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A fourth-order finite difference method for the general one-dimensional nonlinear biharmonic problems of first kind

R.K. Mohanty *

Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110 007, India

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Abstract

We present two new finite difference methods of order two and four in a coupled manner for the general one-dimensional nonlinear biharmonic equation $y^{IV} = f(x, y, y', y'', y''')$ subject to the boundary conditions $y(a) = A_0$, $y'(a) = A_1$, $y(b) = B_0$, $y'(b) = B_1$. In both cases, we use only three grid points and do not require to discretize the boundary conditions. First-order derivative of the solution is obtained as a by-product of the methods. The methods are successfully applied to the problems both in cartesian and polar coordinates. Numerical examples are given to illustrate the methods and their convergence.

MSC: G.1.7

Keywords: Finite difference method; Nonlinear biharmonic equation; Polar coordinate; NBSOR method; Maximum absolute error; Root-mean-square error

1. Introduction

We consider the fourth-order boundary value problem

$$y^{IV}(x) = f(x, y(x), y'(x), y''(x), y'''(x)), \quad a < x < b, \tag{1}$$

subject to the natural boundary conditions

$y(a), y'(a), y(b), y'(b)$ prescribed

* Fax: +91-11-7257049.

E-mail address: rmohanty@himalaya.du.ac.in (R.K. Mohanty)

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or equivalently, $y'(x) = z(x)$,

$$y^{\mathrm{IV}}(x) = f(x, y(x), z(x), y''(x), z''(x)), \quad a < x < b, \tag{2}$$

subject to the natural boundary conditions

$y(a), z(a), y(b), z(b)$ prescribed,

where $-\infty < a \leqslant x \leqslant b < \infty$. We refer to the differential equation (1) or (2) together with the prescribed boundary conditions as the first kind problem. The existence and uniqueness of the first kind problem were discussed in [1].

The interval $[a, b]$ is divided into a set of points with an interval of $h = (b-a)/(N+1)$, N being a positive integer. The finite difference approximation to Eq. (1) is obtained on $[a, b]$ that consists of the central point $x_k = kh$ and the two neighbouring points $x_{k+1} = x_k + h$ and $x_{k-1} = x_k - h$, $k = 1(1)N$, where $x_0 = a$ and $x_{N+1} = b$. A combination of the value of the solution $y(x)$ and the value of its derivative $y'(x)$ are used to derive difference methods at the three grid points. The standard five-point difference formula of Eq. (1) is obtained by using second-order central differences, which requires the use of fictitious points outside of the region $[a, b]$. The accuracy of the numerical solution depends upon the boundary approximation used. The finite difference methods which we present here in a coupled manner are based on only three grid points for both second- and fourth-order methods. i.e. no fictitious points for incorporating the boundary conditions are required. It is mentioned here that, so far, no fourth-order discretizations using three grid points is known for the differential equation (1). Since we need to solve the coupled nonlinear system of equations at each mesh point, the iterative methods of solution are frequently used. Therefore, a solution of coupled system of nonlinear equations is often a very good starting vector for another iteration. Also, a proper choice of the iterative method to be used has a great influence on the amount of computational effort required to solve a given problem. With a slowly converging iterative method, the amount of time required may be so large, even with a very fast computer, as to make the solution of the problem impractical. The systems that are generated by these methods, have a more complicated block structure than those derived from the standard central difference methods. The linear systems have to be solved by block successive overrelaxation (BSOR) method and nonlinear systems have to be solved by Newton's nonlinear block successive overrelaxation (NBSOR) method.

2. The finite difference methods

Let the exact solution values of $y(x)$ and $z(x)$ at the grid point x_k are denoted by Y_k and Z_k, respectively, and y_k and z_k are their approximate solutions, respectively.

Our methods are described as follows: For $k = 1(1)N$, let

$$\bar{y}''_k = (z_{k+1} - z_{k-1})/(2h), \tag{3a}$$

$$\bar{z}''_k = (z_{k+1} - 2z_k + z_{k-1})/(h^2) \tag{3b}$$

and set $\bar{f}_k = f(x_k, y_k, z_k, \bar{y}''_k, \bar{z}''_k)$.

Then the difference method of order two for the given differential equation (1) is given by

$$-2[y_{k+1} - 2y_k + y_{k-1}] + h[z_{k+1} - z_{k-1}] = \frac{h^4}{6}\bar{f}_k \tag{4a}$$

and the corresponding difference method for the derivative $y' = z$ is given by

$$-3[y_{k+1} - y_{k-1}] + h[z_{k+1} + 4z_k + z_{k-1}] = 0. \tag{4b}$$

Also, let

$$\bar{y}''_{k+1} = (3z_{k+1} - 4z_k + z_{k-1})/(2h), \tag{5a}$$

$$\bar{y}''_{k-1} = (-3z_{k-1} + 4z_k - z_{k+1})/(2h), \tag{5b}$$

$$\bar{z}''_{k+1} = -12(y_{k+1} - 2y_k + y_{k-1})/(h^3) + (7z_{k+1} - 2z_k - 5z_{k-1})/(h^2), \tag{5c}$$

$$\bar{z}''_{k-1} = 12(y_{k+1} - 2y_k + y_{k-1})/(h^3) + (7z_{k-1} - 2z_k - 5z_{k+1})/(h^2), \tag{5d}$$

$$\bar{\bar{y}}''_k = 2(y_{k+1} - 2y_k + y_{k-1})/(h^2) - (z_{k+1} - z_{k-1})/(2h), \tag{6a}$$

$$\bar{\bar{y}}''_{k+1} = -(23y_{k+1} - 16y_k - 7y_{k-1})/(2h^2) + (6z_{k+1} + 8z_k + z_{k-1})/h, \tag{6b}$$

$$\bar{\bar{y}}''_{k-1} = -(23y_{k-1} - 16y_k - 7y_{k+1})/(2h^2) - (6z_{k-1} + 8z_k + z_{k+1})/h \tag{6c}$$

and set

$$\bar{f}_{k+1} = f(x_{k+1}, y_{k+1}, z_{k+1}, \bar{y}''_{k+1}, \bar{z}''_{k+1}),$$

$$\bar{f}_{k-1} = f(x_{k-1}, y_{k-1}, z_{k-1}, \bar{y}''_{k-1}, \bar{z}''_{k-1}),$$

$$\bar{\bar{f}}_{k+1} = f(x_{k+1}, y_{k+1}, z_{k+1}, \bar{\bar{y}}''_{k+1}, \bar{z}''_{k+1}),$$

$$\bar{\bar{f}}_{k-1} = f(x_{k-1}, y_{k-1}, z_{k-1}, \bar{\bar{y}}''_{k-1}, \bar{z}''_{k-1}).$$

Finally, let

$$\bar{\bar{z}}''_k = \bar{z}''_k - \frac{h}{104}(\bar{\bar{f}}_{k+1} - \bar{\bar{f}}_{k-1}) \tag{7}$$

and set $\bar{\bar{f}}_k = f(x_k, y_k, z_k, \bar{\bar{y}}''_k, \bar{\bar{z}}''_k)$.

