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Gábor Hetyei

Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28223, United States

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ABSTRACT

We find the winning strategy for a class of truncation games played on words. As a consequence of the present author's recent results on some of these games we obtain new formulas for Bernoulli numbers and polynomials of the second kind and a new combinatorial model for the number of connected permutations of given rank. For connected permutations, the decomposition used to find the winning strategy is shown to be bijectively equivalent to King's decomposition, used to recursively generate a transposition Gray code of the connected permutations.

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0. Introduction

In a recent paper [4] the present author introduced a class of progressively finite games played on ranked posets, where each move of the winning strategy is unique and the positions satisfy the following uniformity criterion: each position of a given rank may be reached from the same number of positions of a given higher rank in a single move. As a consequence, the kernel positions of a given rank may be counted by subtracting from the number of all positions the appropriate multiples of the kernel positions of lower ranks. The main example in [4] is the *original Bernoulli game*, a truncation game played on pairs of words of the same length, for which the number of kernel positions of rank n is a signed factorial multiple of the Bernoulli number of the second kind b_n . Similarly to this game, most examples mentioned in [4] are also truncation games played on words, where the partial order is defined by taking initial segments and the rank is determined by the length of the words involved.

In this paper we consider a class of *strongly Bernoulli type truncation games* played on words, for which we do not require the uniformity condition on the rank to be satisfied. We show that for

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E-mail address: ghetyei@uncc.edu.

such games, the winning strategy may be found by decomposing each kernel position as a concatenation of *elementary kernel factors*. This decomposition is unique. All truncation games considered in [4] (including the ones played on pairs or triplets of words) are isomorphic to a strongly Bernoulli type truncation game. For most of these examples, the elementary kernel factors of a given type are also easy to enumerate. Thus we may obtain explicit summation formulas and non-alternating recurrences for numbers which were expressed in [4] as coefficients in a generating function or by alternating recurrences. The explicit summation formulas are obtained by considering the entire unique decomposition of each kernel position, the non-alternating recurrence is obtained by considering the removal of the last elementary kernel factor only. Thus we find some new identities for the Bernoulli polynomials and numbers of the second kind, and shed new light on King's [6] decomposition of "indecomposable" permutations.

The paper is structured as follows. After the Preliminaries, the main unique decomposition theorem is stated in Section 2. In the subsequent sections we consider games to which this result is applicable: we show they are isomorphic to strongly Bernoulli type truncation games, we find formulas expressing their elementary kernel factors of a given type, and use these formulas to express the number of kernel positions as an explicit sum and by a non-alternating recurrence. Most detail is given for the original Bernoulli game in Section 3, omitted details in other sections are replaced by references to the appropriate part of this section. As a consequence of our analysis of the original Bernoulli game, we obtain an explicit summation formula of the Bernoulli numbers of the second kind, expressing them as a sum of entries of the same sign. We also obtain a non-alternating recurrence for their absolute values.

In Section 4 we consider a restriction of the original Bernoulli game to a set of positions, where the kernel positions are identifiable with the *connected* or *indecomposable* permutations forming an algebra basis of the Malvenuto–Reutenauer Hopf algebra [9]. For these the recurrence obtained by the removal of the last elementary kernel factor is numerically identical to the recurrence that may be found in King's [6] recursive construction of a transposition Gray code for the connected permutations. We show that this is not a coincidence: there is a bijection on the set of permutations, modulo which King's recursive step corresponds to the removal of the last elementary kernel factor in the associated *place-based non-inversion tables* (a variant of the usual inversion tables). Our result inspires another systematic algorithm to list all connected permutations of a given order, and a new combinatorial model for the numbers of connected permutations of order n , in which this number arises as the total weight of all permutations of order $n - 2$, such that the highest weight is associated to the permutations having the most *strong fixed points* (being thus the "least connected").

Section 5 contains the consequences of our main result to Bernoulli polynomials of the second kind. Here we observe that we obtain the coefficients of these polynomials when we expand them in the basis $\{\binom{x+1}{n} : n \geq 0\}$, and obtain a new formula for the Bernoulli numbers of the second kind.

Finally, in Section 6 we consider the *flat Bernoulli game*, whose kernel positions have the generating function $t/((1-t)(1-\ln(1-t)))$ and conclude the section with an intriguing conjecture that for a long random initial word a novice player could not decrease the chance of winning below 50% by simply removing the last letter in the first move.

1. Preliminaries

1.1. Progressively finite games

A progressively finite two-player game is a game whose positions may be represented by the vertices of a directed graph that contains no directed cycle nor infinite path, the edges represent valid moves. Thus the game always ends after a finite number of moves. The players take alternate turns to move along a directed edge to a next position, until one of them reaches a *winning position* with no edge going out: the player who moves into this position is declared a winner, the next player is unable the move.

The winning strategy for a progressively finite game may be found by calculating the *Grundy number* (or Sprague–Grundy number) of each position, the method is well known, a sample reference is [15, Chapter 11]. The positions with Grundy number zero are called *kernel positions*. A player has

a winning strategy exactly when he or she is allowed to start from a non-kernel position. All games considered in this paper are progressively finite.

1.2. The original Bernoulli game and its generalizations

In [4] the present author introduced the *original Bernoulli game* as the following progressively finite two-player game. The positions of rank $n > 0$ in the game are all pairs of words $(u_1 \cdots u_n, v_1 \cdots v_n)$ such that

- (i) the letters u_1, \dots, u_n and v_1, \dots, v_n are positive integers;
- (ii) for each $i \geq 1$ we have $1 \leq u_i, v_i \leq i$.

A valid move consists of replacing the pair $(u_1 \cdots u_n, v_1 \cdots v_n)$ with $(u_1 \cdots u_m, v_1 \cdots v_m)$ for some $m \geq 1$ satisfying $u_{m+1} \leq v_j$ for $j = m + 1, \dots, n$. The name of the game refers to the following fact [4, Theorem 2.2].

Theorem 1.1. For $n \geq 1$, the number κ_n of kernel positions of rank n in the original Bernoulli game is given by

$$\kappa_n = (-1)^{n-1} (n + 1)! b_n,$$

where b_n is the n -th Bernoulli number of the second kind.

Here the Bernoulli number of the second kind b_n is obtained by substituting zero into the Bernoulli polynomial of the second kind $b_n(x)$, given by the generating function

$$\sum_{n=0}^{\infty} \frac{b_n(x)}{n!} t^n = \frac{t(1+t)^x}{\ln(1+t)}, \tag{1}$$

see Roman [10, p. 116]. Note that [10, p. 114] Jordan’s [5, p. 279] earlier definition of the Bernoulli polynomial of the second kind $\phi_n(x)$ is obtained by dividing $b_n(x)$ by $n!$.

The proof of Theorem 1.1 depends on a few simple observations which were generalized in [4] to a class of *Bernoulli type games on posets* (see [4, Definition 3.1]). The set of positions P in these games is a partially ordered set with a unique minimum element $\hat{0}$ and a rank function $\rho : P \rightarrow \mathbb{N}$ such that for each $n \geq 0$ the set P_n of positions of rank n have finitely many elements. The valid moves satisfy the following criteria:

- (i) Each valid move is from a position of higher rank to a position of lower rank. The set of positions reachable from a single position is a chain.
- (ii) If y_1 and y_2 are both reachable from x in a single move and $y_1 < y_2$ then y_1 is reachable from y_2 in a single move.
- (iii) For all $m < n$ there is a number $\gamma_{m,n}$ such that each y of rank m may be reached from exactly $\gamma_{m,n}$ elements of rank n in a single move.

For such games, it was shown in [4, Proposition 3.3], the numbers κ_n of kernel positions of rank n satisfy the recursion formula

$$|P_n| = \kappa_n + \sum_{m=0}^{n-1} \kappa_m \cdot \gamma_{m,n}. \tag{2}$$

2. Winning a strongly Bernoulli type truncation game

Let Λ be an alphabet and let us denote by Λ^* the free monoid generated by Λ , i.e., set

$$\Lambda^* := \{v_1 \cdots v_n : n \geq 0, \forall i (v_i \in \Lambda)\}.$$

Note that Λ^* contains the empty word ε .

Definition 2.1. Given a subset $M \subseteq \Lambda^* \setminus \{\varepsilon\}$, we define the *truncation game induced by M* as the game whose positions are the elements of Λ^* , and whose valid moves consist of all truncations $v_1 \cdots v_n \rightarrow v_1 \cdots v_i$ such that $v_{i+1} \cdots v_n \in M$.

Note that $\varepsilon \notin M$ guarantees that the truncation game induced by M is progressively finite, we may define the *rank* of each position as the length of each word. This rank decreases after each valid move.

