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In Memoriam—Lev Brutman (1939–2001)

1. Part I

Lev M. Brutman was born in Moscow, in the former Soviet Union, on September 25, 1939. He completed his mathematical studies at Moscow State University in 1965. In 1973, he emigrated to Israel.

With Nira Dyn as his advisor, Lev obtained a Ph.D. degree at Tel-Aviv University in 1982. The title of his dissertation was "Operators of polynomial interpolation and alternation: properties, optimality and applications".

After graduating, Lev and his family moved to Haifa where he obtained a position at the University of Haifa. He was an active and respected faculty member of the University of Haifa until his death. He participated in numerous international conferences and visited many universities and research institutes. He is the author of two books and more than 30 papers.

Lev, the father of five children, was a very quiet and peaceful man. He had many friends, and I am proud to have been one of them.

Lev Brutman died after a serious illness, on August 25, 2001, in his chosen homeland, Israel.

2. Part II

In this part, I discuss two areas to which Brutman materially contributed. Namely, the Erdős–Bernstein conjecture on optimal nodes for polynomial interpolation, and Newman's rational approximation to $|x|$.

Of course, this choice mirrors my personal taste, but I hope that it also shows some of the characteristic features of *his* mathematical taste.

2.1. The Erdős–Bernstein conjecture on optimal interpolation

To set the stage, here are some standard definitions. X is an *interpolation array (triangular matrix)* if

$$X = (x_{kn} : k = 1, 2, \dots, n; n \in \mathbb{N})$$

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with $x_{kn} \neq x_{jn}$ for $k \neq j$ and all n. The corresponding *Lagrange* (*or algebraic*) *interpolation polynomials* for a function f are

$$L_n(f, X, x) := \sum_{k=1}^{n} f(x_{kn}) \ell_{kn}(X, x), \quad n \in \mathbb{N},$$

where $\ell_{kn}(X, \cdot)$ is the unique polynomial of degree $n-1$ with $\ell_{kn}(X, x_{jn}) = \delta_{kj}$ $(1 \leqslant k, j \leqslant n, n \in \mathbb{N})$. In the investigation of the convergence of $L_n(f, X)$ to f as $n \to \infty$, the *Lebesgue functions*

$$\lambda_n(X, x) := \sum_{k=1}^{n} |\ell_{kn}(X, x)|, \quad n \in \mathbb{N},$$

and *Lebesgue constants*

$$\Lambda_n(X) := \max_{x \in I} \lambda_n(X, x), \quad n \in \mathbb{N},$$

turn out to be fundamental. Here I is usually the smallest interval containing all the nodes.

In 1978, Brutman published [B7] which contains his fairly precise description of the Lebesgue function of Lagrange interpolation at the Chebyshev nodes

$$T = \left(\cos \frac{2k-1}{2n} \pi : k = 1, 2, \ldots, n; n \in \mathbb{N} \right).$$

This paper is his first in a series of works concerned with the Lebesgue function and Lagrange, trigonometric, and complex interpolation.

In his paper [B32], jointly with Pinkus, they investigate the *minimal Lebesgue constant of interpolation on the unit disc* $D := \{z : |z| \leqslant 1\}$, with the interpolation array $Z = (z_{kn} : k = 1, 2, \ldots, n; n \in \mathbb{N})$ having all its entries on the boundary of that disc. They prove Erdős' conjecture that the *optimal array* Z^* is made up of equally spaced nodes, that is, $Z^* = E = E(\alpha_n) := (\exp(2k\pi i/n + \alpha_n) : k = 1, 2, \ldots, n; n \in \mathbb{N})$, with (α_n) an arbitrary real sequence. Moreover, as shown already in 1921 by Gronwall (see [3]),

$$\Lambda_n(E) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sin \frac{2k-1}{2n} \pi}, \quad n \in \mathbb{N}.$$

The method of proof in [B32] goes back to the papers of Kilgore [4] and de Boor and Pinkus [2], on the solution of the Erdős–Bernstein conjecture, where the algebraic and trigonometric cases were settled. Actually, the optimal nodes for the *trigonometric* case are again the equidistant nodes, and, for them the Lebesgue constant is well-known.