Then the difference method of order four for the differential equation and the corresponding difference method for the derivative $y' = z$ are given by

$$-2(y_{k+1} - 2y_k + y_{k-1}) + h(z_{k+1} - z_{k-1}) = \frac{h^4}{90}(\bar{\bar{f}}_{k+1} + \bar{\bar{f}}_{k-1} + 13\bar{\bar{f}}_k) \tag{8a}$$

and

$$-3(y_{k+1} - y_{k-1}) + h(z_{k+1} + 4z_k + z_{k-1}) = \frac{h^4}{60}(\bar{f}_{k+1} - \bar{f}_{k-1}). \tag{8b}$$

Note that, y_0, z_0, y_{N+1} and z_{N+1} are prescribed. It is convenient to express the above finite difference schemes in block tridiagonal matrix form. If the differential equation (1) is linear, the resulting block tridiagonal linear system can be solved using the block successive overrelaxation (BSOR) method; in the nonlinear case, the system can be solved using the Newton's nonlinear block successive overrelaxation (NBSOR) method (see [4,5]).

3. Derivation of the difference schemes and block iterative methods

We next discuss the derivation of the difference methods and block iterative methods. For derivation of the methods, we simply follow the approaches given in [2,3,8–11].

At the grid point x_k, the given differential equation (2) can be written as

$$y_k^{\mathrm{IV}} = f(x_k, y_k, z_k, y_k'', z_k'') = f_k, \quad k = 1(1)N. \tag{9a}$$

Similarly,

$$f_{k+1} = f(x_{k+1}, y_{k+1}, z_{k+1}, y_{k+1}'', z_{k+1}''), \quad k = 1(1)N \tag{9b}$$

and

$$f_{k-1} = f(x_{k-1}, y_{k-1}, z_{k-1}, y_{k-1}'', z_{k-1}''), \quad k = 1(1)N. \tag{9c}$$

In the following, we set $H = \partial f/\partial z''$.

Using Taylor expansion about the point x_k, we first obtain

$$-2(y_{k+1} - 2y_k + y_{k-1}) + h(z_{k+1} - z_{k-1}) = \frac{h^4}{6} f_k + T_1, \tag{10}$$

where $T_1 = \mathrm{O}(h^6)$.

From (3a) and (3b), we have $\bar{y}_k'' = y_k'' + \mathrm{O}(h^2)$ and $\bar{z}_k'' = z_k'' + \mathrm{O}(h^2)$. Now, replacing $y_k'' = \bar{y}_k'' + \mathrm{O}(h^2)$ and $z_k'' = \bar{z}_k'' + \mathrm{O}(h^2)$ in (9a), we find that $\bar{f}_k = f_k + \mathrm{O}(h^2)$. Thus, the second-order approximation for the difference equation given by (4a) is straightforward and by the help of (10) we can verify that the local truncation error associated with the difference equation (4a) is of $\mathrm{O}(h^2)$. Similarly, the difference equation (4b) approximates the derivative $y' = z$ with $\mathrm{O}(h^2)$ accuracy.

Similarly, by the help of Taylor expansion, we obtain

$$-2(y_{k+1} - 2y_k + y_{k-1}) + h(z_{k+1} - z_{k-1}) = \frac{h^4}{90}(f_{k+1} + f_{k-1} + 13 f_k) + T_2, \tag{11}$$

where $T_2 = \mathrm{O}(h^8)$.

Now, we need $\mathrm{O}(h^2)$-approximation for z_{k+1}''. Let

$$\bar{z}_{k+1}'' = \frac{1}{h^3}(a_{10} y_k + a_{11} y_{k+1} + a_{12} y_{k-1}) + \frac{1}{h^2}(b_{10} z_k + b_{11} z_{k+1} + b_{12} z_{k-1}). \tag{12}$$

Expanding each term on the right-hand side of (11) in Taylor series about the point x_k and equating the coefficients of h^p, $(p = -3, -2, -1, 0$ and $1)$ to zero, we get

$$(a_{10},\ a_{11},\ a_{12},\ b_{10},\ b_{11},\ b_{12}) = (24,\ -12,\ -12,\ -2,\ 7,\ -5).$$

Thus, we obtain

$$\begin{aligned} \bar{z}_{k+1}'' &= \frac{-12}{h^3}(y_{k+1} - 2y_k + y_{k-1}) + \frac{1}{h^2}(7z_{k+1} - 2z_k - 5z_{k-1}) \\ &= z_{k+1}'' - \frac{5h^2}{12} y_k^{\mathrm{V}} + \mathrm{O}(h^3 + h^4). \end{aligned} \tag{13a}$$

Similarly,

$$\bar{z}''_{k-1} = \frac{12}{h^3}(y_{k+1} - 2y_k + y_{k-1}) + \frac{1}{h^2}(7z_{k-1} - 2z_k - 5z_{k+1})$$
$$= z''_{k-1} - \frac{5h^2}{12} y_k^{\mathrm{V}} + \mathrm{O}(-h^3 + h^4). \tag{13b}$$

Further, from (5a) and (5b), we have $\bar{y}''_{k\pm1} = y''_{k\pm1} + \mathrm{O}(h^2 + h^3)$.

Now, it is straightforward to verify that $\bar{f}_{k\pm1} = f_{k\pm1} + \mathrm{O}(h^2 \pm h^3)$ and the difference equation (8b) approximates the derivative $y' = z$ with $\mathrm{O}(h^4)$ accuracy.