Definition 2.2. Given $M \subset \Lambda^* \setminus \{\varepsilon\}$, and $P \subseteq \Lambda^*$, we say that P is *M -closed* if for all $v_1 \cdots v_n \in \Lambda^* \setminus \{\varepsilon\}$, $v_1 \cdots v_n \in P$ and $v_{i+1} \cdots v_n \in M$ imply $v_1 \cdots v_i \in P$. For an M -closed P , the *restriction of the truncation game induced by M to P* is the game whose positions are the elements of P and whose valid moves consist of all truncations $v_1 \cdots v_n \rightarrow v_1 \cdots v_i$ such that $v_{i+1} \cdots v_n \in M$ and $v_1 \cdots v_n \in P$. We denote this game by (P, M) , and call it the *truncation game induced by M on P* .

Clearly the definition of being M -closed is equivalent to saying that the set P is closed under making valid moves.

Definition 2.3. We say that $M \subset \Lambda^* \setminus \{\varepsilon\}$ induces a *Bernoulli type truncation game* if for all pairs of words $\underline{u}, \underline{v} \in \Lambda^* \setminus \{\varepsilon\}$, $\underline{u}\underline{v} \in M$ and $\underline{v} \in M$ imply $\underline{u} \in M$. If M is also closed under taking nonempty initial segments, i.e., $v_1 \cdots v_n \in M$ implies $v_1 \cdots v_m \in M$ for all $m \in \{1, \dots, n\}$ then we say that M induces a *strongly Bernoulli type truncation game*. If M induces a (strongly) Bernoulli type truncation game, we call also (P, M) a (strongly) Bernoulli type truncation game for each M -closed $P \subseteq \Lambda^*$.

Every strongly Bernoulli type truncation game is also a Bernoulli type truncation game. The converse is not true: consider for example the set M of all words of positive even length. It is easy to see that the truncation game induced by M is Bernoulli type, but it is not strongly Bernoulli type since M is not closed under taking initial segments of odd length.

Remark 2.4. The definition of a Bernoulli type truncation game is *almost* a special case of the Bernoulli type games on posets defined in [4, Definition 3.1]. Each M -closed $P \subseteq \Lambda^*$ is partially ordered by the relation $v_1 \cdots v_m < v_1 \cdots v_n$ for all $m < n$, the unique minimum element of this poset is ε , and the length function is a rank function for this partial order. For this poset and rank function, the set of valid moves satisfies conditions (i) and (ii) listed in Subsection 1.2. Only the “uniformity” condition (iii) and the finiteness of $|P_n|$ do not need to be satisfied. These conditions were used in [4] to prove Eq. (2) and count the kernel positions of rank n “externally”. In this section we will show that the kernel positions of a strongly Bernoulli type truncation game on words may be described “internally” in a manner that will allow their enumeration when each $|P_n|$ is finite. The question whether the results presented in this section may be generalized to all Bernoulli type truncation games remains open. All examples of Bernoulli games played on words in [4] are isomorphic to strongly Bernoulli type truncation games, we will prove this for most of them in this paper, the remaining examples are left to the reader. Together with the results in [4], we thus obtain two independent ways to count the same kernel positions in these games. Comparing the results in [4] with the results in the present paper yields explicit formulas for the coefficients in the Taylor expansion of certain functions.

In the rest of the section we set $P = \Lambda^*$ and just find the winning strategy for the truncation game induced by M . Only the formulas counting the kernel positions will change when we change the set P in the subsequent sections, the decomposition of the kernel positions will not. First we define some *elementary kernel positions* in which the second player may win after at most one move by the first player.

Definition 2.5. The word $v_1 \cdots v_n \in \Lambda^* \setminus \{\varepsilon\}$ is an *elementary kernel position* if it satisfies $v_1 \cdots v_n \notin M$, but for all $m < n$ we have $v_1 \cdots v_m \in M$.

In particular, for $n = 1$, v_1 is an elementary kernel position if and only if $v_1 \notin M$. Our terminology is justified by the following two statements.

Remark 2.6. A position v_1 is a winning position, if and only if it is an elementary kernel position. Otherwise it is not a kernel position at all.

Lemma 2.7. For $n > 1$, starting from an elementary kernel position $v_1 \cdots v_n$, the first player is either unable to move, or is able to move only to a position where the second player may win in a single move.

Proof. There is nothing to prove if the first player is unable to move. Otherwise, by $v_1 \cdots v_n \notin M$, the first player is unable to move to the empty word. Thus, after his or her move, we arrive in a $v_1 \cdots v_m$ where $1 \leq m \leq n - 1$. Thus $v_1 \cdots v_m \in M$ holds, the second player may now move to the empty word right away. \square

Next we show that the set of kernel positions in a strongly Bernoulli type truncation game on Λ^* is closed under the concatenation operation.

Proposition 2.8. Let $\underline{u} := u_1 \cdots u_m$ be a kernel position of length $m \geq 1$ in a strongly Bernoulli truncation game induced by M . Then an arbitrary position $\underline{v} := v_1 \cdots v_n$ of length $n \geq 1$ is a kernel position if and only if the concatenation $\underline{u}\underline{v}$ is also a kernel position.

Proof. Assume first that \underline{v} is a kernel position. We instruct the second player to play the winning strategy that exists for \underline{v} as long as the length of the word truncated from $\underline{u}\underline{v}$ at the beginning of his or her move is greater than m . For pairs of words longer than m , the validity of a move is determined without regard to the letters in the first m positions. By playing the winning strategy for \underline{v} as long as possible, the second player is able to force the first player into a position where the first player is either unable to move, or will be the first to move to a word of length less than m , say $u_1 \cdots u_k$. The validity of this move implies $u_{k+1} \cdots u_m v_1 \cdots v_i \in M$ for some $i \geq 0$. By the strong Bernoulli property we obtain $u_{k+1} \cdots u_m \in M$ and moving from $u_1 \cdots u_m$ to $u_1 \cdots u_k$ is also a valid move. We may thus pretend that the first player just made the first move from $u_1 \cdots u_m$ and the second player may win by following the winning strategy that exists for \underline{u} .

For the converse, assume that \underline{v} is not a kernel position. In this case we may instruct the first player to play the strategy associated to \underline{v} as long as possible, forcing the second player into a position where he or she is either unable to move, or ends up making a move equivalent to a first move starting from \underline{u} . Now the original first player becomes the second player in this subsequent game, and is able to win. Therefore, in this case the concatenation $\underline{u}\underline{v}$ is not a kernel position either. \square

Using all results in this section we obtain the following structure theorem.

Theorem 2.9. A word $\underline{v} \in \Lambda^* \setminus \{\varepsilon\}$ is a kernel position in a strongly Bernoulli type truncation game, if and only if it may be obtained by the concatenation of one or several elementary kernel positions. Such a decomposition, if it exists, is unique.

Proof. The elementary kernel positions are kernel positions by Remark 2.6 and Lemma 2.7. Repeated use of Proposition 2.8 yields that a pair of words obtained by concatenating several elementary kernel positions is also a kernel position.

For the converse assume that $\underline{v} := v_1 \cdots v_n$ is a kernel position. We prove by induction on n that this position is either an elementary kernel position or may be obtained by concatenating several elementary kernel positions. Let m be the least index for which $v_1 \cdots v_m \notin M$ holds, such an m exists, otherwise the first player is able to move to ε and win in the first move. It follows from the definition that the $v_1 \cdots v_m$ is an elementary kernel position. If $m = n$ then we are done, otherwise applying Proposition 2.8 to $v_1 \cdots v_n = (v_1 \cdots v_m) \cdot (v_{m+1} \cdots v_n)$ yields that $v_{m+1} \cdots v_n$ must be a kernel position. We may apply the induction hypothesis to $v_{m+1} \cdots v_n$.

The uniqueness of the decomposition may also be shown by induction on n . Assume that $v_1 \cdots v_n$ is a kernel position and thus arises as a concatenation of one or several elementary kernel positions. Let $v_1 \cdots v_m$ be the leftmost factor in this concatenation. By Definition 2.5, m is the least index such $v_1 \cdots v_m \notin M$ is satisfied. This determines the leftmost factor uniquely. Now we may apply our induction hypothesis to $v_{m+1} \cdots v_n$. \square

3. The original Bernoulli game

When we want to apply Theorem 2.9 to the original Bernoulli game, we encounter two minor obstacles. The first obstacle is that the rule defining a valid move from $(u_1 \cdots u_n, v_1 \cdots v_n)$ makes an exception for the letters $u_1 = v_1 = 1$, and does not allow their removal. The second obstacle is that the game is defined on pairs of words. Both problems may be easily remedied by changing the alphabet to $\Lambda = \mathbb{P} \times \mathbb{P} \times \mathbb{P} = \mathbb{P}^3$ where \mathbb{P} is the set of positive integers.

