum per mass m of the solar sail and its load. Then we have

$$p^t = \frac{mE}{c^2 f} - \frac{2GmJL}{c^4 f r^3}, \qquad p^\phi = \frac{mL}{r^2} + \frac{2GmJE}{c^4 f r^3},$$
$$p^r = m\frac{dr}{d\tau}. \tag{2}$$

In the absence of the SRP, $p^2 = -m^2c^2$, which yields

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - \left(c^2 + \frac{L^2}{r^2}\right) f - \frac{4GJEL}{c^4 r^3}. \tag{3}$$

Differentiating this with respect to τ gives the radial component of the 4-acceleration:

$$a^r = \frac{d^2r}{d\tau^2} + \frac{GM}{r^2} - \frac{L^2}{r^3} + \frac{3G(c^2ML^2 - 2JEL)}{c^4 r^4}. \tag{4}$$

Note that this can also be found by taking the covariant derivative of the velocity 4-vector.

We will now include the effects of the SRP. We assume that the backreaction of the electromagnetic radiation on the background geometry is negligible so that it acts on the solar sail only via the SRP. We can say that objects move in the *photo-gravitational field* of the sun. Even though the coordinate r does not measure the proper distance, the surface area of a sphere is still given by $4\pi r^2$ (up to linear order in J). This means that the acceleration due to the SRP is given by the same expression as in the Newtonian approximation, which is

$$a^r = \frac{\kappa}{r^2}, \quad \kappa \equiv \frac{\eta L_S}{2\pi c\sigma}. \tag{5}$$

For more details of this, see for example [3,16]. Note that we are restricting ourselves to the case in which the surface of the solar sail is directly facing the sun. In (5), σ is the mass per area of the solar sail, which is a key design parameter that determines the solar sail performance [3,19,20]. Note that we will use values for σ which are larger than that of the solar sail on its own, since we are taking into account the mass of the load that is being transported. The coefficient η represents the efficiency of the solar sail used to account for the imperfect reflectivity of the sail and the sail billowing. Typically, the conservative value for the solar sail efficiency is $\eta = 0.85$. In (5), $L_S = 3.842 \times 10^{26}$ W is the solar luminosity and $c = 2.998 \times 10^8$ m/s is the speed of light. In the Newtonian approximation, the radially outwards force due to the SRP effectively reduces the mass of the sun to be $\tilde{M} \equiv M - \kappa/G$, where $M = 1.99 \times 10^{30}$ kg is the sun's actual mass. However, we wish to emphasize that this effective renormalization of the sun's mass does not carry over to the general relativistic framework, since both M and $\tilde{M}$ appear in the orbital equations of the solar sail.

Equating the expressions for a^r given in (4) and (5) and taking the first integral gives

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - \left(c^2 + \frac{L^2}{r^2}\right) f - \frac{2\kappa}{r} - \frac{4GJEL}{c^4 r^3}. \tag{6}$$

From (6) and the ϕ equation in (2), we finally find the orbital equation to be

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\left[\frac{E^2}{c^2} - \left(c^2 + \frac{L^2}{r^2}\right) f - \frac{2\kappa}{r} - \frac{4GJEL}{c^4 r^3}\right]}{\left(Lfr - 2J\frac{E}{c^2}\right)^2}\, r^6 f^2. \tag{7}$$

2.2. Deflection of solar sails

We will first consider the deflection of the solar sail due to the curved spacetime for the case of $J = 0$. Namely, we will first neglect the effect of frame dragging. Before the solar sail is deployed at the distance of closest approach $r = r_0$, the gravitational attraction of the sun causes the speed of the spaceship to increase as it gets closer to the sun. The Helios deep space probes would have traveled at the record speed of about 70 km/s at 0.3 AU. This enables us to extrapolate (using conservation of energy within the Newtonian approximation) that the following sampling of speeds v_0 are feasible for the near future: $v_0 = 133$ km/s at $r_0 = 0.1$ AU, $v_0 = 188$ km/s at $r_0 = 0.05$ AU, $v_0 = 298$ km/s at $r_0 = 0.02$ AU, and $v_0 = 420$ km/s at 0.01 AU.