Next, we obtain $\mathrm{O}(h^4)$-approximation for y''_k. Let

$$\bar{\bar{y}}''_k = \frac{1}{h^2}(a_{20}y_k + a_{21}y_{k+1} + a_{22}y_{k-1}) + \frac{1}{h}(b_{20}z_k + b_{21}z_{k+1} + b_{22}z_{k-1}). \tag{14}$$

With the help of Taylor expansion, from (14), we find that if

$$(a_{20}, a_{21}, a_{22}, b_{20}, b_{21}, b_{22}) = (-4, 2, 2, 0, \tfrac{-1}{2}, \tfrac{1}{2}),$$

then

$$\bar{\bar{y}}''_k = \frac{2}{h^2}(y_{k+1} - 2y_k + y_{k-1}) - \frac{1}{2h}(z_{k+1} - z_{k-1}) = y''_k + \mathrm{O}(h^4). \tag{15a}$$

Similarly,

$$\bar{\bar{y}}''_{k\pm1} = \frac{-1}{2h^2}(23y_{k\pm1} - 16y_k - 7y_{k\mp1}) \pm \frac{1}{h}(6z_{k\pm1} + 8z_k + z_{k\mp1}) = y''_{k\pm1} + \mathrm{O}(h^4). \tag{15b}$$

From (13) and (15) it follows that $\bar{\bar{f}}_{k\pm1}$ provide $\mathrm{O}(h^2)$-approximations for $f_{k\pm1}$ and

$$\bar{\bar{f}}_{k\pm1} = f_{k\pm1} - \frac{5h^2}{12} y_k^{\mathrm{V}} H_k + \mathrm{O}(\pm h^3 + h^4). \tag{16}$$

Next, we seek an approximation for z''_k in the form as given by (7). Let

$$\bar{\bar{z}}''_k = \bar{z}''_k + ah(\bar{\bar{f}}_{k+1} - \bar{\bar{f}}_{k-1}), \tag{17}$$

where 'a' is a free parameter to be determined.

With the help of approximations (3b) and (16), from (17) we obtain

$$\bar{\bar{z}}''_k = z''_k + \frac{h^2}{12}(1 + 24a)y_k^{\mathrm{V}} + \mathrm{O}(h^4). \tag{18}$$

From (15a) and (18) it follows that $\bar{\bar{f}}_k$ provides an $\mathrm{O}(h^2)$-approximation for f_k, and

$$\bar{\bar{f}}_k = f_k + \frac{h^2}{12}(1 + 24a)y_k^{\mathrm{V}} H_k + \mathrm{O}(h^4). \tag{19}$$

With the help of (16) and (19), from (8a) and (11), we obtain the local truncation error (LTE_2) associated with the difference scheme (8a) as

$$\mathrm{LTE}_2 = \frac{-h^6}{360}(1 + 104a)y_k^{\mathrm{V}} H_k + \mathrm{O}(h^8). \tag{20}$$

The proposed difference method (8a) to be of $\mathrm{O}(h^4)$, the coefficients of h^6 in (20) must be zero, thus we obtain $a = -\frac{1}{104}$ and the local truncation error reduces to $\mathrm{LTE}_2 = \mathrm{O}(h^8)$.

Whenever $f(x,y,z,y'',z'')$ is nonlinear, the difference equations (4) or (8) form a coupled nonlinear block system. To solve such a system we could apply the Newton's NBSOR method. To define the NBSOR method, we first write (4) or (8) in the form

$$\Phi(\boldsymbol{y},\boldsymbol{z})=\boldsymbol{0}, \tag{21a}$$

$$\Psi(\boldsymbol{y},\boldsymbol{z})=\boldsymbol{0}, \tag{21b}$$

where

$$\boldsymbol{y}=\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_N\end{bmatrix},\quad \boldsymbol{z}=\begin{bmatrix} z_1\\ z_2\\ \vdots\\ z_N\end{bmatrix},\quad \Phi(\boldsymbol{y},\boldsymbol{z})=\begin{bmatrix} \phi_1(\boldsymbol{y},\boldsymbol{z})\\ \phi_2(\boldsymbol{y},\boldsymbol{z})\\ \vdots\\ \phi_N(\boldsymbol{y},\boldsymbol{z})\end{bmatrix},\quad \Psi(\boldsymbol{y},\boldsymbol{z})=\begin{bmatrix} \psi_1(\boldsymbol{y},\boldsymbol{z})\\ \psi_2(\boldsymbol{y},\boldsymbol{z})\\ \vdots\\ \psi_N(\boldsymbol{y},\boldsymbol{z})\end{bmatrix}.$$

Let

$$\boldsymbol{J}=\begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12}\\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22}\end{bmatrix}$$

be the Jacobian of Φ and Ψ, which is the $2N$th-order block tridiagonal matrix where

$$\boldsymbol{T}_{11}=\frac{\partial(\phi_1,\phi_2,\ldots,\phi_N)}{\partial(y_1,y_2,\ldots,y_N)}=\begin{bmatrix} \dfrac{\partial\phi_1}{\partial y_1} & \dfrac{\partial\phi_1}{\partial y_2} & & & \boldsymbol{0}\\ \dfrac{\partial\phi_2}{\partial y_1} & \dfrac{\partial\phi_2}{\partial y_2} & \dfrac{\partial\phi_2}{\partial y_3} & & \\ & \boldsymbol{0} & & \dfrac{\partial\phi_N}{\partial y_{N-1}} & \dfrac{\partial\phi_N}{\partial y_N}\end{bmatrix},$$

$$\boldsymbol{T}_{12}=\frac{\partial(\phi_1,\phi_2,\ldots,\phi_N)}{\partial(z_1,z_2,\ldots,z_N)},\quad \boldsymbol{T}_{21}=\frac{\partial(\psi_1,\psi_2,\ldots,\psi_N)}{\partial(y_1,y_2,\ldots,y_N)}\quad\text{and}\quad \boldsymbol{T}_{22}=\frac{\partial(\psi_1,\psi_2,\ldots,\psi_N)}{\partial(z_1,z_2,\ldots,z_N)},$$

are the Nth-order tridiagonal matrices.