Lemma 3.1. *The original Bernoulli game is isomorphic to the strongly Bernoulli type truncation game induced by*

$$M = \{(p_1, u_1, v_1) \cdots (p_n, u_n, v_n) : p_1 \neq 1, u_1 \leq v_1, \dots, v_n\},$$

on the set of positions

$$P = \{(1, u_1, v_1) \cdots (n, u_n, v_n) : 1 \leq u_i, v_i \leq i\} \subset (\mathbb{P}^3)^*.$$

The isomorphism is given by sending each pair of words $(u_1 \cdots u_n, v_1 \cdots v_n) \in (\mathbb{P}^2)^*$ into the word $(1, u_1, v_1)(2, u_2, v_2) \cdots (n, u_n, v_n) \in (\mathbb{P}^3)^*$.

Theorem 2.9 provides a new way of counting the kernel positions of rank n in the game (P, M) defined in Lemma 3.1. Each kernel position $(1, u_1, v_1) \cdots (n, u_n, v_n)$ may be uniquely written as a concatenation of elementary kernel positions. Note that these elementary kernel positions do not need to belong to the set of valid positions P . However, we are able to independently describe and count all elementary kernel positions that may appear in a concatenation factorization of a valid kernel position $(1, u_1, v_1) \cdots (n, u_n, v_n)$ and contribute the segment $(i, u_i, v_i) \cdots (j, u_j, v_j)$ to it. We call such a pair an *elementary kernel factor of type (i, j)* and denote the number of such factors by $\kappa(i, j)$. Note that for $i = 1$ we must have $j = 1$ and $(1, 1, 1)$ is the only elementary kernel factor of type $(1, 1)$. Thus we have $\kappa(1, 1) = 1$.

Lemma 3.2. *For $2 \leq i \leq j$, a word $(i, u_i, v_i) \cdots (j, u_j, v_j) \in (\mathbb{P}^3)^*$ is an elementary kernel factor of type (i, j) if and only if it satisfies the following criteria:*

- (i) for each $k \in \{i, i + 1, \dots, j\}$ we have $1 \leq u_k, v_k \leq k$;
- (ii) we have $u_i > v_j$;
- (iii) for all $k \in \{i, i + 1, \dots, j - 1\}$ we have $u_i \leq v_k$.

In fact, condition (i) states the requirement for a valid position for the letters at the positions i, \dots, j , whereas conditions (ii) and (iii) reiterate the appropriately shifted variant of the definition of an elementary kernel position. A word $(1, u_1, v_1) \cdots (n, u_n, v_n)$ that arises by concatenating $(1, u_1, v_1) \cdots (i_1, u_{i_1}, v_{i_1}), (i_1 + 1, u_{i_1+1}, v_{i_1+1}) \cdots (i_2, u_{i_2}, v_{i_2}),$ and so on, $(i_{k+1}, u_{i_{k+1}}, v_{i_{k+1}}) \cdots (n, u_n, v_n)$ belongs to P if and only if each factor $(i_s + 1, u_{i_s+1}, v_{i_s+1}) \cdots (i_{s+1}, u_{i_{s+1}}, v_{i_{s+1}})$ (where $0 \leq s \leq k, i_0 = 0$ and $i_{k+1} = n$) satisfies conditions (i) and (ii) in Lemma 3.2 with $i = i_s + 1$ and $j = i_{s+1}$. We obtain the unique factorization as a concatenation of elementary kernel positions if and only if each factor $(u_{i_s+1}, v_{i_s+1}) \cdots (u_{i_{s+1}}, v_{i_{s+1}})$ also satisfies condition (iii) in Lemma 3.2 with $i = i_s + 1$ and $j = i_{s+1}$. Using the description given in Lemma 3.2 it is easy to calculate the numbers $\kappa(i, j)$.

Lemma 3.3. For $2 \leq i \leq j$, the number of elementary kernel factors of type (i, j) is

$$\kappa(i, j) = (j - i)!^2 \binom{j}{i} \binom{j}{i - 2}.$$

Proof. There is no other restriction on v_{i+1}, \dots, v_j than the inequality given in condition (i) of Lemma 3.2. These numbers may be chosen in $(i + 1)(i + 2) \cdots j = j!/i!$ ways. Let us denote the value of u_i by u , this must satisfy $1 \leq u \leq i$. However, $v_j < u_i$ may only be satisfied if u is at least 2. In that case v_j may be selected in $(u - 1)$ ways, and each v_k (where $i \leq k \leq j - 1$) may be selected in $(k + 1 - u)$ ways (since $u_i \leq v_k \leq k$). Thus the values of v_i, \dots, v_j may be selected in $(u - 1) \cdot (i + 1 - u)(i + 2 - u) \cdots (j - u) = (u - 1) \cdot (j - u)! / (i - u)!$ ways. We obtain the formula

$$\kappa(i, j) = \sum_{u=2}^i (u - 1) \cdot \frac{j!(j - u)!}{i!(i - u)!} = (j - i)!^2 \binom{j}{i} \sum_{u=2}^i \binom{u - 1}{u - 2} \cdot \binom{j - u}{i - u}.$$

Replacing the binomial coefficients with symbols

$$\binom{\binom{n}{k}}{k} := \binom{n + k - 1}{k},$$

counting the k -element multisets on an n -element set, we may rewrite the last sum as

$$\sum_{u=2}^i \binom{\binom{2}{u - 2}}{u - 2} \cdot \binom{\binom{j - i + 1}{i - u}}{i - u} = \binom{\binom{j - i + 3}{i - 2}}{i - 2}.$$

Thus we obtain

$$\kappa(i, j) = (j - i)!^2 \binom{j}{i} \binom{\binom{j - i + 3}{i - 2}}{i - 2},$$

which is obviously equivalent to the stated equation. \square

Once we have selected the length of the elementary kernel factors in the unique decomposition of a kernel position, we may select each kernel factor of a given type independently. Thus we obtain the following result.

Theorem 3.4. For $n \geq 1$, the number κ_n of kernel positions of rank n in the original Bernoulli game is given by

$$\kappa_n = \sum_{k=0}^{n-2} \sum_{1=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k (i_{j+1} - i_j - 1)!^2 \binom{i_{j+1}}{i_j + 1} \binom{i_{j+1}}{i_j - 1}.$$

Proof. Consider the isomorphic game (P, M) given in Lemma 3.1. Assuming that the elementary kernel factors cover the positions 1 through 1, $2 = i_0 + 1$ through i_1 , $i_1 + 1$ through i_2 , and so on, $i_k + 1$ through $i_{k+1} = n$, we obtain the formula

$$\kappa_n = \kappa(1, 1) \sum_{k=0}^{n-1} \sum_{1=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k \kappa(i_j + 1, i_{j+1}),$$

from which the statement follows by $\kappa(1, 1) = 1$ and Lemma 3.3. \square

Comparing Theorem 3.4 with Theorem 1.1 we obtain the following formula for the Bernoulli numbers of the second kind.

Corollary 3.5. For $n \geq 2$ the Bernoulli numbers of the second kind are given by

$$b_n = (-1)^{n-1} \frac{1}{(n+1)!} \sum_{k=0}^{n-2} \sum_{1=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k (i_{j+1} - i_j - 1)!^2 \binom{i_{j+1}}{i_j + 1} \binom{i_{j+1}}{i_j - 1}. \tag{3}$$

Example 3.6. For $n = 4$, Eq. (3) yields

$$\begin{aligned} b_4 = & \frac{-1}{5!} \left((3-1)!^2 \binom{4}{2} \binom{4}{0} + (1-1)!^2 \binom{2}{2} \binom{2}{0} (3-2)!^2 \binom{4}{3} \binom{4}{1} \right. \\ & + (2-1)!^2 \binom{3}{2} \binom{3}{0} (3-3)!^2 \binom{4}{4} \binom{4}{2} \\ & \left. + (1-1)!^2 \binom{2}{2} \binom{2}{0} (2-2)!^2 \binom{3}{3} \binom{3}{1} (3-3)!^2 \binom{4}{4} \binom{4}{2} \right) = -\frac{19}{30}. \end{aligned}$$

Thus $b_4/4! = -19/720$, which agrees with the number tabulated by Jordan [5, p. 266].

As n increases, the number of terms in (3) increases exponentially. However, we are unaware of any other explicit formula expressing the Bernoulli numbers of the second kind as a sum of terms of the same sign.

Lemma 3.3 may also be used to obtain a recursion formula for the number of kernel positions of rank n in the original Bernoulli game.