From the metric (1) with $J = 0$, we find that the proper time interval is related to the coordinate time interval by

$$d\tau = dt\sqrt{f - \frac{1}{c^2 f}\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{c^2}\left(\frac{d\phi}{dt}\right)^2}. \tag{8}$$

Using this, we can express the angular momentum parameter L as

$$L = \frac{v_0 r_0}{\sqrt{f_0 - v_0^2/c^2}}, \tag{9}$$

where $f_0 \equiv f|_{r=r_0}$. Since we are restricting ourselves to the case in which the force due to the SRP is purely in the radial direction, L is still a conserved quantity. However, since the SRP is transferring energy to the solar sail, E is no longer a conserved quantity. In particular, E is the energy parameter of the solar sail at $r = r_0$ as measured by a distant observer. Since $dr/d\tau = 0$ at $r = r_0$, we find that

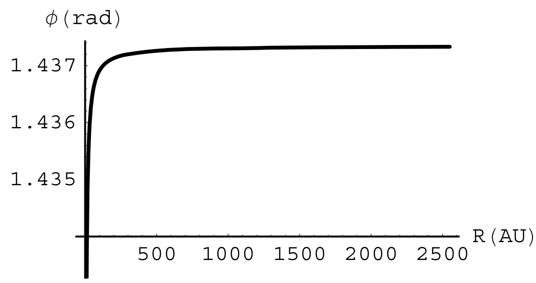

Fig. 2. The angular position ϕ versus the heliocentric distance R for a solar sail starting out at $r_0 = 0.05$ AU with an initial speed of $v_0 = 188$ km/s.

$$E = c\sqrt{c^2 - \frac{2G\tilde{M}}{r_0} + \frac{L^2}{r_0^2} f_0}. \tag{10}$$

From (7) with $J = 0$, we find that the angular position of the solar sail as a function of the heliocentric distance R is given by

$$\phi = L\int_{r_0}^{R} \frac{dr}{r^2\sqrt{h}}, \quad h \equiv 2G\tilde{M}\left(\frac{1}{r} - \frac{1}{r_0}\right) + L^2\left(\frac{f_0}{r_0^2} - \frac{f}{r^2}\right), \tag{11}$$

where we have taken $\phi = 0$ at $r = r_0$. This yields

$$\phi = \alpha^{-3/2}\left(\arcsin(by) - \frac{\pi}{2}\right) - \frac{2GM}{c^2}\sqrt{\frac{1}{b^2} - y^2} + \mathcal{O}\left(c^{-4}\right), \tag{12}$$

where

$$y = \frac{f(R)}{R} - \frac{G\tilde{M}}{\alpha L^2}, \quad \alpha = 1 - \frac{2G^2M\tilde{M}}{c^2L^2},$$
$$b = \frac{c\alpha L^2}{\sqrt{(E^2 - c^4)\alpha L^2 + c^2G^2\tilde{M}^2}}. \tag{13}$$

Note that ϕ can also be expressed in terms of an elliptic integral of the first kind.

Fig. 2 shows a numerical integration of ϕ versus R for $r_0 = 0.05$ AU and $v_0 = 188$ km/s. Clearly most of the deflection of the solar sail occurs when it is in the vicinity of the sun. General relativity predicts that the solar sail will undergo a larger deflection than does the Newtonian approximation. Although the resulting difference in angle is rather small, this can translate into a large discrepancy $d = R(\phi - \phi_N)$ in the location of the solar sail for long-range missions. As shown in Fig. 3, d dramatically increases for closer flybys, approaching as much as half a million kilometers for a solar sail deployed at $r_0 = 0.02$ AU with $v_0 = 298$ km/s and traveling to $R = 2550$ AU. For a solar sail deployed at $r_0 = 0.01$ AU with $v_0 = 420$ km/s, the deflection increases to a million kilometers.

In order to disentangle the contribution to d due to the kinematic effects of special relativity (within the Newtonian framework for gravity) from the effects of curved spacetime, we include the discrepancy between the special relativistic and Newtonian positions in Fig. 4 for the example of $v_0 = 298$ km/s at $r_0 = 0.02$ AU. While both types of effects are enhanced when the solar sail is deployed closer to the sun, it can be seen that the effects of curved spacetime dominate over those of special relativity.