Now, the matrix equation for NBSOR method is given by

$$\begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12}\\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22}\end{bmatrix}\begin{bmatrix} \Delta\boldsymbol{y}\\ \Delta\boldsymbol{z}\end{bmatrix}=\begin{bmatrix} -\Phi\\ -\Psi\end{bmatrix}, \tag{22}$$

where $\Delta\boldsymbol{y}$ and $\Delta\boldsymbol{z}$ are any intermediate values and with any initial approximations $(\boldsymbol{y}^{(0)}, \boldsymbol{z}^{(0)})$ of $(\boldsymbol{y}_k, \boldsymbol{z}_k)$, $k=1(1)N$, we define

$$\boldsymbol{y}^{(n+1)}=\boldsymbol{y}^{(n)}+\Delta\boldsymbol{y}^{(n)},\quad n=0,1,2,\ldots, \tag{23a}$$

$$\boldsymbol{z}^{(n+1)}=\boldsymbol{z}^{(n)}+\Delta\boldsymbol{z}^{(n)},\quad n=0,1,2,\ldots\,. \tag{23b}$$

System (22) can be solved for $\Delta\boldsymbol{y}^{(n+1)}$ and $\Delta\boldsymbol{z}^{(n+1)}$ by using block SOR method (inner iterative method) as follows:

$$\boldsymbol{T}_{11}\Delta\boldsymbol{y}^{(n+1)}=\omega[-\Phi(\boldsymbol{y}^{(n)},\boldsymbol{z}^{(n)})-\boldsymbol{T}_{12}\Delta\boldsymbol{z}^{(n)}]+(1-\omega)\boldsymbol{T}_{11}\Delta\boldsymbol{y}^{(n)}, \tag{24a}$$

$$\boldsymbol{T}_{22}\Delta\boldsymbol{z}^{(n+1)}=\omega[-\Psi(\boldsymbol{y}^{(n)},\boldsymbol{z}^{(n)})-\boldsymbol{T}_{21}\Delta\boldsymbol{y}^{(n+1)}]+(1-\omega)\boldsymbol{T}_{22}\Delta\boldsymbol{z}^{(n)}, \tag{24b}$$

where $\omega\in(0,2)$ is a relaxation parameter and $n=0,1,2,\ldots$. The above system of equations can be solved by using tridiagonal solver (see [6,7]). Then by using outer iterative method (23), we can calculate $\boldsymbol{y}^{(n+1)}$ and $\boldsymbol{z}^{(n+1)}$, $n=0,1,2,\dots$. In order for this method to converge it is sufficient that the initial approximation $(y^{(0)},z^{(0)})$ be 'close' to the solution.

If the differential equation (1) is linear, then the difference method (4) or (8) in matrix form can be written as

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix}, \tag{25}$$

where $\boldsymbol{A}_{11},\boldsymbol{A}_{12},\boldsymbol{A}_{21}$ and $\boldsymbol{A}_{22}$ are the Nth-order tridiagonal matrices, $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are vectors consisting of right-hand side functions and some boundary conditions associated with the block system given by (25).

Relative to the partitioning (25), the BSOR method is defined by

$$\boldsymbol{A}_{11}\boldsymbol{y}^{(n+1)} = \omega[-\boldsymbol{A}_{12}\boldsymbol{z}^{(n)} + \boldsymbol{d}_1] + (1-\omega)\boldsymbol{A}_{11}\boldsymbol{y}^{(n)}, \quad n=0,1,2,\dots, \tag{26a}$$

$$\boldsymbol{A}_{22}\boldsymbol{z}^{(n+1)} = \omega[-\boldsymbol{A}_{21}\boldsymbol{y}^{(n+1)} + \boldsymbol{d}_2] + (1-\omega)\boldsymbol{A}_{22}\boldsymbol{z}^{(n)}, \quad n=0,1,2,\dots, \tag{26b}$$

where ω is a real number known as the relaxation factor. With $\omega=1$, the BSOR method reduces to the block Gauss–Seidel method. If $\omega>1$ or $\omega<1$, we have over relaxation or under relaxation, respectively.

4. Convergence analysis

Let us consider the fourth-order difference methods (8a) and (8b), when applied to a model equation $y^{\mathrm{IV}}=f(x)$ can be written as

$$(y_{k-1}+y_k+y_{k+1}-3y_k)+\frac{h}{2}(z_{k-1}-z_{k+1})=\frac{-h^4}{180}(f_{k+1}+f_{k-1}+13f_k), \tag{27a}$$

$$\frac{3}{h}(y_{k-1}-y_{k+1})+(z_{k-1}+z_k+z_{k+1}+3z_k)=\frac{h^3}{60}(f_{k+1}-f_{k-1}) \tag{27b}$$

which can be written in block form as

$$\begin{bmatrix} (\boldsymbol{L}-3\boldsymbol{I}) & \frac{h}{2}\boldsymbol{M} \\ \frac{3}{h}\boldsymbol{M} & (\boldsymbol{L}+3\boldsymbol{I}) \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix}, \tag{28}$$

where

$$\boldsymbol{L} = \begin{bmatrix} 1 & 1 & & \mathbf{0} \\ 1 & 1 & 1 & \\ & \ddots & \ddots & \ddots \\ \mathbf{0} & & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{M} = \begin{bmatrix} 0 & -1 & & \mathbf{0} \\ 1 & 0 & -1 & \\ & \ddots & \ddots & \ddots \\ \mathbf{0} & & 1 & 0 \end{bmatrix}$$

and $\boldsymbol{y},\boldsymbol{z}$ are two N-dimensional solution vectors and $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ are vectors consisting of right-hand side functions and some boundary values associated with (27). The BSOR method for the scheme (28) is

$$\boldsymbol{y}^{(n+1)} = (1-\omega)\boldsymbol{y}^{(n)} - \frac{\omega h}{2}(\boldsymbol{L}-3\boldsymbol{I})^{-1}\boldsymbol{M}\boldsymbol{z}^{(n)} + \omega(\boldsymbol{L}-3\boldsymbol{I})^{-1}\boldsymbol{d}_1, \tag{29a}$$

$$z^{(n+1)} = (1-\omega)z^{(n)} - \frac{3\omega}{h}(L+3I)^{-1}My^{(n+1)} + \omega(L+3I)^{-1}d_2. \tag{29b}$$

Then the associated block SOR and block Jacobi iteration matrices of (29) are

$$L_\omega = \begin{bmatrix} (1-\omega)I & \dfrac{-\omega h}{2}(L-3I)^{-1}M \\ \dfrac{-3\omega}{h}(L+3I)^{-1}M & (1-\omega)I \end{bmatrix} \tag{30a}$$

and

$$B = \begin{bmatrix} \mathbf{0} & \dfrac{-h}{2}(L-3I)^{-1}M \\ \dfrac{-3}{h}(L+3I)^{-1}M & \mathbf{0} \end{bmatrix}. \tag{30b}$$