Proposition 3.7. For $n \geq 2$, the number κ_n of kernel positions of rank n in the original Bernoulli game satisfies the recursion formula

$$\kappa_n = \sum_{i=1}^{n-1} \kappa_i (n-i-1)!^2 \binom{n}{i+1} \binom{n}{i-1}.$$

Proof. Consider again the isomorphic game (P, M) given in Lemma 3.1. Assume the last elementary kernel factor is $(i+1, u_{i+1}, v_{i+1}) \cdots (n, u_n, v_n)$ where $i \geq 1$. Removing it we obtain a kernel position of rank i . Conversely, concatenating an elementary kernel factor $(i+1, u_{i+1}, v_{i+1}) \cdots (n, u_n, v_n)$ to a kernel position of rank i yields a kernel position of rank n . Thus we have

$$\kappa_n = \sum_{i=0}^{n-1} \kappa_i \cdot \kappa(i+1, n), \tag{4}$$

and the statement follows by Lemma 3.3. \square

Comparing Proposition 3.7 with Theorem 1.1 we obtain the following recursion formula for absolute values of the Bernoulli numbers of the second kind.

$$|b_n| = \frac{1}{n+1} \sum_{i=1}^{n-1} |b_i| (n-i-1)! \binom{n}{i-1} \text{ holds for } n \geq 2. \tag{5}$$

Equivalently, Jordan’s [5] Bernoulli numbers of the second kind $b_n/n!$ satisfy

$$\left| \frac{b_n}{n!} \right| = \sum_{i=1}^{n-1} \left| \frac{b_i}{i!} \right| \frac{i}{(n+1)(n-i+1)(n-i)} \text{ for } n \geq 2. \tag{6}$$

Remark 3.8. Since the sign of b_n for $n \geq 1$ is $(-1)^{n-1}$, and substituting $x = 0$ in (1) gives

$$\sum_{n \geq 0} \frac{b_n}{n!} t^n = \frac{t}{\ln(1-t)},$$

it is easy to verify that (6) could also be derived from the following equation, satisfied by the generating function of the numbers b_n :

$$\frac{d}{dt} \left(t \cdot \frac{t}{\ln(1-t)} \right) + 1 - t = \frac{d}{dt} \left(\frac{t}{\ln(1-t)} \right) \cdot ((1-t)\ln(1-t) + t).$$

However, it seems hard to guess that this equation will yield a nice recursion formula.

4. Decomposing the indecomposable permutations

Definition 4.1. The *instant Bernoulli game* is the restriction of the original Bernoulli game to the set of positions $\{(12 \cdots n, v_1 \cdots v_n) : n \geq 1\}$.

Lemma 4.2. *Equivalently, we may define the set of positions of the instant Bernoulli game as the set of words $v_1 \cdots v_n$ satisfying $n \geq 1$ and $1 \leq v_i \leq i$ for all i . A valid move consists of replacing $v_1 \cdots v_n$ with $v_1 \cdots v_m$ for some $m \geq 1$ such that $m + 1 \leq v_{m+1}, v_{m+2}, \dots, v_n$ holds.*

Lemma 4.2 offers the simplest possible way to visualize the instant Bernoulli game, even if this is not a form in which the applicability of Theorem 2.9 could be directly seen. For that purpose we need to note that the isomorphism of games stated in Lemma 3.1 may be restricted to the set of positions of the instant Bernoulli game, and we obtain the following representation.

Lemma 4.3. *The instant Bernoulli game is isomorphic to the strongly Bernoulli type truncation game induced by*

$$M = \{(p_1, u_1, v_1) \cdots (p_n, u_n, v_n) : p_1 \neq 1, u_1 \leq v_1, \dots, v_n\},$$

on the set of positions

$$P = \{(1, 1, v_1) \cdots (n, n, v_n) : 1 \leq u_i, v_i \leq i\} \subset (\mathbb{P}^3)^*.$$

Unless otherwise noted, we will use the simplified representation stated in Lemma 4.2. The kernel positions of the instant Bernoulli game are identifiable with the primitive elements of the Malvenuto–Reutenauer Hopf algebra, as it was mentioned in the concluding remarks of [4]. We call this game the instant Bernoulli game because this is a game in which one of the players wins instantly: either there is no valid move and the second player wins instantly, or the first player may select the least $m \geq 1$ satisfying $m + 1 \leq v_{m+1}, v_{m+2}, \dots, v_n$ and move to $v_1 \cdots v_m$, thus winning instantly. The kernel positions are identical to the winning positions in this game. The recursion formula (2) may be rewritten as

$$n! = \kappa_n + \sum_{m=1}^{n-1} \kappa_m (n-m)!$$

(we start the summation with κ_1 since the first letter cannot be removed), and the generating function of the numbers κ_n is easily seen to be

$$\sum_{n=1}^{\infty} \kappa_n t^n = 1 - \frac{1}{\sum_{n=0}^{\infty} n! t^n}. \tag{7}$$

The numbers $\{\kappa_n\}_{n \geq 0}$ are listed as sequence A003319 in the On-Line Encyclopedia of Integer Sequences [11], and count the number of *connected* or *indecomposable* permutations of $\{1, 2, \dots, n\}$.

A permutation $\pi \in S_n$ is *connected* if there is no $m < n$ such that π takes the set $\{1, \dots, m\}$ into itself. The kernel positions of the instant Bernoulli game are directly identifiable with the connected permutations in more than one ways. One way is mentioned at the end of [4], we may formalize that bijection using two variants of the well-known *inversion tables* (see, for example [7, Section 5.1.1] or [13, Section 1.3]).

Definition 4.4. Given a permutation $\pi \in S_n$ we define its *letter-based non-inversion table* as the word $v_1 \cdots v_n$ where $v_j = 1 + |\{i < j: \pi^{-1}(i) < \pi^{-1}(j)\}|$.

For example, for $\pi = 693714825$ the letter-based non-inversion table is 121351362. This is obtained by adding 1 to all entries in the usual definition of an inversion table [13, Section 1.3] of the permutation $\tilde{\pi} = 417396285$, defined by $\tilde{\pi}(i) = n + 1 - \pi(i)$ and taking the reverse of the resulting word. In particular, for $\tilde{\pi} = 417396285$ we find the inversion table (1, 5, 2, 0, 4, 2, 0, 1, 0) in [13, Section 1.3]. Our term *letter-based* refers to the fact that here we associate the letter j to v_j and not the place j .

A variant of the notion of letter-based non-inversion table is the place-based non-inversion table.

Definition 4.5. Given a permutation $\pi \in S_n$ we define its *place-based non-inversion table (PNT)* as the word $v_1 \cdots v_n$ where $v_j = 1 + |\{i < j: \pi(i) < \pi(j)\}|$.

Obviously the PNT of a permutation π equals the letter-based non-inversion table of π^{-1} . For example, for $\pi = 583691472$ the PNT is 121351362. We have $v_7 = 3 = 1 + 2$ because $\pi(7) = 4$ is preceded by two letters $\pi(i)$ such that $(\pi(i), \pi(7))$ is not an inversion. Any PNT $v_1 \cdots v_n$ is a word satisfying $1 \leq i \leq v_i$.

Lemma 4.6. A position $v_1 \cdots v_n$ in the instant Bernoulli game is a kernel position if and only if it is the place-based (letter-based) non-inversion table of a connected permutation.

Proof. We prove the place-based variant of the lemma, the letter-based version follows immediately since the set of connected permutations is closed under taking inverses. It is easy to verify that the place-based non-inversion table $v_1 \cdots v_n$ of a permutation π satisfies $m + 1 \leq v_{m+1}, \dots, v_n$ if and only if π takes the set $\{1, \dots, m\}$ into itself. Thus the first player has no valid move if and only if π is connected. \square

The study of connected permutations goes back to the work of Comtet [2,3], for a reasonably complete list of references we refer to the entry A003319 in the On-Line Encyclopedia of Integer Sequences [11]. It was shown by Poirier and Reutenauer [12] that the connected permutations form a free algebra basis of the Malvenuto–Reutenauer Hopf-algebra, introduced by Malvenuto and Reutenauer [9]. The same statement appears in dual form in the work of Aguiar and Sottile [1].

Although the instant Bernoulli game is very simple, Theorem 2.9 offers a nontrivial analysis of its kernel positions, allowing to identify a unique structure on each connected permutation. We begin with stating the following analogue of Theorem 3.4.

Theorem 4.7. The number κ_n of connected permutations of rank n is given by

$$\kappa_n = \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \dots < i_{k+1} = n} \prod_{j=1}^k (i_{j+1} - i_j - 1)! \cdot i_j.$$

Proof. By Lemma 4.6, κ_n is the number of kernel positions of rank n in the instant Bernoulli game. The fact that this number is equal to the expression on the right-hand side may be shown similarly to the proof of Theorem 3.4. Consider the equivalent representation of the instant Bernoulli game given in Lemma 4.2. Note that this is obtained from the representation given in Lemma 4.3 by deleting the

“redundant coordinates” i, i from each letter (i, i, v_i) . Given an arbitrary kernel position $v_1 \cdots v_n$, the first letter $v_1 = 1$ corresponds to an elementary kernel factor of type $(1, 1)$ and we have $\kappa(1, 1) = 1$. For $2 \leq i \leq j$, by abuse of terminology, let us call $v_i \cdots v_j$ an elementary kernel factor of type (i, j) if it corresponds to an elementary kernel factor in the equivalent representation in Lemma 4.3. The elementary kernel factors of type (i, j) are then exactly those words $v_i \cdots v_j$ for which $i \leq v_i, \dots, v_{j-1}$ and $v_j < i$ hold. Thus their number is

$$\kappa(i, j) = (j - i)! \cdot (i - 1). \tag{8}$$

The statement now follows from the obvious formula

$$\kappa_n = \kappa(1, 1) \cdot \sum_{k=1}^{n-1} \sum_{1=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=1}^k \kappa(i_j + 1, i_{j+1}). \quad \square$$

In analogy to Proposition 3.7, we may also use (8) to obtain a recursion formula for the number of connected permutations. We end up with a formula that was first discovered by King [6, Theorem 4].