We will now consider the effect of frame dragging due to the slow rotation of the sun, which is sometimes referred to as gravitomagnetism. The speed of the outer layer of the sun at its equator is $v \approx 2000$ m/s at the equatorial radius of $R_0 \approx 7 \times 10^8$ m. If we make the gross assumption that the core of the sun rotates with the same angular speed, then the angular momentum of the sun

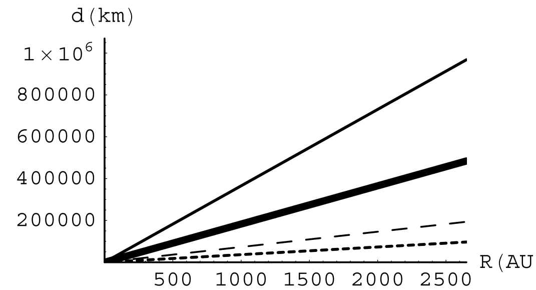

Fig. 3. The discrepancy d in the location of the solar sail versus the heliocentric distance R for the following sets of initial conditions: $v_0 = 133$ km/s at $r_0 = 0.1$ AU (short-dashed line), $v_0 = 188$ km/s at $r_0 = 0.05$ AU (long-dashed line), $v_0 = 298$ km/s at $r_0 = 0.02$ AU (bold line), and $v_0 = 420$ km/s at $r_0 = 0.01$ AU (solid line).

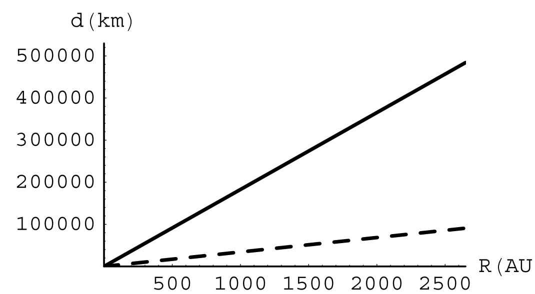

Fig. 4. The discrepancy in the location as predicted by special relativity versus Newtonian mechanics (dashed line) and general relativity versus Newtonian mechanics (solid line) for $v_0 = 298$ km/s, $r_0 = 0.02$ AU.

is given by $J = \frac{2}{5}MvR_0 \approx 10^{42}$ kg m^2/s. Using perturbation techniques, from (7) we find the angular position of the solar sail can be expressed as

$$\phi \approx L\int_{r_0}^{R} \frac{dr}{r^2\sqrt{h}} \times \left[1 + \frac{2GEJ}{c^4L}\left(\frac{1}{fr} - \frac{1}{f_0r_0} + \frac{L^2}{hr^3} - \frac{L^2}{hr_0^3} - \frac{L^3}{Ev_0r_0^4}\right)\right]. \tag{14}$$

For a solar sail traveling from $r_0 = 0.02$ AU at $v_0 = 298$ km/s, frame dragging causes the location at $R = 2550$ AU to be altered by approximately 438 kilometers. For $r_0 = 0.01$ AU at $v_0 = 420$ km/s, the deflection due to frame dragging increases to about 1240 kilometers. The direction of the deflection depends on whether the solar sail is in a prograde or retrograde orbit relative to the rotation of the sun.

3. Other effects of curved spacetime

3.1. Redshift factor

Besides deflection, the curvature of spacetime gives rise to a number of lesser effects, such as the slowing down of the passage of time near the sun. For example, an observer on Earth at 1 AU measures about 16 seconds more per year than does an observer at $r = 0.02$ AU. This phenomenon leads to a redshift in the wavelength of sunlight:

$$\frac{\lambda_\infty - \lambda}{\lambda} = \frac{1}{\sqrt{f}} - 1, \tag{15}$$

where λ is the wavelength measured by an observer at the heliocentric distance r, and λ_∞ is the wavelength measured by a distant observer. It has been shown that the minimum thickness of the solar sail that provides maximum reflectivity depends on the wavelength of the solar radiation, as well as on the temperature [19,20]. In particular, for fixed temperature, the optimum thickness of the solar sail increases with the wavelength. According to the redshift formula (15), the wavelength λ decreases as one gets closer to the sun where most of the acceleration occurs, which implies that the optimum thickness of the solar sail may also decrease. However, even at $r = 0.01$ AU, the redshift is only 10^{-6}, which has a negligible effect on the optimum thickness of the solar sail. The redshift effect also leads to an enhancement of the SRP close to the sun which is a negligible.