From the SOR theory, we know that if η is an eigenvalue of B then λ is an eigenvalue of L_ω,

$$\text{where} \quad (\lambda+\omega-1)^2 = \lambda\omega^2\eta^2. \tag{31}$$

To determine η, we let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a partitioned eigenvector of B, then we have

$$\frac{-h}{2}(L-3I)^{-1}Mv_2 = \eta v_1, \tag{32a}$$

$$\frac{-3}{h}(L+3I)^{-1}Mv_1 = \eta v_2 \tag{32b}$$

and on eliminating v_2, we obtain

$$\tfrac{3}{2}(L-3I)^{-1}M(L+3I)^{-1}Mv_1 = \eta^2 v_1. \tag{33}$$

Now the rate of convergence of the BSOR method is dependent on the eigenvalues of B which are given by

$$\tfrac{3}{2}\tau = \eta^2, \tag{34}$$

where τ are the eigenvalues of $(L^2-9I)^{-1}M^2$ and $(L-3I)^{-1}M$ and $(L+3I)^{-1}M$ have coincident eigenvectors. Since the eigenvalues of L and M are $1+2\cos(j\pi/(N+1))$ and $2\,\mathrm{i}\cos(j\pi/(N+1))$, $j=1,2,\ldots,N$, respectively, then B has purely imaginary eigenvalues and hence from (34) the eigenvalues τ of L_ω are real and positive satisfying $0<\tau<\bar{\tau}=S((L^2-9I)^{-1}M^2)$, where $S(\mathbf{A})$ denotes the spectral radius of $\mathbf{A}$.

Hence we can determine the optimal parameter as

$$\omega_0 = \frac{2}{1+\sqrt{1-(3/2)\tau}}, \tag{35}$$

where $\tau = S((L^2-9I)^{-1}M^2)$. Thus we can determine the convergence factor

$$\bar{\lambda} = \omega_0 - 1 = \frac{1-\sqrt{1-(3/2)\tau}}{1+\sqrt{1-(3/2)\tau}}. \tag{36}$$

For convergence we must have $|\bar{\lambda}|<1$ to give the range

$$0<\tau<\tfrac{2}{3}. \tag{37}$$

5. Application to problems in polar coordinates

Consider a class of singular fourth-order linear ordinary differential equation of the form

$$\Delta^4 u \equiv \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\alpha}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 u = f(r), \quad 0 < r < 1, \quad \alpha = 1 \text{ and } 2$$

or equivalently,

$$u^{\mathrm{IV}} = b(r)u''' + c(r)u'' + d(r)u' + f(r), \quad 0 < r < 1, \tag{38}$$

where $b(r) = -2\alpha/r$, $c(r) = \alpha(2-\alpha)/r^2$, $d(r) = \alpha(\alpha-2)/r^3$, and for $\alpha = 1$ and 2, $\Delta^2 = (\mathrm{d}^2/\mathrm{d}r^2) + (\alpha/r)(\mathrm{d}/\mathrm{d}r)$ represents one-dimensional Laplacian operator in cylindrical and spherical coordinates, respectively.

The boundary conditions are given by

$$u(0) = A_0, \quad u'(0) = A_1, \quad u(1) = B_0, \quad u'(1) = B_1, \tag{39a}$$

or

$$u'(0) = 0, \quad u'''(0) = 0, \quad u(1) = B_0, \quad u'(1) = B_1, \tag{39b}$$

where A_0, B_0, A_1 and B_1 are constants. For $\alpha = 1$ and 2, Eq. (38) represents fourth-order ordinary differential equation in cylindrical and spherical symmetry, respectively. The numerical solution of the differential equation (38) can be obtained by using five grid points. But so far, no second- and fourth-order difference methods using three grid points are known for the singular equation (38). The difficulties were experienced in the past, especially, for the fourth-order numerical solution of the fourth-order ordinary differential equations in polar coordinates. The solution usually deteriorates in the neighbourhood of the singularity $r = 0$. In this section we refine our procedure in such a way that the solutions retain the order and accuracy everywhere including the region in the vicinity of the singularity $r = 0$.

Now replacing the variables x, y, z by r, u, u' and applying the difference scheme (4) to the singular equation (38), we obtain a second-order difference method as

$$-2(u_{k+1} - 2u_k + u_{k-1}) + h(u'_{k+1} - u'_{k-1}) = \frac{h^4}{6}[b_k \bar{u}'''_k + c_k \bar{u}''_k + d_k u'_k + f_k], \tag{40a}$$

$$-3(u_{k+1} - u_{k-1}) + h(u'_{k+1} + 4u'_k + u'_{k-1}) = 0, \quad k = 1(1)N, \tag{40b}$$

where $b_k = b(r_k)$, $c_k = c(r_k)$, $d_k = d(r_k)$, $f_k = f(r_k)$, and $\bar{u}'''_k$, $\bar{u}''_k$ are already defined in (3).

Similarly, replacing the variables x, y, z by r, u, u' and applying the fourth-order difference scheme (8b) to the singular equation (38), we obtain

$$\begin{aligned}
&-2(u_{k+1} - 2u_k + u_{k-1}) + h(u'_{k+1} - u'_{k-1}) \\
&\quad = \frac{h^4}{90}[13(b_k \bar{u}'''_k + c_k \bar{\bar{u}}''_k + d_k u'_k + f_k) \\
&\qquad + (1 - hp_0)(b_{k+1}\bar{u}'''_{k+1} + c_{k+1}\bar{\bar{u}}''_{k+1} + d_{k+1}u'_{k+1} + f_{k+1}) \\
&\qquad + (1 + hp_0)(b_{k-1}\bar{u}'''_{k-1} + c_{k-1}\bar{\bar{u}}''_{k-1} + d_{k-1}u'_{k-1} + f_{k-1})], \quad k = 1(1)N,
\end{aligned} \tag{41}$$

where

$$p_0=\frac{-\alpha}{4r_k},\quad b_{k\pm1}=b(r_{k\pm1}),\quad c_{k\pm1}=c(r_{k\pm1}),\ d_{k\pm1}=d(r_{k\pm1}),\ f_{k\pm1}=f(r_{k\pm1}).$$

Note that, scheme (41) fails when the solution is to be determined at $k=1$, although the scheme is of $\mathrm{O}(h^4)$. We overcome this difficulty by modifying the method (41) in such a way that the solutions retain the order and accuracy even in the vicinity of the singularity $r=0$.