Proposition 4.8 (King). *For $n \geq 2$, the number κ_n of connected permutations of rank n satisfies the recursion formula*

$$\kappa_n = \sum_{i=1}^{n-1} \kappa_i (n - i - 1)! i.$$

The proof may be presented the same way as for Proposition 3.7, by removing the last elementary kernel factor of type (i, n) , using informal notion of an elementary kernel factor as in the proof of Theorem 4.7. King’s proof is worded differently, but may be shown to yield a bijectively equivalent decomposition.

Lemma 4.9. *The induction step presented in King’s proof of Proposition 4.8 is equivalent to the removal of the last elementary kernel factor in the place-based non-inversion table of $\tilde{\sigma}(1)\tilde{\sigma}(2) \cdots \tilde{\sigma}(n)$. Here $\tilde{\sigma}(i) = n + 1 - \sigma(n + 1 - i)$.*

Proof. Let $\sigma(1) \cdots \sigma(n)$ be the connected permutation considered in King’s proof, and let $v_1 \cdots v_n$ be the PNT of $\tilde{\sigma}(1)\tilde{\sigma}(2) \cdots \tilde{\sigma}(n)$. King’s proof first identifies $\sigma(1) = r$. This is equivalent to setting $v_n = n + 1 - r$. King then defines $\pi(1) \cdots \pi(n - 1)$ as the permutation obtained by deleting $\sigma(1)$ and subtracting 1 from all letters greater than r . Introducing $\tilde{\pi}(i) = n - \pi(n - i)$, the permutation $\tilde{\pi}(1) \cdots \tilde{\pi}(n - 1)$ is obtained from $\tilde{\sigma}(1)\tilde{\sigma}(2) \cdots \tilde{\sigma}(n)$ by deleting the last letter $n + 1 - r$ and by decreasing all letters greater than $n + 1 - r$ by one. The PNT of $\tilde{\pi}(1) \cdots \tilde{\pi}(n - 1)$ is thus $v_1 \cdots v_{n-1}$. King then defines j as the largest j such that $\pi(\{1, \dots, j\}) = \{1, \dots, j\}$. This is equivalent to finding the least $n - j$ such that $\tilde{\pi}(\{n - j, n - j + 1, \dots, n - 1\}) = \{n - j, n - j + 1, \dots, n - 1\}$. Using the proof of Lemma 4.6, this is easily seen to be equivalent to finding the smallest $n - j$ such that $v_{n-j} = n - j$ and for all $n - j \leq k \leq n - 1$ we have $v_k \geq n - j$. King defines $\beta(\pi)$ as the permutation obtained from π by removing $\pi(1) \cdots \pi(j)$ and then subtracting j from each element. Correspondingly, we may define $\tilde{\beta}(\tilde{\pi})$ as the permutation obtained from $\tilde{\pi}$ by removing $\tilde{\pi}(n - j) \cdots \tilde{\pi}(n - 1)$. The PNT of $\tilde{\beta}(\tilde{\pi})$ is then $v_1 \cdots v_{n-j-1}$, representing a kernel position in the instant Bernoulli game. This is the kernel position of the least rank that is reachable from $v_1 \cdots v_{n-1}$. In terms of elementary kernel factors, the removal of v_n makes the first player able to remove the rest of the last elementary kernel factor in a single valid move, we only need to show that the first player cannot move to a position $v_1 \cdots v_k$ where $r \leq k \leq s$ for some elementary kernel factor $v_r \cdots v_s$. Assume by way of contradiction that such a move is possible. By definition of a valid move, we then have $k \leq v_s$, implying $r \leq v_s$, in contradiction with the definition of the elementary kernel factor $v_r \cdots v_s$. Therefore $v_1 \cdots v_{n-j-1}$ is obtained from $v_1 \cdots v_n$ by removing exactly the last elementary kernel factor. \square

King [6] uses the removal of the last elementary kernel factor to recursively define a *transposition Gray code* of all connected permutations of a given rank. A transposition Gray code is a list of permutations such that subsequent elements differ by a transposition. Using place-based non-inversion tables, not only the last elementary kernel factor is easily identifiable, but the entire unique decomposition into elementary kernel factors is transparent. This gives rise to a new way to systematically list all connected permutations. The resulting list is not a transposition Gray code, but it is fairly easy to generate.

To explain the construction, consider the connected permutation $\pi = 251376948$. Its letter-based non-inversion table is $v_1 \cdots v_8 = 121355748$ whose decomposition into elementary kernel factors is $1 \cdot 21 \cdot 3 \cdot 5574 \cdot 8$. For $i < j$, each elementary kernel factor of type (i, j) begins with i , all entries in the factor are at least i , except for the last letter which is less than i . For $i = 1$, 1 is a special elementary kernel factor, for $i > 1$ a kernel factor of type (i, i) is a positive integer less than i .

Definition 4.10. Given a connected permutation π , we define its *elevation* $E(\pi)$ as the permutation whose PNT is obtained from the PNT of π as follows: for each elementary kernel factor of type (i, j) , increase the last letter in the factor to j .

For example, the PNT of the elevation of 251376948 is $1 \cdot 23 \cdot 4 \cdot 5578 \cdot 9$, thus $E(\pi)$ is 123465789. The PNT of $E(\pi)$ is written as a product of factors, such that each factor $u_i \cdots u_j$ ends with j , and all letters after u_j are more than j . We may use this observation to prove that each factor $u_i \cdots u_j$ ends with a j that is a *strong fixed point* j .

Definition 4.11. A number $i \in \{1, \dots, n\}$ is a strong fixed point of a permutation σ of $\{1, \dots, n\}$ if $\sigma(i) = i$ and $\sigma(\{1, \dots, i\}) = \{1, \dots, i\}$. We denote the set of strong fixed points of σ by $SF(\sigma)$.

Remark 4.12. The definition of a strong fixed point may be found in Stanley’s book [13, Chapter 1, Exercise 32b], where it is stated that the number $g(n)$ of permutations of rank n with no strong fixed points has the generating function

$$\sum_{n \geq 0} g(n)t^n = \frac{\sum_{n \geq 0} n!t^n}{1 + t \sum_{n \geq 0} n!t^n}.$$

Lemma 4.13. Let $v_1 \cdots v_n$ be the PNT of a permutation σ . Then j is a strong fixed point of σ if and only if $v_j = j$ and for all $k > j$ we have $v_k > j$.

In fact, the condition $\forall j (k > j \implies v_k > j)$ is easily seen to be equivalent to $\sigma(\{1, \dots, j\}) = \{1, \dots, j\}$. Assuming this is satisfied, j is a fixed point of σ if and only if $v_j = j$. As a consequence of Lemma 4.13 the last letters of the elementary kernel factors of the PNT of π mark strong fixed points of $E(\pi)$. The converse is not necessarily true: in our example 7 is a strong fixed point of $E(\pi)$; however, no elementary kernel factor of the PNT of π ends with v_7 . On the other hand, v_1 is always a special elementary kernel factor by itself and the last elementary kernel factor must end at v_n , thus 1 and n must always be strong fixed points of $E(\pi)$. The numbers 1 and n are also special in the sense that $i \in \{1, n\}$ is a strong fixed point if and only if it is a fixed point.

Theorem 4.14. Let $\sigma \in S_n$ be a permutation satisfying $\sigma(1) = 1$ and $\sigma(n) = n$ and let the strong fixed points of σ be $1 = i_0 < i_1 < \cdots < i_{k+1} = n$. Then there are exactly $(i_1 + 1) \cdots (i_k + 1)$ connected permutations π whose elevation is σ .