3.2. Time duration of voyage

The proper time duration of a voyage in the reference frame of a solar sail traveling from $r = r_0$ to R can be found from (6), (9) and (10) to be

$$\Delta\tau = \int_{r_0}^{R} \frac{dr}{\sqrt{h}}, \tag{16}$$

where h is given by (11). This is generally less than the duration of the same voyage as measured by a distant observer (which can approximately be taken to be someone on Earth), which is

$$\Delta t = \int_{r_0}^{R} \frac{dr}{\sqrt{f}} \sqrt{\frac{1}{c^2 f} + \left(1 + \frac{L^2}{c^2 r^2}\right)\frac{1}{h}}. \tag{17}$$

For example, a 35.5 year-long voyage of a solar sail beginning from $r_0 = 0.02$ AU with $v_0 = 298$ km/s takes about 12 minutes longer from the point of view of a distant observer. For a 25 year voyage beginning from $r_0 = 0.01$ AU with $v_0 = 420$ km/s, the time discrepancy is about 17 minutes.

3.3. Cruising velocity

The radial and tangential components of the velocity of the solar sail as measured by a distant observer at rest relative to the sun are given by

$$v_r = \sqrt{f}\left[\frac{1}{c^2 f} + \left(1 + \frac{L^2}{c^2 r^2}\right)\frac{1}{h}\right]^{-1/2},$$
$$v_\phi = L\frac{\sqrt{f}}{r}\left[\frac{h}{c^2 f} + 1 + \frac{L^2}{c^2 r^2}\right]^{-1/2}. \tag{18}$$

For our example of a solar sail beginning at $r_0 = 0.02$ AU with $v_0 = 298$ km/s, the cruising velocity is about 340 km/s, almost entirely in the radial direction. While the tangential component is essentially the same as in the Newtonian approximation, the radial component of the velocity is faster by about 2.33 m/s, which is a difference that remains constant throughout most of the voyage and therefore has a cummulative effect. For the case of $r_0 = 0.01$ AU with $v_0 = 420$ km/s, the cruising velocity is about 480 km/s and the radial component is faster by about 1.65 m/s.

4. Conclusions

We have considered various general relativistic effects on long-range trajectories of solar sails. Small deviations in the initial trajectories of solar sails that are deployed near the sun can translate to large effects in the long run. For example, a solar sail deployed at 0.02 AU can be deflected by about half a million kilometers by the time it gets to 2550 AU, while for a deployment at 0.01 AU the deflection would be about a million kilometers. This deflection is primarily due to the curvature of spacetime near the sun, while the kinematic effects of special relativity contribute to a lesser degree. Frame dragging due to the slow rotation of the sun can result in a deflection of more than 400 kilometers for a solar sail deployed at 0.02 AU, and more than one thousand kilometers for deployment at 0.01 AU. A number of lesser effects of general relativity include the redshifting of sunlight, the slowing down of the passage of time near the sun, and a slightly increased radial component of the cruising velocity.

We have restricted ourselves to an idealized model of the solar sail. For example, we have not taken into account the surface degradation that occurs close to the sun, due to the heating of the surface [1,12] and the interactions of the solar sail material atoms and nuclei with ultra-violet radiation, energetic electrons, protons and α-particles ejected by the sun [20,21]. This, along with a non-trivial temperature dependence of the optical parameters [22], lead to changes in the reflectivity and absorption of the solar sail which can have an effect on its trajectory. These effects are difficult to model or to calibrate from Earth and are highly sensitive to the solar sail surface characteristics, as well as the stochastic solar activity. The general relativistic effects discussed here will still be present and can be straightforwardly incorporated into more complex models which take into account these or any other factors.

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