In the *algebraic* case, neither the optimal array X^* nor the Lebesgue constants $\Lambda_n(X^*)$ are known. However, using some basic facts proved in [2,4] and the analysis of the Lebesgue constants for the Chebyshev nodes T (see [B7]), one can get the value of $\Lambda_n(X^*)$ (but *not* X^* itself) within $o(1)$. Namely,

$$\Lambda_n(X^*) = \frac{2}{\pi} \log n + \chi + o(1), \quad n \to \infty.$$

Here, $\chi := \frac{2}{\pi}(\gamma + \log\frac{4}{\pi}) = 0.521251\ldots$, with $\gamma = 0.577215\ldots$ the Euler constant (see Vértesi [6]). For other details, see [B21] or [7].

2.2. Questions concerning Newman's rational approximation to the function $|x|$

The function $|x|$ plays a central role in approximation theory. To give an example, Lebesgue's proof of the Weierstrass theorem is based on the approximability (by polynomials) of $|x|$. Quantitatively speaking, as was proved by Bernstein, the exact order of approximation to $|x|$ by polynomials of degree $\leqslant n$ in $[-1,1]$ is $1/n$ (see [1]).

In contrast to this, Newman [5] demonstrated that *rational* approximation to $|x|$ is more efficient. Namely, his classical result is as follows:

$$\tfrac{1}{2}e^{-9\sqrt{n}} \leqslant ||x| - r_n(x)| \leqslant 3e^{-\sqrt{n}}, \quad x \in [-1,1], \ n \geqslant 4.$$

Here,

$$r_n(x) := x\frac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)},$$

$$p_n(x) := \sum_{k=0}^{n-1}(x + \zeta^k), \quad \zeta := \exp(-n^{-1/2}).$$

Newman's striking theorem generated a great deal of interest, including [B22,B27,B29,B30] by Brutman, mostly with Passow, in which they investigated Newman's rational function $r_n(\cdot) = r_n(X,\cdot)$ when, more generally, $p_n(x) = \prod_{k=1}^{n}(x + x_{kn})$ for some *arbitrary nodes* satisfying $0 < x_{1n} < x_{2n} < \cdots < x_{nn} \leqslant 1$, $n \in \mathbb{N}$.

In what follows, I collect some of their interesting and surprising statements:

(1) It is easy to see that $r_n(X,x)$ is a rational function of degree $(n)/(n)$ if n is even and of degree $(n+1)/(n-1)$ if n is odd; $r_n(X,x)$ interpolates $|x|$ at the points $\{\pm x_{1n}, \pm x_{2n}, \pm x_{nn}, 0\}$.

(2) $r_n(X,x)$ is an *increasing* function of x in $(0,\infty)$, while $r_n(X,x)/x^2$ is *decreasing* in $(0,\infty)$.

(3) With $S_n := \sum_{k=1}^{n} x_{kn}$, the condition

$$\lim_{n\to\infty} S_n = \infty$$

is *necessary and sufficient* for the pointwise convergence of $r_n(X,x)$ to $|x|$ on $(-\infty,\infty)$. Moreover,

$$e_n(X,x) := ||x| - r_n(X,x)| \leqslant \frac{2}{S_n}, \quad -1 \leqslant x \leqslant 1.$$

In particular, when X is the uniform array $(\frac{j}{n} : j = 1,2,\ldots,n; n \in \mathbb{N})$, then $S_n = \frac{n+1}{2}$, whence $e_n \leqslant \frac{4}{n+1}$. This result is in sharp contrast to the classical divergence result of Bernstein [1] for algebraic Lagrange interpolation to $|x|$.

Other related results are in [B22,B27,B29,B30] and in their references where also many interesting questions and problems await the interested reader, demonstrating the old adage "*ars longa, vita brevis*".

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