Proof. Assume $E(\pi) = \sigma$ and the PNT of π is the product of elementary factors of type $(1, 1), (j_0 + 1, j_1), (j_1 + 1, j_2), \dots, (j_l + 1, j_{l+1})$, where $1 = j_0 < j_1 < \cdots < j_{l+1} = n$. As we have seen above, $\{j_1, \dots, j_l\}$ must be a subset of $\{i_1, \dots, i_k\}$. This condition is also sufficient since we may decompose the PNT of σ as $u_1 \cdot (u_{j_0+1} \cdots u_{j_1}) \cdots (u_{j_l+1}, u_{j_{l+1}})$, and decrease the value of each $u_{j_t} = j_t$ (where

$t = 1, 2, \dots, l + 1$) independently to any number that is at most j_{t-1} . Note that each $u_{j_t+1} = j_t + 1$, and the required inequalities for all other u_j s are automatically satisfied as a consequence of having selected the j_t s from among the strong fixed points. Thus we obtain the PNT of a connected permutation, whose kernel factors are of type $(1, 1), (j_0 + 1, j_1), (j_1 + 1, j_2), \dots, (j_l + 1, j_{l+1})$. Therefore the number of permutations π satisfying $E(\pi) = \sigma$ is

$$\sum_{l=0}^k \sum_{\{j_1, \dots, j_l\} \subseteq \{i_1, \dots, i_k\}} j_1 \cdots j_l = (i_1 + 1) \cdots (i_k + 1). \quad \square$$

The proof of Theorem 4.14 suggests a straightforward way to list the PNTs of all connected permutations of rank n :

- (1) List all words $u_1 \cdots u_n$ satisfying $u_1 = 1, u_n = n$ and $1 \leq u_i \leq i$ for all i . These are the PNTs of all permutations of rank n , of which 1 and n are fixed points.
- (2) For each $u_1 \cdots u_n$, identify the places of strong fixed points by finding all i s such that $u_i = i$ and $u_k > i$ for all $k > i$.
- (3) For each $u_1 \cdots u_n$ select a subset $\{j_1, \dots, j_l\}$ of the set of strong fixed points satisfying $1 < j_1 < \dots < j_l < n$ and decrease the values of each u_{j_t} to any number in $\{1, \dots, j_{t-1}\}$. Output these as the PNTs of connected permutations.

Steps and (1) and (3) involve nothing more than listing words using some lexicographic order, step (2) may be performed after reading each word once.

As a consequence of Theorem 4.14 we obtain the following formula for the number of connected permutations of rank $n \geq 2$:

$$\kappa_n = \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1, \sigma(n)=n}} \prod_{i \in \text{SF}(\sigma) \setminus \{1, n\}} (i + 1).$$

After removing the redundant letters $\sigma(1) = 1$ and $\sigma(n) = n$ and decreasing all remaining letters by 1, we obtain that

$$\kappa_n = \sum_{\sigma \in S_{n-2}} \prod_{i \in \text{SF}(\sigma)} (i + 2) \quad \text{holds for } n \geq 2. \tag{9}$$

Eq. (9) offers a new combinatorial model for the numbers counting the connected permutations of rank $n \geq 2$: it is the total weight of all permutations of rank $n - 2$, using a weighting which assigns the most value to those permutations which have the most strong fixed points and are thus in a sense the farthest from being connected.

5. The polynomial Bernoulli game of the second kind, indexed by x

This game is defined in [4] on triplets of words $(u_1 \cdots u_n, v_1 \cdots v_n, w_1 \cdots w_n)$ for $n \geq 0$ such that $1 \leq u_i \leq i, 1 \leq v_i \leq i + 1$ and $1 \leq w_i \leq x$ hold for $i \geq 1$, furthermore we require $w_i \leq w_{i+1}$ for all $i \leq n - 1$. A valid move consists of replacing $(u_1 \cdots u_n, v_1 \cdots v_n, w_1 \cdots w_n)$ with $(u_1 \cdots u_m, v_1 \cdots v_m, w_1 \cdots w_m)$ for some $m \geq 0$ satisfying $w_{m+1} = w_{m+2} = \dots = w_n = x$ and $u_{m+1} < v_j$ for $j = m + 1, \dots, n$. Theorem 2.9 is applicable to this game, because of the following isomorphism.

Lemma 5.1. *Let $\Lambda = \mathbb{P} \times \mathbb{P} \times \{1, \dots, x\}$ where $x \in \mathbb{P}$. The polynomial Bernoulli game, indexed by x is isomorphic to the strongly Bernoulli type truncation game, induced by*

$$M := \{(u_1, v_1, x) \cdots (u_n, v_n, x) : u_1 < v_1, \dots, v_n\}$$

on the set of positions

$$P := \{(u_1, v_1, w_1) \cdots (u_n, v_n, w_n) : 1 \leq u_i \leq i, 1 \leq v_i \leq i + 1, w_1 \leq \dots \leq w_n\}.$$

This isomorphism is given by sending each triplet $(u_1 \cdots u_n, v_1 \cdots v_n, w_1 \cdots w_n) \in \mathbb{P}^* \times \mathbb{P}^* \times \{1, \dots, x\}^*$ into $(u_1, v_1, w_1) \cdots (u_n, v_n, w_n) \in (\mathbb{P} \times \mathbb{P} \times \{1, \dots, x\})^*$.

Theorem 5.2. The number κ_n of kernel positions of rank n in the polynomial Bernoulli game of the second kind, indexed by x is

$$\begin{aligned} \kappa_n &= \sum_{m=0}^{n-1} \binom{x+m-2}{m} m!(m+1)! \\ &\quad \times \sum_{k=0}^{n-m-1} \sum_{m=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k (i_{j+1} - i_j - 1)!^2 \binom{i_{j+1}}{i_j + 1} \binom{i_{j+1} + 1}{i_j} \\ &\quad + \binom{x+n-2}{n} n!(n+1)!. \end{aligned}$$

Proof. Consider the isomorphic game (P, M) , given in Lemma 5.1. Since in a valid move all truncated letters (u_j, v_j, w_j) satisfy $w_j = x$, we have to distinguish two types of elementary kernel factors: those which contain a letter (u_i, v_i, w_i) with $w_i < x$ and those which do not. If the elementary kernel factor contains a (u_i, v_i, w_i) with $w_i < x$, it must consist of the single letter (u_i, v_i, w_i) . We call such a factor an *elementary kernel factor of type $(i; w_i)$* . Clearly, their number is

$$\kappa(i; w_i) = i(i+1), \tag{10}$$

since $u_i \in \{1, \dots, i\}$ and $v_i \in \{1, \dots, i+1\}$ may be selected independently. The elementary kernel factors containing only x in their w -component of their letters are similar to the ones considered in Lemma 3.2. We call an elementary kernel factor of type $(i, j; x)$ an elementary kernel factor $(u_i, v_i, x) \cdots (u_j, v_j, x)$. A calculation completely analogous to the one in Lemma 3.3 shows that their number is

$$\kappa(i, j; x) = \sum_{u=1}^i u \frac{(j-u)!j!}{(i-u)!i!} = (j-i)!^2 \binom{j}{i} \binom{j+1}{i-1}. \tag{11}$$

Because of $w_1 \leq \dots \leq w_n$, the factors of type $(i; w_i)$ must precede the factors of type $(i, j; x)$. Thus we obtain

$$\begin{aligned} \kappa_n &= \sum_{m=0}^{n-1} \sum_{1 \leq w_1 \leq \dots \leq w_m \leq x-1} \prod_{i=1}^m \kappa(i; w_i) \sum_{k=0}^{n-m-1} \sum_{m=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k \kappa(i_j + 1, i_{j+1}; x) \\ &\quad + \sum_{1 \leq w_1 \leq \dots \leq w_n \leq x-1} \prod_{i=1}^n \kappa(i; w_i) \end{aligned}$$

The statement now follows from (10), (11), from $\prod_{i=1}^m i(i+1) = m!(m+1)!$, and from the fact that the number of words $w_1 \cdots w_m$ satisfying $1 \leq w_1 \leq \dots \leq w_m \leq x-1$ is

$$\binom{\binom{x-1}{m}}{m} = \binom{x+m-2}{m}. \quad \square$$

We already know [4, Theorem 4.2] that we also have

$$\kappa_n = (-1)^n (n+1)! b_n(-x)$$

for all positive integer x . Since two polynomial functions are equal if they agree for infinitely many substitutions, we obtain a valid expansion of the polynomial $(-1)^n (n+1)! b_n(-x)$. Substituting $-x$ into x and rearranging yields the expansion of $b_n(x)$ in the basis $\{\binom{x+1}{n}; n \geq 0\}$.

Corollary 5.3. *Introducing $c_{n,n} = n!$ and*

$$c_{n,m} = \frac{(-1)^{n-m} m!(m+1)!}{(n+1)!} \sum_{k=0}^{n-m-1} \sum_{m=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k (i_{j+1} - i_j - 1)!^2 \binom{i_{j+1}}{i_j} \binom{i_{j+1} + 1}{i_j}$$

for $0 \leq m < n$, we have

$$b_n(x) = \sum_{m=0}^n c_{n,m} \binom{x+1}{m}.$$

Example 5.4. For $n = 2$, Corollary 5.3 gives

$$\begin{aligned} b_2(x) &= \frac{0!1!}{3!} \left(1!^2 \binom{2}{1} \binom{3}{0} + 0!^2 \binom{1}{1} \binom{2}{0} 0!^2 \binom{2}{2} \binom{3}{1} \right) \\ &\quad - \frac{1!2!}{3!} \binom{x+1}{1} 0!^2 \binom{2}{2} \binom{3}{1} + \binom{x+1}{2} 2! \\ &= \frac{5}{6} - (x+1) + (x+1)x = x^2 - \frac{1}{6}. \end{aligned}$$

Thus $b_2(x)/2! = x^2/2 - 1/12$ which agrees with the formula given in [5, §92].