We consider the following approximations:

$$b_{k\pm1}=b_k\pm hb'_k+\frac{h^2}{2}b''_k+\mathrm{O}(\pm h^3+h^4), \tag{42a}$$

$$c_{k\pm1}=c_k\pm hc'_k+\frac{h^2}{2}c''_k+\mathrm{O}(\pm h^3+h^4), \tag{42b}$$

$$d_{k\pm1}=d_k\pm hd'_k+\frac{h^2}{2}d''_k+\mathrm{O}(\pm h^3+h^4), \tag{42c}$$

$$f_{k\pm1}=f_k\pm hf'_k+\frac{h^2}{2}f''_k+\mathrm{O}(\pm h^3+h^4). \tag{42d}$$

Now using approximation (42) in (41) and neglecting the higher-order terms, we can rewrite (41) in compact operator form as

$$\begin{aligned}
&-2(u_{k+1}-2u_k+u_{k-1})+h(u'_{k+1}-u'_{k-1})\\
&\quad=\frac{h^2}{90}[24p_0b_k-24b'_k+18c_k+h^2(12p_0b''_k+8p_0c'_k-4c''_k)]\delta_r^2u_k\\
&\qquad+\frac{h^3}{180}[30p_0c_k-30c'_k+15p_0h^2c''_k](2\mu_r\delta_r)u_k\\
&\qquad+\frac{h^4}{90}[30c'_k-30p_0c_k+15d_k+h^2(-15p_0c''_k+d''_k-2p_0d'_k)]u'_k\\
&\qquad+\frac{h^2}{90}[15b_k+h^2(b''_k-2p_0b'_k+7c'_k-7p_0c_k+d_k)]\delta_r^2u'_k\\
&\qquad+\frac{h^3}{180}[24b'_k-24p_0b_k-3c_k+h^2(-12p_0b''_k+5c''_k-10p_0c'_k+2d'_k-2p_0d_k)](2\mu_r\delta_r)u'_k\\
&\qquad+\frac{h^4}{90}[15f_k+h^2(f''_k-2p_0f'_k)],\quad k=1(1)N.
\end{aligned} \tag{43a}$$

Similarly, using the difference scheme (8b), a fourth-order approximation for the derivative u' for the singular equation (38) in compact form may be written as

$$\begin{aligned}
&-3(u_{k+1}-u_{k-1})+h(u'_{k+1}+4u'_k+u'_{k-1})\\
&\quad=\frac{-2h}{5}b_k\delta_r^2u_k+\frac{h^3}{30}(c_k+b'_k)\delta_r^2u'_k+\frac{h^2}{60}[12b_k+h^2(c'_k+d_k)](2\mu_r\delta_r)u'_k\\
&\qquad+\frac{h^5}{30}(d'_ku'_k+f'_k),\quad k=1(1)N.
\end{aligned} \tag{43b}$$

Finite difference equation (40) or (43) alongwith the boundary conditions (39a) give a $2N\times2N$ linear system of equations for the unknowns $u_1,u_2,\ldots,u_N,u'_1,u'_2,\ldots,u'_N$.

If the boundary conditions of type (39b) are used, then $r=0$ is a part of the solution space and the solution is to be determined at this point. In this case, we need two extra difference equations valid at $r=0$.

For simplicity, let us consider the case when $\alpha=2$. Then the differential equation (38) reduces to

$$u^{\mathrm{IV}}+\frac{4}{r}u'''=f(r). \tag{44}$$

Since $u'(0)=u'''(0)=0$, in the limit, at $r=0$, Eq. (44) may be written as

$$5u^{\mathrm{IV}}=f(0) \tag{45}$$

and the corresponding difference equation of $O(h^2)$ and of $O(h^4)$ at $r=0$ are given by

$$2(u_0-u_1)+hu_1'=\frac{h^4}{60}f_0 \tag{46}$$

and

$$2(u_0-u_1)+hu_1'=\frac{h^4}{900}[15f_0+h^2f_0''], \tag{47}$$

respectively, where we have used the condition $u_0'=u_0'''=0$, i.e. $u_1=u_{-1}$ and $u_1'=-u_{-1}'$.

Equations (41) alongwith (46), or (44) alongwith (47) produce a $(2N+1)\times(2N+1)$ linear system of equations which can be solved for the unknowns $u_0,u_1,\ldots,u_N,u_1',u_2',\ldots,u_N'$.

Now consider the coupled nonlinear singular equations of the form

$$u^{\mathrm{IV}}=a(r)[u'v''+v'u'']+f(r),\quad 0<r<1, \tag{48a}$$

$$v^{\mathrm{IV}}=-a(r)u'u''+g(r),\quad 0<r<1, \tag{48b}$$

where $a(r)=1/r$ and at the end points $u(0)$, $v(0)$, $u'(0)$, $v'(0)$, $u(1)$, $v(1)$, $u'(1)$ and $v'(1)$ are known. The above system of equations represent model equations of equilibrium for a load symmetrical about the centre (see [12]).

Second-order difference methods for u,v,u' and v' for solving the system (48) are straightforward and are given by

$$-2(u_{k+1}-2u_k+u_{k-1})+h(u_{k+1}'-u_{k-1}')=\frac{h^4}{6}[a_k(u_k'\bar{v}_k''+v_k'\bar{u}_k'')+f_k], \tag{49a}$$

$$-2(v_{k+1}-2v_k+v_{k-1})+h(v_{k+1}'-v_{k-1}')=\frac{h^4}{6}[-a_ku_k'\bar{u}_k''+g_k], \tag{49b}$$

$$-3(u_{k+1}-u_{k-1})+h(u_{k+1}'+4u_k'+u_{k-1}')=0, \tag{49c}$$

$$-3(v_{k+1}-v_{k-1})+h(v_{k+1}'+4v_k'+v_{k-1}')=0, \tag{49d}$$

where $a_k=a(r_k)$, $f_k=f(r_k)$ and $g_k=g(r_k)$.