We may also obtain a new formula for the Bernoulli numbers of the second kind by substituting $x = 0$ into Corollary 5.3. We obtain $b_n = c_{n,0} + c_{n,1}$, i.e.,

$$\begin{aligned} b_n &= \frac{(-1)^n}{(n+1)!} \sum_{k=0}^{n-1} \sum_{0=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k (i_{j+1} - i_j - 1)!^2 \binom{i_{j+1}}{i_j} \binom{i_{j+1} + 1}{i_j} \\ &\quad + \frac{(-1)^{n-1} \cdot 2}{(n+1)!} \sum_{k=0}^{n-2} \sum_{1=i_0 < i_1 < \dots < i_{k+1}=n} \prod_{j=0}^k (i_{j+1} - i_j - 1)!^2 \binom{i_{j+1}}{i_j} \binom{i_{j+1} + 1}{i_j} \end{aligned} \tag{12}$$

for $n \geq 2$.

6. The flat Bernoulli game

This game is defined in [4] on words $u_1 \dots u_n$ for $n \geq 0$ such that each $u_i \in \mathbb{P}$ satisfies $1 \leq u_i \leq i$. A valid move consists of replacing $u_1 \dots u_n$ with $u_1 \dots u_m$ if $m \geq 1$ and $u_{m+1} < u_j$ holds for all $j > m + 1$. In analogy to Lemma 3.1, we have the following result.

Lemma 6.1. *The flat Bernoulli game is isomorphic to the strongly Bernoulli type truncation game induced by*

$$M = \{(p_1, u_1) \dots (p_n, v_n) : p_1 \neq 1, u_1 < u_2, \dots, u_n\},$$

on the set of positions

$$P = \{(1, u_1) \dots (n, u_n) : 1 \leq u_i \leq i\} \subset (\mathbb{P}^2)^*.$$

The isomorphism is given by sending each word $u_1 \dots u_n \in \mathbb{P}^*$ into the word $(1, u_1)(2, u_2) \dots (n, u_n) \in (\mathbb{P}^2)^*$.

Theorem 6.2. *For $n \geq 2$, the number κ_n of kernel positions of rank n in the flat Bernoulli game is*

$$\kappa_n = \sum_{k=0}^{\lfloor (n-3)/2 \rfloor} \sum_{\substack{1=i_0 < i_1 < \dots < i_{k+1}=n \\ i_{j+1}-i_j \geq 2}} \prod_{j=0}^k (i_{j+1} - i_j - 2)! \binom{i_{j+1}}{i_j}.$$

Proof. Consider isomorphic representation given in Lemma 6.1. Note first that, in any kernel position $(1, u_1) \cdots (n, u_n)$, the letter $u_1 = (1, 1)$ is an elementary kernel factor of type $(1, 1)$ and we have $\kappa(1, 1) = 1$. For $2 \leq i < j$, let $\kappa(i, j)$ be the number of elementary kernel factors $(i, u_i) \cdots (j, u_j)$ of type (i, j) . A calculation completely analogous to the one in Lemma 3.3 shows

$$\kappa(i, j) = \sum_{u=1}^i u \frac{(j-1-u)!}{(j-1-i)!} = (j-1-i)! \binom{j}{i-1}. \tag{13}$$

Note that for $i \geq 2$ there is no elementary kernel factor of type (i, i) since removing the last letter only is always a valid move, provided at least one letter is left. The statement now follows from Eq. (13) and the obvious formula

$$\kappa_n = \kappa(1, 1) \cdot \sum_{k=0}^{\lfloor (n-3)/2 \rfloor} \sum_{\substack{1=i_0 < i_1 < \dots < i_{k+1}=n \\ i_{j+1}-i_j \geq 2}} \prod_{j=0}^k \kappa(i_j+1, i_{j+1}). \quad \square$$

Introducing $m_j := i_j - i_{j-1} - 1$ for $j \geq 1$ and shifting the index k by 1, we may rewrite the equation in Theorem 6.2 as

$$\kappa_n = n \cdot \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \sum_{\substack{m_1 + \dots + m_k = n-1 \\ m_1, \dots, m_k \geq 2}} \binom{n-1}{m_1, \dots, m_k} (m_1 - 2)! \cdots (m_k - 2)!. \tag{14}$$

A more direct proof of this equation follows from Corollary 6.6 below.

Example 6.3. For $n = 5$, (14) yields

$$\kappa_5 = 5 \left(\binom{4}{4} 2! + \binom{4}{2, 2} 0! 0! \right) = 40.$$

Thus $\kappa_5/5! = 1/3$ which agrees with the number given in [4, Table 1].

We already know [4, Proposition 7.3] that the exponential generating function of the numbers κ_n is

$$\sum_{n=1}^{\infty} \frac{\kappa_n}{n!} t^n = \frac{t}{(1-t)(1-\ln(1-t))}. \tag{15}$$

Just like in Section 4, we may use place-based non-inversion tables to find a permutation enumeration model for the numbers κ_n .

Lemma 6.4. Let $u_1 \cdots u_n$ be the PNT of a permutation $\pi \in S_n$. Then, for all $i < j$, $\pi(i) < \pi(j)$ implies $u_i < u_j$. The following partial converse is also true: $u_i < u_{i+1}, \dots, u_j$ implies $\pi(i) < \pi(i+1), \dots, \pi(j)$.

Proof. If $\pi(i) < \pi(j)$ then the set $\{k < i: \pi(k) < \pi(i)\}$ is a proper subset of $\{k < j: \pi(k) < \pi(j)\}$ (the index i belongs only to the second subset). Thus $u_i < u_j$. The converse may be shown by induction on $j - i$. For $j = i + 1$, $\pi(i) > \pi(i + 1)$ implies that the set $\{k < i + 1: \pi(k) < \pi(i + 1)\}$ is a subset of $\{k < i: \pi(k) < \pi(i)\}$, thus $u_i \geq u_{i+1}$. Therefore $u_i < u_{i+1}$ implies $\pi(i) < \pi(i + 1)$. Assume now that $u_i \leq u_{i+1}, \dots, u_j$ holds and that we have already shown $\pi(i) < \pi(i + 1), \dots, \pi(j - 1)$. Assume, by way of contradiction, that $\pi(i) > \pi(j)$ holds. Then there is no k satisfying $i < k < j$ and $\pi(k) < \pi(j)$ thus $\{k < j: \pi(k) < \pi(j)\}$ is a subset of $\{k < i: \pi(k) < \pi(i)\}$, implying $u_i \geq u_j$, a contradiction. Therefore we obtain $\pi(i) < \pi(j)$. \square

Corollary 6.5. Let $u_1 \cdots u_n$ be the PNT of a permutation $\pi \in S_n$. Then $u_i \cdots u_j$ satisfies $u_i < u_{i+1}, \dots, u_{j-1}$ and $u_i \geq u_j$ if and only if $\pi(j) < \pi(i) < \pi(i + 1), \dots, \pi(j - 1)$ holds.

Corollary 6.6. Let $u_1 \cdots u_n$ be the PNT of a permutation $\pi \in S_n$. Then $u_1 \cdots u_n$ is a kernel position in the flat Bernoulli game, if and only if there exists a set of indices $1 = i_0 < i_1 < \cdots < i_{k+1} = n$ such that for each $j \in \{0, \dots, k\}$ we have $\pi(i_{j+1}) < \pi(i_j + 1) < \pi(i_j + 2), \pi(i_j + 3), \dots, \pi(i_{j+1} - 1)$.

Eq. (14) also follows from Corollary 6.6. In fact, there are n ways to select $\pi(1)$. Then, introducing $m_j := i_j - i_{j-1} - 1$ for $j \geq 1$, we have $\binom{n-1}{m_1, \dots, m_k}$ ways to select the partitioning

$$\{1, \dots, n\} \setminus \pi(1) = \bigsqcup_{j=0}^k \pi(\{i_j + 1, \dots, i_{j+1}\})$$

and, for each j there are $(i_{j+1} - i_j - 2)! = (m_j - 2)!$ ways to select the partial permutation $\pi(i_j + 1) \cdots \pi(i_{j+1})$. Both Eq. (14) and Corollary 6.6 suggest looking at the numbers

$$K_n = \kappa_{n+1}/(n + 1) = \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\substack{m_1 + \dots + m_k = n \\ m_1, \dots, m_k \geq 2}} \binom{n}{m_1, \dots, m_k} (m_1 - 2)! \cdots (m_k - 2)! \quad \text{for } n \geq 0. \quad (16)$$

It is easy to check the following statement.

Proposition 6.7. K_n is the number of kernel positions of rank n in the exception-free variant of the flat Bernoulli game, where removing the entire word if $u_1 < u_2, \dots, u_n$ is also a valid move, and the empty word is a valid position.

Corollary 6.6 may be rephrased as follows.

Corollary 6.8. K_n is the number of those permutations $\pi \in S_n$ for which there exists a set of indices $0 = i_0 < i_1 < \cdots < i_{k+1} = n$ such that for each $j \in \{0, \dots, k\}$ we have $\pi(i_{j+1}) < \pi(i_j + 1) < \pi(i_j + 2), \pi(i_j + 3), \dots, \pi(i_{j+1} - 1)$.

The generating function of the numbers K_n is

$$\sum_{n=0}^{\infty} \frac{K_n}{n!} t^n = \frac{1}{(1-t)(1-\ln(1-t))}. \quad (17)$$

This formula may be derived not only from $K_n = \kappa_{n+1}/(n + 1)$ and (15), but also from Corollary 6.8 and the compositional formula for exponential generating functions [14, Theorem 5.5.4]. We only need to observe that

$$\frac{1}{(1-t)(1-\ln(1-t))} = \frac{1}{1-t} \circ (t + (1-t)\ln(1-t)),$$

where

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} \frac{n! t^n}{n!}$$

is the exponential generating function of linear orders, whereas

$$t + (1-t)\ln(1-t) = -t\ln(1-t) - (-\ln(1-t-t)) = \sum_{n=1}^{\infty} \frac{t^{n+1}}{n} - \sum_{n=2}^{\infty} \frac{t^n}{n} = \sum_{n=2}^{\infty} \frac{(n-2)! t^n}{n!}$$

is the exponential generating function of linear orders of $\{1, \dots, n\}$, listing 1 last and 2 first.

By taking the antiderivative on both sides of (17) we obtain

$$\sum_{n=0}^{\infty} \frac{K_n}{(n+1)!} t^{n+1} = \int \frac{1}{(1-t)(1-\ln(1-t))} dt = \ln(1-\ln(1-t)) + K_{-1}.$$

Introducing $K_{-1} := 0$, the numbers K_{-1}, K_0, K_1, \dots are listed as sequence A089064 in the On-Line Encyclopedia of Integer Sequences [11]. There we may also find the formula

$$K_n = (-1)^n \sum_{k=1}^{n+1} s(n+1, k) \cdot (k-1)! \tag{18}$$

expressing them in terms of the Stirling numbers of the first kind. Using the well-known formulas

$$\sum_{k=1}^n s(n+1, k)x^k = x(x-1)\cdots(x-n) \quad \text{and} \quad n! = \int_0^\infty x^n e^{-x} dx,$$

Eq. (18) is equivalent to

$$K_n = (-1)^n \int_0^\infty (x-1)\cdots(x-n)e^{-x} dx. \tag{19}$$

This formula may be directly verified by substituting it into the left-hand side of (17) and obtaining

$$\int_0^\infty e^{-x} \sum_{n=0}^\infty \binom{x-1}{n} (-t)^n dx = \int_0^\infty e^{-x} (1-t)^{x-1} dx = \frac{1}{(1-t)(1-\ln(1-t))}.$$

We conclude this section with an intriguing conjecture. By inspection of (15) and (17) we obtain the following formula.

Lemma 6.9. For $n \geq 1$,

$$a_n := (-1)^n \frac{(\kappa_{n+1} - (n+1) \cdot \kappa_n)}{n+1} = (-1)^n (K_n - n \cdot K_{n-1})$$

is the coefficient of $t^n/n!$ in $1/(1-\ln(1+t))$.

The numbers a_0, a_1, \dots are listed as sequence A006252 in the On-Line Encyclopedia of Integer Sequences [11]. The first 11 entries are positive, then $a_{12} = -519312$ is negative, the subsequent entries seem to have alternating signs. The conjecture that this alternation continues indefinitely, may be rephrased as follows.

Conjecture 6.10. For $n \geq 12$ we have $n \cdot \kappa_{n-1} > \kappa_n$. Equivalently, $n \cdot K_{n-1} > K_n$ holds for $n \geq 11$.

We may call Conjecture 6.10 the *novice's chance*. Imagine that the first player asks a novice friend to replace him or her for just the first move in a flat Bernoulli game starting from a random position of rank $n \geq 12$. If Conjecture 6.10 is correct then novice could simply remove the last letter, because the number of non-kernel positions in which this is the first move of the first player's winning strategy still exceeds the number of all kernel positions. We should note that for the original Bernoulli game a novice has no such chance. In that game the removal of a single letter at the end of both words is not always a valid move, but we could advise our novice to remove the last letters at the end of both words if this is a valid move and make a random valid move otherwise. Our novice would have a chance if

$$\kappa_{n-1} \cdot \left(n^2 - \binom{n-1}{2} \right) = \kappa_{n-1} \cdot \binom{n+1}{2} \geq \kappa_n$$

was true for all large n . However, it is known [5, §93] that we have

$$\frac{n-2}{n} \left| \frac{b_{n-1}}{(n-1)!} \right| < \left| \frac{b_n}{n!} \right| < \frac{n-1}{n} \left| \frac{b_{n-1}}{(n-1)!} \right|, \text{ implying} \\ (n-2)(n+1)\kappa_{n-1} < \kappa_n < (n-1)(n+1)\kappa_{n-1}. \tag{20}$$

On the page of A006252 in [11] we find that the coefficient of $t^n/n!$ in $1/(1 - \ln(1+t))$ is

$$\frac{(-1)^n(\kappa_{n+1} - (n+1) \cdot \kappa_n)}{n+1} = (-1)^n(K_n - n \cdot K_{n-1}) = \sum_{k=0}^n s(n, k)k! \tag{21}$$

Equivalently,

$$\frac{(-1)^n(\kappa_{n+1} - (n+1) \cdot \kappa_n)}{n+1} = (-1)^n(K_n - n \cdot K_{n-1}) = \int_0^\infty x(x-1)\cdots(x-n+1)e^{-x} dx. \tag{22}$$

Eqs. (21) and (22) may be verified the same way as the analogous formulas (18) and (19). Therefore we may rewrite Conjecture 6.10 as follows:

$$(-1)^n \int_0^\infty x(x-1)\cdots(x-n+1)e^{-x} dx > 0 \text{ holds for } n \geq 11. \tag{23}$$

This form indicates well the complication that arises, compared to the original Bernoulli game. To prove (20), Jordan [5, §93] uses the formula

$$\frac{b_n}{n!} = \int_0^1 \binom{x}{n} dx$$

and is able to use the mean value theorem to compare $b_n/n!$ with $b_{n+1}/(n+1)!$, because the function $\binom{x}{n}$ does not change sign on the interval $(0, 1)$. Proving Eq. (23) is equivalent to a similar estimate of the change of the integral $(-1)^n \int_0^\infty (x-1)\cdots(x-n)e^{-x} dx$ as we increase n , however, this integrand does change the sign several times on the interval $(0, \infty)$.

7. Concluding remarks

Conjecture 6.10, if true, would be an intriguing example of a sequence “finding its correct signature pattern” after a relatively long “exceptional initial segment”. Many such examples seem to exist in analysis, and it is perhaps time for combinatorialists to start developing a method of proving some of them.

Some of the most interesting questions arising in connection with this paper seem to be related to the instant Bernoulli game, presented in Section 4. The fact that our decomposition into elementary kernel factors is bijectively equivalent to King’s [6] construction raises the suspicion that this decomposition may also have an algebraic importance beyond the combinatorial one. This suspicion is underscored by the fact that the correspondence between our decomposition and King’s is via some modified inversion table, whereas Aguiar and Sottile [1] highlight the importance of the weak order to the structure of the Malvenuto–Reutenauer Hopf algebra, first pointed out by Loday and Ronco [8]. The weak order is based on comparing the sets of inversions of two permutations. Depending the way we choose the basis of the self-dual Malvenuto–Reutenauer Hopf algebra, expressing one of the product and coproduct seems easy in terms of place-based non-inversion tables, whereas the other seems very difficult. If we choose the representation considered by Poirier and Reutenauer [12] where connected permutations form the free algebra basis, then the product of two permutations is easily expressed in terms of PNTs, thus the elementary kernel factor decomposition might indicate the presence of a larger algebra “looming on the horizon” in which the multiplicative indecomposables of the Malvenuto–Reutenauer Hopf algebra become decomposable.

We should also mention that the decomposition that is equivalent to the removal of the last elementary kernel factor is only the first phase in King's construction [6], a lot of hard work is done afterwards to find the transposition Gray code, while recursing on these reduction steps. Our presentation allows to better visualize King's entire "rough" decomposition "at once" and thus may be suitable to attack the open question of finding an adjacent transposition Gray code.

Finally, the degenerate Bernoulli game indexed with (p, q) [4, §6] can also be shown to be isomorphic to a strongly Bernoulli type truncation game. For this game, the number of kernel positions of rank n is $(-q)^n(n+1)!\beta_n(p/q, 0)$ [4, Theorem 6.2], where $\beta_n(p/q)$ is a degenerate Bernoulli number. We leave the detailed analysis of this game to a future occasion.

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