Using the same technique, as in the case of linear difference scheme (43), difference methods of $O(h^4)$ for u, v, u' and v' for solving the system (48) in compact operator form are given by

$$\begin{aligned}-2(u_{k+1}-2u_k+u_{k-1})+h(u_{k+1}'-u_{k-1}')=&\,P_1[15f_k+h^2f_k'']\\&+10P_2[a^*(2\mu_r\delta_r)(u_k'v_k')+ha_k'(2+\delta_r^2)(u_k'v_k')]\end{aligned}$$

$$
\begin{aligned}
&- P_4[a^*(2\mu_r\delta_r)u'_k + ha'_k(2+\delta_r^2)u'_k](2\mu_r\delta_r)v_k\\
&- P_4[a^*(2\mu_r\delta_r)v'_k + ha'_k(2+\delta_r^2)v'_k](2\mu_r\delta_r)u_k\\
&- P_3[(2a^* - \tfrac{13}{2}a_k)u'_k + ha'_k(2\mu_r\delta_r)u'_k + a_k\delta_r^2u'_k]\delta_r^2v_k\\
&- P_3[(2a^* - \tfrac{13}{2}a_k)v'_k + ha'_k(2\mu_r\delta_r)v'_k + a_k\delta_r^2v'_k]\delta_r^2u_k\\
&+ P_2[(10a^* - \tfrac{13}{2}a_k)(2\mu_r\delta_r)u'_k + 10ha'_k(2+\delta_r^2)u'_k]v'_k\\
&+ P_2[(10a^* - \tfrac{13}{2}a_k)(2\mu_r\delta_r)v'_k + 10ha'_k(2+\delta_r^2)v'_k]u'_k\\
&+ P_2[a_k(2\mu_r\delta_r)u'_k + 2ha'_ku'_k]\delta_r^2v'_k\\
&+ P_2[a_k(2\mu_r\delta_r)v'_k + 2ha'_kv'_k]\delta_r^2u'_k, \qquad (50a)
\end{aligned}
$$

$$
\begin{aligned}
-2(v_{k+1} - 2v_k + v_{k-1}) + h(v'_{k+1} - v'_{k-1}) = {}& P_1[15g_k + h^2g''_k]\\
&- 5P_2[a^*(2\mu_r\delta_r)(u'_k)^2 + ha'_k(2+\delta_r^2)(u'_k)^2]\\
&+ P_4[a^*(2\mu_r\delta_r)u'_k + ha'_k(2+\delta_r^2)u'_k](2\mu_r\delta_r)u_k\\
&+ P_3[(2a^* - \tfrac{13}{2}a_k)u'_k + ha'_k(2\mu_r\delta_r)u'_k + a_k\delta_r^2u'_k]\delta_r^2u_k\\
&- P_2[(10a^* - \tfrac{13}{2}a_k)(2\mu_r\delta_r)u'_k + 10ha'_k(2+\delta_r^2)u'_k]u'_k\\
&- P_2[a_k(2\mu_r\delta_r)u'_k + 2ha'_ku'_k]\delta_r^2u'_k, \qquad (50b)
\end{aligned}
$$

$$
\begin{aligned}
-3(u_{k+1} - u_{k-1}) + h(u'_{k+1} + 4u'_k + u'_{k-1}) = {}& Q_1f'_k\\
&+ Q_2[2a_k(2\mu_r\delta_r)u'_k\cdot(2\mu_r\delta_r)v'_k + 4a_k(u'_k\delta_r^2v'_k + v'_k\delta_r^2u'_k)\\
&+ 2ha'_ku'_k(2\mu_r\delta_r)v'_k + 2ha'_kv'_k(2\mu_r\delta_r)u'_k], \qquad (50c)
\end{aligned}
$$

$$
\begin{aligned}
-3(v_{k+1} - v_{k-1}) + h(v'_{k+1} + 4v'_k + v'_{k-1}) = {}& Q_1g'_k\\
&- Q_2[a_k(2\mu_r\delta_r)u'_k\cdot(2\mu_r\delta_r)u'_k + 2ha'_ku'_k(2\mu_r\delta_r)u'_k\\
&+ 4a_ku'_k\delta_r^2u'_k], \qquad (50d)
\end{aligned}
$$

where we denote

$$
P_1 = \frac{h^4}{90},\quad P_2 = \frac{h^3}{90},\quad P_3 = \frac{2h^2}{45},\quad P_4 = \frac{h^2}{12},\quad Q_1 = \frac{h^5}{30},\quad Q_2 = \frac{h^3}{120},
$$

$$
a_k = a(r_k),\quad f_k = f(r_k),\quad g_k = g(r_k),\quad a^* = a_k + \frac{h^2}{2}a''_k,\ \text{etc.},
$$

and $\delta_r u_k = (u_{k+1/2} - u_{k-1/2})$ and $\mu_r u_k = \frac{1}{2}(u_{k+1/2} + u_{k-1/2})$ are central and average difference operators with respect to r-direction, respectively.

Note that, schemes (40), (43), (49) and (50) are free from the terms $1/(k \pm 1)$ hence very easily solved for $k=1(1)N$ in the region $(0,1)$. In Section 3, we have already discussed the NBSOR method for scalar general nonlinear equation. In a similar manner, NBSOR method with four unknowns can be expressed for the coupled nonlinear differential equations. The resulting block tridiagonal systems (40) and (43) can be solved using BSOR method and (49) and (50) can be solved using NBSOR method.

Table 1
Problem 1: The RMSE

h		Scheme (40) $\alpha=1$	Scheme (40) $\alpha=2$	Scheme (43) $\alpha=1$	Scheme (43) $\alpha=2$
$\frac{1}{8}$	u	0.6737 (−03)	0.2214 (−02)	0.9155 (−04)	0.2005 (−03)
	u'	0.2137 (−02)	0.1081 (−01)	0.3365 (−03)	0.5835 (−03)
$\frac{1}{16}$	u	0.1501 (−03)	0.6109 (−03)	0.6562 (−05)	0.1492 (−04)
	u'	0.4995 (−03)	0.3859 (−02)	0.2584 (−04)	0.5486 (−04)
$\frac{1}{32}$	u	0.3576 (−04)	0.1590 (−03)	0.4273 (−06)	0.1047 (−05)
	u'	0.1206 (−03)	0.1337 (−02)	0.1746 (−05)	0.4508 (−05)
$\frac{1}{64}$	u	0.8769 (−05)	0.4051 (−04)	0.2690 (−07)	0.7053 (−07)
	u'	0.2972 (−04)	0.4648 (−03)	0.1118 (−06)	0.3382 (−06)

6. Numerical illustrations

To illustrate our methods and to demonstrate computationally its convergence, we have solved four problems, two linear using BSOR method and two nonlinear using NBSOR method whose exact solutions are known to us. In each case, we took the unit length $[0,1]$ as our region of integration. The right-hand side functions and boundary conditions may be obtained using the exact solutions. The initial vectors $\vec{0}$ are used in all cases and the iterations were stopped when the average absolute error tolerance $\leqslant 10^{-12}$ was achieved. Theoretically, it is difficult to calculate the value of ω in general case. Therefore, while solving nonlinear equations, we have considered only five inner iterations and the value of $\omega = 1$. However, for linear model equation we are able to provide the value of ω. All computations were carried out using double precision arithmetic at the Computer Service Centre, University of Delhi.

Problem 1. The problem is to solve (38) subject to the boundary conditions (39a). The exact solution is $u = r^4 \sin r$. The root mean square errors (RMSE) are tabulated in Table 1 for $\alpha = 1$ and 2.

Problem 2. The problem is to solve (44) subject to the boundary conditions of the form (39b). The exact solution is $u = \cosh r$. The RMSE are tabulated in Table 2.

Problem 3. $y^{\mathrm{IV}} = y(y' + y'' + y''') + f(x)$, $0 < x < 1$, subject to the natural boundary conditions prescribed. The exact solution is $y = \mathrm{e}^{2x}$. The maximum absolute errors (MAE) and RMSE are tabulated in Table 3.

Problem 4. The system of nonlinear equations (48) are to be solved subjected to the natural boundary conditions prescribed. The exact solutions are $u=\cos r$ and $v=\mathrm{e}^r$. The MAE and RMSE are tabulated in Table 4.

Now for the model problem $y^{\mathrm{IV}} = (12 + 48x^2 + 16x^4)\mathrm{e}^{x^2}$, $0 < x < 1$ when applied to the coupled difference equation (8) subjected to appropriate natural boundary conditions with the exact solution

Table 2
Problem 2: The RMSE

	Schemes (46) and (40)		Schemes (47) and (43)	
h	u	u'	u	u'
$\frac{1}{8}$	0.2683 (−04)	0.1101 (−03)	0.1048 (−04)	0.3090 (−04)
$\frac{1}{16}$	0.6982 (−05)	0.3701 (−04)	0.1028 (−05)	0.2896 (−05)
$\frac{1}{32}$	0.1782 (−05)	0.1263 (−04)	0.9225 (−07)	0.2390 (−06)
$\frac{1}{64}$	0.4433 (−06)	0.4305 (−05)	0.7747 (−08)	0.1573 (−07)

Table 3
Problem 3: The MAE and RMSE

		Scheme (4)		Scheme (8)	
h		MAE	RMSE	MAE	RMSE
$\frac{1}{8}$	y	0.2314 (−02)	0.1600 (−02)	0.4665 (−04)	0.3216 (−04)
	y'	0.9420 (−02)	0.5697 (−02)	0.2166 (−03)	0.1257 (−03)
$\frac{1}{16}$	y	0.5684 (−03)	0.3757 (−03)	0.3151 (−05)	0.2080 (−05)
	y'	0.2205 (−02)	0.1330 (−02)	0.1285 (−04)	0.7554 (−05)
$\frac{1}{32}$	y	0.1412 (−03)	0.9165 (−04)	0.2007 (−06)	0.1299 (−06)
	y'	0.5425 (−03)	0.3236 (−03)	0.7923 (−06)	0.4631 (−06)

Table 4
Problem 4: The MAE and RMSE

		Scheme (49)		Scheme (50)	
h		MAE	RMSE	MAE	RMSE
$\frac{1}{8}$	u	0.1098 (−04)	0.7702 (−05)	0.5593 (−05)	0.3832 (−05)
	v	0.5124 (−05)	0.3483 (−05)	0.1590 (−05)	0.1115 (−05)
	u'	0.3968 (−04)	0.2648 (−04)	0.2551 (−04)	0.1369 (−04)
	v'	0.1867 (−04)	0.1308 (−04)	0.6140 (−05)	0.3893 (−05)
$\frac{1}{16}$	u	0.2825 (−05)	0.1859 (−05)	0.5396 (−06)	0.3600 (−06)
	v	0.1330 (−05)	0.8757 (−06)	0.1154 (−06)	0.7678 (−07)
	u'	0.9905 (−05)	0.6484 (−05)	0.2582 (−05)	0.1303 (−05)
	v'	0.4165 (−05)	0.3083 (−05)	0.4324 (−06)	0.2678 (−06)
$\frac{1}{32}$	u	0.7063 (−06)	0.4572 (−06)	0.4609 (−07)	0.3042 (−07)
	v	0.3355 (−06)	0.2175 (−06)	0.7753 (−08)	0.5079 (−08)
	u'	0.2533 (−05)	0.1599 (−05)	0.2296 (−06)	0.1106 (−06)
	v'	0.1048 (−05)	0.7562 (−06)	0.2891 (−07)	0.1768 (−07)

Table 5
Model problem

h	Block Gauss–Seidel		Block SOR		τ_{est}	ω_{est}
	τ	No. of iterations	ω_0	No. of iterations		
$\frac{1}{4}$	0.4969	91	1.34	30	0.5	1.333
$\frac{1}{8}$	0.6328	441	1.63	67	0.625	1.600
$\frac{1}{16}$	0.6565	1762	1.7961	133	0.656	1.777
$\frac{1}{32}$	0.6648	6722	1.8932	276	0.664	1.880
$\frac{1}{64}$	0.6666	25260	1.9450	561	0.666	1.939

$y=e^{x^2}$, we can derive estimates for ω and τ from the formula $\omega_{est}=2/(1+ch)$. For values of $c=2$, we obtain the result for $\tau_{est}=\frac{2}{3}(1-c^2h^2)$, which is confirmed by the results given in Table 5.

7. Concluding remarks

The numerical results confirm that the finite difference methods obtained from the new discretizations technique outlined in Section 2 do yield second- and fourth-order convergence for the solution and its derivative of the fourth-order ordinary differential equations. It is mentioned here that for meaningful local truncation errors the partial derivatives of $f(x,y,z,y'',z'')$ with respect to x should be continuous atleast twice and four times in $[a,b]$ for the second- and fourth-order finite difference methods, respectively. Difference formulas for mesh points near a boundary are obtained without the use of fictitious points, thereby eliminating the usual difficulty encountered in using central difference methods. Further, it is shown that the structured block matrix systems discussed can be solved in an efficient manner by the BSOR and NBSOR method. During computation we found that for convergent results our difference schemes require a large number of iterations, thus the order drops in some cases due to the effect of round-off errors, but numerical oscillations do not appear throughout the computation.

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